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Sylvie Monniaux

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NAVIER-STOKES EQUATIONS IN ARBITRARY DOMAINS:
THE FUJITA-KATO SCHEME

SYLVIE MONNIAUX

Abstract. Navier-Stokes equations are investigated in a functional setting in 3D open sets Ω, bounded or not, without assuming any regularity of the boundary ∂Ω. The main idea is to find a correct definition of the Stokes operator in a suitable Hilbert space of divergence-free vectors and apply the Fujita-Kato method, a fixed point procedure, to get a local strong solution.

1. Introduction

Since the pioneering work by Leray [3] in 1934, there have been several studies on solutions of Navier-Stokes equations

\[
\begin{aligned}
\frac{\partial u}{\partial t} - \Delta u + \nabla \pi + (u \cdot \nabla) u &= 0 \quad \text{in } ]0, T[ \times \Omega, \\
\text{div } u &= 0 \quad \text{in } ]0, T[ \times \Omega, \\
 u &= 0 \quad \text{on } ]0, T[ \times \partial \Omega, \\
u(0) &= u_0 \quad \text{in } \Omega.
\end{aligned}
\]

Fujita and Kato [2] in 1964 gave a method to construct so called mild solutions in smooth domains Ω, producing local (in time) smooth solutions of \((NS)\) in a Hilbert space setting. These solutions are global in time if the initial value \(u_0\) is small enough in a certain sense. The case of non smooth domains has been studied by Deuring and von Wahl [1] in 1995 where they considered domains \(\Omega \subset \mathbb{R}^3\) with Lipschitz boundary \(\partial \Omega\). They found local smooth solutions using results contained in Shen’s PhD thesis [4]. Their method does not cover the critical space case as in [2]. One of the difficulty there was to understand the Stokes operator, and in particular its domain of definition.

In Section 2, we give a “universal” definition of the Stokes operator, for any domain \(\Omega \subset \mathbb{R}^3\) (Definition 2.3). In Section 3, we construct a mild solution of \((NS)\) with a method similar to Fujita-Kato’s [2] (Theorem 3.2) for initial values \(u_0\) in the critical space \(D(A^{1/4})\). We show in Section 4 that this mild solution is a strong solution, i.e. \((NS)\) is satisfied almost everywhere.

2. The Stokes operator

Let \(\Omega\) be an open set in \(\mathbb{R}^3\). The space

\[
L^2(\Omega)^3 = \{u = (u_1, u_2, u_3); u_i \in L^2(\Omega), \ i = 1, 2, 3\}
\]

endowed with the scalar product

\[
\langle u, v \rangle = \int_\Omega u \cdot v = \sum_{i=1}^3 \int_\Omega u_i \overline{v_i}
\]

is a Hilbert space. Define

\[
\mathcal{G} = \{\nabla p; p \in L^2_{\text{loc}}(\Omega) \text{ and } \nabla p \in L^2(\Omega)^3\};
\]

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the set $G$ is a closed subspace of $L^2(\Omega)^3$. Let

$$\mathcal{H} = G^\perp = \{ u \in L^2(\Omega)^3; \langle u, \nabla p \rangle = 0, \ \forall p \in H^1(\Omega) \}.$$  

The space $\mathcal{H}$, endowed with the scalar product $\langle \cdot, \cdot \rangle$ is a Hilbert space. We have the following Hodge decomposition

$$L^2(\Omega)^3 = \mathcal{H} \oplus G.$$

We denote by $\mathbb{P}$ the projection from $L^2(\Omega)^3$ onto $\mathcal{H}$: $\mathbb{P}$ is the usual Helmholtz projection. We denote by $J$ the canonical injection $\mathcal{H} \hookrightarrow L^2(\Omega)^3$: $J' = \mathbb{P}$ ($J'$ being the adjoint of $J$) and $\mathbb{P}J$ is the identity on $\mathcal{H}$. Let now $\mathcal{D}(\Omega)^3 = \mathcal{C}^\infty(\Omega)^3$ and

$$D = \{ u \in \mathcal{D}(\Omega)^3; \text{div} u = 0 \}.$$

It is clear that $D$ is a closed subspace of $\mathcal{D}(\Omega)^3$. We denote by $J_0 : D \hookrightarrow \mathcal{D}(\Omega)^3$ the canonical injection: $J_0 \subset J$. Let $\mathbb{P}_1$ be the adjoint of $J_0$: $\mathbb{P}_1 = J'_0 : \mathcal{D}(\Omega)^3 \rightarrow D'$. We have $\mathbb{P}_1 \subset \mathbb{P}$. The following theorem characterizes the elements in $\ker \mathbb{P}_1$.

