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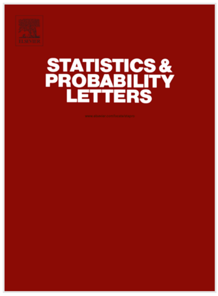

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A note on birth-death processes with catastrophes*

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Abstract

For a birth-death process defined on the state-space $\mathcal{S} = \{0, 1, 2, \ldots\}$ and subject to catastrophes the first effective catastrophe occurrence time is considered. The Laplace transform of its probability density function, expectation and variance are determined.

Keywords: birth-death processes; catastrophes; occurrence times.

AMS 2000 Subject Classification: 60J27, 60J80

1 Introduction

Great attention has been paid in the literature to the description of the evolution of systems modeled via discrete state-space random processes such as populations evolving in random environments or queueing and service systems under various operating protocols. More recently, certain systems have been studied also assuming that they may be subject to catastrophes. The contributions to the area in which the present paper belongs are too numerous to be exhaustively listed here. Hence, we limit ourselves to recalling the following ones: The results concerning *(i)* the distribution of the extinction time for a linear birth-death process subject to catastrophes and, in particular, the determination of necessary and sufficient conditions for population extinction to occur (see Lee, 2000, and Brockwell, 1985 and 1986); *(ii)* the studies on the transient and equilibrium behaviors of immigration-birth-death process with catastrophes (see Chao and Zheng, 2003, Kyriakidis, 1994, Renshaw and Chen, 1997, and Swift, 1997); *(iii)* the analysis of birth-death processes under the influence of Poisson time-distributed or state-dependent catastrophes (see Bartoszynski *et al.*, 1989, Peng *et al.*, 1993, Van Doorn and Zeifman, 2005); *(iv)* the study of the joint distribution of catastrophe time and process state at the catastrophe time, for diffusion processes and Markov chains (see Berman and Frydman, 1996); *(v)* the determination of

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transient and limiting distributions as well as other quantities of interest for continuous-time Markov chains subject to catastrophes (see Chen *et al.*, 2004, Economou and Fakinos, 2003, Kyriakidis, 2001, 2002 and 2004, Pakes, 1997, Stirzaker, 2001, Swift, 2000, and Switkes, 2004); *(vi)* analysis of the effect of catastrophes and jumps in the case of $M/M/1$ and other Markov queueing systems (see Chen and Renshaw, 1997 and 2004, and Di Crescenzo *et al.*, 2003), including the case when the number of initially present customers is random (see Krishna Kumar and Arivudainambi, 2000). Two recent papers are also of interest: Stirzaker (2006) looks at hitting times for a general Markov processes subject to catastrophes, whereas Stirzaker (2007) deals with an even more general model where the process is switched to another state at Poisson event times, and the change of state is governed by a stochastic matrix.

The classical paradigm about processes subject to catastrophes assumes that disasters occur according to Poisson processes or according to more general counting processes. The effect of a catastrophe is to reset the state of the process to zero, the system being immediately able to evolve afresh. In our set-up, the catastrophe effect is not observable if the state of the process is 0. Hence, it is of interest to investigate the first occurrence of an "effective catastrophe", namely a disaster that occurs when the state of the process is not 0. This is precisely the object of the present paper.

In the sequel, we shall recall certain useful results, such as the relations among transition probabilities in the presence and in the absence of catastrophes. More specifically, in Section 2 we shall introduce a birth-death process with catastrophes $\{N(t);\, t \geq 0\}$ with state space $\mathcal{S} = \{0, 1, 2, \ldots\}$, and shall outline various functional relations that allow to describe $N(t)$ in terms of the birth-death process $\{\widehat{N}(t); t \geq 0\}$ defined on the same state-space $\mathcal{S}$ and characterized by the same birth and death rates as $N(t)$, but for which catastrophes are absent.

The problem of the first occurrence of a catastrophe is considered in Section 3, where the Laplace transform of the pdf of catastrophe's first-occurrence time and its mean and variance are obtained.

2 Background

Let $\{N(t);\, t \geq 0\}$ be a birth-death process with catastrophes defined on the state-space $\mathcal{S} = \{0, 1, 2, \ldots\}$, such that transitions occur according to the following scheme:
(i) $n \to n+1$ with rate α_n, for $n = 0, 1, \ldots$,
(ii) $n \to n-1$ with rate β_n, for $n = 2, 3, \ldots$,
(iii) $1 \to 0$ with rate $\beta_1 + \xi$,
(iv) $n \to 0$ with rate ξ, for $n = 2, 3, \ldots$.

Hence, births occur with rates α_n, deaths with rates β_n, and catastrophes with rate ξ, the effect of each catastrophe being the instantaneous transition to the reflecting state 0. For all $j, n \in \mathcal{S}$ and $t > 0$ the transition probabilities

$$p_{j,n}(t) = \mathrm{P}\{N(t) = n \,|\, N(0) = j\}$$

satisfy the following system of forward equations:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} p_{j,0}(t) = -(\alpha_0 + \xi)\, p_{j,0}(t) + \beta_1\, p_{j,1}(t) + \xi, \\
&\frac{\mathrm{d}}{\mathrm{d}t} p_{j,n}(t) = -(\alpha_n + \beta_n + \xi)\, p_{j,n}(t) + \alpha_{n-1}\, p_{j,n-1}(t) + \beta_{n+1}\, p_{j,n+1}(t), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n = 1, 2, \ldots,
\end{aligned} \tag{1}$$

with initial condition

$$\lim_{t \downarrow 0} p_{j,n}(t) = \delta_{j,n} = \begin{cases} 1, & n = j \\ 0, & \text{otherwise.} \end{cases}$$

