The shortest Clopper--Pearson confidence interval for binomial probability

Journal:	<i>Communications in Statistics - Simulation and Computation</i>
Manuscript ID:	LSSP-2009-0041.R2
Manuscript Type:	Review Article
Date Submitted by the Author:	23-Sep-2009
Complete List of Authors:	Zielinski, Wojciech; Warsaw University of Life Sciences, Econometrics and Statistics
Keywords:	binomial proportion, confidence interval, shortest confidence interval
Abstract:	In the paper there is shown the existence of the shortest confidence interval for binomial probability. The method of obtaining such interval is presented as well.
Note: The following files were submitted by the author for peer review, but cannot be converted to PDF. You must view these files (e.g. movies) online.	
Zielinski_v3.tex

The shortest Clopper–Pearson confidence interval for binomial probability

Wojciech Zieliński

Department of Econometrics and Statistics
Warsaw University of Life Sciences
Nowoursynowska 159, 02-776 Warszawa, Poland
e-mail: wojtek.zielinski@statystyka.info

Summary. The existence of the shortest confidence interval for binomial probability is shown. The method of obtaining such an interval is presented as well.

Key words: binomial proportion, confidence interval, shortest confidence intervals

AMS 2000 Subject Classification: 62F25

Consider the binomial statistical model

$$(\{0, 1, \dots, n\}, \{Bin(n, \pi), 0 < \pi < 1\}),$$

where $Bin(n, \pi)$ denotes the binomial distribution with pdf

$$\binom{n}{x} \pi^x (1 - \pi)^{n-x}, \quad x = 0, 1, \dots, n.$$

Let X denote a binomial $Bin(n, \pi)$ random variable. A confidence interval for the probability π at the confidence level γ is of the form (Clopper and Pearson, 1934)

$$(F^{-1}(X, n - X + 1; \gamma_1); F^{-1}(X + 1, n - X; \gamma_2)),$$

where $\gamma_1, \gamma_2 \in (0, 1)$ are such that $\gamma_2 - \gamma_1 = \gamma$ and $F^{-1}(a, b; \alpha)$ is the α quantile of the beta distribution with parameters (a, b) , i.e

$$P_{\pi} \{ \pi \in (F^{-1}(X, n - X + 1; \gamma_1); F^{-1}(X + 1, n - X; \gamma_2)) \} \geq \gamma, \quad \forall \pi \in (0, 1).$$

For $X = 0$ the left end is taken to be 0, and for $X = n$ the right end is taken to be 1.

Clopper and Pearson (1934) in their construction used $\gamma_1 = (1 - \gamma)/2$ (see also Brown et al. (2001)), i.e. they applied the rule of symmetric division of $1 - \gamma$ to both sides of the interval. The length of the confidence interval was not considered as a criterion. It is of interest to find the shortest confidence interval. So we want to find γ_1 and γ_2 such that the confidence interval is the shortest possible.

The problem of shortest confidence intervals was seldom considered in the past (Crow 1956, Blyth and Hutchinson 1960, Blyth and Still 1983, Casella 1986). Unfortunately, the solutions proposed are either complicated or randomization is applied. In what follows, a much simpler solution of the problem is presented. Consider the length of the confidence interval when $X = x$ is observed,

$$d(\gamma_1, x) = F^{-1}(x + 1, n - x; \gamma + \gamma_1) - F^{-1}(x, n - x + 1; \gamma_1).$$

Let x be given. We want to find $0 < \gamma_1 < 1 - \gamma$ such that $d(\gamma_1, x)$ is minimal. The derivative of $d(\gamma_1, x)$ with respect to γ_1 equals (in what follows $B(\cdot, \cdot)$ denotes the beta function)

$$\begin{aligned} \frac{\partial d(\gamma_1, x)}{\partial \gamma_1} &= B(x + 1, n - x)(1 - F^{-1}(x + 1, n - x; \gamma + \gamma_1))^{1+x-n} F^{-1}(x + 1, n - x; \gamma + \gamma_1)^{-x} \\ &\quad - B(x, n - x + 1)(1 - F^{-1}(x, n - x + 1; \gamma_1))^{x-n} F^{-1}(x, n - x + 1; \gamma_1)^{1-x}. \end{aligned}$$

