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CARTESIAN CLOSED 2-CATEGORIES AND PERMUTATION EQUIVALENCE IN HIGHER-ORDER REWRITING

TOM HIRSCHOWITZ

ABSTRACT. We propose a semantics for permutation equivalence in higher-order rewriting. This semantics takes place in cartesian closed 2-categories, and is proved sound and complete.

1. INTRODUCTION

It is known since the end of the 80's that 2-categories with finite products provide a semantics for term rewriting [3]. *Higher-order rewriting* [10, 17, 14, 15] is a framework for specifying rewrite systems on terms with variable binding. Many results from standard term rewriting have been generalised to higher-order rewriting, notably normalisation or confluence results. An important tool for confluence results is the notion of *permutation equivalence*, which was generalised to the higher-order case by Bruggink [1]. He defines a calculus of *proof terms* for specifying reductions in a higher-order rewrite system.

We here propose a categorical semantics for a variant of this calculus, in terms of *cartesian closed 2-categories*. We first define *cartesian closed 2-signatures*, which generalise higher-order rewrite systems, and organise them into a category Sig . We then construct an adjunction

$$(1.1) \quad \text{Sig} \begin{array}{c} \xrightarrow{\mathcal{H}} \\ \perp \\ \xleftarrow{\mathcal{W}} \end{array} 2\text{CCCat},$$

where 2CCCat is the category of small cartesian closed 2-categories. From a given higher-order rewrite system S , the functor \mathcal{H} constructs a cartesian closed 2-category, whose 2-cells are Bruggink's proof terms modulo permutation equivalence, which we prove is the free cartesian closed 2-category generated by S .

We review a number of examples and non-examples, and sketch an extension to deal with the latter.

Related work. Our cartesian closed 2-signatures are a 2-dimensional refinement of *cartesian closed sketches* [16, 4, 9]. Bruggink's calculus of permutation equivalence is close in spirit to Hilken's 2-categorical semantics of the simply-typed λ -calculus [7], but technically different and generalised to arbitrary higher-order rewrite systems. Capriotti [2] proposes a semantics of so-called *flat* permutation equivalence in sesquicategories. More related work is discussed in Section 4.2.

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2. CARTESIAN CLOSED SIGNATURES AND CATEGORIES

We start by recalling the well-known, or at least folklore, adjunction between what we here call *(cartesian closed) 1-signatures* and cartesian closed categories.

For any set X , define *types* over X by the grammar:

$$A, B, \dots \in \mathcal{L}_0(X) \quad ::= \quad x \mid 1 \mid A \times B \mid B^A,$$

with $x \in X$.

Proposition 1. \mathcal{L}_0 defines a monad on Set .

Let the set of *sequents* over a set X be $\mathcal{S}_0(X) = \mathcal{L}_0(X)^* \times \mathcal{L}_0(X)$, i.e., sequents are pairs of a list of types and a type. The assignment $X \mapsto \mathcal{S}_0(X)$ extends to an endofunctor on Set .

Definition 1. A 1-signature consists of a set X_0 of sorts, and an $\mathcal{S}_0(X_0)$ -indexed set X_1 of operations, or equivalently a map $X_1 \rightarrow \mathcal{S}_0(X_0)$.

A morphism of 1-signatures $(X_0, X_1) \rightarrow (Y_0, Y_1)$ is a pair (f_0, f_1) where $f_i: X_i \rightarrow Y_i$ such that

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & Y_1 \\ \downarrow & & \downarrow \\ \mathcal{S}_0(X_0) & \xrightarrow{\mathcal{S}_0(f_0)} & \mathcal{S}_0(Y_0) \end{array}$$

commutes. Morphisms compose in the obvious way, and we have:

Proposition 2. Composition of morphisms is associative and unital, and hence 1-signatures and their morphisms form a category Sig_1 .

There is a well-known adjunction

$$\begin{array}{ccc} & \mathcal{H}_1 & \\ & \curvearrowright & \\ \text{Sig}_1 & \xleftrightarrow{\perp} & \text{CCCat} \\ & \curvearrowleft & \\ & \mathcal{W}_1 & \end{array}$$

between 1-signatures and the category CCCat of small cartesian closed categories (with chosen structure) and (strict) cartesian closed functors.

The functor \mathcal{W}_1 sends a cartesian closed category \mathcal{C} to the signature with sorts \mathcal{C}_0 , its set of objects, and with operations $A_1, \dots, A_n \rightarrow A$ the set $\mathcal{C}(\llbracket A_1 \times \dots \times A_n \rrbracket, \llbracket A \rrbracket)$, where $\llbracket - \rrbracket$ denotes the function $\mathcal{L}_0(\mathcal{C}_0) \rightarrow \mathcal{C}_0$ defined by induction:

$$(2.1) \quad \begin{aligned} \llbracket c \rrbracket &= c & c \in \mathcal{C}_0 \\ \llbracket 1 \rrbracket &= 1 \\ \llbracket A \times B \rrbracket &= \llbracket A \rrbracket \times \llbracket B \rrbracket \\ \llbracket B^A \rrbracket &= \llbracket B \rrbracket^{\llbracket A \rrbracket}. \end{aligned}$$

Conversely, given a 1-signature X , consider the simply-typed λ -calculus with base types in X_0 and constants in X_1 . Terms modulo $\beta\eta$ form a category $\mathcal{H}_1(X)$ with objects all types over X_0 and morphisms $A \rightarrow B$ all terms of type B with one free variable of type A .

A less often formulated observation, which is useful to us, is that the adjunction $\mathcal{H}_1 \dashv \mathcal{W}_1$ decomposes into two adjunctions

$$\begin{array}{ccccc}
& & \mathcal{L}_1 & & \mathcal{F}_1 \\
& \swarrow & \xrightarrow{\quad} & \searrow & \xrightarrow{\quad} \\
\text{Sig}_1 & \xrightarrow{\quad} & \mathcal{L}_1\text{-Alg} & \xrightarrow{\quad} & \text{CCCat}, \\
& \nwarrow & \xleftarrow{\quad} & \nwarrow & \xleftarrow{\quad} \\
& & \mathcal{U}_1 & & \mathcal{V}_1
\end{array}$$

where $\mathcal{L}_1\text{-Alg}$ is the category of algebras for the monad \mathcal{L}_1 defined as follows (and \mathcal{L}_1 is shorthand for the functor $X \mapsto (\mathcal{L}_1(X), \mu)$).

For any 1-signature X , let $\mathcal{L}_1(X)$ denote the 1-signature with

- as sorts the set X_0 , and
- as operations $\Gamma \vdash A$ the λ -terms $\Gamma \vdash M : A$, modulo $\beta\eta$.

\mathcal{L}_1 extends to an endofunctor on Sig_1 , whose action on morphisms of 1-signatures $X \xrightarrow{f} Y$ substitutes constants $c \in X_1$ with $f_1(c)$. We obtain

Proposition 3. \mathcal{L}_1 is a monad on Sig_1 .

The functor \mathcal{V}_1 sends any cartesian closed category \mathcal{C} to the \mathcal{L}_1 -algebras $(\mathcal{C}_0, \mathcal{C}_1)$ defined as follows. First, \mathcal{C}_0 is the set of objects of \mathcal{C} . It has a canonical \mathcal{L}_0 -algebra structure, say $h_0 : \mathcal{L}_0(\mathcal{C}_0) \rightarrow \mathcal{C}_0$, obtained by interpreting type constructors in \mathcal{C} as in (2.1). Extending this to contexts G by $h_0(G) = \prod_i h_0(G_i)$, let the operations in $\mathcal{C}_1(G, A)$ be the 1-cells in $\mathcal{C}(h_0(G), h_0(A))$. Beware: the domain and codomain of such an operation are really G and A , not $h_0(G)$ and $h_0(A)$. Similarly, interpreting the λ -calculus in \mathcal{C} , the 1-signature $(\mathcal{C}_0, \mathcal{C}_1)$ has a canonical \mathcal{L}_1 -algebra structure, say $h_1 : \mathcal{L}_1(\mathcal{C}_0, \mathcal{C}_1) \rightarrow (\mathcal{C}_0, \mathcal{C}_1)$:

$$\begin{aligned}
h_1(G \vdash x_i : G_i) &= \pi_i \\
h_1(G \vdash () : 1) &= ! \\
h_1(G \vdash c(M_1, \dots, M_n)) &= c \circ (h_1(M_1), \dots, h_1(M_n)) \\
h_1(G \vdash \lambda x : A. M : B^A) &= \varphi(h_1(G, x : A \vdash M : B)) \\
h_1(G \vdash MN : B) &= ev \circ (h_1(M), h_1(N)) \\
h_1(G \vdash (M, N) : A \times B) &= (h_1(M), h_1(N)) \\
h_1(G \vdash \pi M : A) &= \pi \circ M \\
h_1(G \vdash \pi' M : A) &= \pi' \circ M,
\end{aligned}$$

where $!$ is the unique morphism $h_0(G) \rightarrow 1$, φ is the bijection $\mathcal{C}(h_0(G, A), h_0(B)) \cong \mathcal{C}(h_0(G), h_0(B^A))$, and ev is the structure morphism $h_0(B^A \times A) \rightarrow h_0(B)$.