Denote by $\{\widehat{N}(t);\, t \geq 0\}$ the time-homogeneous birth-death process obtained from $N(t)$ by removing the possibility of catastrophes, i.e. by setting $\xi = 0$. The transition probabilities

$$\widehat{p}_{j,n}(t) = \mathrm{P}\{\widehat{N}(t) = n \,|\, \widehat{N}(0) = j\}, \qquad j, n \in \mathcal{S}, \quad t \geq 0$$

then satisfy the system of forward equations obtained from (1) by setting $\xi = 0$, with initial condition $\lim_{t \downarrow 0} \widehat{p}_{j,n}(t) = \delta_{j,n}$.

Hereafter, we shall restrict our attention to non-explosive processes $\widehat{N}(t)$, i.e. we shall assume that $\sum_{n=0}^{+\infty} \widehat{p}_{j,n}(t) = 1$ for all $j \in \mathcal{S}$ and $t \geq 0$.

We note that some descriptors of $N(t)$ can be expressed in terms of the corresponding ones of $\widehat{N}(t)$. Indeed, making use of the forward equations for probabilities $p_{j,n}(t)$ and for $\widehat{p}_{j,n}(t)$, for all $j, n \in \mathcal{S}$ and $t > 0$ we have (see for instance Eq. (2.2) of Pakes, 1997):

$$p_{j,n}(t) = e^{-\xi t}\, \widehat{p}_{j,n}(t) + \xi \int_0^t e^{-\xi \tau}\, \widehat{p}_{0,n}(\tau)\, \mathrm{d}\tau. \tag{2}$$

Moreover, by setting

$$\pi_{j,n}(\lambda) := \int_0^{+\infty} e^{-\lambda t}\, p_{j,n}(t)\, \mathrm{d}t, \qquad \widehat{\pi}_{j,n}(\lambda) := \int_0^{+\infty} e^{-\lambda t}\, \widehat{p}_{j,n}(t)\, \mathrm{d}t, \qquad \lambda > 0,$$

from (2) it follows (see Eq. (3.11) of Pakes, 1997)

$$\pi_{j,n}(\lambda) = \widehat{\pi}_{j,n}(\lambda + \xi) + \frac{\xi}{\lambda}\, \widehat{\pi}_{0,n}(\lambda + \xi), \qquad \lambda > 0. \tag{3}$$

Due to Eq. (3), the steady state distribution

$$q_n := \lim_{t \to +\infty} p_{j,n}(t), \qquad j, n \in \mathcal{S}$$

can be expressed as (see for instance Eq. (2.3) of Pakes, 1997):

$$q_n = \xi\, \widehat{\pi}_{0,n}(\xi), \qquad n \in \mathcal{S}. \tag{4}$$

From the assumed non explosivity of $\widehat{N}(t)$ and from (4) it follows that $N(t)$ possesses a steady-state distribution, with $\sum_{n=0}^{+\infty} q_n = 1$.

Let us now define the first-visit time of $N(t)$ to state 0 as

$$T_{j,0} := \inf\{t \geq 0 : \, N(t) = 0\}, \qquad N(0) = j \in \{1, 2, \ldots\},$$

and denote its pdf by

$$g_{j,0}(t) = \frac{\mathrm{d}}{\mathrm{d}t} \mathrm{P}\{T_{j,0} \leq t\}.$$

Moreover, for the corresponding birth-death process $\widehat{N}(t)$ in the absence of catastrophes, we shall denote by $\widehat{T}_{j,0}$ the first-visit time, and by $\widehat{g}_{j,0}(t)$ its pdf. Hereafter, $\gamma_{j,0}(\lambda)$ and $\widehat{\gamma}_{j,0}(\lambda)$ will denote the Laplace transforms of $g_{j,0}(t)$ and $\widehat{g}_{j,0}(t)$, respectively. It is not hard to make use of probabilistic arguments to see that these functions are related. Indeed, since for $j = 1, 2, \ldots$ the random variable $\min\{\widehat{T}_{j,0}, Z\}$ has the same distribution as $T_{j,0}$, where Z is an exponentially distributed r.v. independent of $\widehat{T}_{j,0}$ with mean ξ^{-1}, one has:

$$g_{j,0}(t) = e^{-\xi t}\, \widehat{g}_{j,0}(t) + \xi\, e^{-\xi t} \left[1 - \int_0^t \widehat{g}_{j,0}(\tau)\, \mathrm{d}\tau\right], \qquad t > 0, \tag{5}$$

$$\gamma_{j,0}(\lambda) = \frac{\lambda}{\lambda + \xi}\, \widehat{\gamma}_{j,0}(\lambda + \xi) + \frac{\xi}{\lambda + \xi}, \qquad \lambda > 0. \tag{6}$$

For $j = 1, 2, \ldots$ mean and variance of $T_{j,0}$ can be expressed as

$$\mathrm{E}(T_{j,0}) = \frac{1}{\xi} \left[1 - \widehat{\gamma}_{j,0}(\xi)\right], \tag{7}$$

$$\mathrm{Var}(T_{j,0}) = \frac{1}{\xi^2} \left[1 - \widehat{\gamma}^2_{j,0}(\xi) + 2\xi\, \frac{\mathrm{d}}{\mathrm{d}\xi} \widehat{\gamma}_{j,0}(\xi)\right]. \tag{8}$$

Use of Eqs. (4) to (8) will be made in Section 3 in order to analyze the problem of the first occurrence of an effective catastrophe.

