The shortest Clopper–Pearson confidence interval for binomial probability

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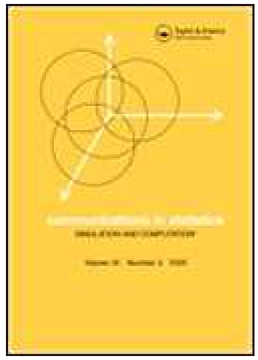

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Abstract:	In the paper there is shown the existence of the shortest confidence interval for binomial probability. The method of obtaining such interval is presented as well.

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The shortest Clopper–Pearson confidence interval for binomial probability

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Summary. The existence of the shortest confidence interval for binomial probability is shown. The method of obtaining such an interval is presented as well.

Key words: binomial proportion, confidence interval, shortest confidence intervals
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Consider the binomial statistical model

$$(\{0,1,\ldots,n\},\{Bin(n,\pi),0<\pi<1\}),$$

where $Bin(n,\pi)$ denotes the binomial distribution with pdf

$$\binom{n}{x}\pi^x(1-\pi)^{n-x},\quad x=0,1,\ldots,n.$$

Let X denote a binomial $Bin(n,\pi)$ random variable. A confidence interval for the probability π at the confidence level γ is of the form (Clopper and Pearson, 1934)

$$\left(F^{-1}(X,n-X+1;\gamma_1);F^{-1}(X+1,n-X;\gamma_2)\right),$$

where $\gamma_1,\gamma_2\in(0,1)$ are such that $\gamma_2-\gamma_1=\gamma$ and $F^{-1}(a,b;\alpha)$ is the α quantile of the beta distribution with parameters (a,b), i.e

$$P_\pi\left\{\pi\in\left(F^{-1}(X,n-X+1;\gamma_1);F^{-1}(X+1,n-X;\gamma_2)\right)\right\}\geq\gamma,\quad\forall\pi\in(0,1).$$

For $X=0$ the left end is taken to be 0, and for $X=n$ the right end is is taken to be 1.

Clopper and Pearson (1934) in their construction used $\gamma_1=(1-\gamma)/2$ (see also Brown et al. (2001)), i.e. they applied the rule of symmetric division of $1-\gamma$ to both sides of the interval. The length of the confidence interval was not considered as a criterion. It is of interest to find the shortest confidence interval. So we want to find γ_1 and γ_2 such that the confidence interval is the shortest possible.

The problem of shortest confidence intervals was seldom considered in the past (Crow 1956, Blyth and Hutchinson 1960, Blyth and Still 1983, Casella 1986). Unfortunately, the solutions proposed are either complicated or randomization is applied. In what follows, a much simpler solution of the problem is presented.

Consider the length of the confidence interval when $X=x$ is observed,

$$d(\gamma_1,x)=F^{-1}(x+1,n-x;\gamma+\gamma_1)-F^{-1}(x,n-x+1;\gamma_1).$$

Let x be given. We want to find $0<\gamma_1<1-\gamma$ such that $d(\gamma_1,x)$ is minimal. The derivative of $d(\gamma_1,x)$ with respect to γ_1 equals (in what follows $B(\cdot,\cdot)$ denotes the beta function)

$$\frac{\partial d(\gamma_1,x)}{\partial\gamma_1}=B(x+1,n-x)(1-F^{-1}(x+1,n-x;\gamma+\gamma_1))^{1+x-n}F^{-1}(x+1,n-x;\gamma+\gamma_1)^{-x}$$
$$-B(x,n-x+1)(1-F^{-1}(x,n-x+1;\gamma_1))^{x-n}F^{-1}(x,n-x+1;\gamma_1)^{1-x}.$$

Let

$$LHS(\gamma_1, x) = \frac{[1 - F^{-1}(x+1, n-x; \gamma+\gamma_1)]^{n-x-1} F^{-1}(x+1, n-x; \gamma+\gamma_1)^x}{B(x+1, n-x)},$$
$$RHS(\gamma_1, x) = \frac{[1 - F^{-1}(x, n-x+1; \gamma_1)]^{n-x} F^{-1}(x, n-x+1; \gamma_1)^{x-1}}{B(x, n-x+1)}.$$

Then

$$\frac{\partial d(\gamma_1, x)}{\partial \gamma_1} = \frac{1}{LHS(\gamma_1, x)} - \frac{1}{RHS(\gamma_1, x)}.$$

Because

$$F^{-1}(x, n-x+1; 0) = 0 \quad \text{and} \quad F^{-1}(x+1, n-x; 1) = 1,$$

for $1 < x < n-1$ we have

$$\text{if } \gamma_1 \to 0 \text{ then } LHS(\gamma_1, x) > 0 \text{ and } RHS(\gamma_1, x) \to 0^+,$$
$$\text{if } \gamma_1 \to 1-\gamma \text{ then } LHS(\gamma_1, x) \to 0^+ \text{ and } RHS(\gamma_1, x) > 0.$$

Therefore, the equation

$$\frac{\partial d(\gamma_1, x)}{\partial \gamma_1} = 0 \qquad (*)$$

has a solution.