Let

$$LHS(\gamma_1, x) = \frac{[1 - F^{-1}(x+1, n-x; \gamma + \gamma_1)]^{n-x-1} F^{-1}(x+1, n-x; \gamma + \gamma_1)^x}{B(x+1, n-x)},$$

$$RHS(\gamma_1, x) = \frac{[1 - F^{-1}(x, n-x+1; \gamma_1)]^{n-x} F^{-1}(x, n-x+1; \gamma_1)^{x-1}}{B(x, n-x+1)}.$$

Then

$$\frac{\partial d(\gamma_1, x)}{\partial \gamma_1} = \frac{1}{LHS(\gamma_1, x)} - \frac{1}{RHS(\gamma_1, x)}.$$

Because

$$F^{-1}(x, n-x+1; 0) = 0 \quad \text{and} \quad F^{-1}(x+1, n-x; 1) = 1,$$

for $1 < x < n-1$ we have

$$\begin{aligned} &\text{if } \gamma_1 \rightarrow 0 \text{ then } LHS(\gamma_1, x) > 0 \text{ and } RHS(\gamma_1, x) \rightarrow 0^+, \\ &\text{if } \gamma_1 \rightarrow 1 - \gamma \text{ then } LHS(\gamma_1, x) \rightarrow 0^+ \text{ and } RHS(\gamma_1, x) > 0. \end{aligned}$$

Therefore, the equation

$$\frac{\partial d(\gamma_1, x)}{\partial \gamma_1} = 0 \quad (*)$$

has a solution.

It is easy to see that $LHS(\cdot, x)$ and $RHS(\cdot, x)$ are unimodal and concave on the interval $(0, 1 - \gamma)$. Hence, the solution of $(*)$ is unique. Let γ_1^* denote the solution. Because $\frac{\partial d(\gamma_1, x)}{\partial \gamma_1} < 0$ for $\gamma_1 < \gamma_1^*$ and $\frac{\partial d(\gamma_1, x)}{\partial \gamma_1} > 0$ for $\gamma_1 > \gamma_1^*$, we have $d(\gamma_1^*, x) = \inf\{d(\gamma_1, x) : 0 < \gamma_1 < 1 - \gamma\}$.

It is interesting to note that for even n and $x = n/2$ the only solution of $(*)$ is $\gamma_1^* = (1 - \gamma)/2$, i.e. the symmetric confidence interval is the shortest one. Indeed, in that case

$$LHS(\gamma_1, x) = \frac{[1 - F^{-1}(x+1, x; \gamma + \gamma_1)]^{n-x-1} F^{-1}(x+1, x; \gamma + \gamma_1)^x}{B(x+1, x)},$$

$$RHS(\gamma_1, x) = \frac{[1 - F^{-1}(x, x+1; \gamma_1)]^{n-x} F^{-1}(x, x+1; \gamma_1)^{x-1}}{B(x, x+1)}.$$

Making use of the known equality

$$F^{-1}(a, b; q) = 1 - F^{-1}(b, a; 1 - q),$$

we can rewrite equation $(*)$ in the form

$$\left[\frac{F^{-1}(x, x+1; \gamma_1)}{F^{-1}(x, x+1; 1 - \gamma - \gamma_1)} \right]^{x-1} = \left[\frac{F^{-1}(x+1, x; \gamma + \gamma_1)}{F^{-1}(x+1, x; 1 - \gamma_1)} \right]^x.$$

It is not difficult to check that the solution of the last equation is $\gamma_1^* = (1 - \gamma)/2$.