3 First occurrence of effective catastrophe

Descriptions of various stochastic processes subject to catastrophes occurring according to Poisson processes are present in the literature (see for instance Pakes, 1997). Customarily, a catastrophe is assumed to occur even when the process is in state zero, and thus remains there. On the contrary, in the present paper we shall restrict our considerations to the case when a catastrophe is able to change to zero any positive state of the process, thus excluding catastrophes such that the zero-state is left unchanged. Such catastrophes will be denoted by us as "effective". Hence, in our approach catastrophes occurring while the process is in state zero will not be taken into account. Consequently, in our set-up the catastrophes first-occurrence time is no longer exponentially distributed.

Let us denote by $C_{j,0}$ the first occurrence time of an effective catastrophe, when $N(0) = j$, with $j \in \mathcal{S}$. An effective catastrophe, or shortly "a catastrophe" from now on, produces a transition, with rate ξ, from any state $n > 0$ to the reflecting state 0. Hence, certain transitions from 1 to 0 may be due to the occurrence of a catastrophe (with rate ξ), whereas the remaining transitions are due to a death (with rate β_1).

In order to investigate on the features of $C_{j,0}$, let us refer to a modified birth-death process with catastrophes that will be denoted as $\{M(t);\, t \geq 0\}$. This is assumed to be defined on the state-space $\{-1, 0, 1, \ldots\}$. Its behavior is identical to that of $N(t)$, the only difference being that the effect of a catastrophe from state $n > 0$ is a jump from n to the absorbing state -1. In other words, the allowed transitions of $M(t)$ are the following:

(i) $n \to n+1$ with rate α_n, for $n = 0, 1, \ldots$,

(ii) $n \to n-1$ with rate β_n, for $n = 1, 2, \ldots$,

(iii) $n \to -1$ with rate ξ, for $n = 1, 2, \ldots$.

For all $t \geq 0$ and $j \in \mathcal{S}$, $n \in \{-1, 0, 1, \ldots\}$, let us now consider the transition probabilities of the modified process

$$h_{j,n}(t) = \mathrm{P}\{M(t) = n \,|\, M(0) = j\}.$$

The link between $M(t)$ and $C_{j,0}$ is evident by noting that the transitions of $M(t)$ from $n > 0$ to -1 corresponds to the transitions of $N(t)$ from n to 0 due to a catastrophe. Hence, denoting by $d_{j,0}(t)$ the density of $C_{j,0}$ for all $t > 0$ we have:

$$\mathrm{P}(C_{j,0} > t) \equiv \int_t^{+\infty} d_{j,0}(\tau)\,\mathrm{d}\tau = \sum_{n=0}^{+\infty} h_{j,n}(t) = 1 - h_{j,-1}(t), \qquad j \in \mathcal{S}. \tag{9}$$

Moreover, for all $j \in \mathcal{S}$ the following system of forward equations holds:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} h_{j,-1}(t) = \xi\,[1 - h_{j,-1}(t) - h_{j,0}(t)], \\
&\frac{\mathrm{d}}{\mathrm{d}t} h_{j,0}(t) = -\alpha_0\, h_{j,0}(t) + \beta_1\, h_{j,1}(t), \\
&\frac{\mathrm{d}}{\mathrm{d}t} h_{j,n}(t) = -(\alpha_n + \beta_n + \xi)\, h_{j,n}(t) + \alpha_{n-1}\, h_{j,n-1}(t) + \beta_{n+1}\, h_{j,n+1}(t), \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n = 1, 2, \ldots,
\end{aligned} \tag{10}$$

with initial condition

$$h_{j,n}(0) = \delta_{j,n}.$$

Let us denote by $\eta_{j,n}(\lambda)$ the Laplace transform of $h_{j,n}(t)$. In the following theorem we shall express $\eta_{j,n}(\lambda)$ in terms of $\widehat{\pi}_{j,n}(\lambda)$.

Theorem 3.1 *For all $j \in \mathcal{S}$ and $\lambda > 0$ we have:*

$$\eta_{j,-1}(\lambda) = \frac{\xi}{\lambda + \xi} \left[\frac{1}{\lambda} - \frac{\widehat{\pi}_{j,0}(\lambda + \xi)}{1 - \xi\, \widehat{\pi}_{0,0}(\lambda + \xi)} \right], \tag{11}$$

$$\eta_{j,n}(\lambda) = \widehat{\pi}_{j,n}(\lambda + \xi) + \xi\, \widehat{\pi}_{0,n}(\lambda + \xi)\, \frac{\widehat{\pi}_{j,0}(\lambda + \xi)}{1 - \xi\, \widehat{\pi}_{0,0}(\lambda + \xi)}, \qquad n = 0, 1, \ldots. \tag{12}$$

Proof. The proof goes as follows. First, the Laplace transform of a solution of the differential system satisfied by the transition probabilities such that initial conditions are fulfilled is determined. It is then shown that this is indeed the correct probabilistic solution by retrospectively justifying the mentioned formal procedure via a direct probabilistic construction.

