On Alexander–Conway polynomials of two-bridge links

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Conway polynomials of two-bridge links

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Abstract

We give necessary conditions for a polynomial to be the Conway polynomial of a two-bridge link. As a consequence, we obtain simple proofs of the classical theorems of Murasugi and Hartley. We give a modulo 2 congruence for links, which implies the classical modulo 2 Murasugi congruence for knots. We also give sharp bounds for the coefficients of the Conway and Alexander polynomials of a two-bridge link. These bounds improve and generalize those of Nakanishi and Suketa.

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1 Introduction

In this paper, we study the problem of determining whether a given polynomial is the Conway polynomial of a two-bridge link or knot. For small degrees, this problem can be solved by an exhaustive search of possible two-bridge links. Here, however, we give necessary conditions on the coefficients of the polynomial, which can be tested for high degree polynomials.

In section 2 we present Siebenmann's description of the Conway polynomial of a two-bridge link. We obtain a characterization of modulo 2 two-bridged Conway polynomials with the help of the Fibonacci polynomials f_k defined by:

$$f_0 = 0, f_1 = 1, f_{n+2}(z) = zf_{n+1}(z) + f_n(z),\ n \in \mathbf{Z}. \qquad (1)$$

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Theorem 2.3. *Let $\nabla(z) \in \mathbf{Z}[z]$ be the Conway polynomial of a rational link (or knot). There exists a Fibonacci polynomial $f_D(z)$ such that $\nabla(z) \equiv f_D(z) \pmod 2$.*

We give a simple method (Algorithm 2.5) that determines this Fibonacci polynomial.

In section 3, we obtain inequalities for the coefficients of the Conway polynomials of links (or knots) denoted by

$$\nabla_m(z) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} c_{m-2k} z^{m-2k}.$$

Theorem 3.3. *For $k \geq 0$,*

$$|c_{m-2k}| \leq \binom{m-k}{k} |c_m| .$$

If equality holds for some positive integer $k < \lfloor \frac{m}{2} \rfloor$, then it holds for all integers. In this case, the link is isotopic to a link of Conway form $C(2, -2, 2, \dots, (-1)^{m+1}2)$ or $C(2, 2, \dots, 2)$, up to mirror symmetry.

When $|c_m| \neq 1$, we have the following sharper bounds:

Theorem 3.6. *Let $g \geq 1$ be the greatest prime divisor of c_m, and let $k \neq 0$. Then*

$$|c_{m-2k}| \leq \Big(\binom{m-k-1}{k} + \frac{1}{g} \big(\binom{m-k-1}{k-1} - 1 \big) \Big) |c_m| + 1.$$

Equality holds for links of Conway forms $C(2g, 2, 2, \dots, 2)$ and $C(2g, -2, 2, \dots, (-1)^{m+1}\,2)$.

We also obtain the following trapezoidal property:

Theorem 3.7. *Let K be a two-bridge link (or knot). Let*

$$\nabla_K = c_m \Big(\sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^i \alpha_i f_{m-2i+1} \Big), \quad \alpha_0 = 1$$

be its Conway polynomial expressed in the Fibonacci basis. Then we have

1. $\alpha_j \geq 0,\ j = 0, \dots, \lfloor \frac{m}{2} \rfloor$.
2. *If $\alpha_i = 0$ for some $i > 0$ then $\alpha_j = 0$ for $j \geq i$.*

In section 4, we apply our results to the Alexander polynomials. Theorem 2.3 provides an easy proof of a congruence of Murasugi [21] for two-bridge knots. Moreover, we also obtain a congruence for the Hosokawa polynomials of two-bridge links.

Then, as a consequence of Theorem 3.7, we obtain a simple proof of both the Murasugi alternating theorem ([19, 20]), and the Hartley trapezoidal theorem ([7], see also [9]).

We conclude this section by giving bounds for the coefficients of the Alexander coefficients. These bounds improve those of Nakanishi and Suketa for Alexander polynomials of two-bridge knots (see [22, Theorems 2 and 3]). Moreover, they are sharp and hold for any k.

We prove that the conditions on the Conway coefficients are better than the conditions on the Alexander coefficients deduced from them.

In section 5, we conclude our paper with the following convexity conjecture:

Conjecture 5.2. *Let* $P(t) = a_0 - a_1(t+t^{-1}) + a_2(t^2+t^{-2}) - \cdots + (-1)^n a_n(t^n+t^{-n})$ *be the Alexander polynomial of a two-bridge knot. Then there exists an integer* $k \le n$ *such that* $(a_0, \ldots, a_k)$ *is convex and* $(a_k, \ldots, a_n)$ *is concave.*

We have tested this conjecture for all two-bridge knots with 20 crossings or fewer.

2 Conway polynomial

Any oriented two-bridge link can be put in the form shown in Figure 1. It will be denoted by $C(2b_1, 2b_2, \ldots, 2b_m)$ with $b_i \neq 0$ for all i, including the indicated orientation (see [13, p. 26], [15, 11]). This is a two-component link if and only if m is odd.

Its Conway polynomial ∇_m is then given by the Siebenmann method (see [23, 5]).