\mathcal{L}_1 -algebras are much like cartesian closed categories whose objects are freely generated by their set of sorts. A perhaps useful analogy here is with multicategories \mathcal{M} , seen as being close to monoidal categories whose objects are freely generated by those of \mathcal{M} by tensor and unit. Here, the functor \mathcal{F}_1 sends any \mathcal{L}_1 -algebra (X, h) to the cartesian closed category with

- objects the types over X_0 , i.e., $\mathcal{L}_0(X_0)$,
- morphisms $A \rightarrow B$ the set of operations in $X_1(A, B)$.

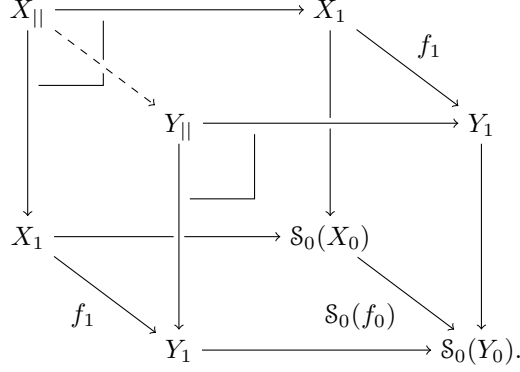
This canonically forms a cartesian closed category, with structure induced by the \mathcal{L}_1 -algebra structure. We define it in more detail in dimension 2 in Section 7.2.

3. CARTESIAN CLOSED 2-SIGNATURES

Given a 1-signature X , let $X_{||}$ denote the set of pairs of parallel operations, i.e., pairs of operations M, N above the same sequent. Otherwise said, $X_{||}$ is the pullback

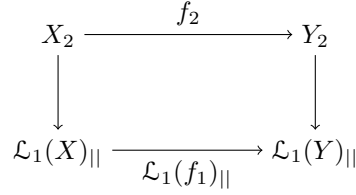
$$\begin{array}{ccc}
X_{||} & \xrightarrow{\quad} & X_1 \\
\downarrow & \lrcorner & \downarrow \\
X_1 & \xrightarrow{\quad} & \mathfrak{S}_0(X_0).
\end{array}$$

Any morphism $f: X \rightarrow Y$ of 1-signatures yields a function $f_{||}: X_{||} \rightarrow Y_{||}$, via the dashed arrow (obtained by universal property of pullback) in



Definition 2. A 2-signature consists of a 1-signature X , plus a set X_2 of reduction rules with a function $X_2 \rightarrow \mathcal{L}_1(X)_{||}$.

A morphism of 2-signatures $(X, X_2) \rightarrow (Y, Y_2)$ is a pair (f, f_2) where $f: X \rightarrow Y$ is a morphism of 1-signatures and $f_2: X_2 \rightarrow Y_2$ makes the diagram



commute. We obtain:

Proposition 4. Composition of morphisms is associative and unital, and hence 2-signatures and their morphisms form a category Sig .

4. EXAMPLES

4.1. Higher-order rewrite systems. The prime example of a 2-signature is that for the pure λ -calculus: it has a sort t and operations

$$a: t \times t \rightarrow t \qquad \ell: t^t \rightarrow t,$$

with a reduction rule β above the pair

$$x: t^t, y: t \vdash a(\ell(x), y), x(y): t$$

in $\mathcal{L}_1(\{t\}, \{\ell, a\})_{||}$. Categorically, this will yield a 2-cell

$$\begin{array}{ccc} & t \times t & \\ \ell \times t \nearrow & \Downarrow \beta & \searrow a \\ t^t \times t & & t. \\ & \text{ev} & \end{array}$$

This is an example of a *higher-order rewrite system* in the sense of Nipkow [14]. Nipkow's definition is formally different, but his higher-order rewrite systems are in bijection with 2-signatures $h: X_2 \rightarrow \mathcal{L}_1(X)_{||}$ such that for all rules $r \in X_2$, letting $(\Gamma \vdash M, N: A) = h(r)$:

- M is not a variable,
- A is a sort,
- each variable occurring in Γ occurs free in M .

These restrictions help formulating and proving decidability problems on higher-order rewrite systems, whose extension to our setting we leave open.

Let us now anticipate over our main results below and state our soundness and completeness theorem. Given a higher-order rewrite system X , i.e., a 2-signature satisfying the above conditions, let $\mathcal{R}(X)$ be the following locally-preordered 2-category. It has:

- objects are types in $\mathcal{L}_0(X_0)$;
- morphisms $A \rightarrow B$ are λ -terms in $\mathcal{L}_1(X)(A \vdash B)$, modulo $\beta\eta$;
- given two parallel morphisms M and N , there is one 2-cell $M \rightarrow N$ exactly when there is a sequence of reductions $M \rightarrow^* N$ in the usual sense [14].

Proposition 5. $\mathcal{R}(X)$ is 2-cartesian closed.

$\mathcal{R}(X)$ and $\mathcal{H}(X)$ have the same objects and morphisms. But because our inference rules for forming reductions are the same as deduction rules for proving the existence of a reduction in the usual sense, we may send any reduction $P: M \rightarrow N$ to the unique reduction $M \rightarrow N$ in $\mathcal{R}(X)$.

Theorem 1 (Soundness and completeness). *This defines an identity-on-objects, identity-on-morphisms, locally full cartesian closed 2-functor $\mathcal{R}(X) \xrightarrow{1} \mathcal{H}(X)$.*

4.2. Theories with binding. Understanding reduction rules as equations, it is easy to define the free cartesian closed category generated by a 2-signature. This yields an adjunction

$$(4.1) \quad \text{Sig} \begin{array}{c} \xrightarrow{\mathcal{H}'} \\ \perp \\ \xleftarrow{\mathcal{W}'} \end{array} \text{CCCat.}$$

This adjunction provides a categorical semantics for theories with binding, which is more general than other approaches by Fiore and Hur [6], Hirschowitz and Maggesi [8], and Zsidó [18], and which is in line with Lambek's seminal paper [11].

If I understand correctly, the motivation for Fiore and Hur's subtle approach is the will to explain the λ -calculus by strictly less than itself. The present framework does not obey this specification, and instead tends to view the λ -calculus as a universal (parameterised) theory with binding.

We end this section by giving a formal construction of the adjunction (4.1). Cartesian closed categories form a full, reflective subcategory of 2CCCat , via the functor $\mathcal{J}: 2\text{CCCat} \rightarrow \text{CCCat}$ sending a cartesian closed 2-category \mathcal{C} to the cartesian closed category with:

- objects those of \mathcal{C} ,
- morphisms those of \mathcal{C} , modulo the congruence generated by $f \sim g$ iff there exists a 2-cell $f \rightarrow g$.

Here, $\mathcal{J}(\mathcal{C})$ is thought of as the free cartesian closed 2-category with trivial 2-cells (i.e., 0 or 1). The desired adjunction is obtained by composing the adjunctions

$$\text{Sig} \begin{array}{c} \xrightarrow{\mathcal{H}} \\ \perp \\ \xleftarrow{\mathcal{W}} \end{array} 2\text{CCCat} \begin{array}{c} \xrightarrow{\mathcal{J}} \\ \perp \\ \xleftarrow{\quad} \end{array} \text{CCCat.}$$

4.3. Non-examples. Non-examples are given by calculi whose reduction semantics is defined on terms modulo a so-called *structural congruence*, e.g., CCS [12], or the π -calculus [5, 13].

For example, consider the CCS term $(a \mid 0) \mid \bar{a}$. In CCS, it is *structurally* equivalent to $(a \mid \bar{a}) \mid 0$, which then reduces to $0 \mid 0$.

In order to account for this, we would have to consider a 2-signature with reduction rules for structural congruence, here $(M_1 \mid M_2) \mid M_3 \rightarrow M_1 \mid (M_2 \mid M_3)$ for associativity, and $M \mid N \rightarrow N \mid M$ for commutativity. But then, these reductions count as proper reductions, which departs from the desired computational behaviour. For example, the term $a \mid a$ has an infinite reduction sequence, using commutativity.

Anticipating the development in the next sections, a potential solution is to extend 2-signatures to *2-theories*. For any 2-signature X , let X_{\parallel} denote the set of pairs of reduction rules r, s with a common type $G \vdash M \rightarrow N : A$. A 2-theory is a 2-signature X , together with a set of equations between parallel reductions, i.e., a subset X_3 of $\mathcal{L}(X)_{\parallel}$.

