A Capacity Constrained Production-Inventory System with Stochastic Demand and Production Times

Sven Axsäter

A Capacity Constrained Production-Inventory System with Stochastic Demand and Production Times

Journal:	*International Journal of Production Research*
Manuscript ID:	TPRS-2009-IJPR-0477.R1
Manuscript Type:	Original Manuscript
Date Submitted by the Author:	13-Aug-2009
Complete List of Authors:	Axsäter, Sven; Lund University, Industrial Management and Logistics
Keywords:	INVENTORY MANAGEMENT, PRODUCTION PLANNING
Keywords (user):	

A Capacity Constrained Production-Inventory System with Stochastic Demand and Production Times

Sven Axsäter
Lund University

Revised August 2009

Abstract

This paper considers a simple model of a capacity constrained production-inventory system with Poisson demand. The system is controlled by an S policy. The production time for a unit is modeled as a gamma distributed stochastic variable. Using M/G/1 queuing theory it is very easy to evaluate holding and backorder costs and optimize the ordering policy. The suggested model may be useful when evaluating investments in production.

Keywords: Production-inventory management, Capacity constrained, Stochastic

1 Introduction

In this paper we consider a simple integrated production-inventory system facing Poisson demand. The production capacity is constrained and the production time is stochastic and follows a gamma distribution. Our purpose is to illustrate in a simple way the impact of both capacity limitations and variations in the production times. Our model may be a useful tool when evaluating different capacity investments, which may affect both the mean and the variance of the production time. The impact of capacity investments can, of course, also be evaluated by simulation. However, the tool we provide is, in general, much simpler and quicker to use. The planning of capacity investments has been considered in several papers. A recent review of strategic capacity planning is given by Geng and Jiang (2009). Vits et al. (2007) provide a model to study myopic and long-term process change strategies.

In this paper it is assumed that there are no ordering costs, so there is no need for batch-ordering. The production system can handle only one item at a time. The system is controlled by a so-called S policy. When the inventory position (stock on hand plus outstanding orders and minus backorders) has reached S no more orders are triggered until a new demand occurs. By outstanding orders we mean orders that have not yet been delivered. We consider standard holding and backorder costs and optimize the sum of these costs with respect to S.

The paper analyzes the impact of stochastic variations in production times. In a sense this is related to some earlier papers that consider variations in the production load due to batch quantities and how these variations will affect the lead-times. Larger batch quantities will reduce the average production time but, on the other hand, increase the variations in the production load. See e.g., Jönsson and Silver (1985), Karmarkar (1987, 1993), Axsäter (1980, 2006), and Zipkin (1986, 2000). Another more recent paper dealing with a related problem is Pac et al. (2009).

The outline of the paper is as follows. In Section 2 we give a detailed problem formulation. Section 3 describes our solution technique. In Section 4 we provide some numerical results and discuss how they can be interpreted. Finally we give some concluding remarks in Section 5.

2 Problem formulation

We consider the simple production-inventory system in Figure 1.

Insert Figure 1 Production-inventory system

The inventory is facing Poisson demand and is replenishing from production according to an S policy, i.e., there are no setup costs that motivate batch-ordering. However, the production system can only process a single unit at a time, so it is not possible to have more than one order in process. If the inventory position is strictly less than S an order for one unit is triggered. Production is started as soon as the production of possible preceding units is finished. If there is no unit in production, production is started immediately. Production times are independent stochastic variables following a gamma distribution. The gamma distribution has several advantages in this context. The production time is always positive and we can fit the distribution to any mean and variance. Furthermore the demand during the stochastic production time will get a negative binomial distribution, which is easy to deal with. (See Section 3.2.) The gamma distribution is available in various software packages, e.g., in Excel. See Tijms (1994) for a comparison with other distributions. Components and/or raw materials needed in production can be obtained without any delay. Consequently, there are no holding costs associated with a unit before the production starts. Evidently, each unit produced incurs holding costs corresponding to the stochastic production time. However, the average holding cost in production for an item is clearly independent of the ordering policy. We shall therefore disregard this holding cost. We consider, however, standard linear holding and backorder costs associated with time in inventory and customer waiting time.