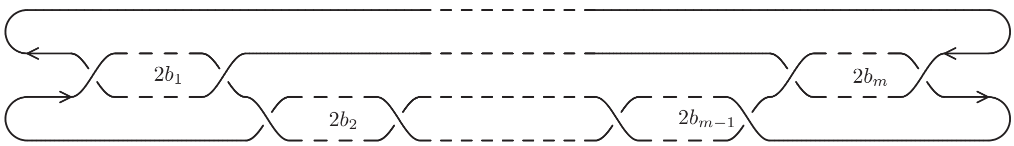

Figure 1: Oriented two-bridge links (m odd)

Theorem 2.1 (Siebenmann, [23, 5]) *Let* $\nabla_m = \nabla_m(z)$ *be the Conway polynomial of the oriented two-bridge link (or knot) of Conway form* $C(2b_1, -2b_2, \ldots, (-1)^{m+1}2b_m)$.

Let $\nabla_{-1} = 0,\ \nabla_0 = 1$. *Then* $\nabla_m = b_m z \nabla_{m-1} + \nabla_{m-2}$ *for* $m \ge 1$.

When $z = 1$, this is the classical Euler continuant polynomial (see [14]). When all the b_i are equal to 1, we obtain the Fibonacci polynomials.

Example 2.2. **The torus links** $\mathrm{T}(2, m)$. *The Conway polynomial of the torus link* $\mathrm{T}(2, m) = C(2, -2, \ldots, (-1)^{m+1}2)$ *is the Fibonacci polynomial* $f_m(z)$ *(see [12, 17]).*

Consequently, the following result gives in fact a characterization of modulo 2 Conway polynomials of two-bridge links.

Theorem 2.3. *Let* ∇_m *be the Conway polynomial of a two-bridge link. Then there exists a Fibonacci polynomial* f_D *such that* $\nabla_m \equiv f_D \pmod 2$.

Proof. Let us write $(a, b) \equiv (c, d) \pmod 2$ when $a \equiv c \pmod 2$ and $b \equiv d \pmod 2$. We will show by induction on m that there exist integers D and $e = \pm 1$ such that $(\nabla_{m-1}, \nabla_m) \equiv (f_{D-e}, f_D) \pmod 2$.

The result is true for $m = 0$ as $(\nabla_{-1}, \nabla_0) = (0, 1) = (f_0, f_1)$, that is $D = e = 1$.

Suppose that $(\nabla_{m-1}, \nabla_m) \equiv (f_{D-e}, f_D) \pmod 2$, with $e = \pm 1$ for some $m \geq 0$. Then we have $\nabla_{m+1} = b_{m+1} z \nabla_m + \nabla_{m-1}$.

If $b_{m+1} \equiv 0 \pmod 2$ then $\nabla_{m+1} \equiv \nabla_{m-1} \equiv f_{D-e} \pmod 2$ and $(\nabla_m, \nabla_{m+1}) \equiv (f_D, f_{D-e})$.

If $b_{m+1} \equiv 1 \pmod 2$ then $\nabla_{m+1} \equiv z f_D + f_{D-e} \equiv f_{D+e} \pmod 2$, and consequently $(\nabla_m, \nabla_{m+1}) \equiv (f_D, f_{D+e})$. ☐

Example 2.4. *The Pretzel knot* 8_5 *has Conway polynomial* $1 - z - 3z^4 - z^6 \equiv f_1 + f_3 + f_7 \pmod 2$. *By theorem 2.3 it is not a two-bridge knot.*

From the proof of Theorem 2.3, we deduce a fast algorithm for the determination of the integer D such that $\nabla_K \equiv f_D \pmod 2$, see also [3].

Algorithm 2.5. *Let* K *be a two-bridge link (or knot) of Conway form* $C(2b_1, 2b_2, \ldots, 2b_m)$. *Let us define the sequences of integers* e_i *and* D_i, $i = 0, \ldots, m$, *by*

$$e_0 = 1,\ D_0 = 1,\ e_{i+1} = -(-1)^{b_{i+1}} e_i,\ D_{i+1} = D_i + e_{i+1}.$$

Then we have $\nabla(z) \equiv f_D(z) \pmod 2$ *where* $D = |D_m|$.

This algorithm may be useful for the study of Lissajous knots. Jones, Przytycki, and Lamm proved that the Conway polynomial of a two-bridge Lissajous knot satisfies the congruence $\nabla(z) \equiv 1 \pmod 2$, that is $D = 0$ (see [2, 8, 18]).

3 Inequalities for the coefficients of the Conway polynomial

We shall need the following explicit notation for Conway polynomials:

$$\nabla_m(z) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} c_{m-2k}(b_1, \ldots, b_m) z^{m-2k}.$$

Thus, the Siebenmann formula (Theorem 2.1) means that

$$c_{m-2k}(b_1, \ldots, b_m) = b_m \cdot c_{m-1-2k}(b_1, \ldots, b_{m-1}) + c_{m-2k}(b_1, \ldots, b_{m-2}). \tag{2}$$

Remark 3.1. For the torus link $T(2, m) = C(2, -2, \ldots, (-1)^{m+1} 2)$, all the b_i are equal to 1, and an easy induction shows that $c_{m-2k}(1, \ldots, 1) = \binom{m-k}{k}$. consequently, we obtain the following expression for the Fibonacci polynomials:

$$f_{m+1}(z) = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-k}{k} z^{m-2k} \quad \text{for} \quad m \geq 0.$$

This means that the Fibonacci polynomials can be read on the diagonals of the [the] Pascal's triangle. When $z = 1$, we recover the classical Lucas identity ZZ

$$F_m = \sum_{k=0}^{\lfloor \frac{m}{2} \rfloor} \binom{m-k}{k},$$

where F_m are the Fibonacci numbers ($F_0 = 0,\ F_1 = 1, F_{n+1} = F_n + F_{n-1}$).