The main adjunction announced above (1.1) extends to an adjunction between 2-theories and cartesian closed 2-categories. Using equations, we may specify that any reduction $M \rightarrow M$ using only structural rules be the identity on M , and consider the computational behaviour of a 2-category to consist of its non-invertible morphisms, as proposed by Hilken [7]. A question is whether for a given calculus this can be done with finitely many equations.

5. A 2-LAMBDA-CALCULUS

We now begin the construction of Adjunction (1.1). We start in this section by defining a monad \mathcal{L} on Sig , which we will use to factor Adjunction (1.1) as

$$\text{Sig} \begin{array}{c} \xrightarrow{\mathcal{L}} \\ \perp \\ \xleftarrow{\mathcal{U}} \end{array} \mathcal{L}\text{-Alg} \begin{array}{c} \xrightarrow{\mathcal{F}} \\ \perp \\ \xleftarrow{\mathcal{V}} \end{array} 2\text{CCCat},$$

where:

- $\mathcal{L}\text{-Alg}$ is the category of \mathcal{L} -algebras,
- $\mathcal{L} : \text{Sig} \rightarrow \mathcal{L}\text{-Alg}$ is a shortcut for $X \mapsto (\mathcal{L}^2 X \xrightarrow{\mu} \mathcal{L} X)$,
- $\mathcal{U}(\mathcal{L} X \xrightarrow{h} X) = X$,
- 2CCCat is the category of cartesian closed 2-categories, which we define in Section 6.

The left-hand adjunction holds by \mathcal{L} being a monad, thus we concentrate in Section 7 on establishing the right-hand one.

But for now, let us define the monad \mathcal{L} .

5.1. Syntax. Given a 2-signature $X = ((X_0, X_1), h : X_2 \rightarrow \mathcal{L}_1(X)_{\parallel})$ (actually $\mathcal{L}_1(X)$ is $\mathcal{L}_1(X_0, X_1)$), we construct a new 2-signature $\mathcal{L}(X)$, whose reduction rules represent reduction sequences in the “higher-order rewrite system” defined by X , modulo permutation equivalence. The 2-signature $\mathcal{L}(X)$ has the same base 1-signature (X_0, X_1) , and as reduction rules the terms of a 2λ -calculus (in the sense of Hilken [7]) modulo permutation equivalence, which we now define.

First, terms, called *reductions*, are defined by induction in Figure 1. The typing judgement has the shape $\Gamma \vdash P : M \rightarrow N : A$, where A is a type in $\mathcal{L}_0(X_0)$, Γ is a list of pairs of a variable and a type, with no variable appearing more than once, M and N are terms of type $\Gamma \vdash A$ modulo $\beta\eta$, and P is a reduction. In the sequel, we often forget the variables in such pairs $(\Gamma \vdash A)$, and identify them with sequents in $\mathcal{S}_0(X_0)$.

When clear from context, we abbreviate substitutions $[M_1/x_1, \dots, M_n/x_n]$ by $[M_1, \dots, M_n]$. For a context G , G_i denotes its i th type. Also, for $(M, N) \in \mathcal{L}_1(X)_{\parallel}$, we let $X(M, N)$ be the set of all reduction rules $r \in X_2$ such that $h(r) = (M, N)$. We write $X(\Gamma \vdash M, N : A)$ to indicate the common type of M and N . Similarly, $X(G \vdash A)$ denotes the set of operations in X_1 above $G \vdash A$.

$$\begin{array}{c}
\dfrac{\dots \quad \Gamma \vdash P_i : M_i \rightarrow N_i : G_i \quad \dots}{\Gamma \vdash r(P_1, \dots, P_n) : M[M_1, \dots, M_n] \rightarrow N[N_1, \dots, N_n] : A} \quad (r \in X(G \vdash M, N : A)) \\
\\
\dfrac{\Gamma \vdash P : M_1 \rightarrow M_2 : A \quad \Gamma \vdash Q : M_2 \rightarrow M_3 : A}{\Gamma \vdash P ;_{M_2} Q : M_1 \rightarrow M_3 : A} \quad \Gamma, x : A, \Delta \vdash x : x \rightarrow x : A \\
\\
\Gamma \vdash () : () \rightarrow () : 1 \\
\\
\dfrac{\Gamma \vdash P_1 : M_1 \rightarrow N_1 : G_1 \quad \dots \quad \Gamma \vdash P_n : M_n \rightarrow N_n : G_n}{\Gamma \vdash c(P_1, \dots, P_n) : c(M_1, \dots, M_n) \rightarrow c(N_1, \dots, N_n) : A} \quad (c \in X_1(G \vdash A)) \\
\\
\dfrac{\Gamma, x : A \vdash P : M \rightarrow N : B}{\Gamma \vdash \lambda x : A. P : \lambda x : A. M \rightarrow \lambda x : A. N : B^A} \\
\\
\dfrac{\Gamma \vdash P : M \rightarrow M' : B^A \quad \Gamma \vdash Q : N \rightarrow N' : A}{\Gamma \vdash PQ : MN \rightarrow M'N' : B} \\
\\
\dfrac{\Gamma \vdash P : M \rightarrow M' : A \quad \Gamma \vdash Q : N \rightarrow N' : B}{\Gamma \vdash (P, Q) : (M, N) \rightarrow (M', N') : A \times B} \\
\\
\dfrac{\Gamma \vdash P : M \rightarrow N : A \times B}{\Gamma \vdash \pi_{A,B} P : \pi_{A,B} M \rightarrow \pi_{A,B} N : A} \quad \dfrac{\Gamma \vdash P : M \rightarrow N : A \times B}{\Gamma \vdash \pi'_{A,B} P : \pi'_{A,B} M \rightarrow \pi'_{A,B} N : B}
\end{array}$$

FIGURE 1. Reductions

5.2. **Substitution.** Next, we define substitution, which has “type”

$$(5.1) \quad \dfrac{\Gamma \vdash Q : N \rightarrow N' : \Delta \quad \Delta \vdash P : M \rightarrow M' : A}{\Gamma \vdash P[Q] : M[N] \rightarrow M'[N'] : A},$$

i.e., given a reduction P and a tuple of reductions Q , it produces a reduction of the indicated type, which we denote $P[Q]$. Here, we denote by $\Gamma \vdash Q : N \rightarrow N' : \Delta$ a tuple of reductions $\Gamma \vdash Q_i : N_i \rightarrow N'_i : \Delta_i$, for $1 \leq i \leq |\Delta|$.

The definition is a bit tricky:

- first we define *left whiskering*, which has “type”

$$\dfrac{\Gamma \vdash Q : N \rightarrow N' : \Delta \quad \Delta \vdash M : A}{\Gamma \vdash M[Q] : M[N] \rightarrow M'[N'] : A},$$

- then we define *right whiskering*, which has “type”

$$\dfrac{\Gamma \vdash N : \Delta \quad \Delta \vdash P : M \rightarrow M' : A}{\Gamma \vdash P[N] : M[N] \rightarrow M'[N] : A},$$

(where N denotes a tuple),

- then we define substitution by

$$P[Q] = (P[N] ;_{M'[N]} M'[Q]).$$

There is of course another legitimate definition, namely

$$P[Q] = (M[Q] ;_{M[N']} P[N']).$$

The two will be equated by permutation equivalence in the next section.

Left whiskering is defined by induction, with $\Delta = (x_1 : A_1, \dots, x_n : A_n)$ and $Q = (Q_1, \dots, Q_n)$, by:

$$\begin{aligned}
() [Q] &= () \\
x_i [Q] &= Q_i \\
c(M_1, \dots, M_p) [Q] &= c(M_1 [Q], \dots, M_p [Q]) \\
(\lambda x : B. M) [Q] &= \lambda x : B. (M [Q, x]) \quad (\text{for } x \notin \text{dom}(\Delta)) \\
(MN) [Q] &= (M [Q] N [Q]) \\
(M, N) [Q] &= (M [Q], N [Q]) \\
(\pi_{A,B} M) [Q] &= \pi_{A,B} (M [Q]) \\
(\pi'_{A,B} M) [Q] &= \pi'_{A,B} (M [Q]).
\end{aligned}$$

Right whiskering is defined by induction, with $\Delta = (x_1 : A_1, \dots, x_n : A_n)$ and $N = (N_1, \dots, N_n)$, by:

$$\begin{aligned}
(r(P_1, \dots, P_p)) [N] &= r(P_1 [N], \dots, P_p [N]) \\
(P_1 ;_{M''} P_2) [N] &= (P_1 [N] ;_{M'' [N]} P_2 [N]) \\
() [N] &= () \\
x_i [N] &= N_i \\
c(P_1, \dots, P_p) [N] &= c(P_1 [N], \dots, P_p [N]) \\
(\lambda x : B. P') [N] &= \lambda x : B. (P' [N, x]) \quad (\text{for } x \notin \text{dom}(\Delta)) \\
(P_1 P_2) [N] &= (P_1 [N] P_2 [N]) \\
(P_1, P_2) [N] &= (P_1 [N], P_2 [N]) \\
(\pi_{A,B} P') [N] &= \pi_{A,B} (P' [N]) \\
(\pi'_{A,B} P') [N] &= \pi'_{A,B} (P' [N]).
\end{aligned}$$

Definition 3. Let $P [Q] = (P [N] ;_{M' [N]} M' [Q])$.