We start assuming $j = 0$. Taking the Laplace transform of second and third equations in (10), we obtain:

$$\begin{aligned}&(\lambda+\alpha_0)\,\eta_{0,0}(\lambda)-1=\beta_1\,\eta_{0,1}(\lambda),\\&(\lambda+\alpha_n+\beta_n+\xi)\,\eta_{0,n}(\lambda)=\alpha_{n-1}\,\eta_{0,n-1}(\lambda)+\beta_{n+1}\,\eta_{0,n+1}(\lambda),\qquad n>0.\end{aligned}\tag{13}$$

We now look for a solution of the form

$$\eta_{0,n}(\lambda)=A(\lambda)\,\pi_{0,n}(\lambda),\qquad n=0,1,\dots,\tag{14}$$

with $A(\lambda)$ to be determined. From (13) and (14), we obtain:

$$\begin{aligned}&A(\lambda)\,(\lambda+\alpha_0)\,\pi_{0,0}(\lambda)-1=\beta_1\,A(\lambda)\,\pi_{0,1}(\lambda),\\&(\lambda+\alpha_n+\beta_n+\xi)\,\pi_{0,0}(\lambda)=\alpha_{n-1}\,\pi_{0,n-1}(\lambda)+\beta_{n+1}\,\pi_{0,n+1}(\lambda),\qquad n>0.\end{aligned}\tag{15}$$

Moreover, by taking the Laplace transform of (1), for $j = 0$ it follows that:

$$\begin{aligned}&(\lambda+\alpha_0+\xi)\,\pi_{0,0}(\lambda)-1=\beta_1\,\pi_{0,1}(\lambda)+\frac{\xi}{\lambda},\\&(\lambda+\alpha_n+\beta_n+\xi)\,\pi_{0,0}(\lambda)=\alpha_{n-1}\,\pi_{0,n-1}(\lambda)+\beta_{n+1}\,\pi_{0,n+1}(\lambda),\qquad n>0.\end{aligned}\tag{16}$$

Comparing Eqs. (15) with Eqs. (16) one has:

$$A(\lambda)=\frac{\lambda}{\lambda+\xi-\lambda\,\xi\,\pi_{0,0}(\lambda)}.\tag{17}$$

Making use of (3) in (14), with $A(\lambda)$ given in (17), we obtain Eq. (12) for $j = 0$.

Next, let $j > 0$. Taking the Laplace transform of second and third equation in (10), we have:

$$\begin{aligned}&(\lambda+\alpha_0)\,\eta_{j,0}(\lambda)=\beta_1\,\eta_{j,1}(\lambda),\\&(\lambda+\alpha_n+\beta_n+\xi)\,\eta_{j,n}(\lambda)=\alpha_{n-1}\,\eta_{j,n-1}(\lambda)+\beta_{n+1}\,\eta_{j,n+1}(\lambda),\\&\qquad\qquad n>0,\ \ n\neq j,\\&(\lambda+\alpha_j+\beta_j+\xi)\,\eta_{j,j}(\lambda)-1=\alpha_{j-1}\,\eta_{j,j-1}(\lambda)+\beta_{j+1}\,\eta_{j,j+1}(\lambda).\end{aligned}\tag{18}$$

We look for a solution of the form

$$\eta_{j,n}(\lambda)=B(\lambda)\,\pi_{j,n}(\lambda)+C(\lambda)\,\pi_{0,n}(\lambda),\qquad n=0,1,\dots,\tag{19}$$

with $B(\lambda)$ and $C(\lambda)$ to be determined. Substuting Eq. (19) in (18) and recalling the Laplace transform of (1) for $j = 1, 2, \dots$, one has:

$$B(\lambda)=1,\qquad C(\lambda)=\frac{\xi\,[\lambda\,\pi_{j,0}(\lambda)-1]}{\lambda+\xi-\xi\,\lambda\,\pi_{0,0}(\lambda)}.\tag{20}$$

Hence, making use of (3) in (19), with $B(\lambda)$ and $C(\lambda)$ given in (20), some straightforward calculations lead us to Eq. (12) for $j = 1, 2, \ldots$. Furthermore, taking the Laplace transform of the first equation in (10) we obtain:

$$\eta_{j,-1}(\lambda) = \frac{\xi}{\lambda + \xi}\left[\frac{1}{\lambda} - \eta_{j,0}(\lambda)\right].$$

Hence, making use of (12) for $n = 0$, Eq. (11) finally follows.

Note that (11) and (12) give $\sum_{n=-1}^{+\infty} h_{j,n}(t) = 1$ for all $t \geq 0$ and $j \in \mathcal{S}$.

We shall now show that (11) and (12) are actually the Laplace transforms of the transition probabilities. We consider separately the cases $n = -1$, $n = 0$ and $n = 1, 2, \ldots$.