**Theorem 2.1 (de Rahm).** Let $T \in \mathcal{D}'(\Omega)^3$ such that $\mathbb{P}_1 T = 0$ in $D'$. Then there exists $S \in (\mathcal{C}^\infty(\Omega))'$ such that $T = \nabla S$. Conversely, if $T = \nabla S$ with $S \in (\mathcal{C}^\infty(\Omega))'$, then $\mathbb{P}_1 T = 0$ in $D'$.

We denote by $H^1_0(\Omega)^3$ the closure of $\mathcal{D}(\Omega)^3$ with respect to the scalar product $(u, v) \mapsto \langle u, v \rangle_1 = \langle u, v \rangle + \sum_{i=1}^3 (\partial_i u, \partial_i v)$. By Sobolev embeddings, we have $H^1_0(\Omega)^3 \hookrightarrow L^3(\Omega)^3$. Define

$$V = \mathcal{H} \cap H^1_0(\Omega)^3.$$

The space $V$ is a closed subspace of $H^1_0(\Omega)^3$; endowed with the scalar product $\langle \cdot, \cdot \rangle_1$, $V$ is a Hilbert space. The canonical injection $J : V \hookrightarrow H^1_0(\Omega)^3$ is the restriction of $J$ to $V$. Let $H^{-1}(\Omega)^3 = (H^1_0(\Omega)^3)'$: $\mathbb{P}_1$ maps $H^{-1}(\Omega)^3$ to $V'$: the restriction of $\mathbb{P}_1$ to $H^{-1}(\Omega)^3$ is $\mathbb{P}$, the adjoint of $J$. On $V \times V$ we define now the form $a$ by

$$a(u, v) = \sum_{i=1}^3 (\partial_i Jv, \partial_i Jv) : a \text{ is a bilinear, symmetric, } \delta + a \text{ is a coercive form on } V \times V \text{ for all } \delta > 0,$$

then defines a bounded self-adjoint operator $A_0 : V \rightarrow V'$ by $(A_0 u)(v) = a(u, v)$ with $\delta + A_0$ invertible for all $\delta > 0$.

**Proposition 2.2.** For all $u \in V$, $A_0 u = \mathbb{P}(-\Delta^D_0)J u$, where $\Delta^D_0$ denotes the Dirichlet-Laplacian on $H^1_0(\Omega)^3$.

**Proof.** For all $u, v \in V$, we have

$$(A_0 u)(v) \overset{(1)}{=} a(u, v) \overset{(2)}{=} \sum_{i=1}^3 \langle \partial_i Jv, \partial_i Jv \rangle \overset{(3)}{=} \langle (-\Delta^D_0)J u, J v \rangle_{H^{-1}, H^1_0} \overset{(4)}{=} \langle \mathbb{P}(-\Delta^D_0)J u, v \rangle_{V', V}.$$

The first two equalities come from the definition of $A_0$ and $a$. The third equality comes from the definition of the Dirichlet-Laplacian on $H^1_0(\Omega)^3$ and the fact that for $v \in V$, $J v = v$. The last equality is due to $J v = \mathbb{P} v$ in $V'$ for all $v \in H^{-1}(\Omega)^3$. This shows that $A_0 u$ and $\mathbb{P}(-\Delta^D_0)J u$ are two continuous linear forms on $V$ which coincide on $V$, they are then equal. $\square$