Proposition 6. Given reductions P and Q as above, the capture-avoiding substitution $P [Q]$ is a well-typed reduction $\Gamma \vdash P [Q] : M [N] \rightarrow M' [N'] : A$.

Similarly, there is a weakening operation with “type”

$$\frac{\Gamma \vdash P : M \rightarrow N : A}{\Gamma, x : B \vdash P : M \rightarrow N : A.} \quad (x \notin \Gamma)$$

5.3. Permutation equivalence. We now define *permutation equivalence* on reductions, by the equations in Figures 3 and 4, in Appendix A. The *congruence* rules in Figure 3 are bureaucratic: they just say that permutation equivalence is a congruence. The *category* rules make reductions of a given type $\Gamma \vdash A$ into a category. In Figure 4, the *beta* and *eta* rules mirror the term-level beta and eta rules. Finally, the *lifting* rules lift composition of reductions towards toplevel.

So, $\mathcal{L}(X)$ has sorts X_0 , operations X_1 , and as reduction rules in $\mathcal{L}(X)(G \vdash M, N : A)$ all reductions $G \vdash P : M \rightarrow N : A$, modulo the equations.

This easily extends to:

Proposition 7. \mathcal{L} is a functor $\text{Sig} \rightarrow \text{Sig}$.

Now, consider $\mathcal{L}\mathcal{L}(X)$. We define a mapping $\mu_X : \mathcal{L}\mathcal{L}(X) \rightarrow \mathcal{L}(X)$, by induction on reductions. The typing rule for reduction rules specialises to:

$$\frac{(R \in \mathcal{L}(X)(G \vdash M, N : A)) \quad \Gamma \vdash P_1 : M_1 \rightarrow N_1 : G_1 \quad \dots \quad \Gamma \vdash P_n : M_n \rightarrow N_n : G_n}{\Gamma \vdash R(P_1, \dots, P_n) : M[M_1, \dots, M_n] \rightarrow N[N_1, \dots, N_n] : A}$$

We set $\mu(R(P_1, \dots, P_n)) = R[\mu(P_1), \dots, \mu(P_n)]$. The other cases just propagate the substitution:

$$\begin{aligned}
P ; Q &\mapsto \mu(P) ; \mu(Q) \\
x &\mapsto x \\
() &\mapsto () \\
c(P_1, \dots, P_n) &\mapsto c(\mu(P_1), \dots, \mu(P_n)) \\
\lambda x : A. P &\mapsto \lambda x : A. \mu(P) \\
PQ &\mapsto \mu(P)\mu(Q) \\
(P, Q) &\mapsto (\mu(P), \mu(Q)) \\
\pi P &\mapsto \pi(\mu(P)) \\
\pi' P &\mapsto \pi'(\mu(P)).
\end{aligned}$$

Lemma 1. *This defines a natural transformation $\mu : \mathcal{L}^2 \rightarrow \mathcal{L}$, which makes the diagram*

$$\begin{array}{ccc}
\mathcal{L}^3 & \xrightarrow{\mathcal{L}\mu} & \mathcal{L}^2 \\
\mu\mathcal{L} \downarrow & & \downarrow \mu \\
\mathcal{L}^2 & \xrightarrow{\mu} & \mathcal{L}
\end{array}$$

commute.

Similarly, there is a natural transformation $\eta : id \rightarrow \mathcal{L}$, sending each $r \in X(G \vdash M, N : A)$ to the reduction $G \vdash r(x_1, \dots, x_n) : M \rightarrow N : A$, and we have:

Lemma 2. *The diagram*

$$\begin{array}{ccccc}
& & \eta\mathcal{L} & & \\
& & \rightarrow & & \\
\mathcal{L} & & \mathcal{L}^2 & \xleftarrow{\mathcal{L}\eta} & \mathcal{L} \\
& \searrow & \downarrow \mu & \swarrow & \\
& & \mathcal{L} & &
\end{array}$$

commutes.

Corollary 1. *(\mathcal{L}, μ, η) is a monad on Sig .*

A crucial result is:

Proposition 8. *For all $\Gamma \vdash Q : N \rightarrow N' : \Delta$ and $\Delta \vdash P : M \rightarrow M' : A$, we have:*

$$\Gamma \vdash P[Q] \equiv (M[Q] ;_{M[N']} P[N']) : M[N] \rightarrow M'[N'] : A.$$

Proof. We proceed by induction on P . Most cases are bureaucratic. Consider for instance $P = c(P_1, \dots, P_p)$. Then, by definition:

$$P[Q] = (c(P_1[N], \dots, P_p[N]) ;_{c(M'_1[N], \dots, M'_p[N])} c(M'_1[Q], \dots, M'_p[Q])).$$

By the third lifting rule, this is \equiv -related to

$$c(P_1[N] ;_{M'_1[N]} M'_1[Q], \dots, P_p[N] ;_{M'_p[N]} M'_p[Q]).$$

By p applications of the induction hypothesis, we obtain

$$c(M_1[Q] ;_{M_1[N']} P_1[N'], \dots, M_p[Q] ;_{M_p[N']} P_p[N']),$$

which by lifting again yields the desired result:

$$c(M_1[Q], \dots, M_p[Q]) ;_{c(M_1[N'], \dots, M_p[N'])} c(P_1[N'], \dots, P_p[N']).$$

The case where something actually happens is $P = r(P_1, \dots, P_p)$, with $r \in X(G \vdash M_0, M'_0 : A)$ and each $\Delta \vdash P_i : M_i \rightarrow M'_i : G_i$. Then, the left-hand side is

$$r(P_1[N], \dots, P_p[N]) ;_{M_0[M_1, \dots, M_p][N]} M'_0[M'_1, \dots, M'_p][Q].$$

By lifting, omitting indices of vertical compositions, we have

$$r(P_1[N], \dots, P_p[N]) \equiv r(M_1[N], \dots, M_p[N]); M'_0[P_1[N], \dots, P_p[N]].$$

Observing that $M'_0[M'_1, \dots, M'_p][Q] = M'_0[M'_1[Q], \dots, M'_p[Q]]$, the whole is \equiv -related to

$$\begin{aligned} & r(M_1[N], \dots, M_p[N]); \\ & M'_0[P_1[N], \dots, P_p[N]]; \\ & M'_0[M'_1[Q], \dots, M'_p[Q]], \end{aligned}$$

i.e., by lifting (inductively):

$$\begin{aligned} & r(M_1[N], \dots, M_p[N]); \\ & M'_0[(P_1[N]; M'_1[Q]), \dots, (P_p[N]; M'_1[Q])]. \end{aligned}$$

This is by induction hypothesis \equiv -related to

$$\begin{aligned} & r(M_1[N], \dots, M_p[N]); \\ & M'_0[(M_1[Q]; P_1[N']), \dots, (M_1[Q]; P_p[N'])], \end{aligned}$$

i.e., by lifting again to

$$\begin{aligned} & r(M_1[N], \dots, M_p[N]); \\ & M'_0[M_1[Q], \dots, M_1[Q]]; \\ & M'_0[P_1[N'], \dots, P_p[N']]. \end{aligned}$$

The second lifting rule then yields

$$\begin{aligned} & r(M_1[Q], \dots, M_p[Q]); \\ & M'_0[P_1[N'], \dots, P_p[N']]. \end{aligned}$$

And then the first lifting rule yields

$$\begin{aligned} & M_0[M_1[Q], \dots, M_1[Q]]; \\ & r(M_1[N'], \dots, M_p[N']); \\ & M'_0[P_1[N'], \dots, P_p[N']], \end{aligned}$$

so, by the second lifting rule again:

$$\begin{aligned} & M_0[M_1[Q], \dots, M_1[Q]]; \\ & r(P_1[N'], \dots, P_p[N']), \end{aligned}$$

i.e., the right-hand side. □

6. CARTESIAN CLOSED 2-CATEGORIES

6.1. Definition. In a 2-category \mathcal{C} , a diagram $A \xleftarrow{p} C \xrightarrow{q} B$ is a *product diagram* iff for all object D , the induced functor

$$\mathcal{C}(D, C) \xrightarrow{\Delta} \mathcal{C}(D, C) \times \mathcal{C}(D, C) \xrightarrow{\mathcal{C}(D, p) \times \mathcal{C}(D, q)} \mathcal{C}(D, A) \times \mathcal{C}(D, B)$$

is an isomorphism. Because this family of functors is 2-natural in D , the inverse functors will also be 2-natural.