(i) Let $n = -1$. For all $t > 0$ one has

$$h_{0,-1}(t) = \int_0^t \alpha_0\, e^{-\alpha_0 \tau}\, h_{1,-1}(t-\tau)\, \mathrm{d}\tau, \tag{21}$$

$$h_{j,-1}(t) = \int_0^t \xi e^{-\xi\tau}\left[1 - \int_0^\tau \widehat{g}_{j,0}(\theta)\, \mathrm{d}\theta\right] \mathrm{d}\tau + \int_0^t e^{-\xi\tau}\, \widehat{g}_{j,0}(\tau)\, h_{0,-1}(t-\tau)\, \mathrm{d}\tau \qquad (j = 1, 2, \ldots). \tag{22}$$

Eq. (21) expresses the circumstance that any transition from 0 to -1 is accompanied by a transition through 1. In turn, Eq. (22) holds since, starting from $j \geq 1$, one has $M(t) = -1$ if and only if either a catastrophe occurs before $\widehat{N}(t)$ hits 0, or $\widehat{N}(t)$ hits 0 before t without any catastrophe, going then from 0 to -1 during the remaining time. By Laplace transforming (21) and (22) one obtains

$$\eta_{0,-1}(\lambda) = \frac{\alpha_0}{\alpha_0 + \lambda}\, \eta_{1,-1}(\lambda),$$

$$\eta_{j,-1}(\lambda) = \frac{\xi}{\lambda(\lambda+\xi)}\left[1 - \widehat{\gamma}_{j,0}(\lambda+\xi)\right] + \widehat{\gamma}_{j,0}(\lambda+\xi)\, \eta_{0,-1}(\lambda) \quad (j = 1, 2, \ldots),$$

and hence

$$\begin{aligned}
\eta_{0,-1}(\lambda) &= \frac{\alpha_0}{\alpha_0 + \lambda}\, \frac{\xi}{\lambda(\lambda+\xi)}\, \frac{1 - \widehat{\gamma}_{1,0}(\lambda+\xi)}{1 - \dfrac{\alpha_0}{\alpha_0+\lambda}\, \widehat{\gamma}_{1,0}(\lambda+\xi)}, \\
\eta_{j,-1}(\lambda) &= \frac{\xi}{\lambda(\lambda+\xi)}\left\{1 - \frac{\lambda}{\alpha_0 + \lambda}\, \frac{\widehat{\gamma}_{j,0}(\lambda+\xi)}{1 - \dfrac{\alpha_0}{\alpha_0+\lambda}\, \widehat{\gamma}_{1,0}(\lambda+\xi)}\right\} \quad (j = 1, 2, \ldots).
\end{aligned} \tag{23}$$

Eqs. (23) are equivalent to Eq. (11). Indeed, for $\widehat{N}(t)$ one has

$$\widehat{p}_{j,0}(t) = \int_0^t \widehat{g}_{j,0}(\tau)\, \widehat{p}_{0,0}(t-\tau)\, \mathrm{d}\tau \qquad (j = 1, 2, \ldots)$$

so that

$$\widehat{\gamma}_{j,0}(\lambda) = \frac{\widehat{\pi}_{j,0}(\lambda)}{\widehat{\pi}_{0,0}(\lambda)} \qquad (j = 1, 2, \ldots). \tag{24}$$

Furthermore,

$$\widehat{p}_{0,0}(t) = \int_0^t \alpha_0\, e^{-\alpha_0 \tau}\, \widehat{p}_{1,0}(t-\tau)\,\mathrm{d}\tau + e^{-\alpha_0 t},$$

yielding

$$\widehat{\pi}_{1,0}(\lambda) = \frac{\alpha_0+\lambda}{\alpha_0}\,\widehat{\pi}_{0,0}(\lambda) - \frac{1}{\alpha_0}. \tag{25}$$

From (24) and (25) one obtaines

$$1 - \frac{\alpha_0}{\alpha_0+\lambda}\,\widehat{\gamma}_{1,0}(\lambda+\xi) = \frac{1-\xi\,\widehat{\pi}_{0,0}(\lambda+\xi)}{(\alpha_0+\lambda)\,\widehat{\pi}_{0,0}(\lambda+\xi)}, \tag{26}$$

$$1 - \widehat{\gamma}_{1,0}(\lambda+\xi) = \frac{1-(\lambda+\xi)\,\widehat{\pi}_{0,0}(\lambda+\xi)}{\alpha_0\,\widehat{\pi}_{0,0}(\lambda+\xi)}. \tag{27}$$

Eq. (11) finally follows by using (24), (26) and (27) in (23).

(ii) Let $n=0$. For all $t>0$,

$$h_{0,0}(t) = \int_0^t \alpha_0\, e^{-\alpha_0\tau}\, h_{1,0}(t-\tau)\,\mathrm{d}\tau + e^{-\alpha_0 t}, \tag{28}$$

$$h_{j,0}(t) = \int_0^t e^{-\xi\tau}\,\widehat{g}_{j,0}(\tau)\, h_{0,0}(t-\tau)\,\mathrm{d}\tau \qquad (j=1,2,\ldots). \tag{29}$$

Eq. (28) expresses the partition of transitions from 0 to 0 into those characterized by at least one transition from 0 to 1 and those that never leave the state 0. Eq. (29) expresses the transitions to 0 for the first time in the absence of catastrophe, followed by transitions from 0 to 0. Use of Laplace transform in (28) and (29) yields

$$\eta_{0,0}(\lambda) = \frac{\alpha_0}{\alpha_0+\lambda}\,\eta_{1,0}(\lambda) + \frac{1}{\alpha_0+\lambda},$$
$$\eta_{j,0}(\lambda) = \widehat{\gamma}_{j,0}(\lambda+\xi)\,\eta_{0,0}(\lambda) \qquad (j=1,2,\ldots)$$

and hence

$$\begin{aligned}\eta_{0,0}(\lambda) &= \frac{1}{\alpha_0+\lambda}\left[1-\frac{\alpha_0}{\alpha_0+\lambda}\,\widehat{\gamma}_{1,0}(\lambda+\xi)\right]^{-1},\\ \eta_{j,0}(\lambda) &= \widehat{\gamma}_{j,0}(\lambda+\xi)\,\frac{1}{\alpha_0+\lambda}\left[1-\frac{\alpha_0}{\alpha_0+\lambda}\,\widehat{\gamma}_{1,0}(\lambda+\xi)\right]^{-1} \qquad (j=1,2,\ldots).\end{aligned} \tag{30}$$

Use of (24) and (26) in (30) leads one to (12) written for $n=0$.

