Navier-Stokes equations in arbitrary domains: the Fujita-Kato scheme

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NAVIER-STOKES EQUATIONS IN ARBITRARY DOMAINS : THE FUJITA-KATO SCHEME

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Abstract. Navier-Stokes equations are investigated in a functional setting in 3D open sets Ω, bounded or not, without assuming any regularity of the boundary $\partial\Omega$. The main idea is to find a correct definition of the Stokes operator in a suitable Hilbert space of divergence-free vectors and apply the Fujita-Kato method, a fixed point procedure, to get a local strong solution.

1. Introduction

Since the pioneering work by Leray [3] in 1934, there have been several studies on solutions of Navier-Stokes equations

$$(NS)\quad \begin{cases} \frac{\partial u}{\partial t} - \Delta u + \nabla \pi + (u \cdot \nabla) u &= 0 \quad \text{in} \quad]0,T[\times\Omega, \\ \operatorname{div} u &= 0 \quad \text{in} \quad]0,T[\times\Omega, \\ u &= 0 \quad \text{on} \quad]0,T[\times\partial\Omega, \\ u(0) &= u_0 \quad \text{in} \quad \Omega. \end{cases}$$

Fujita and Kato [2] in 1964 gave a method to construct so called mild solutions in smooth domains Ω, producing local (in time) smooth solutions of (NS) in a Hilbert space setting. These solutions are global in time if the initial value u_0 is small enough in a certain sense. The case of non smooth domains has been studied by Deuring and von Wahl [1] in 1995 where they considered domains $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary $\partial\Omega$. They found local smooth solutions using results contained in Shen's PhD thesis [4]. Their method does not cover the critical space case as in [2]. One of the difficulty there was to understand the Stokes operator, and in particular its domain of definition.

In Section 2, we give a "universal" definition of the Stokes operator, for any domain $\Omega \subset \mathbb{R}^3$ (Defintion 2.3). In Section 3, we construct a mild solution of (NS) with a method similar to Fujita-Kato's [2] (Theorem 3.2) for initial values u_0 in the critical space $D(A^{\frac{1}{4}})$. We show in Section 4 that this mild solution is a strong solution, *i.e.* (NS) is satisfied almost everywhere.

2. The Stokes operator

Let Ω be an open set in $\mathbb{R}^3$. The space

$$L^2(\Omega)^3 = \{u = (u_1, u_2, u_3); u_i \in L^2(\Omega),\ i = 1, 2, 3\}$$

endowed with the scalar product

$$\langle u, v\rangle = \int_\Omega u \cdot \overline{v} = \sum_{i=1}^3 \int_\Omega u_i\, \overline{v_i}$$

is a Hilbert space. Define

$$\mathcal{G} = \{\nabla p; p \in L^2_{loc}(\Omega) \text{ and } \nabla p \in L^2(\Omega)^3\};$$

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the set $\mathcal{G}$ is a closed subspace of $L^2(\Omega)^3$. Let

$$\mathcal{H} = \mathcal{G}^\perp = \left\{u \in L^2(\Omega)^3; \langle u, \nabla p\rangle = 0,\ \forall p \in H^1(\Omega)\right\}.$$

The space $\mathcal{H}$, endowed with the scalar product $\langle\cdot,\cdot\rangle$ is a Hilbert space. We have the following Hodge decomposition

$$L^2(\Omega)^3 = \mathcal{H} \overset{\perp}{\oplus} \mathcal{G}.$$

We denote by $\mathbb{P}$ the projection from $L^2(\Omega)^3$ onto $\mathcal{H}$: $\mathbb{P}$ is the usual Helmoltz projection. We denote by J the canonical injection $\mathcal{H} \hookrightarrow L^2(\Omega)^3$: $J' = \mathbb{P}$ (J' beeing the adjoint of J) and $\mathbb{P}J$ is the identity on $\mathcal{H}$. Let now $\mathscr{D}(\Omega)^3 = \mathscr{C}_c^\infty(\Omega)^3$ and

$$\mathcal{D} = \{u \in \mathscr{D}(\Omega)^3; \mathrm{div} u = 0\}.$$

It is clear that $\mathcal{D}$ is a closed subspace of $\mathscr{D}(\Omega)^3$. We denote by $J_0 : \mathcal{D} \hookrightarrow \mathscr{D}(\Omega)^3$ the canonical injection : $J_0 \subset J$. Let $\mathbb{P}_1$ be the adjoint of J_0 : $\mathbb{P}_1 = J_0' : \mathscr{D}'(\Omega)^3 \to \mathcal{D}'$. We have $\mathbb{P}_1 \subset \mathbb{P}$. The following theorem characterizes the elements in $\ker \mathbb{P}_1$.