Similarly, an object 1 of \mathcal{C} is *terminal* iff for all D the unique functor

$$\mathcal{C}(D, 1) \xrightarrow{!} 1$$

is an isomorphism (where the right-hand 1 is the terminal category).

Definition 4. A 2-category with finite products, or *fp 2-category*, is a 2-category \mathcal{C} , equipped with a terminal object and a 2-functor

$$\mathcal{C} \times \mathcal{C} \xrightarrow{\times} \mathcal{C},$$

plus, for all A and B , a product diagram

$$A \xleftarrow{p} A \times B \xrightarrow{q} B.$$

In such an fp 2-category \mathcal{C} , given objects A and B , an *exponential* for them is a pair of an object B^A and a morphism $ev: A \times B^A \rightarrow B$, such that for all D , the functor

$$\begin{array}{ccc}
 & \mathcal{C}(A, A) \times \mathcal{C}(D, B^A) & \xrightarrow{\times} & \mathcal{C}(A \times D, A \times B^A) \\
 (id_{A^!}, id) \nearrow & & & \searrow \mathcal{C}(A \times D, ev) \\
 \mathcal{C}(D, B^A) & & & \mathcal{C}(A \times D, B)
 \end{array}$$

is an isomorphism. As above, because this family of functors is 2-natural in D , the inverse functors will also be 2-natural.

Definition 5. A cartesian closed 2-category, or *cartesian closed 2-category*, is an fp 2-category, equipped with a choice of exponentials for all pairs of objects. The category 2CCCat has cartesian closed 2-categories as objects, and strictly structure-preserving functors between them as morphisms.

7. MAIN ADJUNCTION

7.1. Right adjoint. Given a cartesian closed 2-category \mathcal{C} , define $\mathcal{V}(\mathcal{C}) = (\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2)$ as follows. First, let as in Section 2 $(\mathcal{C}_0, \mathcal{C}_1) = \mathcal{V}_1(\mathcal{C})$, and recall the canonical \mathcal{L}_0 and \mathcal{L}_1 -algebra structures h_0 and h_1 . Let then the reduction rules in $\mathcal{C}_2(G \vdash M, N : A)$ be the 2-cells in $\mathcal{C}(h_0(G), h_0(A))(h_1(M), h_1(N))$, abbreviated to $\mathcal{C}(G, A)(M, N)$ in the sequel.

This signature $\mathcal{V}\mathcal{C}$ has a canonical \mathcal{L} -algebra structure $h_2: \mathcal{L}(\mathcal{V}\mathcal{C}) \rightarrow \mathcal{V}\mathcal{C}$, which we define by induction over terms in Figure 2. In the case for λ , φ denotes the structure isomorphism $\mathcal{C}((\prod \Gamma) \times A, B) \cong \mathcal{C}(\prod \Gamma, B^A)$.

In order for the definition to make sense as a morphism $\mathcal{L}(\mathcal{V}\mathcal{C}) \rightarrow \mathcal{V}\mathcal{C}$, we have to check its compatibility with the equations. We have first:

Lemma 3. For all $\Delta \vdash Q : N \rightarrow N' : \Gamma$ and $\Gamma \vdash P : M \rightarrow M' : A$ in $\mathcal{L}(\mathcal{V}\mathcal{C})$,

$$\begin{array}{ccc}
 \Delta \begin{array}{c} \xrightarrow{M[N]} \\ \Downarrow h_2(P[Q]) \\ \xrightarrow{M'[N']} \end{array} A & = & \Delta \begin{array}{c} \xrightarrow{N} \\ \Downarrow h_2(Q) \\ \xrightarrow{N'} \end{array} \Gamma \begin{array}{c} \xrightarrow{M} \\ \Downarrow h_2(P) \\ \xrightarrow{M'} \end{array} A.
 \end{array}$$

Proof. By induction on P and the axioms for cartesian closed 2-categories. \square

Lemma 4. Any two equated reductions are mapped to the same 2-cell in \mathcal{C} .

Proof. We proceed by induction on the proof of the considered equation. The rules of Figure 3 hold because, in \mathcal{C} , vertical composition is associative and unital, and equality is a congruence. The beta rule is less easy, so we spell it out.

The left-hand reduction is interpreted in \mathcal{C} as

$$\begin{array}{ccc}
 & (\varphi M, N) & \\
 & \curvearrowright & \\
 \prod \Gamma & \Downarrow (\varphi P, Q) & B^A \times A \xrightarrow{ev} B \\
 & \curvearrowleft & \\
 & (\varphi M', N') &
 \end{array}$$

which is equal to

$$\begin{aligned}
& (G \vdash x_i : x_i \rightarrow x_i : G_i) \mapsto (id_{\pi_i} : \pi_i \rightarrow \pi_i : \prod G \rightarrow G_i) \\
& (G \vdash () : () \rightarrow () : 1) \mapsto (id_! : ! \rightarrow ! : \prod G \rightarrow 1) \\
& (\Gamma \vdash c(P_1, \dots, P_n) : c(M_1, \dots, M_n) \rightarrow c(N_1, \dots, N_n) : A) \mapsto \\
& \quad (M_1, \dots, M_n) \\
& \quad \prod \Gamma \begin{array}{c} \curvearrowright \\ \Downarrow P \\ \curvearrowleft \end{array} \prod G \xrightarrow{c} A \quad (c \in \mathcal{C}_1(G, A), P = (P_1, \dots, P_n)) \\
& \quad (N_1, \dots, N_n) \\
& (\Gamma \vdash r(P_1, \dots, P_n) : M[M_1, \dots, M_n] \rightarrow N[N_1, \dots, N_n] : A) \mapsto \\
& \quad (M_1, \dots, M_n) \\
& \quad \prod \Gamma \begin{array}{c} \curvearrowright \\ \Downarrow P \\ \curvearrowleft \end{array} \prod G \begin{array}{c} \xrightarrow{M} \\ \Downarrow r \\ \xrightarrow{N} \end{array} A \quad (P = (P_1, \dots, P_n)) \\
& \quad (N_1, \dots, N_n) \\
& (G \vdash P ;_{M_2} Q : M_1 \rightarrow M_3 : A) \mapsto \prod G \begin{array}{c} \xrightarrow{M_1} \\ \Downarrow P \\ \Downarrow Q \\ \xrightarrow{M_3} \end{array} A \\
& (\Gamma \vdash \lambda x : A. P : \lambda x : A. M \rightarrow \lambda x : A. N : B^A) \mapsto \varphi(P : M \rightarrow N : (\prod \Gamma) \times A \rightarrow B) \\
& \quad (M, N) \\
& (\Gamma \vdash PQ : MN \rightarrow M'N' : B) \mapsto \prod \Gamma \begin{array}{c} \curvearrowright \\ \Downarrow (P,Q) \\ \curvearrowleft \end{array} B^A \times A \xrightarrow{ev} B \\
& \quad (M', N') \\
& (\Gamma \vdash (P, Q) : (M, N) \rightarrow (M', N') : A \times B) \mapsto \prod \Gamma \begin{array}{c} \curvearrowright \\ \Downarrow (P,Q) \\ \curvearrowleft \end{array} A \times B \\
& \quad (M', N') \\
& (\Gamma \vdash \pi_{A,B} P : \pi_{A,B} M \rightarrow \pi_{A,B} N : A) \mapsto \prod \Gamma \begin{array}{c} \xrightarrow{M} \\ \Downarrow P \\ \xrightarrow{N} \end{array} A \times B \xrightarrow{\pi} A \\
& (\Gamma \vdash \pi'_{A,B} P : \pi'_{A,B} M \rightarrow \pi'_{A,B} N : B) \mapsto \prod \Gamma \begin{array}{c} \xrightarrow{M} \\ \Downarrow P \\ \xrightarrow{N} \end{array} A \times B \xrightarrow{\pi'} B
\end{aligned}$$

FIGURE 2. The \mathcal{L} -algebra structure on $\mathcal{V}(\mathcal{C})$

$$\begin{array}{ccccc}
& & (id, N) & & \varphi M \times A \\
& \curvearrowright & \rightarrow & \curvearrowright & \\
\Pi \Gamma & & \Downarrow (id, Q) & & \Downarrow \varphi P \times A \\
& \curvearrowleft & \Pi \Gamma \times A & \curvearrowleft & B^A \times A \\
& & (id, N') & & \varphi M' \times A
\end{array}
\longrightarrow B \xrightarrow{ev} B$$

which is in turn equal (by cartesian closedness of \mathcal{C}) to:

$$\begin{array}{ccccc}
& & (id, N) & & M \\
& \curvearrowright & \rightarrow & \curvearrowright & \\
\Pi \Gamma & & \Downarrow (id, Q) & & \Downarrow P \\
& \curvearrowleft & \Pi \Gamma \times A & \curvearrowleft & B \\
& & (id, N') & & M'
\end{array}$$

and hence to the right-hand side of the equation by Lemma 3. The other beta and eta rules similarly hold by the properties of products, internal homs, and terminal object in \mathcal{C} .