For $x = 1$ we have

$$LHS(\gamma_1, 1) = \frac{[1 - F^{-1}(2, n-1; \gamma + \gamma_1)]^{n-2} F^{-1}(2, n-1; \gamma + \gamma_1)}{B(2, n-1)},$$

$$RHS(\gamma_1, 1) = \frac{[1 - F^{-1}(1, n; \gamma_1)]^{n-1}}{B(1, n)} = n(1 - \gamma_1)^{\frac{n-1}{n}}.$$

It is seen that

$$\begin{aligned} &\text{if } \gamma_1 \rightarrow 0 \text{ then } LHS(\gamma_1, 1) \rightarrow \frac{[1 - F^{-1}(2, n-1; \gamma)]^{n-2} F^{-1}(2, n-1; \gamma)}{B(2, n-1)} \text{ and } RHS(\gamma_1, x) \rightarrow n, \\ &\text{if } \gamma_1 \rightarrow 1 - \gamma \text{ then } LHS(\gamma_1, x) \rightarrow 0 \text{ and } RHS(\gamma_1, x) \rightarrow n\gamma^{\frac{n-1}{n}}. \end{aligned}$$

and

$$RHS(\gamma_1, 1) > LHS(\gamma_1, 1) \quad \text{for } 0 < \gamma_1 < 1 - \gamma.$$

So $\frac{\partial d(\gamma_1, 1)}{\partial \gamma_1} > 0$ and hence $d(\gamma_1, 1)$ achieves its minimal value for $\gamma_1 = 0$. For $x = 1$ the shortest confidence interval is the one-sided one. Similarly, for $x = n - 1$ the length $d(\gamma_1, n - 1)$ achieves its minimal value for $\gamma_1 = 1 - \gamma$ and the shortest confidence interval is also one-sided.

The value of γ_1^* for given γ , n and x may be found numerically. Tables 1-3 give those values for $\gamma = 0.95$. Clopper-Pearson confidence intervals are also shown.

Note that if $x > n/2$, then

$$\begin{aligned} \gamma_1^*(x) &= (1 - \gamma) - \gamma_1^*(n - x), \\ left_{short}(x) &= 1 - right_{short}(n - x), \\ right_{short}(x) &= 1 - left_{short}(n - x), \\ left_{sym}(x) &= 1 - right_{sym}(n - x), \\ right_{sym}(x) &= 1 - left_{sym}(n - x). \end{aligned}$$

Table 1. $n = 20$

x	Shortest c.i.				Clopper-Pearson c.i.		
	γ_1^*	$left_{short}$	$right_{short}$	$length_{short}$	$left_{sym}$	$right_{sym}$	$length_{sym}$
0	0.00000	0.00000	0.13911	0.13911	0.00000	0.16843	0.16843
1	0.00000	0.00000	0.21611	0.21611	0.00127	0.24873	0.24747
2	0.00125	0.00261	0.28393	0.28132	0.01235	0.31698	0.30463
3	0.00561	0.01839	0.34998	0.33159	0.03207	0.37893	0.34686
4	0.00966	0.04318	0.41249	0.36931	0.05733	0.43661	0.37928
5	0.01302	0.07344	0.47156	0.39812	0.08657	0.49105	0.40447
6	0.01587	0.10763	0.52766	0.42004	0.11893	0.54279	0.42386
7	0.01840	0.14496	0.58118	0.43622	0.15391	0.59219	0.43828
8	0.02071	0.18496	0.63234	0.44738	0.19119	0.63946	0.44827
9	0.02288	0.22733	0.68126	0.45393	0.23058	0.68472	0.45414
10	0.02500	0.27196	0.72804	0.45609	0.27196	0.72804	0.45609

Table 2. $n = 100$

x	Shortest c.i.				Clopper-Pearson c.i.		
	γ_1^*	$left_{short}$	$right_{short}$	$length_{short}$	$left_{sym}$	$right_{sym}$	$length_{sym}$
5	0.00823	0.01236	0.10507	0.09272	0.01643	0.11284	0.09640
10	0.01399	0.04462	0.16978	0.12516	0.04901	0.17622	0.12722
15	0.01690	0.08234	0.22989	0.14755	0.08645	0.23531	0.14885
20	0.01880	0.12299	0.28733	0.16434	0.12666	0.29184	0.16519
25	0.02023	0.16564	0.34287	0.17723	0.16878	0.34655	0.17777
30	0.02140	0.20984	0.39692	0.18708	0.21241	0.39982	0.18741
35	0.02241	0.25534	0.44972	0.19438	0.25729	0.45185	0.19456
40	0.02332	0.30197	0.50139	0.19942	0.30330	0.50279	0.19950
45	0.02417	0.34965	0.55203	0.20238	0.35032	0.55272	0.20240
50	0.02500	0.39832	0.60168	0.20336	0.39832	0.60168	0.20336