At this stage we assume that the distributions of the stochastic production time and the demand are given. Our problem is then to choose the order-up-to level S in order to minimize the sum of expected holding and backorder costs.

We introduce the following notation:

λ = intensity of the Poisson demand,

μ = average production time per unit,

σ = standard deviation of the production time per unit,

ρ = $\lambda\mu$ = traffic density,

c = σ/μ = coefficient of variation for the production time,

h = holding cost per unit and unit time,

b = backorder cost per unit and unit time,

S = order-up-to level,

C = expected costs per unit of time.

Evidently we must require that $\rho = \lambda\mu < 1$. The gamma distribution that characterizes the production time for a unit is completely specified by the two parameters μ and σ.

3 Cost evaluation and optimization

3.1 The gamma distribution

Let us first for completeness define the gamma distribution, which has the density

$$g(x) = \frac{\xi(\xi x)^{r-1} e^{-\xi x}}{\Gamma(r)}, \quad x \geq 0. \tag{1}$$

The two parameters r and ξ are both positive and $\Gamma(r)$ is the gamma function

$$\Gamma(r) = \int_0^\infty x^{r-1} e^{-x} dx\,. \tag{2}$$

Given μ and σ the parameters r and ξ are uniquely determined as

$$r = (\mu/\sigma)^2, \tag{3}$$

$$\xi = \mu/\sigma^2. \tag{4}$$

It is useful to note that $\Gamma(r) = (r-1)\Gamma(r-1)$ and that

$$\int_0^\infty x^\alpha e^{-\beta x} dx = \frac{\Gamma(\alpha+1)}{\beta^{(\alpha+1)}}, \tag{5}$$

for $\alpha > -1$ and $\beta > 0$.

3.2 Distribution of the demand during the production time

Next we consider the stochastic demand during the stochastic production time. The demand process and the stochastic production time are independent. Let q_j be the probability for demand j. We obtain

$$q_j = \int_0^\infty \frac{\xi(\xi x)^{r-1} e^{-\xi x}}{\Gamma(r)} \frac{(\lambda x)^j e^{-\lambda x}}{j!} dx. \tag{6}$$

Using (5) we get

$$q_j = \frac{(\xi)^r \lambda^j}{\Gamma(r) j!} \frac{\Gamma(r+j)}{(\lambda+\xi)^{r+j}}, \tag{7}$$

i.e., for $j = 0$ we obtain

$$q_0 = \frac{\xi^r}{(\lambda+\xi)^r}, \tag{8}$$

and for $j > 0$,

$$q_j = \frac{\xi^r}{(\lambda+\xi)^r} \frac{\lambda^j}{(\lambda+\xi)^j} \frac{r(r+1)\dots(r+j-1)}{j!}. \tag{9}$$

This means that q_j has a negative binomial distribution that is easy to deal with. When, for a given μ, σ approaches 0, the distribution in (8) and (9) will, as expected, approach a Poisson distribution. However, for $\sigma = 0$, or σ very small, it is then computationally much more efficient to replace (8) and (9) by the Poisson distribution, i.e., to set

$$q_j = \frac{(\lambda r/\xi)^j e^{-\lambda r/\xi}}{j!} = \frac{(\lambda\mu)^j e^{-\lambda\mu}}{j!}. \qquad (10)$$

Recall that we have defined the traffic density $\rho = \lambda\mu$, and the coefficient of variation for the production time $c = \sigma/\mu$. The distribution q_j in (9) is expressed in terms of three parameters λ, μ, and σ. However, it is easy to see that it can be specified completely by the two parameters ρ and c. Just note that $\lambda/\xi = \rho c^2$ and $r = 1/c^2$.

3.3 The associated queuing system

Note now first that the optimal solution must have $S \geq 0$. Clearly, any $S \leq 0$ will give zero holding costs. Furthermore, $S < 0$ must give higher backorder costs than $S = 0$.

Assume then that we start with S items in stock. It is obvious that each demand will trigger a corresponding production order to be produced as soon as all previous orders have been produced. The orders in production or waiting to be produced constitute a so-called M/G/1 queuing system, i.e., we have Poisson arrivals and a production time that is not exponential. Using a standard approach we can derive the steady-state distribution for the queue length. See e.g., Grimmett and Stirzaker (1987) for details.