In the next result, we deduce some properties of the coefficient $c_{m-2k}(b_1, \ldots, b_m)$, considered as a polynomial in the m variables $b_1, \ldots, b_m$.

Proposition 3.2. *Let $\mathcal{C}(m,k)$, $m \geq 2k$, be the set of all monomials* $\dfrac{b_1 \cdots b_m}{b_{i_1} b_{i_1+1} \cdots b_{i_k} b_{i_k+1}}$, *where $i_h+1 < i_{h+1}$. Let $\mathcal{C}_j(m,k)$ be the subset of all monomials of $\mathcal{C}(m,k)$ that are relatively prime to b_j. Then we have*

1. *The set $\mathcal{C}(m,k)$ has $\binom{m-k}{k}$ elements.*
2. *The polynomial $c_{m-2k}(b_1, \ldots, b_m)$ is the sum of all monomials of $\mathcal{C}(m,k)$.*
3. *If $k \neq 0$, then the monomials of $\mathcal{C}(m,k)$ do not have a common divisor except 1.*
4. *The number of elements of $\mathcal{C}_j(m,k)$ is at least $\binom{m-1-k}{k-1}$.*
5. *If $k \geq 2$, then the monomials of $\mathcal{C}_j(m,k)$ do not have a common divisor except 1.*

Proof.

1. By induction on m.

   We have $\mathcal{C}(m,0) = \{\, b_1 \cdots b_m \,\}$, $\mathcal{C}(2,1) = \{1\}$ and $\mathcal{C}(3,1) = \{b_1,\ b_3\}$. Hence the result is true for $k = 0$, and also for $m \leq 3$.

   Let us suppose the result true for $m-1$ and $m-2$. We can suppose $k \neq 0$. If a monomial of $\mathcal{C}(m,k)$ is not a multiple of b_m, then it is not a multiple of b_{m-1} either, and consequently it is an element of $\mathcal{C}(m-2,k-1)$. Therefore, we have the following partition of $\mathcal{C}(m,k)$ for $k \neq 0$:

   $$\mathcal{C}(m,k) = b_m \cdot \mathcal{C}(m-1,k) \bigsqcup \mathcal{C}(m-2,k-1),$$

   and then

   $$\operatorname{card} \mathcal{C}(m,k) = \operatorname{card} \mathcal{C}(m-1,k) + \operatorname{card} \mathcal{C}(m-2,k-1) = \binom{m-1-k}{k} + \binom{m-1-k}{k-1} = \binom{m-k}{k}.$$

2. By induction on m. Using our partition of $\mathcal{C}(m,k)$, we see that the sum of the monomials of $\mathcal{C}(m,k)$ is $b_m \cdot c_{m-1-2k}(b_1, \ldots, b_{m-1}) + c_{m-2k}(b_1, \ldots, b_{m-2})$.

   By Siebenmann's formula, this polynomial is equal to $c_{m-2k}(b_1, \ldots, b_m)$.
3. If $k \neq 0$, then for every integer $i \leq m$, there is an element of $\mathcal{C}(m,k)$ which is not divisible by b_i. Hence the GCD of the elements of $\mathcal{C}(m,k)$ is 1.

4. Let $\mathbf{b} = (1, \ldots, 1, 0, 1, \ldots, 1) \in \mathbf{R}^m$ where $b_j = 0$, and $b_k = 1$ for $k \neq j$.

   Let us define the polynomials g_n, for $n \leq m$ by $g_n(z) = \nabla_n(\mathbf{b})(z)$. The number of elements of $\mathcal{C}_j(m,k)$ is the coefficient $c_{m-2k}(\mathbf{b})$ of $g_m(z)$.

   If $j = 1$, then we have $g_1 = 0$, $g_2 = 1$, and $g_n = z g_{n-1} + g_{n-2}$ for $n \geq 2$. Then, an easy induction shows that $g_n = f_{n-1}$.

   If $j > 1$, then we have $g_1 = f_2, \ \ldots, \ g_{j-1} = f_j, \ g_j = f_{j-1}$, and $g_{n+1} = z g_n + g_{n-1}$ for $n \geq j$.