The lifting rules hold by (particular cases of) the interchange law in \mathcal{C} and functoriality of the structural isomorphisms

$$\mathcal{C}(A \times B, C) \cong \mathcal{C}(B, C^A) \quad \text{and} \quad \mathcal{C}(C, A \times B) \cong \mathcal{C}(C, A) \times \mathcal{C}(C, B),$$

which concludes the proof. \square

This assignment extends to cartesian closed functors and we have:

Proposition 9. \mathcal{V} is a functor $2\text{CCCat} \rightarrow \text{Sig}$.

7.2. Left adjoint. Given an \mathcal{L} -algebra $h: \mathcal{L}(X) \rightarrow X$, we now construct a cartesian closed 2-category $\mathcal{F}(X, h)$. It has:

- objects the types in $\mathcal{L}_0(X_0)$;
- 1-cells $A \rightarrow B$ the terms in $\mathcal{L}_1(X_0, X_1)(A, B)$;
- 2-cells $M \rightarrow N: A \rightarrow B$ the reduction rules in $X_2(M, N)$.

We then must define the cartesian closed 2-category structure, and we start with the 2-category structure. Composition of 1-cells $A \xrightarrow{M} B \xrightarrow{N} C$ is defined to be $A \xrightarrow{N[M]} C$. Vertical composition of 2-cells

$$\begin{array}{ccc}
& M_1 & \\
& \curvearrowright & \\
A & \xrightarrow{M_2} & B \\
& \curvearrowleft & \\
& M_3 & \\
& \Downarrow \alpha & \\
& \Downarrow \beta &
\end{array}$$

is given by $h(\eta(\alpha);_{M_2} \eta(\beta))$.

Horizontal composition of 2-cells

$$(7.1) \quad \begin{array}{ccccc}
& M & & N & \\
& \curvearrowright & & \curvearrowright & \\
A & \xrightarrow{\quad} & B & \xrightarrow{\quad} & C \\
& \curvearrowleft & & \curvearrowleft & \\
& M' & & N' & \\
& \Downarrow \alpha & & \Downarrow \beta &
\end{array}$$

is obtained as $h(\beta(\eta(\alpha)))$.

To show that this yields a 2-category structure, the only non obvious point is the interchange law. We deal with it using the following series of results. First, consider the *left whiskering*

$$\begin{array}{ccccc}
& & M & & \\
& \curvearrowright & \downarrow \alpha & \curvearrowleft & \\
A & & & & B \xrightarrow{N} C \\
& \curvearrowleft & & \curvearrowright & \\
& & M' & &
\end{array}$$

of a 2-cell α by a 1-cell N , i.e., the composition $id_N \circ \alpha = h((h(N))(\eta(\alpha)))$.

Lemma 5. *We have: $h((h(N))(\eta(\alpha))) = h(N[\eta(\alpha)])$.*

Proof. Indeed, consider the term $N(\eta(\eta(\alpha)))$ in $\mathcal{L}(\mathcal{L}(X))$. Its images by $h \circ \mathcal{L}(h)$ and $h \circ \mu$ coincide, and are respectively $h((h(N))(\eta(\alpha)))$, i.e., $id_N \circ \alpha$, and $h(N[\eta(\alpha)])$. \square

Similarly, consider the *right whiskering*

$$\begin{array}{ccccc}
& & & & N \\
& & & \curvearrowright & \\
A \xrightarrow{M} & B & & \downarrow \gamma & C \\
& & & \curvearrowleft & \\
& & & & N'
\end{array}$$

of a 2-cell γ by a 1-cell M , i.e., the composition $\gamma \circ id_N = h(\gamma(\eta(h(M))))$.

Lemma 6. *We have: $h(\gamma(\eta(h(M)))) = h(\gamma(M))$.*

Proof. Consider $(\eta\gamma)(\eta M)$ in $\mathcal{L}(\mathcal{L}(X))$. Its images by $h \circ \mathcal{L}(h)$ and $h \circ \mu$ coincide, and are respectively $h(\gamma(\eta(h(M))))$ and $h(\gamma(M))$. \square

Now, we prove that the two sensible ways of mimicking horizontal composition using whiskering coincide with actual horizontal composition:

Lemma 7. *For any cells as in (7.1),*

$$(\beta \circ id_M) ; (id_{N'} \circ \alpha) = \beta \circ \alpha = (id_N \circ \alpha) ; (\beta \circ id_{M'}).$$

Proof. Consider first the reduction $\eta(\beta(M)) ; \eta(N'[\eta(\alpha)])$ in $\mathcal{L}(\mathcal{L}(X))$. Taking $h \circ \mathcal{L}(h)$ and $h \circ \mu$ as above respectively yields

- $h(\eta(h(\beta(M)))) ; \eta(h(N'[\alpha]))$, and
- $h(\beta(M) ; N'[\eta(\alpha)]) = h(\beta(\eta(\alpha)))$,

hence the left-hand equality. Then consider $\eta(N[\eta(\alpha)]) ; \eta(\gamma(M'))$. Evaluating as before yields the right-hand equality. \square

Finally, consider any configuration like:

$$\begin{array}{ccccc}
& & M & & \\
& \curvearrowright & \downarrow \alpha & \curvearrowleft & \\
A & \xrightarrow{M'} & & \downarrow \beta & B \xrightarrow{N} C \\
& \curvearrowleft & & & \\
& & M'' & &
\end{array}$$

Lemma 8. *We have $(id_N \circ \alpha) ; (id_N \circ \beta) = id_N \circ (\alpha ; \beta)$.*

Proof. Consider $\eta(N[\eta(\alpha)]) ; \eta(N[\eta(\beta)])$. Evaluating yields equality of

- $h(\eta(h(N[\eta(\alpha)]))) ; \eta(h(N[\eta(\beta)]))$, i.e., the left-hand side, and
- $h(N[\eta(\alpha)] ; N[\eta(\beta)])$, i.e., $h(N[\eta(\alpha) ; \eta(\beta)])$ by lifting.

But now consider $N[\eta(\alpha) ; \eta(\beta)]$. Evaluating yields equality of

- $h(N[\eta(\alpha) ; \eta(\beta)])$, as above, and
- $h(N[\eta(h(\eta(\alpha) ; \eta(\beta)))])$, i.e., $h(N[\eta(\alpha) ; \beta])$ (where $\alpha ; \beta$ denotes vertical composition in our candidate 2-category), i.e., the right-hand side. \square

Lemma 9. *The interchange law holds, i.e., for all reduction rules as in*

$$\begin{array}{ccc}
& M_1 & N_1 \\
A & \begin{array}{c} \xrightarrow{M_2} \\ \Downarrow \alpha \\ \xrightarrow{\quad} \\ \Downarrow \beta \\ \xrightarrow{M_3} \end{array} & B & \begin{array}{c} \xrightarrow{N_2} \\ \Downarrow \gamma \\ \xrightarrow{\quad} \\ \Downarrow \theta \\ \xrightarrow{N_3} \end{array} & C,
\end{array}$$

we have

$$(\gamma; \theta) \circ (\alpha; \beta) = (\gamma \circ \alpha); (\theta \circ \beta).$$

Proof. By the previous results, we have

$$\begin{aligned}
& (\gamma; \theta) \circ (\alpha; \beta) \\
&= ((\gamma; \theta) \circ M_1); (N_3 \circ (\alpha; \beta)) \\
&= (\gamma \circ M_1); (\theta \circ M_1); (N_3 \circ \alpha); (N_3 \circ \beta) \\
&= (\gamma \circ M_1); (N_2 \circ \alpha); (\theta \circ M_2); (N_3 \circ \beta) \\
&= (\gamma \circ \alpha); (\theta \circ \beta).
\end{aligned}$$

□

Now, let us show cartesian closedness. We have a bijection of hom-sets $\mathcal{L}_1(X)(C \vdash A \times B) \cong \mathcal{L}_1(X)(C \vdash A) \times \mathcal{L}_1(X)(C \vdash B)$, given by

$$\begin{array}{ccc}
\mathcal{L}_1(X)(C \vdash A \times B) & \rightarrow & \mathcal{L}_1(X)(C \vdash A) \times \mathcal{L}_1(X)(C \vdash B) \\
M & \mapsto & \pi M, \pi' M
\end{array}$$

and

$$\begin{array}{ccc}
\mathcal{L}_1(X)(C \vdash A) \times \mathcal{L}_1(X)(C \vdash B) & \rightarrow & \mathcal{L}_1(X)(C \vdash A \times B) \\
M, N & \mapsto & (M, N).
\end{array}$$

These are mutually inverse thanks to the beta and eta rules for products in the simply-typed λ -calculus.