Theorem 2.1 (de Rahm)**.** *Let $T \in \mathscr{D}'(\Omega)^3$ such that $\mathbb{P}_1 T = 0$ in $\mathcal{D}'$. Then there exists $S \in (\mathscr{C}_c^\infty(\Omega))'$ such that $T = \nabla S$. Conversely, if $T = \nabla S$ with $S \in (\mathscr{C}_c^\infty(\Omega))'$, then $\mathbb{P}_1 T = 0$ in $\mathcal{D}'$.*

We denote by $H_0^1(\Omega)^3$ the closure of $\mathscr{D}(\Omega)^3$ with respect to the scalar product $(u,v) \mapsto \langle u,v\rangle_1 = \langle u,v\rangle + \sum_{i=1}^3 \langle \partial_i u, \partial_i v\rangle$. By Sobolev embeddings, we have $H_0^1(\Omega)^3 \hookrightarrow L^6(\Omega)^3$. Define

$$\mathcal{V} = \mathcal{H} \cap H_0^1(\Omega)^3.$$

The space $\mathcal{V}$ is a closed subspace of $H_0^1(\Omega)^3$; endowed with the scalar product $\langle\cdot,\cdot\rangle_1$, $\mathcal{V}$ is a Hilbert space. The canonical injection $\tilde{J} : \mathcal{V} \hookrightarrow H_0^1(\Omega)^3$ is the restriction of J to $\mathcal{V}$. Let $H^{-1}(\Omega)^3 = (H_0^1(\Omega)^3)'$; $\mathbb{P}_1$ maps $H^{-1}(\Omega)^3$ to $\mathcal{V}'$: the restriction of $\mathbb{P}_1$ to $H^{-1}(\Omega)^3$ is $\tilde{\mathbb{P}}$, the adjoint of $\tilde{J}$. On $\mathcal{V} \times \mathcal{V}$ we define now the form a by $a(u,v) = \displaystyle\sum_{i=1}^3 \langle \partial_i \tilde{J}u, \partial_i \tilde{J}v\rangle$: a is a bilinear, symmetric, $\delta + a$ is a coercive form on $\mathcal{V} \times \mathcal{V}$ for all $\delta > 0$, then defines a bounded self-adjoint operator $A_0 : \mathcal{V} \to \mathcal{V}'$ by $(A_0 u)(v) = a(u,v)$ with $\delta + A_0$ invertible for all $\delta > 0$.

Proposition 2.2. *For all $u \in \mathcal{V}$, $A_0 u = \tilde{\mathbb{P}}(-\Delta_D^\Omega)\tilde{J}u$, where Δ_D^Ω denotes the Dirichlet-Laplacian on $H_0^1(\Omega)^3$.*

Proof. For all $u, v \in \mathcal{V}$, we have

$$\begin{aligned}
(A_0 u)(v) &\overset{(1)}{=} a(u,v) \overset{(2)}{=} \sum_{i=1}^3 \langle \partial_i \tilde{J}u, \partial_i \tilde{J}v\rangle \\
&\overset{(3)}{=} \langle (-\Delta_D^\Omega)\tilde{J}u, \tilde{J}v\rangle_{H^{-1},H_0^1} \\
&\overset{(4)}{=} \langle \tilde{\mathbb{P}}(-\Delta_D^\Omega)\tilde{J}u, v\rangle_{\mathcal{V}',\mathcal{V}}.
\end{aligned}$$

The first two equalities come from the definition of A_0 and a. The third equality comes from the definition of the Dirichlet-Laplacian on $H_0^1(\Omega)^3$ and the fact that for $v \in \mathcal{V}$, $\tilde{J}v = v$. The last equality is due to $\tilde{J}'\varphi = \tilde{\mathbb{P}}\varphi$ in $\mathcal{V}'$ for all $\varphi \in H^{-1}(\Omega)^3$. This shows that $A_0 u$ and $\tilde{\mathbb{P}}(-\Delta_D^\Omega)\tilde{J}u$ are two continuous linear forms on $\mathcal{V}$ which coïncide on $\mathcal{V}$, they are then equal. □

Definition 2.3. The operator A defined on its domain $D(A) = \{u \in \mathcal{V}; A_0 u \in \mathcal{H}\}$ by $Au = A_0 u$ is called the Stokes operator.