**Definition 2.3.** The operator $A$ defined on its domain $D(A) = \{ u \in V; A_0 u \in \mathcal{H} \}$ by $Au = A_0 u$ is called the Stokes operator.
Remark 2.5. Since $H^1_0(\Omega)^3 \hookrightarrow L^6(\Omega)^3$, it is clear by interpolation and dualization that $\mathbb{P}_1$ maps $L^p(\Omega)^3$ for $\frac{2}{3} \leq p \leq 2$, $0 \leq s \leq \frac{1}{2}$ and $s = -\frac{1}{2} + \frac{2}{2p}$. Since $A$ is self-adjoint, one has $(\delta + A_0)^{-s}D(A^*)' = \{(\delta + A_0)^{-s}u; u \in D(A^*)\} = \mathcal{H}$. In particular, $(\delta + A_0)^{-\frac{1}{2}}\mathbb{P}_1$ maps $L^2(\Omega)^3$ into $\mathcal{H}$.

3. Mild solution to the Navier-Stokes system

Let $T > 0$.

Define the following Banach space

$$
\mathcal{E}_T = \left\{ u \in \mathcal{C}([0,T]; D(A^+) \cap \mathcal{C}([0,T]; D(A^+))) \right. \\
\left. \text{such that } \sup_{0 < s < T} \| s^{\frac{1}{2}} A^+ u(s) \|_{\mathcal{H}} + \sup_{0 < s < T} \| s A^+ u(s) \|_{\mathcal{H}} + \sup_{0 < s < T} \| s A^+ u'(s) \|_{\mathcal{H}} < \infty \right\}
$$

endowed with the norm

$$
\| u \|_{\mathcal{E}_T} = \sup_{0 < s < T} \| s^{\frac{1}{2}} A^+ u(s) \|_{\mathcal{H}} + \sup_{0 < s < T} \| s A^+ u(s) \|_{\mathcal{H}} + \sup_{0 < s < T} \| s A^+ u'(s) \|_{\mathcal{H}}.
$$

Let $\alpha$ be defined by $\alpha(t) = e^{-tA}u_0$ where $u_0 \in D(A^+)$. Then $\alpha \in \mathcal{E}_T$. Indeed, it is clear that $\alpha \in \mathcal{C}([0,T]; D(A^+))$. We also have that $t^{\frac{1}{2}}A^+ \alpha(t) = t^{\frac{1}{2}}A^+ e^{-tA}u_0$ is bounded on $(0,T)$ since $(e^{-tA})_{t \geq 0}$ is an analytic semigroup. Moreover, one has $\alpha'(t) = -Ac^{-tA}u_0$ which yields to $tA^+ \alpha'(t) = -tAc^{-tA}A^+u_0$ continuous on $[0,T]$, bounded in $\mathcal{H}$. For $u, v \in \mathcal{E}_T$, we define now

$$
\Phi(u, v)(t) = \int_0^t e^{-(t-s)A}(\mathbb{P}_1)((u(s) \cdot \nabla)v(s) + (v(s) \cdot \nabla)u(s))ds, \quad 0 < t < T.
$$

Proposition 3.1. The transform $\Phi$ is bilinear, symmetric, continuous from $\mathcal{E}_T \times \mathcal{E}_T$ to $\mathcal{E}_T$ and the norm of $\Phi$ is independent of $T$.

Proof. The fact that $\Phi$ is bilinear and symmetric is clear. Moreover, $\Phi(u, v) = e^{-A}*f$, where $f$ is defined by

$$
f(s) = \left(-\frac{1}{2}\mathbb{P}_1\right)((u(s) \cdot \nabla)v(s) + (v(s) \cdot \nabla)u(s)), \quad s \in [0,T].
$$

For $u, v \in \mathcal{E}_T$, it is clear that $(u(s) \cdot \nabla)v(s) + (v(s) \cdot \nabla)u(s) \in L^2(\Omega)^3$ and therefore $(\delta + A_0)^{-\frac{1}{2}}f(s) \in \mathcal{H}$ with $\sup_{0 < s < T} s^\frac{1}{2} \|(\delta + A_0)^{-\frac{1}{2}}f(s)\|_{\mathcal{H}} \leq c \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T}$. We have then

$$
\Phi(u, v) = e^{-A}*f = (\delta + A)^{\frac{1}{2}}e^{-A}*((\delta + A_0)^{-\frac{1}{2}}f)
$$