Consider the number of waiting orders when an order has just been finished and delivered to inventory. If there are k orders waiting we say that the state of the queuing system is k. Let

p_j = steady-state probability that the state is k (0, 1, 2, ...).

First we have

$$p_0 = 1 - \lambda\mu = 1 - \rho, \qquad (11)$$

i.e., the ratio of time when no production is taking place. To see this note that $\lambda\mu$ is the average production time during a time unit. Consequently, $1 - \lambda\mu$ is the ratio of time when no production is going on. Next we note that

$$p_j = p_0 q_j + \sum_{i=1}^{j+1} p_i q_{j-i+1}\,, \tag{12}$$

i.e., if we are in state 0 the next production concerns an order triggered by the next demand, but if the state is i > 0 the next production concerns one of the waiting orders.

Reformulating (12) we get

$$p_{j+1} = \frac{1}{q_0}\left[p_j - p_0 q_j - \sum_{i=1}^{j} p_j q_{j-i+1}\right] \qquad j = 0, 1, 2, \ldots \tag{13}$$

i.e., we can easily determine the steady-state distribution recursively. Evidently $p_j \to 0$ as $j \to \infty$ because $\rho < 1$. (In (13) the sum is defined to be zero for j = 0.)

We note that p_0 only depends on ρ. Because q_j depends only on the parameters ρ and c this must then be the case also for all p_j.

Let us conclude at this stage that it is very easy to determine the steady-state distribution of the number of outstanding orders. The distribution is completely specified by the parameters ρ and c. First we get the distribution of the demand during the production time from (8) and (9). Then we simply apply (11) and (13). Note that the distribution of the number of outstanding orders is independent of S.

3.4 Evaluation of costs and optimization of S

We are now ready to evaluate the expected costs for a given $S \geq 0$. If there are k outstanding orders, the inventory level is S – k. Consequently, the expected costs for a given S can be expressed as

$$C = \sum_{k=0}^{\infty} p_k \left[(S-k)^+ h + (S-k)^- b\right], \tag{14}$$

where $x^+ = \max(x, 0)$ and $x^- = \max(-x, 0)$.

It is easy to see that C is a convex function of S. Consequently, to find the optimal solution we simply evaluate S = 0, 1, ... The first local optimum is also the global optimum.

4 Numerical results and discussion

We have shown that the optimal policy and the optimal costs (for given holding and backorder costs) only depend on the traffic density ρ and the coefficient of variation c. So, for example, $\lambda = 1$, $\mu = 0.8$, and $\sigma = 0.4$ give exactly the same results as $\lambda = 10$, $\mu = 0.08$, and $\sigma = 0.04$, because in both cases $\rho = 0.8$ and c = 0.5. Furthermore, concerning the costs it is obvious that it is only the ratio between the backorder cost b and the holding cost h that is of interest. If both costs are changed by a certain percentage, this will change the optimal costs by the same percentage but the optimal policy is not affected. Therefore, in our numerical study we assume that h =1. Two different values of the backorder cost b = 5 and b = 20 are considered in Table 1 and Table 2 respectively.

Insert Table 1 Optimal policies and costs for h = 1 and b = 5.

Insert Table 2 Optimal policies and costs for h = 1 and b = 20.

In both tables we have evaluated the costs for the same four values of ρ and five values of c. It is not surprising that the costs will increase both with ρ and c. From the tables it is also obvious that the additional cost when increasing c is much larger when ρ is high, and similarly the additional cost when increasing ρ is much larger when c is high. All such tendencies are stronger with the higher backorder cost in Table 2.

We believe that our simple model may be useful in connection with evaluation of investments in production. Such an investment will in general reduce the average production time μ and also the standard deviation of the production time σ. Clearly this means that ρ is reduced. The coefficient of variation c may both decrease and increase. If, for example, μ and σ are reduced by the same percentage, this means that c is unchanged. Our model shows how the inventory costs are affected by the investment. If μ is reduced we will also get lower holding costs for the time in production. These costs are normally proportional to $1/\mu$ and easy to evaluate. The total reduction of holding and backorder costs should then be compared to the cost for a considered investment.