It is easy to see that $LHS(\cdot, x)$ and $RHS(\cdot, x)$ are unimodal and concave on the interval $(0, 1-\gamma)$. Hence, the solution of $(*)$ is unique. Let γ_1^* denote the solution. Because $\frac{\partial d(\gamma_1,x)}{\partial \gamma_1} < 0$ for $\gamma_1 < \gamma_1^*$ and $\frac{\partial d(\gamma_1,x)}{\partial \gamma_1} > 0$ for $\gamma_1 > \gamma_1^*$, we have $d(\gamma_1^*, x) = \inf\{d(\gamma_1, x) : 0 < \gamma_1 < 1-\gamma\}$.

It is interesting to note that for even n and $x = n/2$ the only solution of $(*)$ is $\gamma_1^* = (1-\gamma)/2$, i.e. the symmetric confidence interval is the shortest one. Indeed, in that case

$$LHS(\gamma_1, x) = \frac{[1 - F^{-1}(x+1, x; \gamma+\gamma_1)]^{n-x-1} F^{-1}(x+1, x; \gamma+\gamma_1)^x}{B(x+1, x)},$$
$$RHS(\gamma_1, x) = \frac{[1 - F^{-1}(x, x+1; \gamma_1)]^{n-x} F^{-1}(x, x+1; \gamma_1)^{x-1}}{B(x, x+1)}.$$

Making use of the known equality

$$F^{-1}(a, b; q) = 1 - F^{-1}(b, a; 1-q),$$

we can rewrite equation $(*)$ in the form

$$\left[\frac{F^{-1}(x, x+1; \gamma_1)}{F^{-1}(x, x+1; 1-\gamma-\gamma_1)}\right]^{x-1} = \left[\frac{F^{-1}(x+1, x; \gamma+\gamma_1)}{F^{-1}(x+1, x; 1-\gamma_1)}\right]^x.$$

It is not difficult to check that the solution of the last equation is $\gamma_1^* = (1-\gamma)/2$.

For $x = 1$ we have

$$LHS(\gamma_1, 1) = \frac{[1 - F^{-1}(2, n-1; \gamma+\gamma_1)]^{n-2} F^{-1}(2, n-1; \gamma+\gamma_1)}{B(2, n-1)},$$
$$RHS(\gamma_1, 1) = \frac{[1 - F^{-1}(1, n; \gamma_1)]^{n-1}}{B(1, n)} = n(1-\gamma_1)^{\frac{n-1}{n}}.$$

It is seen that

$$\text{if } \gamma_1 \to 0 \text{ then } LHS(\gamma_1, 1) \to \frac{[1 - F^{-1}(2, n-1; \gamma)]^{n-2} F^{-1}(2, n-1; \gamma)}{B(2, n-1)} \text{ and } RHS(\gamma_1, x) \to n,$$
$$\text{if } \gamma_1 \to 1-\gamma \text{ then } LHS(\gamma_1, x) \to 0 \text{ and } RHS(\gamma_1, x) \to n\gamma^{\frac{n-1}{n}}.$$

and

$$RHS(\gamma_1, 1) > LHS(\gamma_1, 1) \quad \text{for } 0 < \gamma_1 < 1 - \gamma.$$

So $\frac{\partial d(\gamma_1,1)}{\partial \gamma_1} > 0$ and hence $d(\gamma_1, 1)$ achieves its minimal value for $\gamma_1 = 0$. For $x = 1$ the shortest confidence interval is the one-sided one. Similarly, for $x = n - 1$ the length $d(\gamma_1, n - 1)$ achieves its minimal value for $\gamma_1 = 1 - \gamma$ and the shortest confidence interval is also one-sided.

The value of γ_1^* for given γ, n and x may be found numerically. Tables 1-3 give those values for $\gamma = 0.95$. Clopper-Pearson confidence intervals are also shown.