Theorem 2.4. *The Stokes operator is self-adjoint in $\mathcal{H}$, generates an analytic semi-group $(e^{-tA})_{t\geq 0}$, $D(A^{\frac12})=\mathcal{V}$ and satisfies*

$$\begin{array}{rcl} D(A) & = & \{u\in\mathcal{V}\ ;\ \exists\pi\in(\mathbb{C}_c^\infty(\Omega))' : \nabla\pi\in H^{-1}(\Omega) \text{ and } -\Delta u+\nabla\pi\in\mathcal{H}\} \\ Au & = & -\Delta u+\nabla\pi. \end{array}$$

Remark 2.5. Since $H_0^1(\Omega)^3\hookrightarrow L^6(\Omega)^3$, it is clear by interpolation and dualization that $\mathbb{P}_1$ maps $L^p(\Omega)^3$ to $D(A^s)'$ for $\frac65\leq p\leq 2$, $0\leq s\leq\frac12$ and $s=-\frac34+\frac{3}{2p}$. Since A is self-adjoint, one has $(\delta+A_0)^{-s}D(A^s)'=\{(\delta+A_0)^{-s}u; u\in D(A^s)'\}=\mathcal{H}$. In particular, $(\delta+A_0)^{-\frac14}\mathbb{P}_1$ maps $L^{\frac32}(\Omega)^3$ into $\mathcal{H}$.

3. Mild solution to the Navier-Stokes system

Let $T>0$.

Define the following Banach space

$$\begin{array}{rcl} \mathcal{E}_T & = & \Big\{u\in\mathscr{C}([0,T];D(A^{\frac14})\cap\mathscr{C}^1(]0,T];D(A^{\frac14})) \\ & & \quad \text{such that } \sup\limits_{0<s<T}\|s^{\frac14}A^{\frac12}u(s)\|_{\mathcal{H}}+\sup\limits_{0<s<T}\|sA^{\frac14}u'(s)\|_{\mathcal{H}}<\infty\Big\} \end{array}$$

endowed with the norm

$$\|u\|_{\mathcal{E}_T}=\sup_{0<s<T}\|A^{\frac14}u(s)\|_{\mathcal{H}}+\sup_{0<s<T}\|s^{\frac14}A^{\frac12}u(s)\|_{\mathcal{H}}+\sup_{0<s<T}\|sA^{\frac14}u'(s)\|_{\mathcal{H}}.$$

Let α be defined by $\alpha(t)=e^{-tA}u_0$ where $u_0\in D(A^{\frac14})$. Then $\alpha\in\mathcal{E}_T$. Indeed, it is clear that $\alpha\in\mathscr{C}([0,T];D(A^{\frac14}))$. We also have that $t^{\frac14}A^{\frac12}\alpha(t)=t^{\frac14}A^{\frac14}e^{-tA}A^{\frac14}u_0$ is bounded on $(0,T)$ since $(e^{-tA})_{t\geq0}$ is an analytic semigroup. Moreover, one has $\alpha'(t)=-Ae^{-tA}u_0$ which yields to $tA^{\frac14}\alpha'(t)=-tAe^{-tA}A^{\frac14}u_0$ continuous on $]0,T]$, bounded in $\mathcal{H}$. For $u,v\in\mathcal{E}_T$, we define now

$$\Phi(u,v)(t)=\int_0^t e^{-(t-s)A}(-\tfrac12\mathbb{P}_1)((u(s)\cdot\nabla)v(s)+(v(s)\cdot\nabla)u(s))ds,\quad 0<t<T.$$

Proposition 3.1. *The transform Φ is bilinear, symmetric, continuous from $\mathcal{E}_T\times\mathcal{E}_T$ to $\mathcal{E}_T$ and the norm of Φ is independent of T.*

Proof. The fact that Φ is bilinear and symmetric is clear. Moreover, $\Phi(u,v)=e^{-\cdot A}*f$, where f is defined by

$$f(s)=(-\tfrac12\mathbb{P}_1)((u(s)\cdot\nabla)v(s)+(v(s)\cdot\nabla)u(s)),\quad s\in[0,T].$$