and therefore

$$
\| A^{\frac{1}{2}} \Phi(u, v)(t) \|_\mathcal{H} \leq \int_0^t \| A^{\frac{1}{2}}(\delta + A)^{\frac{1}{2}}e^{-(t-s)A} \|_{\mathcal{L}(\mathcal{H})} \| (\delta + A_0)^{-\frac{1}{2}}f(s) \|_{\mathcal{H}}ds
$$

$$
\leq c \int_0^t \frac{1}{\sqrt{t-s}} \frac{1}{\sqrt{s}} ds \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T}
$$

$$
\leq c \int_0^t \frac{1}{\sqrt{1-\sigma}} \frac{1}{\sqrt{\sigma}} d\sigma \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T}
$$

$$
\leq c \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T}.
$$
Continuity with respect to \( t \in [0,T] \) of \( t \mapsto A^\frac{t}{2}\Phi(u,v)(t) \) is clear once we have proved the boundedness. We also have

\[
\|A^\frac{t}{2}\Phi(u,v)(t)\|_\mathcal{H} \leq \int_0^t \|A^\frac{s}{2}e^{-\frac{s}{2}\mathcal{A}}\|_{\mathcal{X}(\mathcal{H})}\| \mathcal{A}e^{-\frac{s}{2}\mathcal{A}}\|_{\mathcal{H}}ds \\
\leq c \left( \int_0^1 \frac{1}{\mathcal{s}^{\frac{1}{2}}} ds \right) \|u\|_{\mathcal{E}_T} \|v\|_{\mathcal{E}_T}\]

\[
\leq ct^{-\frac{1}{2}} \left( \int_0^1 \frac{1}{(1-\sigma)^{\frac{1}{2}}} ds \right) \|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T}\]

\[
\leq ct^{-\frac{1}{2}}\|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T}.
\]

Continuity with respect to \( t \in [0,T] \) is clear once we have proved the boundedness. To prove the last part of the norm of \( \Phi(u,v) \) in \( \mathcal{E}_T \), we have for \( s \in [0,T] \)

\[
f'(s) = \begin{cases} (-\frac{1}{2} \mathcal{A}^\frac{1}{2})((u^\prime(s) \cdot \nabla)v(s) + (v(s) \cdot \nabla)u^\prime(s)) + (v^\prime(s) \cdot \nabla)u(s) + (v(s) \cdot \nabla)u^\prime(s), \end{cases}
\]

and therefore

\[
\sup_{0 < s < T} \|s^\frac{1}{2}(\delta + A)^{-\frac{1}{2}}f'(s)\|_\mathcal{H} \leq c\|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T}.
\]

We have

\[
\Phi(u,v)(t) = \int_0^t e^{-s\mathcal{A}}f(t-s)ds + \int_0^t e^{-s\mathcal{A}}f(s)ds \quad t \in [0,T],
\]

and therefore

\[
\Phi(u,v)'(t) = e^{-\frac{1}{2}\mathcal{A}}f(u) + \int_0^t (\delta + A)^{\frac{1}{2}}e^{-s(\delta + A)^{\frac{1}{2}}}f'(t-s)ds \\
+ \int_0^t -A(\delta + A)^{\frac{1}{2}}e^{-s(\delta + A)^{\frac{1}{2}}}f(s)ds,
\]

which yields

\[
\|A^{\frac{t}{2}}\Phi(u,v)'(t)\|_\mathcal{H} \leq \frac{c}{\sqrt{t}}\|((\delta + A)^{\frac{1}{2}}f'(\frac{s}{2}))\|_{\mathcal{H}} + c \left( \int_0^1 \frac{1}{s^{\frac{1}{2}}} ds \right) \|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T}\]

\[
+ c \left( \int_0^1 \frac{1}{(1-\sigma)^{\frac{1}{2}}} ds \right) \|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T}\]

\[
\leq \frac{c}{t} \left( \int_0^1 \frac{ds}{s^{\frac{1}{2}}} \right) \|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T}.
\]

This last inequality ensures that \( \Phi(u,v) \in \mathcal{E}_T \) whenever \( u, v \in \mathcal{E}_T \).

\[ \square \]

**Theorem 3.2.** For all \( u_0 \in D(A^{\frac{1}{2}}) \), there exists \( T > 0 \) such that there exists a unique \( u \in \mathcal{E}_T \) solution of \( u = \alpha + \Phi(u,u) \) on \([0,T]\). This function \( u \) is called the mild solution to the Navier-Stokes system.