Note that if $x > n/2$, then

$$\begin{aligned}
&\gamma_1^*(x) = (1-\gamma) - \gamma_1^*(n-x),\\
&left_{short}(x) = 1 - right_{short}(n-x),\\
&right_{short}(x) = 1 - left_{short}(n-x),\\
&left_{sym}(x) = 1 - right_{sym}(n-x),\\
&right_{sym}(x) = 1 - left_{sym}(n-x).
\end{aligned}$$

Table 1. $n = 20$

	Shortest c.i.				Clopper-Pearson c.i.		
x	γ_1^*	$left_{short}$	$right_{short}$	$length_{short}$	$left_{sym}$	$right_{sym}$	$length_{sym}$
0	0.00000	0.00000	0.13911	0.13911	0.00000	0.16843	0.16843
1	0.00000	0.00000	0.21611	0.21611	0.00127	0.24873	0.24747
2	0.00125	0.00261	0.28393	0.28132	0.01235	0.31698	0.30463
3	0.00561	0.01839	0.34998	0.33159	0.03207	0.37893	0.34686
4	0.00966	0.04318	0.41249	0.36931	0.05733	0.43661	0.37928
5	0.01302	0.07344	0.47156	0.39812	0.08657	0.49105	0.40447
6	0.01587	0.10763	0.52766	0.42004	0.11893	0.54279	0.42386
7	0.01840	0.14496	0.58118	0.43622	0.15391	0.59219	0.43828
8	0.02071	0.18496	0.63234	0.44738	0.19119	0.63946	0.44827
9	0.02288	0.22733	0.68126	0.45393	0.23058	0.68472	0.45414
10	0.02500	0.27196	0.72804	0.45609	0.27196	0.72804	0.45609

Table 2. $n = 100$

	Shortest c.i.				Clopper-Pearson c.i.		
x	γ_1^*	$left_{short}$	$right_{short}$	$length_{short}$	$left_{sym}$	$right_{sym}$	$length_{sym}$
5	0.00823	0.01236	0.10507	0.09272	0.01643	0.11284	0.09640
10	0.01399	0.04462	0.16978	0.12516	0.04901	0.17622	0.12722
15	0.01690	0.08234	0.22989	0.14755	0.08645	0.23531	0.14885
20	0.01880	0.12299	0.28733	0.16434	0.12666	0.29184	0.16519
25	0.02023	0.16564	0.34287	0.17723	0.16878	0.34655	0.17777
30	0.02140	0.20984	0.39692	0.18708	0.21241	0.39982	0.18741
35	0.02241	0.25534	0.44972	0.19438	0.25729	0.45185	0.19456
40	0.02332	0.30197	0.50139	0.19942	0.30330	0.50279	0.19950
45	0.02417	0.34965	0.55203	0.20238	0.35032	0.55272	0.20240
50	0.02500	0.39832	0.60168	0.20336	0.39832	0.60168	0.20336

Table 3. $n = 1000$

	Shortest c.i.				Clopper-Pearson c.i.		
x	γ_1^*	$left_{short}$	$right_{short}$	$length_{short}$	$left_{sym}$	$right_{sym}$	$length_{sym}$
50	0.01979	0.03678	0.06473	0.02795	0.03734	0.06539	0.02806
100	0.02165	0.08159	0.11972	0.03812	0.08211	0.12029	0.03818
150	0.02254	0.12797	0.17317	0.04519	0.12843	0.17366	0.04523
200	0.02312	0.17523	0.22574	0.05051	0.17562	0.22616	0.05054
250	0.02356	0.22310	0.27771	0.05460	0.22343	0.27805	0.05462
300	0.02391	0.27146	0.32919	0.05773	0.27172	0.32946	0.05774
350	0.02421	0.32022	0.38027	0.06005	0.32042	0.38047	0.06005
400	0.02449	0.36934	0.43099	0.06165	0.36947	0.43112	0.06165
450	0.02475	0.41879	0.48138	0.06259	0.41885	0.48144	0.06259
500	0.02500	0.46855	0.53145	0.06290	0.46855	0.53145	0.06290

Below we give a short Mathematica program for calculating γ_1^* and the ends of the shortest confidence interval. Of course, one can also use other mathematical or statistical packages (in a similar way) to find the values of γ_1^*.

```
In[1]:= << Statistics'ContinuousDistributions'
        n =. ;
        x =. ;
        q =. ;
        Obet[a_,b_,q_]=Quantile[BetaDistribution[a,b],q];
        Lower[n_,x_,q_]=Obet[x,n-x+1,q];
        Upper[n_,x_,q_]=Obet[x+1,n-x,q];
        Leng[n_,x_,q_,r_]=Upper[n,x,q+r]-Lower[n,x,r];
In[2]:= n =20;(*input n*)
        x =7;(*input x between 2 and n-2*)
        q =0.95 ;(*input confidence level*)
        rr=r/.FindMinimum[Leng[n,x,q,r],{r,0,1-q}][[2]](*output gamma1*)
        Lower[n,x,rr](*output left end*)
        Upper[n,x,q+rr](*output right end*)
```

In Clopper and Pearson's times calculating quantiles of a beta distribution was numerically complicated. Nowadays, it is very easy with the aid of computer software, so using the shortest confidence interval is recommended, especially for small sample sizes n.

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