   Let us write $p(z) \succeq q(z)$ when each coefficient of p is greater than or equal to the corresponding coefficient of q. We have $f_{k+2} \succeq f_k$, and therefore $g_{j+1} = z f_{j-1} + f_j \succeq z f_{j-1} + f_{j-2} = f_j$. Then a simple induction shows that $g_m \succeq f_{m-1}$, and consequently $c_{m-2k}(\mathbf{b}) \geq \binom{m-1-k}{k-1}$.
5. Since $k \geq 2$, then for every $i \neq j$ there is a monomial of $\mathcal{C}_j(m,k)$ which is not divisible by b_i. Consequently, the GCD of the elements of $\mathcal{C}_j(m,k)$ is 1. $\square$

Theorem 3.3. *For $k \geq 0$,*
$$|c_{m-2k}| \leq \binom{m-k}{k} |c_m| .$$
If equality holds for some integer $k < \lfloor \frac{m}{2} \rfloor$, then it holds for all integers. In this case, the link is isotopic to the torus link $T(2,m)$ or to the link $C(2,2,\ldots,2)$, up to mirror symmetry.

Proof. By Proposition 3.2, the number of monomials of $c_{m-2k}(b_1, \ldots, b_m)$ is $\binom{m-k}{k}$. The result follows since no monomial is greater than $|c_m| = |b_1 \cdots b_m|$.

If equality holds for some positive integer $k < \lfloor \frac{m}{2} \rfloor$, then for all i, j, $b_i b_{i+1} = b_j b_{j+1} = \pm 1$, which implies the result. $\square$

Example 3.4. *The knot 10_{145} has Conway polynomial $P = 1 + 5z^2 + z^4$. We have $P \equiv f_5 \pmod 2$, but P does not satisfy the condition $|c_2| \leq 3$, and then 10_{145} is not a two-bridge knot.*

The knot $11n109$ has Conway polynomial $1 + 6z^2 + z^4 - z^6$. It satisfies the bounds of Theorem 3.3: $|c_2| \leq 6$, $|c_4| \leq 5$, but not the equality condition: $c_2 = 6$ whereas $c_4 \neq 5$. Consequently, $11n109$ is not a two-bridge knot.

To prove the refined inequalities of Theorem 3.6, we shall use the following lemma, which generalizes the inequality $a + b \leq ab + 1$, valid for positive integers (see also [22]).

Lemma 3.5. *Let $p_i(\mathbf{x}), i \in \mathcal{S}$ be relatively prime divisors of $p(\mathbf{x}) = x_1 x_2 \cdots x_m$.*
Let $\mathbf{b} = (b_1, \ldots, b_m)$ be a m-tuple of positive integers. Then
$$\sum_{i \in \mathcal{S}} p_i(\mathbf{b}) \leq \Big(\mathrm{card}\,(\mathcal{S}) - 1\Big) p(\mathbf{b}) + 1. \qquad (3)$$

Proof. We do not suppose that the p_i are distinct integers. Let us prove the result by induction on $k = \operatorname{card}(\mathcal{S})$. If $k = 1$, then we have $p_1 = \pm 1$, and the result is true. When all the p_i are equal to 1, the result is true. Otherwise, let x_h be a divisor of some p_i.

Let $\mathcal{S}_1 = \{i \in \mathcal{S} : \ x_h \mid p_i\}$ and $\mathcal{S}_2 = \mathcal{S} - \mathcal{S}_1$. We have $k = k_1 + k_2$, where $k_j = \operatorname{card}(\mathcal{S}_j)$. Let $q_j = \operatorname{GCD}\{p_i, i \in \mathcal{S}_j\}$, then q_1 and q_2 are coprime, and $q_1 q_2$ is a divisor of p.

By induction we obtain for $j = 1, 2$:

$$\sum_{i \in \mathcal{S}_j} p_i(\mathbf{b}) \le q_j(\mathbf{b}) \Big((k_j - 1) \frac{p(\mathbf{b})}{q_j(\mathbf{b})} + 1 \Big) = (k_j - 1) p(\mathbf{b}) + q_j(\mathbf{b}).$$

Adding these two inequalities we get

$$\begin{aligned} \sum_{i \in \mathcal{S}} p_i(\mathbf{b}) &\le (k_1 + k_2 - 1) p(\mathbf{b}) + q_1(\mathbf{b}) + q_2(\mathbf{b}) - p(\mathbf{b}) \\ &\le (k - 1) p(\mathbf{b}) + q_1(\mathbf{b}) q_2(\mathbf{b}) - p(\mathbf{b}) + 1, \end{aligned}$$

which proves the result, since $q_1(\mathbf{b}) q_2(\mathbf{b}) \le p(\mathbf{b})$. $\square$

With this lemma we can prove:

Theorem 3.6. *Let $g \ge 1$ be the greatest prime divisor of c_m, and let $k \ne 0$. Then*

$$|c_{m-2k}| \le \Big(\tbinom{m-k-1}{k} + \tfrac{1}{g} \big(\tbinom{m-k-1}{k-1} - 1 \big) \Big) |c_m| + 1.$$

Equality holds for links of Conway form $C(2g, -2, \ldots, (-1)^{m+1} 2)$ and $C(2g, 2, \ldots, 2)$.