On 2-hom-sets, we have

$$\begin{array}{ccc}
\mathcal{L}(X)(C \vdash M, N: A \times B) & \rightarrow & \mathcal{L}(X)(C \vdash \pi M, \pi N: A) \times \mathcal{L}(X)(C \vdash \pi' M, \pi' N: B) \\
P & \mapsto & \pi P, \pi' P
\end{array}$$

and (omitting C)

$$\begin{array}{ccc}
\mathcal{L}(X)(M_1, N_1: A) \times \mathcal{L}(X)(M_2, N_2: B) & \rightarrow & \mathcal{L}(X)((M_1, M_2), (N_1, N_2): A \times B) \\
P_1, P_2 & \mapsto & (P_1, P_2),
\end{array}$$

which are mutually inverse thanks to the beta and eta rules for products in Figure 4. We use these to define the desired isomorphism (u, v)

$$X_2(C \vdash M, N: A \times B) \cong X_2(C \vdash \pi M, \pi N: A) \times X_2(C \vdash \pi' M, \pi' N: B),$$

as in the diagrams

$$\begin{array}{ccc}
X_2(M, N) & \xrightarrow{u} & X_2(\pi M, \pi N) \times X_2(\pi' M, \pi' N) \\
\eta \downarrow & & \uparrow h \times h \\
\mathcal{L}(X)(M, N) & \xrightarrow{\cong} & \mathcal{L}(X)(\pi M, \pi N) \times \mathcal{L}(X)(\pi' M, \pi' N)
\end{array}$$

and

$$\begin{array}{ccc}
X_2(\pi M, \pi N) \times X_2(\pi' M, \pi' N) & \xrightarrow{v} & X_2(M, N) \\
\eta \times \eta \downarrow & & \uparrow h \\
\mathcal{L}(X)(\pi M, \pi N) \times \mathcal{L}(X)(\pi' M, \pi' N) & \xrightarrow{\cong} & \mathcal{L}(X)(M, N).
\end{array}$$

Starting from $r \in X_2(M, N)$, we obtain

$$v(u(r)) = h(\eta(h(\pi(\eta(r))))), \eta(h(\pi'(\eta(r))))).$$

But consider $(\eta(\pi\eta(r)), \eta(\pi'\eta(r)))$ in $\mathcal{L}(\mathcal{L}X)$; its images by $h \circ \mathcal{L}h$ and $h \circ \mu$ are respectively:

- $h(\eta(h(\pi(\eta r))), \eta(h(\pi'(\eta r))))$, and
- $h(\pi\eta(r), \pi'\eta(r))$, i.e., $h(\eta(r))$, i.e., r ,

which must be equal because h is an \mathcal{L} -algebra, hence $v \circ u = id$.

Conversely, starting from $(r, s) \in X_2(M_1, M_2) \times X_2(N_1, N_2)$, we obtain the pair with components

$$h(\pi(\eta(h(\eta(r), \eta(s)))))) \quad \text{and} \quad h(\pi'(\eta(h(\eta(r), \eta(s))))).$$

Considering $\pi(\eta(\eta(r), \eta(s))) \in \mathcal{L}(\mathcal{L}(X))$, its images by $h \circ \mathcal{L}(h)$ and $h \circ \mu$ are respectively:

- $h(\pi(\eta(h(\eta(r), \eta(s))))))$, and
- $h(\pi(\eta(r), \eta(s))) = h(\eta(r)) = r$.

As above, they must be equal, and by symmetry the second component is s , and we have proved $u \circ v = id$. Similar reasoning for the terminal object and internal homs leads to:

Proposition 10. *This yields a cartesian closed 2-category structure on \mathcal{C} .*

This extends to morphisms of \mathcal{L} -algebras, so we have constructed a functor $\mathcal{F}: \mathcal{L}\text{-Alg} \rightarrow 2\text{CCCat}$.

7.3. Adjunction. Consider any \mathcal{L} -algebra (X, h) . What does $(Y, k) = \mathcal{V}(\mathcal{F}(X, h))$ look like? Sorts in Y_0 are types in $\mathcal{L}_0(X_0)$. Operations $Y_1(G \vdash A)$ are terms in $\mathcal{L}_1(X_0, X_1)(\prod G \vdash A)$. Reduction rules in $Y_2(G \vdash M, N: B)$ are reductions in $\mathcal{L}(X)(\prod G \vdash M', N': B)$, where $M' = M[\pi_1 x/x_1, \dots, \pi_n x/x_n]$ (and similarly for N').

Let η_X send:

- each sort $\iota \in X_0$ to the type $\iota \in \mathcal{L}_0(X_0)$,
- each operation $c \in X(G \vdash A)$ to the term $c(\pi_1 x, \dots, \pi_n x)$, and
- each reduction rule $r \in X_2(G \vdash M, N: A)$ to the reduction $x: \prod G \vdash r(\pi_1 x, \dots, \pi_n x): M' \rightarrow N': A$.

Theorem 2. *This η is a natural transformation which is the unit of an adjunction*

$$\mathcal{L}\text{-Alg} \begin{array}{c} \xrightarrow{\mathcal{F}} \\ \perp \\ \xleftarrow{\mathcal{V}} \end{array} 2\text{CCCat}.$$

Proof. Consider any morphism $f: (X, h) \rightarrow \mathcal{V}(\mathcal{C})$, and let $(Y, k) = \mathcal{V}(\mathcal{F}(X, h))$ and $\mathcal{V}(\mathcal{C}) = (\mathcal{C}_0, \mathcal{C}_1, h_2: \mathcal{C}_2 \rightarrow \mathcal{C}_1)$. We now define a uniquely determined cartesian closed functor $f': \mathcal{F}(X, h) \rightarrow \mathcal{C}$ making the triangle

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \mathcal{V}(\mathcal{F}(X)) \\ & \searrow f & \downarrow \mathcal{V}(f') \\ & & \mathcal{V}(\mathcal{C}) \end{array}$$

commute.

On objects, it is determined by induction: on sorts by f_0 , and on type constructors by the requirement that f' be cartesian closed. On morphisms, it is similarly determined by f_1 and f' being cartesian closed. On 2-cells, define f' to be $f_2: X_2(A \vdash M, N: B) \rightarrow \mathcal{C}(f'(A), f'(B))(f'(M), f'(N))$, which is also the only possible choice from f .

We thus only have to show that f' is cartesian closed, which follows by f being a morphism of \mathcal{L} -algebras. For example, to show that binary products of reductions are preserved, consider $r \in X_2(C \vdash M_1, M_2: A)$ and $s \in X_2(C \vdash N_1, N_2: B)$. Their product in $\mathcal{F}(X)$ is obtained by considering the atomic reductions $x: C \vdash r(x): M_1 \rightarrow M_2: A$ and $x: C \vdash s(x): N_1 \rightarrow N_2: B$ and taking $h(r(x), s(x))$, which is sent by f_2 to $f_2(h(r(x), s(x)))$. But, because f is a morphism of \mathcal{L} -algebras, this is the same as $h_2((f_2(r))(x), (f_2(s))(y))$, which is by definition (i.e., Figure 2) the product $(f_2(r), f_2(s))$ in \mathcal{C} . \square

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APPENDIX A. EQUATIONS ON REDUCTIONS