(iii) Let $n=1,2,\ldots$. For all $t>0$, $h_{j,n}(t)$ satisfy

$$h_{0,n}(t) = \int_0^t \alpha_0\, e^{-\alpha_0\tau}\, h_{1,n}(t-\tau)\,\mathrm{d}\tau, \tag{31}$$

$$h_{j,n}(t) = p_{j,n}(t) - \int_0^t g_{j,0}(\tau)\, p_{0,n}(t-\tau)\,\mathrm{d}\tau + \int_0^t e^{-\xi\tau}\,\widehat{g}_{j,0}(\tau)\, h_{0,n}(t-\tau)\,\mathrm{d}\tau \qquad (j=1,2,\ldots). \tag{32}$$

Eq. (31) holds because for $M(t)$ the transitions from 0 to n imply at least one transition from 0 to 1. Eq. (32) expresses the partition of the transitions into those that have never visited

0 and those that have reached for the first time 0 without any previous catastrophe and that successively go from 0 to n again in the absence of catastrophe. Laplace transforming (31) and (32) yields

$$\eta_{0,n}(\lambda) = \frac{\alpha_0}{\alpha_0+\lambda}\,\eta_{1,n}(\lambda),$$
$$\eta_{j,n}(\lambda) = \pi_{j,n}(\lambda) - \gamma_{j,0}(\lambda)\,\pi_{0,n}(\lambda) + \widehat{\gamma}_{j,0}(\lambda+\xi)\,\eta_{0,n}(\lambda) \qquad (j=1,2,\ldots),$$

from which

$$\eta_{0,n}(\lambda) = \frac{\alpha_0}{\alpha_0+\lambda}\,\frac{\pi_{1,n}(\lambda)-\gamma_{1,0}(\lambda)\,\pi_{0,n}(\lambda)}{1-\dfrac{\alpha_0}{\alpha_0+\lambda}\,\widehat{\gamma}_{1,0}(\lambda+\xi)},$$

$$\eta_{j,n}(\lambda) = \pi_{j,n}(\lambda)-\gamma_{j,0}(\lambda)\,\pi_{0,n}(\lambda) + \widehat{\gamma}_{j,0}(\lambda+\xi)\,\frac{\alpha_0}{\alpha_0+\lambda}\,\frac{\pi_{1,n}(\lambda)-\gamma_{1,0}(\lambda)\,\pi_{0,n}(\lambda)}{1-\dfrac{\alpha_0}{\alpha_0+\lambda}\,\widehat{\gamma}_{1,0}(\lambda+\xi)} \qquad (j=1,2,\ldots). \tag{33}$$

We now remark that

$$\widehat{p}_{0,n}(t) = \int_0^t \alpha_0\, e^{-\alpha_0\tau}\,\widehat{p}_{1,n}(t-\tau)\,\mathrm{d}\tau \qquad (n=1,2,\ldots),$$

yielding

$$\widehat{\pi}_{1,n}(\lambda) = \frac{\alpha_0+\lambda}{\alpha_0}\,\widehat{\pi}_{0,n}(\lambda) \qquad (n=1,2,\ldots). \tag{34}$$

Making use of (3), (6), (24), (25), (26) and (34) in Eqs. (33) finally yields Eq. (12) for $n=1,2,\ldots$ and $j=1,2,\ldots$. ■

Let us now denote by $\Delta_{j,0}(\lambda)$ the Laplace transform of $d_{j,0}(t)$, $j\in\mathcal{S}$.

Proposition 3.1 *For all $j\in\mathcal{S}$ there holds:*

$$\Delta_{j,0}(\lambda) = \gamma_{j,0}(\lambda) - \frac{\lambda}{\lambda+\xi}\,\frac{\widehat{\gamma}_{j,0}(\lambda+\xi)}{1-\xi\,\widehat{\pi}_{0,0}(\lambda+\xi)}, \tag{35}$$

where $\gamma_{0,0}(\lambda)=1$ and $\widehat{\gamma}_{0,0}(\lambda+\xi)=1$.

Proof. Since (9) implies $d_{j,0}(t)=\frac{\mathrm{d}}{\mathrm{d}t}h_{j,-1}(t)$, $j\in\mathcal{S}$, recalling (11) for $j\in\mathcal{S}$ and $\lambda>0$ we have:

$$\Delta_{j,0}(\lambda) = \lambda\,\eta_{j,-1}(\lambda) = \frac{\xi}{\lambda+\xi} - \frac{\lambda}{\lambda+\xi}\,\frac{\xi\,\widehat{\pi}_{j,0}(\lambda+\xi)}{1-\xi\,\widehat{\pi}_{0,0}(\lambda+\xi)}. \tag{36}$$