Proof. If $k = 1$, then by Proposition 3.2 the polynomial $c_{m-2}(b_1, \ldots, b_m)$ is the sum of $m - 1$ coprime monomials. Then, using Lemma 3.5 and the notation $|\mathbf{b}| = (|b_1|, \ldots, |b_m|)$, we get

$$|c_{m-2}| = |c_{m-2}(\mathbf{b})| \le c_{m-2}(|\mathbf{b}|) \le (m-2) c_m(|\mathbf{b}|) + 1 = (m-2) |c_m| + 1.$$

Now, suppose $k \ge 2$. Let g be the greatest prime divisor of the integer $c_m = b_1 \cdots b_m$, and suppose that $g \mid b_j$. Let $\mathcal{M}$ be the set of monomials of $c_{m-2k}(b_1, \ldots, b_m)$, and let $\mathcal{M}_j$ be the subset of monomials of $\mathcal{M}$ that are prime to b_j.

By Proposition 3.2, the monomials of $\mathcal{M}_j$ are relatively prime, and their number N verifies $N \ge \binom{m-1-k}{k-1}$. Using Lemma 3.5 we obtain: $\sum_{p_i \in \mathcal{M}_j} p_i(\mathbf{b}) \le (N-1) \dfrac{|c_m|}{|b_j|} + 1$ and then

$$\begin{aligned} |c_{m-2k}| = \Big| \textstyle\sum_{p_i \in \mathcal{M}} p_i(\mathbf{b}) \Big| &\le \Big(\frac{N-1}{g} + \big(\tbinom{m-k}{k} - N \big) \Big) |c_m| + 1 \\ &= \Big(\tbinom{m-k}{k} - N(1 - \tfrac{1}{g}) - \tfrac{1}{g} \Big) |c_m| + 1 \\ &\le \Big(\tbinom{m-k}{k} - \tbinom{m-1-k}{k-1} (1 - \tfrac{1}{g}) - \tfrac{1}{g} \Big) |c_m| + 1 \\ &= \Big(\tbinom{m-1-k}{k} + \tfrac{1}{g} \big(\tbinom{m-1-k}{k-1} - 1 \big) \Big) |c_m| + 1. \end{aligned}$$

For links of Conway form $C(2g, 2, \ldots, 2)$ or $C(2g, -2, \ldots, (-1)^{m+1}2)$, we have $|\mathbf{b}| = (g, 1, \ldots, 1)$, $N = \binom{m-1-k}{k-1}$, $|c_m| = g$, $|c_{m-2k}| = g\binom{m-1-k}{k} + \binom{m-1-k}{k-1}$, and equality holds. $\square$

We will now express the Conway polynomials of two-bridge links in terms of Fibonacci polynomials, and show that their coefficients are alternating.

Theorem 3.7. *Let K be a two-bridge link (or knot). Let*

$$\nabla_K = c_m \Big(\sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^i \alpha_i f_{m-2i+1} \Big), \quad \alpha_0 = 1$$

be its Conway polynomial written in the Fibonacci basis. Then we have

1. $\alpha_j \geq 0$, $j = 0, \ldots, \lfloor \frac{m}{2} \rfloor$.
2. *If* $\alpha_i = 0$ *for some* $i > 0$ *then* $\alpha_j = 0$ *for* $j \geq i$.

Proof. Let $K = C(2b_1, -2b_2, \ldots, (-1)^{m+1}\, 2b_m)$, with $b_i \neq 0$ for all i, and let ∇_n be the polynomials obtained in the Siebenmann method.

We have $\nabla_0 = f_1$, $\nabla_1 = b_1 f_2$, $\nabla_2 = b_1 b_2 \Big(f_3 - (1 - \frac{1}{b_1 b_2}) f_1 \Big)$.

Let us show by induction that if

$$\nabla_m = b_1 \cdots b_m \Big(\sum_{i=0}^{\lfloor \frac{m}{2} \rfloor} (-1)^i \alpha_i f_{m+1-2i} \Big), \; \nabla_{m-1} = b_1 \cdots b_{m-1} \Big(\sum_{i=0}^{\lfloor \frac{m-1}{2} \rfloor} (-1)^i \beta_i f_{m-2i} \Big)$$

then $\alpha_j \geq \beta_j \geq 0$, and if $\alpha_i = 0$ for some i, then $\alpha_j = 0$ for $j \geq i$.

The result is true for $m = 1$ and for $m = 2$. Using $z f_{m+1-2i} = f_{m+2-2i} - f_{m-2i}$ and $\nabla_{m+1} = b_{m+1} z \nabla_m + \nabla_{m-1}$, we deduce that

$$\nabla_{m+1} = b_1 \cdots b_{m+1} \Big(\sum_{i=0}^{\lfloor \frac{m+1}{2} \rfloor} (-1)^i \gamma_i f_{m+2-2i} \Big),$$

where $\gamma_0 = 1$ and

$$\gamma_i = \alpha_i + (\alpha_{i-1} - \beta_{i-1}) + (1 - \tfrac{1}{b_m b_{m+1}}) \beta_{i-1}, \; i = 1, \ldots, \lfloor \tfrac{m+1}{2} \rfloor. \tag{4}$$

As $|b_m b_{m+1}| \geq 1$, we deduce by induction that $\gamma_i \geq \alpha_i \geq 0$.

Furthermore, if $\gamma_i = 0$, then by Formula (4) $\alpha_i = 0$, and then, by induction, $\alpha_j = \beta_j = 0$ for $j \geq i$. Finally, by Formula (4), we get $\gamma_j = 0$ for $j \geq i$. $\square$

4 Applications to the Alexander polynomial

In this section, we will see that our necessary conditions on the Conway coefficients imply similar necessary conditions on the Alexander coefficients of two-bridge knots and links. These conditions are improvements of the classical results.