For $u,v\in\mathcal{E}_T$, it is clear that $(u(s)\cdot\nabla)v(s)+(v(s)\cdot\nabla)u(s)\in L^{\frac32}(\Omega)^3$ and therefore $(\delta+A_0)^{-\frac14}f(s)\in\mathcal{H}$ with $\sup\limits_{0<s<T}s^{\frac12}\|(\delta+A_0)^{-\frac14}f(s)\|_{\mathcal{H}}\leq c\|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T}$. We have then

$$\Phi(u,v)=e^{-\cdot A}*f=(\delta+A)^{\frac14}e^{-\cdot A}*((\delta+A_0)^{-\frac14}f)$$

and therefore

$$\begin{array}{rcl} \|A^{\frac14}\Phi(u,v)(t)\|_{\mathcal{H}} & \leq & \displaystyle\int_0^t\|A^{\frac14}(\delta+A)^{\frac14}e^{-(t-s)A}\|_{\mathscr{L}(\mathcal{H})}\|(\delta+A_0)^{-\frac14}f(s)\|_{\mathcal{H}}ds \\ & \leq & c\displaystyle\left(\int_0^t\frac{1}{\sqrt{t-s}}\frac{1}{\sqrt{s}}\,ds\right)\|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T} \\ & \leq & c\displaystyle\left(\int_0^1\frac{1}{\sqrt{1-\sigma}}\frac{1}{\sqrt{\sigma}}\,d\sigma\right)\|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T} \\ & \leq & c\|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T}. \end{array}$$

Continuity with respect to $t \in [0,T]$ of $t \mapsto A^{\frac14}\Phi(u,v)(t)$ is clear once we have proved the boundedness. We also have

$$\begin{aligned}\|A^{\frac12}\Phi(u,v)(t)\|_{\mathcal{H}} &\le \int_0^t \|A^{\frac12}(\delta+A)^{\frac14}e^{-(t-s)A}\|_{\mathscr{L}(\mathcal{H})}\|(\delta+A_0)^{-\frac14}f(s)\|_{\mathcal{H}}ds\\ &\le c\left(\int_0^t \frac{1}{(t-s)^{\frac34}}\frac{1}{\sqrt{s}}\,ds\right)\|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T}\\ &\le ct^{-\frac14}\left(\int_0^1 \frac{1}{(1-\sigma)^{\frac34}}\frac{1}{\sqrt{\sigma}}\,d\sigma\right)\|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T}\\ &\le ct^{-\frac14}\|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T}.\end{aligned}$$

Continuity with respect to $t \in]0,T]$ is clear once we have proved the boundedness. To prove the last part of the norm of $\Phi(u,v)$ in $\mathcal{E}_T$, we have for $s \in]0,T[$

$$f'(s) = (-\tfrac12\mathbb{P}_1)((u'(s)\cdot\nabla)v(s) + (u(s)\cdot\nabla)v'(s) + (v'(s)\cdot\nabla)u(s) + (v(s)\cdot\nabla)u'(s))$$

and therefore

$$\sup_{0<s<T}\|s^{\frac54}(\delta+A_0)^{-\frac12}f'(s)\|_{\mathcal{H}} \le c\|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T}.$$

We have

$$\Phi(u,v)(t) = \int_0^{\frac t2} e^{-sA}f(t-s)ds + \int_0^{\frac t2} e^{-(t-s)A}f(s)ds \quad t\in]0,T[,$$

and therefore

$$\begin{aligned}\Phi(u,v)'(t) &= e^{-\frac t2 A}f(\tfrac t2) + \int_0^{\frac t2}(\delta+A)^{\frac12}e^{-sA}(\delta+A_0)^{-\frac12}f'(t-s)ds\\ &\quad + \int_0^{\frac t2} -A(\delta+A)^{\frac14}e^{-(t-s)A}(\delta+A_0)^{-\frac14}f(s)ds,\end{aligned}$$

which yields

$$\begin{aligned}\|A^{\frac14}\Phi(u,v)'(t)\|_{\mathcal{H}} &\le \frac{c}{\sqrt t}\Big\|(\delta+A_0)^{-\frac14}f(\tfrac t2)\Big\|_{\mathcal{H}} + c\left(\int_0^{\frac t2}\frac{1}{s^{\frac12}}\frac{1}{(t-s)^{\frac54}}ds\right)\|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T}\\ &\quad + c\left(\int_0^{\frac t2}\frac{1}{(t-s)^{\frac54}}\frac{1}{s^{\frac12}}ds\right)\|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T}\\ &\le \frac{c}{t}\left(\int_0^{\frac12}\frac{d\sigma}{(1-\sigma)^{\frac54}\sigma^{\frac12}}\right)\|u\|_{\mathcal{E}_T}\|v\|_{\mathcal{E}_T}.\end{aligned}$$

This last inequality ensures that $\Phi(u,v) \in \mathcal{E}_T$ whenever $u,v \in \mathcal{E}_T$. □