Hence, making use of (24), Eq. (36) can be re-written as

$$\Delta_{j,0}(\lambda) = \frac{\xi}{\lambda+\xi} + \frac{\lambda}{\lambda+\xi}\,\widehat{\gamma}_{j,0}(\lambda+\xi) - \frac{\lambda}{\lambda+\xi}\,\frac{\widehat{\gamma}_{j,0}(\lambda+\xi)}{1-\xi\,\widehat{\pi}_{0,0}(\lambda+\xi)}, \qquad j>0,$$

so that, the thesis follows by virtue of (6). ■

Recalling again that $d_{j,0}(t) = \frac{\mathrm{d}}{\mathrm{d}t} h_{j,-1}(t)$, $j \in \mathcal{S}$, from (21) and (22) we immediately obtain

$$d_{0,0}(t) = \int_0^t \alpha_0 \, e^{-\alpha_0 \tau} \, d_{1,0}(t-\tau) \, \mathrm{d}\tau, \tag{37}$$

$$d_{j,0}(t) = \int_0^t e^{-\xi \tau} \, \widehat{g}_{j,0}(\tau) \, d_{0,0}(t-\tau) \, \mathrm{d}\tau + \xi e^{-\xi t} \left[1 - \int_0^t \widehat{g}_{j,0}(\theta) \, \mathrm{d}\theta \right] \qquad (j = 1, 2, \ldots). \tag{38}$$

Eq. (37) expresses the circumstance that the first occurrence time of an effective catastrophe when $N(0) = 0$ can be expressed as the sum of the time to reach state 1 and the time of first occurrence of an effective catastrophe starting from state 1. Moreover, Eq. (38) holds since the sample-paths of $N(t)$ that start from j and undergo the first catastrophe at time t can be partitioned into those that visit 0 for the first time before t in the absence of previous catastrophes and those that do not visit 0 before t. It is interesting to note that Eqs. (5) and (38) express the difference between the first-passage time density through state 0 and the first catastrophe time density.

Proposition 3.2 *For all $j \in \mathcal{S}$ there holds:*

$$\mathrm{E}(C_{j,0}) = \frac{1}{\xi} + \frac{\widehat{\pi}_{j,0}(\xi)}{1 - \xi \, \widehat{\pi}_{0,0}(\xi)}, \tag{39}$$

$$\begin{aligned} \mathrm{Var}(C_{j,0}) = \frac{1}{\xi^2} \Bigg\{ 1 - \frac{\xi^2 \, \widehat{\pi}_{j,0}^2(\xi)}{[1 - \xi \, \widehat{\pi}_{0,0}(\xi)]^2} - \frac{2\xi^2}{1 - \xi \, \widehat{\pi}_{0,0}(\xi)} \frac{\mathrm{d}}{\mathrm{d}\xi} \widehat{\pi}_{j,0}(\xi) \\ - \frac{2\xi^3 \, \widehat{\pi}_{j,0}(\xi)}{[1 - \xi \, \widehat{\pi}_{0,0}(\xi)]^2} \frac{\mathrm{d}}{\mathrm{d}\xi} \widehat{\pi}_{0,0}(\xi) \Bigg\}. \end{aligned} \tag{40}$$

Proof. For $j \in \mathcal{S}$, (39) follows by differentiating the right-hand-side of (35) with respect to λ, taking the limit $\lambda \to 0$, and by recalling (7). Furthermore, Eq. (40) can be similarly obtained by making use of (7) and (8). ■

Hereafter, we shall denote by

$$u_0 = 1, \qquad u_k = \frac{\alpha_0 \alpha_1 \cdots \alpha_{k-1}}{\beta_1 \beta_2 \cdots \beta_k}, \qquad k = 1, 2, \ldots$$

the potential coefficients of birth-death process $\widehat{N}(t)$.

We note that making use of (4) and of the well-known relation $u_j \, \widehat{p}_{j,n}(t) = u_n \, \widehat{p}_{n,j}(t)$, from Eqs. (39) and (40) for $j \in \mathcal{S}$ the following alternative expressions follow:

$$\mathrm{E}(C_{j,0}) = \frac{1}{\xi} \left(1 + \frac{1}{u_j} \frac{q_j}{1 - q_0} \right), \tag{41}$$

$$\mathrm{Var}(C_{j,0}) = \frac{1}{\xi^2}\left[1 - \frac{q_j^2}{u_j^2(1-q_0)^2} + \frac{2}{u_j(1-q_0)}\left(q_i - \xi\,\frac{\mathrm{d}}{\mathrm{d}\xi}q_j\right)\right.$$
$$\left. + \frac{2q_j}{u_j(1-q_0)^2}\left(q_0 - \xi\,\frac{\mathrm{d}}{\mathrm{d}\xi}q_0\right)\right],$$

where q_j's are given in (4). We stress that $C_{j,0}$ is not exponentially distributed, while the almost sure inequality $C_{j,0} \geq Z$ holds, with Z an exponentially distributed r.v. with mean ξ^{-1}.

As pointed out at the beginning of this section, and as is also evident from (36), $C_{j,0}$ is not exponentially distributed. In other words, the following

Hereafter we shall study the behaviour of the mean first catastrophe time for small and for large values of ξ.