Table 3. $n = 1000$

x	Shortest c.i.				Clopper-Pearson c.i.		
	γ_1^*	$left_{short}$	$right_{short}$	$length_{short}$	$left_{sym}$	$right_{sym}$	$length_{sym}$
50	0.01979	0.03678	0.06473	0.02795	0.03734	0.06539	0.02806
100	0.02165	0.08159	0.11972	0.03812	0.08211	0.12029	0.03818
150	0.02254	0.12797	0.17317	0.04519	0.12843	0.17366	0.04523
200	0.02312	0.17523	0.22574	0.05051	0.17562	0.22616	0.05054
250	0.02356	0.22310	0.27771	0.05460	0.22343	0.27805	0.05462
300	0.02391	0.27146	0.32919	0.05773	0.27172	0.32946	0.05774
350	0.02421	0.32022	0.38027	0.06005	0.32042	0.38047	0.06005
400	0.02449	0.36934	0.43099	0.06165	0.36947	0.43112	0.06165
450	0.02475	0.41879	0.48138	0.06259	0.41885	0.48144	0.06259
500	0.02500	0.46855	0.53145	0.06290	0.46855	0.53145	0.06290

Below we give a short Mathematica program for calculating γ_1^* and the ends of the shortest confidence interval. Of course, one can also use other mathematical or statistical packages (in a similar way) to find the values of γ_1^* .

```

In[1]:= << Statistics'ContinuousDistributions'
      n = . ;
      x = . ;
      q = . ;
      Obet[a_,b_,q_]=Quantile[BetaDistribution[a,b],q];
      Lower[n_,x_,q_]=Obet[x,n-x+1,q];
      Upper[n_,x_,q_]=Obet[x+1,n-x,q];
      Leng[n_,x_,q_,r_]=Upper[n,x,q+r]-Lower[n,x,r];
In[2]:= n =20;(*input n*)
      x =7;(*input x between 2 and n-2*)
      q =0.95 ;(*input confidence level*)
      rr=r/.FindMinimum[Leng[n,x,q,r],{r,0,1-q}][[2]](*output gamma1*)
      Lower[n,x,rr](*output left end*)
      Upper[n,x,q+rr](*output right end*)

```

In Clopper and Pearson's times calculating quantiles of a beta distribution was numerically complicated. Nowadays, it is very easy with the aid of computer software, so using the shortest confidence interval is recommended, especially for small sample sizes n .

1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30
31
32
33
34
35
36
37
38
39
40
41
42
43
44
45
46
47
48
49
50
51
52
53
54
55
56
57
58
59
60

References

Brown L. D., Cai T. T., DasGupta A. (2001) Interval Estimation for Binomial Proportion, *Statistical Science*, 16, 101-133

Blyth C. R., Hutchinson D. W. (1960) Table of Neyman-Shortest Unbiased Confidence Intervals for the Binomial Parameter, *Biometrika* 47, 381-391

Blyth C. R., Still H. A. (1983) Binomial Confidence Intervals, *Journal of the American Statistical Association* 78, 108-116

Casella G. (1986) Refining Binomial Confidence Intervals, *The Canadian Journal of Statistics* 14, 113-129

Clopper C. J., Pearson E. S. (1934), The Use of Confidence or Fiducial Limits Illustrated in the Case of the Binomial, *Biometrika* 26, 404-413

Crow J. L. (1956) Confidence Intervals for a Proportion, *Biometrika* 45, 423-435