Theorem 3.2. *For all $u_0 \in D(A^{\frac14})$, there exists $T>0$ such that there exists a unique $u \in \mathcal{E}_T$ solution of $u = \alpha + \Phi(u,u)$ on $[0,T]$. This function u is called the mild solution to the Navier-Stokes system.*

Proof. Let $T>0$. Since $\Phi : \mathcal{E}_T \times \mathcal{E}_T \to \mathcal{E}_T$ is bilinear continuous, it suffices to apply Picard fixed point theorem, as in [2]. The sequence in $\mathcal{E}_T$ $(v_n)_{n\in\mathbb{N}}$ defined by $v_0 = \alpha$ as first term and

$$v_{n+1} = \alpha + \Phi(v_n, v_n), \quad n \in \mathbb{N}$$

converges to the unique solution $u \in \mathcal{E}_T$ of $u = \alpha + \Phi(u,u)$ provided $\|A^{\frac14}u_0\|_{\mathcal{H}}$ is small enough ($\|\alpha\|_{\mathcal{E}_T} < \frac{1}{4\|\Phi\|_{\mathscr{L}(\mathcal{E}_T\times\mathcal{E}_T;\mathcal{E}_T)}}$). In the case where $\|A^{\frac14}u_0\|_{\mathcal{H}}$ is not small (that is, if $\|\alpha\|_{\mathcal{E}_T} \geq \frac{1}{4\|\Phi\|_{\mathscr{L}(\mathcal{E}_T \times \mathcal{E}_T;\mathcal{E}_T)}}$) then for $\varepsilon > 0$, there exists $u_{0,\varepsilon} \in D(A)$ such that $\|A^{\frac{1}{4}}(u_0 - u_{0,\varepsilon})\|_{\mathcal{H}} \leq \varepsilon$. If we take as initial value $u_{0,\varepsilon} \in D(A)$, we have

$$\|\alpha_\varepsilon\|_{\mathcal{E}_T} \leq cT^{\frac{3}{4}}\|Au_{0,\varepsilon}\|_{\mathcal{H}} \xrightarrow[T\to 0]{} 0.$$

Therefore, we can find $T > 0$ such that $\|\alpha\|_{\mathcal{E}_T} < \frac{1}{4\|\Phi\|_{\mathscr{L}(\mathcal{E}_T \times \mathcal{E}_T;\mathcal{E}_T)}}$. □

4. Strong solutions

Let u be the mild solution to the Navier-Stokes system. We show in this section that u in fact satisfies the equations of the Navier-Stokes system in an L^p-sense (for a suitable p). To begin with, we know that $u \in \mathcal{E}_T$ and satisfies

$$u = \alpha + \Phi(u,u) = \alpha + e^{-\cdot A} * \varphi(u),$$

where $\varphi(u) = -\mathbb{P}_1((u \cdot \nabla)u)$ and we have $\|t^{\frac{1}{2}}(u(t)\cdot\nabla)u(t)\|_{\frac{3}{2}} \leq c\|u\|^2_{\mathcal{E}_T}$. Therefore, we get

$$u(0) = \alpha(0) = u_0, \tag{4.1}$$

$$\mathrm{div} u(t) = 0 \text{ in the } L^2-\text{sense for } t \in]0,T[, \tag{4.2}$$

and

$$u' + Au = f \quad \text{ in } \mathscr{C}(]0,T[;\mathcal{V}'),$$

which means that for all $t \in]0,T[$,

$$\mathbb{P}_1(u'(t) - \Delta_D^\Omega u(t) + (u(t)\cdot\nabla)u(t)) = 0.$$

Then, by Theorem 2.1, there exists $(-\pi)(t) \in (\mathscr{C}_c^\infty(\Omega))'$ such that $\nabla\pi(t) \in H^{-1}(\Omega)^3$ and

$$\nabla(-\pi)(t) = u'(t) - \Delta_D^\Omega u(t) + (u(t)\cdot\nabla)u(t) \tag{4.3}$$

and we have for $0 < t < T$

$$-\Delta_D^\Omega u(t) + \nabla\pi(t) = -u'(t) - (u(t)\cdot\nabla)u(t) \in L^3(\Omega)^3 + L^{\frac{3}{2}}(\Omega)^3.$$

The equation (4.3), together with (4.1) and (4.2), give the usual Navier-Stokes equations which are fulfilled in a strong sense (*a.e.*) where we consider the expression $-\Delta u + \nabla\pi$ undecoupled.

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