Proposition 3.3 *(i) If $\widehat{N}(t)$ is positive recurrent then*

$$\lim_{\xi\downarrow 0}\xi\,\mathrm{E}\left(C_{j,0}\right) = \frac{1}{1-\widehat{q}_0}\,, \tag{42}$$

where $\widehat{q}_0 = \lim_{t\to\infty}\widehat{p}_{j,0}(t)$.
(ii) If $\widehat{N}(t)$ is transient then

$$\lim_{\xi\downarrow 0}\left\{\mathrm{E}\left(C_{j,0}\right) - \frac{1}{\xi}\right\} = \sum_{k=j}^{+\infty}\frac{1}{\alpha_k u_k}. \tag{43}$$

Proof. (i) If $\widehat{N}(t)$ is positive recurrent with limiting stationary law $\{\widehat{q}_n\}$, then making use of a Tauberian theorem from (39) we have

$$\lim_{\xi\downarrow 0}\xi\,\mathrm{E}\left(C_{j,0}\right) = 1 + \frac{\lim_{t\to+\infty}\widehat{p}_{j,0}(t)}{1 - \lim_{t\to+\infty}\widehat{p}_{0,0}(t)},$$

from which (42) immediately follows.

(ii) If $\widehat{N}(t)$ is transient, from (39) follows

$$\lim_{\xi\downarrow 0}\left\{\mathrm{E}\left(C_{j,0}\right) - \frac{1}{\xi}\right\} = \int_0^{+\infty}\widehat{p}_{j,0}(t)\mathrm{d}t.$$

Eq. (42) thus follows by recalling that (see, for instance, Karlin and Mc Gregor, 1957) for the transient birth-death process $\widehat{N}(t)$ the Green's function $\int_0^{+\infty}\widehat{p}_{j,0}(t)\mathrm{d}t$ takes the form of the series $\displaystyle\sum_{k=j}^{+\infty}\frac{1}{\alpha_k u_k}$, that is convergent in the present case. ∎

From Proposition 3.3 the following asymptotic behaviour for the mean catastrophe time when ξ is small (infrequent catastrophes) follows:

$$\mathrm{E}\left(C_{j,0}\right) \approx \begin{cases} \dfrac{1}{(1-\widehat{q}_0)\xi}, & \text{if } \widehat{N}(t) \text{ is positive recurrent} \\ \dfrac{1}{\xi} + \displaystyle\sum_{k=j}^{+\infty}\frac{1}{\alpha_k u_k}, & \text{if } \widehat{N}(t) \text{ is transient.} \end{cases}$$

In particular, if $\alpha_k = \alpha$ $(k = 0, 1, \ldots)$ and $\beta_k = \beta$ $(k = 1, 2, \ldots)$ for $\xi \downarrow 0$ we have $\mathrm{E}\,(C_{j,0}) \approx \beta/(\alpha\xi)$ if $\alpha < \beta$, and $\mathrm{E}\,(C_{j,0}) \approx 1/\xi + (\alpha-\beta)^{-1}\,(\beta/\alpha)^j$ if $\alpha > \beta$.

Proposition 3.4 *There holds:*

$$\lim_{\xi\to+\infty} \mathrm{E}\,(C_{0,0}) = \frac{1}{\alpha_0},$$
$$\lim_{\xi\to+\infty} \xi\,\mathrm{E}\,(C_{j,0}) = \begin{cases} 1 + \dfrac{\beta_1}{\alpha_0}, & j = 1 \\ 1, & j = 2, 3, \ldots \end{cases}$$

Proof. Since

$$(\xi + \alpha_0)\widehat{\pi}_{j,0}(\xi) = \delta_{j,0} + \beta_1 \widehat{\pi}_{j,1}(\xi), \qquad j \in \mathcal{S},$$

from (39) we have

$$\mathrm{E}(C_{0,0}) = \frac{1}{\xi} + \frac{\widehat{\pi}_{0,0}(\xi)}{\alpha_0\,\widehat{\pi}_{0,0}(\xi) - \beta_1\,\widehat{\pi}_{0,1}(\xi)},$$
$$\mathrm{E}(C_{j,0}) = \frac{1}{\xi}\left[1 - \frac{\alpha_0\,\widehat{\pi}_{j,0}(\xi) - \beta_1\,\widehat{\pi}_{j,1}(\xi)}{\alpha_0\,\widehat{\pi}_{0,0}(\xi) - \beta_1\,\widehat{\pi}_{0,1}(\xi)}\right] \qquad (j = 1, 2, \ldots).$$

Making use of a Tauberian theorem, we then obtain

$$\lim_{\xi\to+\infty} \mathrm{E}\,(C_{0,0}) = \frac{\lim\limits_{t\to 0} \widehat{p}_{0,0}(t)}{\alpha_0 \lim\limits_{t\to 0} \widehat{p}_{0,0}(t) - \beta_1 \lim\limits_{t\to 0} \widehat{p}_{0,1}(t)},$$
$$\lim_{\xi\to+\infty} \xi\,\mathrm{E}\,(C_{j,0}) = 1 - \frac{\alpha_0 \lim\limits_{t\to 0} \widehat{p}_{j,0}(\xi) - \beta_1 \lim\limits_{t\to 0} \widehat{p}_{j,1}(\xi)}{\alpha_0 \lim\limits_{t\to 0} \widehat{p}_{0,0}(\xi) - \beta_1 \lim\limits_{t\to 0} \widehat{p}_{0,1}(\xi)} \qquad (j = 1, 2, \ldots),$$

immediately yielding the thesis. ∎

Due to Proposition 3.4, for the mean catastrophe time when ξ is large (frequent catastrophes) the following holds:

$$\mathrm{E}\,(C_{j,0}) \approx \begin{cases} \dfrac{1}{\alpha_0}, & j = 0 \\ \dfrac{1}{\xi}\left(1 + \dfrac{\beta_1}{\alpha_0}\right), & j = 1 \\ \dfrac{1}{\xi}, & j = 2, 3, \ldots. \end{cases}$$

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