5 Conclusions

We have suggested a simple model of a production-inventory system. The demand is Poisson and the production time for a unit is modeled as a gamma distributed stochastic variable. The system is controlled by a so-called S policy. Using M/G/1 queuing theory it is very easy to evaluate and optimize holding and backorder costs, which only depend on the traffic density and the coefficient of variation for the production time.

The considered model may be useful when evaluating different investments in the production system, because such investments will, in general, affect holding and backorder costs in a way that is easy to illustrate by our model.[1]

[1] I am grateful to Christian Howard for help with parts of the numerical evaluation.

References

Axsäter, S. 1980. Economic Order Quantities and Variations in Production Load, *International Journal of Production Research*, 18, 359-365.

Axsäter, S. 2006. *Inventory Control*, 2nd edition, Springer, New York.

Geng, N., and Z. Jiang. 2009. A Review on Strategic Capacity Planning for the Semiconductor Manufacturing Industry, *International Journal of Production Research*, 47, 3639-3656.

Grimmett, G. R., and D. R. Stirzaker. 1987. *Probability and Random Processes*, Oxford University Press, New York.

Jönsson, H., and E. A. Silver. 1985. Impact of Processing and Queuing Times on Order Quantities, *Material Flow*, 2, 221-230.

Karmarkar, U. S. 1987. Lot Sizes, Lead Times and In-Process Inventories, *Management Science*, 33, 409-418.

Karmarkar, U. S. 1993. Manufacturing Lead Times, Order Release and Capacity Loading, in S. C. Graves et al. Eds. *Handbooks in OR & MS Vol. 4*, North Holland Amsterdam, 287-329.

Pac, M. F., Alp, O., and T. Tan. 2009, Integrated Workforce Capacity and Inventory Management under Labour Supply Uncertainty, *International Journal of Production Research*, 47, 4281-4304.

Tijms, H. C. 1994. *Stochastic Models: An Algorithmic Approach*, Wiley, Chichester.

Vits, J., Gelders, L., and L. Pintelon. 2007. Manufacturing Process Changes: Myopic and Long-term Planning, *International Journal of Production Research*, 45, 1-28.

Zipkin, P. H. 1986. Models for Design and Control of Stochastic, Multi-Item Batch Production Systems, *Operations Research*, 34, 91-104.

Zipkin, P. H. 2000. *Foundations of Inventory Management*, McGraw-Hill, Singapore.

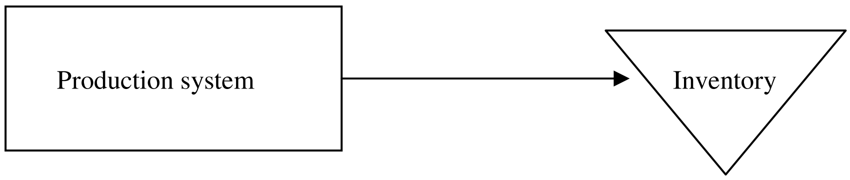

Figure 1 Production-inventory system

Table 1 Optimal policies and costs for h = 1 and b = 5.

		c = 0	**c = 0.5**	**c = 1**	**c = 1.5**	**c = 2**
ρ = 0.9	S	8	11	17	26	40
	C	8.69	10.77	17.01	27.37	41.89
ρ = 0.8	S	4	5	8	12	17
	C	4.25	5.18	8.03	12.71	19.26
ρ = 0.7	S	3	3	5	7	10
	C	2.77	3.33	5.02	7.74	11.57
ρ = 0.6	S	2	2	3	4	6
	C	2.02	2.41	3.44	5.19	7.61

Table 2 Optimal policies and costs for h = 1 and b = 20.

		c = 0	**c = 0.5**	**c = 1**	**c = 1.5**	**c = 2**
ρ = 0.9	S	15	18	28	45	69
	C	14.74	18.28	28.89	46.56	71.29
ρ = 0.8	S	7	9	13	21	31
	C	7.15	8.78	13.62	21.70	32.98
ρ = 0.7	S	5	6	8	12	18
	C	4.65	5.65	8.49	13.30	20.01
ρ = 0.6	S	3	4	5	8	12
	C	3.41	4.00	5.95	9.02	13.39