The Conway and the Alexander polynomials of a knot K will be denoted by

$$\nabla_K(z) = 1 + \tilde{c}_1 z^2 + \cdots + \tilde{c}_n z^{2n}$$

and

$$\Delta_K(t) = a_0 - a_1(t + t^{-1}) + \cdots + (-1)^n a_n (t^n + t^{-n}).$$

The Alexander polynomial $\Delta_K(t)$ is deduced from the Conway polynomial by:

$$\Delta_K(t) = \pm \nabla_K\Big(t^{1/2} - t^{-1/2}\Big).$$

It is often normalized so that a_n is positive. Thanks to this formula, it is not difficult to deduce the Alexander polynomial from the Conway polynomial. If we use the Fibonacci basis, it is even easier to deduce the Conway polynomial of a knot from its Alexander polynomial.

Lemma 4.1. *If* $z = t^{1/2} - t^{-1/2}$, *and* $n \in \mathbf{Z}$ *, then we have the identity*

$$f_{n+1}(z) + f_{n-1}(z) = (t^{1/2})^n + (-t^{-1/2})^n,$$

where the $f_k(z)$ *are the Fibonacci polynomials.*

Proof. Let $A = \begin{bmatrix} z & 1 \\ 1 & 0 \end{bmatrix}$ be the (polynomial) Fibonacci matrix. If $z = t^{1/2} - t^{-1/2}$, then the eigenvalues of A are $t^{1/2}$ and $-t^{-1/2}$, and consequently $\operatorname{tr} A^n = (t^{1/2})^n + (-t^{-1/2})^n$. On the other hand, we have $A^n = \begin{bmatrix} f_{n+1}(z) & f_n(z) \\ f_n(z) & f_{n-1}(z) \end{bmatrix}$, and then $\operatorname{tr} A^n = f_{n+1}(z) + f_{n-1}(z)$. $\square$

From Lemma 4.1, we immediately deduce:

Proposition 4.2. *Let the Laurent polynomial* $P(t)$ *be defined by*

$$P(t) = a_0 - a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) - \cdots + (-1)^n a_n (t^n + t^{-n}).$$

We have

$$P(t) = \sum_{k=0}^{n} (-1)^k (a_k - a_{k+1}) f_{2k+1}(z),$$

where $z = t^{1/2} - t^{-1/2}$ *and* $a_{n+1} = 0$.

Using the substitution $a_0 = \ldots = a_n = 1$, We deduce the following useful formula.

$$f_{2n+1}\big(t^{1/2} - t^{-1/2}\big) = (t^n + t^{-n}) - (t^{n-1} + t^{1-n}) + \cdots + (-1)^n. \tag{5}$$

Then, we deduce a simple proof of an elegant criterion due to Murasugi ([21, 3])

Corollary 4.3 (Murasugi (1971)) *Let $\Delta(t) = a_0 - a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) - \cdots + (-1)^n a_n(t^n + t^{-n})$ be the Alexander polynomial of a two-bridge knot. There exists an integer $k \le n$ such that $a_0, a_1, \ldots, a_k$ are odd, and $a_{k+1}, \ldots, a_n$ are even.*

Proof. If K is a two-bridge knot, its Conway polynomial is a modulo 2 Fibonacci polynomial f_{2k+1}. By Proposition 4.2 we have $f_{2k+1}\big(t^{1/2} - t^{-1/2}\big) = (t^k + t^{-k}) - (t^{k-1} + t^{1-k}) + \cdots + (-1)^k$, and the result follows. □

Remark 4.4. This congruence may be used as a simple criterion to prove that some knots cannot be two-bridge knots. There is a more efficient criterion by Kanenobu [10, 24] using the Jones and Q polynomials.

We also deduce an analogous result for two-component links

Corollary 4.5 (Modulo 2 Hosokawa polynomials of two-bridge links) *Let $\Delta(t) = \big(t^{1/2} - t^{-1/2}\big)\Big(a_0 - a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) - \cdots + (-1)^n a_n(t^n + t^{-n})\Big)$ be the Alexander polynomial of a two-component two-bridge link. Then all the coefficients a_i are even or there exists an integer $k \le n$ such that $a_k, a_{k-2}, a_{k-4}, \ldots$ are odd, and the other coefficients are even.*

Proof. If K is a two-component two-bridge link, its Conway polynomial is an odd Fibonacci polynomial modulo 2, that is of the form $f_{2h}(z)$. An easy induction shows that

$$f_{4k}\big(t^{1/2} - t^{-1/2}\big) = \big(t^{1/2} - t^{-1/2}\big)\big(u_1 + u_3 + \cdots + u_{2k-1}\big)$$

and

$$f_{4k+2}\big(t^{1/2} - t^{-1/2}\big) = \big(t^{1/2} - t^{-1/2}\big)\big(1 + u_2 + \cdots + u_{2k}\big),$$

where $u_j = t^j + t^{-j}$, and the result follows. □

Remark 4.6. This rectifies Satz 4 in [3, p. 186].