**Proof.** Let \( T > 0 \). Since \( \Phi : \mathcal{E}_T \times \mathcal{E}_T \rightarrow \mathcal{E}_T \) is bilinear continuous, it suffices to apply Picard fixed point theorem, as in [3]. The sequence in \( \mathcal{E}_T \) \((v_n)_{n \in \mathbb{N}}\) defined by \( v_0 = \alpha \) as first term and

\[
v_{n+1} = \alpha + \Phi(v_n,v_n), \quad n \in \mathbb{N}
\]

converges to the unique solution \( u \in \mathcal{E}_T \) of \( u = \alpha + \Phi(u,u) \) provided \( \|A^{\frac{1}{2}}u_0\|_\mathcal{H} \) is small enough \((\|u\|_{\mathcal{E}_T} < \frac{1}{\sqrt{T}}\|f(\mathcal{E}_T \times \mathcal{E}_T)\|_{\mathcal{H}})\). In the case where \( \|A^{\frac{1}{2}}u_0\|_\mathcal{H} \) is not small
(that is, if $\|\alpha\|_{E_T} \geq \frac{1}{4\|\Phi\|_{L(E_T \times E_T; E_T)}}$) then for $\varepsilon > 0$, there exists $u_{0,\varepsilon} \in D(A)$ such that $\|A\tilde{\tau}(u_0 - u_{0,\varepsilon})\|_H \leq \varepsilon$. If we take as initial value $u_{0,\varepsilon} \in D(A)$, we have

$$\|\alpha_\varepsilon\|_{E_T} \leq cT^{\frac{1}{2}}\|Au_{0,\varepsilon}\|_H \xrightarrow{T \to 0} 0.$$ 

Therefore, we can find $T > 0$ such that $\|\alpha\|_{E_T} < \frac{1}{4\|\Phi\|_{L(E_T \times E_T; E_T)}}$. □

4. Strong solutions

Let $u$ be the mild solution to the Navier-Stokes system. We show in this section that $u$ in fact satisfies the equations of the Navier-Stokes system in an $L^p$-sense (for a suitable $p$). To begin with, we know that $u \in E_T$ and satisfies

$$u = \alpha + \Phi(u, u) = \alpha + e^{-A\cdot} \varphi(u),$$

where $\varphi(u) = -P_1((u \cdot \nabla)u)$ and we have $\|t^{\frac{1}{2}}(u(t) \cdot \nabla)u(t)\|_{\frac{3}{2}} \leq c\|u\|_{E_T}^2$. Therefore, we get

$$u(0) = \alpha(0) = u_0,$$

$$\text{div}u(t) = 0 \text{ in the } L^2 - \text{sense for } t \in [0, T],$$

and

$$u' + Au = f \quad \text{in } \mathcal{C}([0, T]; \mathcal{V}),$$

which means that for all $t \in [0, T]$,

$$P_1(u'(t) - \Delta_D \Delta u(t) + (u(t) \cdot \nabla)u(t)) = 0.$$ 

Then, by Theorem 2.1, there exists $(-\pi)(t) \in (\mathcal{C}_c^\infty(\Omega))'$ such that $\nabla\pi(t) \in H^{-1}(\Omega)^3$ and

$$\nabla(-\pi)(t) = u'(t) - \Delta_D \Delta u(t) + (u(t) \cdot \nabla)u(t)$$

and we have for $0 < t < T$

$$-\Delta_D \Delta u(t) + \nabla\pi(t) = -u'(t) - (u(t) \cdot \nabla)u(t) \in L^3(\Omega)^3 + L^2(\Omega)^3.$$ 

The equation (4.3), together with (4.1) and (4.2), give the usual Navier-Stokes equations which are fulfilled in a strong sense (a.e.) where we consider the expression $-\Delta u + \nabla\pi$ undecoupled.

References