Current address: CNRS, Université de Savoie

Congruence	
$\frac{\Gamma \vdash P : M \rightarrow N : A}{\Gamma \vdash P \equiv P : M \rightarrow N : A}$	$\frac{\Gamma \vdash P \equiv Q : M \rightarrow N : A}{\Gamma \vdash Q \equiv P : M \rightarrow N : A}$
$\frac{\Gamma \vdash P_1 \equiv P_2 : M \rightarrow N : A \quad \Gamma \vdash P_2 \equiv P_3 : M \rightarrow N : A}{\Gamma \vdash P_1 \equiv P_3 : M \rightarrow N : A}$	
$\frac{\Gamma \vdash P \equiv P' : M_1 \rightarrow M_2 : A \quad \Gamma \vdash Q \equiv Q' : M_2 \rightarrow M_3 : A}{\Gamma \vdash (P ;_{M_2} Q) \equiv (P' ;_{M_2} Q') : M_1 \rightarrow M_3 : A}$	
$(r \in X(G \vdash M, N : A))$	
$\frac{\Gamma \vdash P_1 \equiv Q_1 : M_1 \rightarrow N_1 : G_1 \quad \dots \quad \Gamma \vdash P_n \equiv Q_n : M_n \rightarrow N_n : G_n}{\Gamma \vdash r(P_1, \dots, P_n) \equiv r(Q_1, \dots, Q_n) : M[M_1, \dots, M_n] \rightarrow N[N_1, \dots, N_n] : A}$	
$(c \in X(G \vdash A))$	
$\frac{\Gamma \vdash P_1 \equiv Q_1 : M_1 \rightarrow N_1 : G_1 \quad \dots \quad \Gamma \vdash P_n \equiv Q_n : M_n \rightarrow N_n : G_n}{\Gamma \vdash c(P_1, \dots, P_n) \equiv c(Q_1, \dots, Q_n) : c(M_1, \dots, M_n) \rightarrow c(N_1, \dots, N_n) : A}$	
$\frac{\Gamma, x : A \vdash P \equiv Q : M \rightarrow N : B}{\Gamma \vdash (\lambda x : A. P) \equiv (\lambda x : A. Q) : \lambda x : A. M \rightarrow \lambda x : A. N : B^A}$	
$\frac{\Gamma \vdash P \equiv P' : M \rightarrow M' : B^A \quad \Gamma \vdash Q \equiv Q' : N \rightarrow N' : A}{\Gamma \vdash (PQ) \equiv (P'Q') : MN \rightarrow M'N' : B}$	
$\frac{\Gamma \vdash P \equiv P' : M \rightarrow M' : A \quad \Gamma \vdash Q \equiv Q' : N \rightarrow N' : B}{\Gamma \vdash (P, Q) \equiv (P', Q') : (M, N) \rightarrow (M', N') : A \times B}$	
$\frac{\Gamma \vdash P \equiv Q : M \rightarrow N : A \times B}{\Gamma \vdash (\pi_{A,B} P) \equiv (\pi_{A,B} Q) : \pi_{A,B} M \rightarrow \pi_{A,B} N : A}$	
$\frac{\Gamma \vdash P \equiv Q : M \rightarrow N : A \times B}{\Gamma \vdash (\pi'_{A,B} P) \equiv (\pi'_{A,B} Q) : \pi'_{A,B} M \rightarrow \pi'_{A,B} N : A}$	
Category	
$\frac{\Gamma \vdash P_1 : M_1 \rightarrow M_2 : A \quad \Gamma \vdash P_2 : M_2 \rightarrow M_3 : A \quad \Gamma \vdash P_3 : M_3 \rightarrow M_4 : A}{\Gamma \vdash (P_1 ;_{M_2} (P_2 ;_{M_3} P_3)) \equiv ((P_1 ;_{M_2} P_2) ;_{M_3} P_3) : M_1 \rightarrow M_4 : A}$	
$\frac{\Gamma \vdash P : M \rightarrow N : A}{\Gamma \vdash (P ;_N N) \equiv P : M \rightarrow N : A}$	$\frac{\Gamma \vdash P : M \rightarrow N : A}{\Gamma \vdash (M ;_M P) \equiv P : M \rightarrow N : A}$

FIGURE 3. Equations on reductions (Congruence and category)

Beta and eta	
$\frac{\Gamma, x : A \vdash P : M \rightarrow M' : B \quad \Gamma \vdash Q : N \rightarrow N' : A}{\Gamma \vdash ((\lambda x : A. P)Q) \equiv P[Q/x] : (\lambda x : A. M)N \rightarrow M'[N'/x] : B}$	
$\frac{\Gamma \vdash P \equiv M \rightarrow N : B^A}{\Gamma \vdash P \equiv \lambda x : A. (Px) : M \rightarrow N : B^A} \quad (x \notin \Gamma)$	
$\frac{\Gamma \vdash P : M_1 \rightarrow M_2 : A \quad \Gamma \vdash Q : N_1 \rightarrow N_2 : B}{\Gamma \vdash \pi(P, Q) \equiv P : \pi(M_1, N_1) \rightarrow M_2 : A}$	
$\frac{\Gamma \vdash P : M_1 \rightarrow M_2 : A \quad \Gamma \vdash Q : N_1 \rightarrow N_2 : B}{\Gamma \vdash \pi'(P, Q) \equiv P : \pi'(M_1, N_1) \rightarrow N_2 : A}$	
$\frac{\Gamma \vdash P : (M_1, N_1) \rightarrow (M_2, N_2) : A \times B}{\Gamma \vdash P \equiv (\pi P, \pi' P) : (M_1, N_1) \rightarrow (M_2, N_2) : A \times B} \quad \frac{\Gamma \vdash P : M \rightarrow N : 1}{\Gamma \vdash P \equiv () : M \rightarrow N : 1}$	
Lifting	
$\frac{(r \in X(\Gamma \vdash \langle M_1, M_2 \rangle : A)) \quad \Delta \vdash P : N_1 \rightarrow N_2 : \Gamma \quad \Delta \vdash Q : N_2 \rightarrow N_3 : \Gamma}{\Gamma \vdash r(P ;_{N_2} Q) \equiv M_1[P] ;_{M_1[N_2]} r(Q) : M_1[N_1] \rightarrow M_2[N_3] : A}$	
$\frac{(r \in X(\Gamma \vdash \langle M_1, M_2 \rangle : A)) \quad \Delta \vdash P : N_1 \rightarrow N_2 : \Gamma \quad \Delta \vdash Q : N_2 \rightarrow N_3 : \Gamma}{\Gamma \vdash r(P ;_{N_2} Q) \equiv r(P) ;_{M_2[N_2]} M_2[Q] : M_1[N_1] \rightarrow M_2[N_3] : A}$	
$\frac{\Gamma \vdash P : M_1 \rightarrow M_2 : G \quad \Gamma \vdash Q : M_2 \rightarrow M_3 : G}{\Gamma \vdash (c(P ;_{M_2} Q)) \equiv (c(P) ;_{c(M_2)} c(Q)) : M_1 \rightarrow M_3 : A} \quad (c \in X(G \vdash A))$	
$\frac{\Gamma, x : A \vdash P : M_1 \rightarrow M_2 : B \quad \Gamma, x : A \vdash Q : M_2 \rightarrow M_3 : B}{\Gamma \vdash (\lambda x : A. (P ;_{M_2} Q)) \equiv ((\lambda x : A. P) ;_{\lambda x : A. M_2} (\lambda x : A. Q)) : \lambda x : A. M_1 \rightarrow \lambda x : A. M_3 : B^A}$	
$\frac{\Gamma \vdash P : M_1 \rightarrow M_2 : B^A \quad \Gamma \vdash P' : M_2 \rightarrow M_3 : B^A \quad \Gamma \vdash Q : N_1 \rightarrow N_2 : A \quad \Gamma \vdash Q' : N_2 \rightarrow N_3 : A}{\Gamma \vdash ((P ;_{M_2} P')(Q ;_{N_2} Q')) \equiv ((PQ) ;_{M_2 N_2} (P'Q')) : M_1 N_1 \rightarrow M_3 N_3 : B}$	
$\frac{\Gamma \vdash P : M_1 \rightarrow M_2 : A \quad \Gamma \vdash P' : M_2 \rightarrow M_3 : A \quad \Gamma \vdash Q : N_1 \rightarrow N_2 : B \quad \Gamma \vdash Q' : N_2 \rightarrow N_3 : B}{\Gamma \vdash ((P ;_{M_2} P'), (Q ;_{N_2} Q')) \equiv ((P, Q) ;_{(M_2, N_2)} (P', Q')) : (M_1, N_1) \rightarrow (M_3, N_3) : A \times B}$	
$\frac{\Gamma \vdash P : M_1 \rightarrow M_2 : A \times B \quad \Gamma \vdash Q : M_2 \rightarrow M_3 : A \times B}{\Gamma \vdash (\pi_{A,B}(P ;_{M_2} Q)) \equiv (\pi_{A,B} P ;_{\pi_{A,B} M_2} \pi_{A,B} Q) : M_1 \rightarrow M_3 : A}$	
$\frac{\Gamma \vdash P : M_1 \rightarrow M_2 : A \times B \quad \Gamma \vdash Q : M_2 \rightarrow M_3 : A \times B}{\Gamma \vdash (\pi'_{A,B}(P ;_{M_2} Q)) \equiv (\pi'_{A,B} P ;_{\pi'_{A,B} M_2} \pi'_{A,B} Q) : M_1 \rightarrow M_3 : B}$	

FIGURE 4. Equations on reductions (beta-eta and lifting)