Now, we shall show that Theorem 3.7 implies both Murasugi and Hartley theorems for two-bridge knots:

Theorem 4.7 (Murasugi (1958), Hartley (1979)) *Let*

$$P(t) = a_0 - a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) - \cdots + (-1)^n a_n(t^n + t^{-n}),\ a_n > 0$$

be the Alexander polynomial of a two-bridge knot. There exists an integer $k \le n$ such that $a_0 = a_1 = \ldots = a_k > a_{k+1} > \ldots > a_n$.

Proof. Let K be a two-bridge knot and $\nabla(z) = \alpha_0 f_1 - \alpha_1 f_3 + \cdots + (-1)^n \alpha_n f_{2n+1}$ be its Conway polynomial expressed in the Fibonacci basis. By Theorem 3.7 $\alpha_n \alpha_k \geq 0$ for all k, and if $\alpha_i = 0$ for some i then $\alpha_j = 0$ for $j \leq i$.

Let $\Delta(t) = a_0 - a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) - \cdots + (-1)^n a_n(t^n + t^{-n}),\ a_n > 0$ be the Alexander polynomial of K. We have $\Delta(t) = \varepsilon \nabla(t^{1/2} - t^{-1/2})$, where $\varepsilon = \pm 1$, and then, by Corollary 4.2, $\varepsilon \alpha_k = a_k - a_{k+1}$. We deduce that $\varepsilon \alpha_n = a_n > 0$, and then $a_k - a_{k+1} = \varepsilon \alpha_k \geq 0$ for all k. Consequently we obtain $a_0 \geq a_1 \geq \ldots \geq a_n > 0$.

Furthermore, if $a_k = a_{k-1}$ for some k, then $\alpha_{k-1} = 0$, and consequently $\alpha_{j-1} = 0$ for all $j \leq k$. This implies that for all $j \leq k$, $a_j = a_{j-1}$, which concludes the proof. □

Now, we shall give explicit formulas for Alexander coefficients in terms of Conway coefficients.

Proposition 4.8. *Let $Q(z) = \tilde{c}_0 + \tilde{c}_1 z^2 + \cdots + \tilde{c}_n z^{2n}$ be a polynomial. We have*

$$Q(t^{1/2} - t^{-1/2}) = a_0 - a_1(t + t^{-1}) + a_2(t^2 + t^{-2}) - \cdots + (-1)^n a_n(t^n + t^{-n}),$$

where

$$a_{n-j} = \sum_{k=0}^{j} (-1)^{n-k} \tilde{c}_{n-k} \binom{2n-2k}{j-k}. \tag{6}$$

Proof. It is sufficient to prove Formula (6) for the monomials $Q(z) = z^{2m}$. Let us consider $u_i = t^i + t^{-i}$. By the binomial formula we have

$$\left(t^{1/2} - t^{-1/2}\right)^{2m} = \sum_{k=0}^{m-1} (-1)^k \binom{2m}{k} u_{m-k} + (-1)^m \binom{2m}{m}.$$

and then $a_{n-j} = (-1)^m \binom{2m}{h}$ where $m - h = n - j$. On the other hand, the proposed formula asserts

$$a_{n-j} = \sum_{k=0}^{j} (-1)^{n-k} \tilde{c}_{n-k} \binom{2n-2k}{j-k} = (-1)^m \binom{2m}{h} \quad \text{where} \quad h = m + j - n,$$

which is the same result. □

Remark 4.9. Considering the Fibonacci polynomials $f_{2n+1} = \sum_{k=0}^{n} \binom{2n-k}{k} z^{2n-2k}$, Formulas (5) and (6) give the identity

$$\sum_{k=0}^{j} (-1)^k \binom{2n-k}{k} \binom{2n-2k}{j-k} \ = \ 1, \quad n, j \geq 0.$$

Remark 4.10. Fukuhara [6] gives a converse formula for the c_k in terms of the a_k,

$$\tilde{c}_{n-j} = \sum_{k=0}^{j} (-1)^{n-k} a_{n-k} \tfrac{2n-2k}{2n-j-k} \tbinom{2n-j-k}{2n-2j}.$$

We shall not use this formula.

From the bounds we obtained for Conway coefficients we can deduce a simple proof of the Nakanishi–Suketa bounds ([22, Th. 1, 2]) for the Alexander coefficients. [] ZZ

Corollary 4.11 (Nakanishi–Suketa (1993)) *We have the following sharp inequalities (where all the a_i are positive):*

1. $a_{n-j} \le a_n \Big(\sum_{k=0}^{j} \binom{2n-2k}{j-k} \binom{2n-k}{k} \Big).$
2. $2a_n - 1 \le a_{n-1} \le (4n-2)a_n + 1.$

Proof.

1. Using Formula (6) and Theorem 3.3, we obtain

$$|a_{n-j}| \le \sum_{k=0}^{j} |\tilde{c}_{n-k}| \tbinom{2n-2k}{j-k} \le |a_n| \sum_{k=0}^{j} \tbinom{2n-k}{k} \tbinom{2n-2k}{j-k}. \qquad (7)$$

2. We have $|\tilde{c}_{n-1}| \le \binom{2n-2}{1} |\tilde{c}_n| + 1$ by Theorem 3.6, and $a_{n-1} = \tilde{c}_{n-1} - \binom{2n}{1} \tilde{c}_n$ by Proposition 4.8. We thus deduce

$$|a_{n-1}| \le \tbinom{2n}{1} |\tilde{c}_n| + \tbinom{2n-2}{1} |\tilde{c}_n| + 1 = (4n-2)\,|a_n| + 1. \qquad (8)$$

We also have

$$|a_{n-1}| \ge \tbinom{2n}{1} |\tilde{c}_n| - |\tilde{c}_{n-1}| \ge \tbinom{2n}{1} |\tilde{c}_n| - \tbinom{2n-2}{1} |\tilde{c}_n| - 1 = 2\,|a_n| - 1.$$

The upper bounds (7) and (8) are attained by the knots $C(2, 2, \ldots, 2)$. $\square$

We also have the following sharp bound, which improves the Nakanishi–Suketa third bound ([22, Th. 3])

Theorem 4.12. *If $a_n \neq 1$, then $a_{n-2} \le (8n^2 - 15n + 8)a_n + 2n - 1$. This bound is sharp.*

Proof. From Proposition 4.8 and Theorem 3.6, we get

$$\begin{aligned} |a_{n-2}| &\le \tbinom{2n}{2} |\tilde{c}_n| + \tbinom{2n-2}{1} |\tilde{c}_{n-1}| + \tbinom{2n-4}{0} |\tilde{c}_{n-2}| \\ &\le \tbinom{2n}{2} |\tilde{c}_n| + \tbinom{2n-2}{1} (\tbinom{2n-2}{1} |\tilde{c}_n| + 1) + \Big(\tbinom{2n-3}{2} + \tfrac{1}{g} (\tbinom{2n-3}{1} - 1) \Big) |\tilde{c}_n| + 1 \\ &= (8n^2 - 16n + 10 + \tfrac{2(n-2)}{g})\,|a_n| + 2n - 1. \end{aligned}$$

If $a_n \neq 1$ then $g \geq 2$, and we obtain

$$|a_{n-2}| \leq |a_n|\,(8n^2 - 15n + 8) + 2n - 1. \tag{9}$$

This bound is attained for the knot $C(4, 2, 2, 2, \ldots, 2)$. $\square$

The following example shows that the bounds on the Conway coefficients are better than the bounds on the Alexander coefficients.

Example 4.13. *Let us consider the Conway polynomial $\nabla_K(z) = 1 + 8z^2 + 3z^4 - z^6$ of the knot $K = 13n1862$ (see [1]). It does not verify the bound of theorem 3.3, and then it is not a two-bridge knot. Nevertheless, its Alexander polynomial $\Delta_K(t) = 23 - 19(t+1/t) + 9(t^2 + 1/t^2) - (t^3 + 1/t^3)$ satisfies the bounds of Nakanishi and Suketa, and also the conditions of Murasugi and Hartley. This example shows that the conditions on the Conway coefficients are stronger than the conditions on the Alexander coefficient deduced from them.*

Remarks 4.14.

1. If $g \geq 3$, we obtained an improvement of the inequality (9):
$$a_{n-2} \leq (8n^2 - 16n + 10 + \tfrac{2(n-2)}{g})a_n + 2n - 1.$$
2. For $j = 3$ we obtain
$$\begin{aligned} a_{n-3} &\leq 2/3\ (2\,n-3)\left(8\,n^2 - 24\,n + 25\right) a_n + \tfrac{(3\,n-5)(2\,n-5)}{g} a_n + n\,(2\,n-3) \\ &\leq 1/6\ \left(64\,n^3 - 270\,n^2 + 413\,n - 225\right) a_n + n\,(2\,n-3)\,. \end{aligned}$$
3. Since the inequalities on Conway coefficients are simpler and stronger, we shall not give the inequalities on Alexander coefficients for $j \geq 4$.

5 A conjecture

We have computed the Conway polynomials of the 131 839 two-bridge links and knots with 20 or fewer crossings, using Siebenmann's method. We observed the following property:

Conjecture 5.1. *Let $\nabla_m = c_m\Big(\sum_{i=0}^{\lfloor \frac{m}{2} \rfloor}(-1)^i \alpha_i f_{m+1-2i}\Big)$, $\alpha_0 = 1$, be the Conway polynomial of a two-bridge link (or knot) written in the Fibonacci basis. Then there exists $n \leq \lfloor \frac{m}{2} \rfloor$ such that*

$$0 \leq \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n, \quad \alpha_n \geq \alpha_{n+1} \geq \cdots \geq \alpha_{\lfloor \frac{m}{2} \rfloor} \geq 0.$$

If this conjecture was true, it would imply the following property of Alexander polynomials:

Conjecture 5.2. *Let $P(t) = a_0 - a_1(t+t^{-1}) + a_2(t^2+t^{-2}) - \cdots + (-1)^n a_n(t^n + t^{-n})$ be the Alexander polynomial of a two-bridge knot. Then there exists an integer $k \leq n$ such that $(a_0, \ldots, a_k)$ is convex and $(a_k, \ldots, a_n)$ is concave.*

This property detects many non two-bridged polynomials which are not detected by the other conditions.

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