# ON MEASURES OF THE EDGE UNCOLORABILITY OF GRAPHS WITH MAXIMUM DEGREE THREE

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Abstract. In [5] Kochol study three invariants of graphs measuring how far a graph is from having a proper 3-edge-coloring. We show here how to get his main result easily.

## 1. Introduction

Throughout this note we shall be concerned with connected graphs with maximum degree 3.

Let $\phi : E(G) \rightarrow \{\alpha, \beta, \gamma, \delta\}$ be a proper edge-coloring of $G$. It is often of interest to try to use one color (say $\delta$) as few as possible. When an edge coloring is optimal, following this constraint, we shall say that $\phi$ is $\delta - minimum$. Since any two $\delta-$minimum edge-coloring of $G$ have the same number of edges colored $\delta$ we shall denote by $s(G)$ this number (the *color number* as defined in [8]).

In [1] we gave without proof (in French, see [4] for a translation) results on $\delta$-minimum edge-colorings of graphs with maximum degree three.

An edge coloring of $G$ with $\{\alpha, \beta, \gamma, \delta\}$ is said to be *$\delta$-improper* whenever we only allow edges colored with $\delta$ to be incident. It must be clear that a proper edge coloring (and hence a $\delta-$minimum edge-coloring ) of $G$ is a particular $\delta$-improper edge coloring. For a proper or $\delta$-improper edge coloring $\phi$ of $G$, it will be convenient to denote $E_\phi(x)$ ($x \in \{\alpha, \beta, \gamma, \delta\}$) the set of edges colored with $x$ by $\phi$.

A *strong matching* $C$ in a graph $G$ is a matching $C$ such that there is no edge of $E(G)$ connecting any two edges of $C$, or, equivalently, such that $C$ is the edge-set of the subgraph of $G$ induced on the vertex-set $V(C)$.

## 2. Results

The proof of the following lemma is given in [3] in the context of simple graphs. We extend this result here to graphs with multiple edges (without loops).

**Lemma 2.1.** [2] *Let $\phi$ be $\delta-$improper coloring of $G$. Suppose that $E_\phi(\delta)$ contains two edges $uv$ and $uw$. Then there exists a $\delta$-improper edge coloring $\phi'$ of $G$ such that $E_{\phi'}(\delta) = E_\phi(\delta) - uv$ or $E_{\phi'}(\delta) = E_\phi(\delta) - uw$*

**Proof** If $v = w$, $u$ and $v$ are joined by two edges and one color, $\alpha, \beta$ or $\gamma$, at least is not incident to both vertices. By giving this color to one of the edges joining $u$ and $v$ we get a new $\delta$-improper edge coloring $\phi'$ satisfying our conclusion.

We can thus assume that $v \neq w$. If some color in $\{\alpha, \beta, \gamma\}$ is missing in $u$ and $v$, we can give this color to $uv$ and the result follows. For the same reason, we can

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assume that the 3 above colors are incident to $u$ and $w$. Without loss of generality, we can consider that the two colors $\alpha$ and $\beta$ are incident to $v$ (as well as to $w$) while $u$ is incident to $\gamma$.

Let $P$ be the path alternately colored with $\alpha$ and $\gamma$ with $u$ as end. If $P$ does not end with $v$, then we can exchange the colors $\alpha$ and $\gamma$ on this path leading to a $\delta$-improper edge coloring such that the color $\gamma$ is missing in $u$ and $v$. The color $\gamma$ could be now given to the edge $uv$ leading to a $\delta$-improper edge coloring $\phi^{'}$ satisfying our conclusion. If $P$ ends with $v$, let $P^{'}$ be the path alternately colored with $\alpha$ and $\gamma$ with $w$ as end. This path does not end with $u$ and an exchange of colors on this path leads to a $\delta$-improper edge coloring such that the color $\alpha$ is missing in $u$ and $w$. The color $\alpha$ could be now given to the edge $uw$ leading to a $\delta$-improper edge coloring $\phi^{'}$ satisfying our conclusion. □

As in [5] denote by $\rho(G)$ the minimum number of vertices that must be deleted from $G$ so that the resulting graph is $3-$edge colorable. We give here a short proof of the main result of Kochol [5].

**Theorem 2.2.** [5] *Let $G$ be a graph with maximum degree 3 then $s(G) = \rho(G)$*

**Proof** Let $X \subseteq V(G)$ such that $G-X$ is $3-$edge colorable and let $\phi:\ E(G-X) \rightarrow \{\alpha, \beta, \gamma\}$ be a $3-$edge coloring. We can extend $\phi$ in a $\delta$-improper edge coloring $\phi^{'}$ of the edge set of $G$ by giving the color $\delta$ to the edges incident to the vertices of $X$. By repeated application of Lemma 2.1, we can find a proper edge coloring of $G$ $\phi^{''}$ such that $E_{\phi^{''}}(\delta) \subseteq E_{\phi^{'}}(\delta)$. Hence $|X| \geq |E_{\phi^{''}}(\delta)| \geq s(G)$.

Conversely let $\phi$ be a $\delta-$minimum edge-coloring of $G$. By Lemma 2.1 $E_{\phi}(\delta)$ is a matching. Let $X \subseteq V(G)$ be a minimal set of vertices intersecting each edge of $E_{\phi}(\delta)$. Then $G - X$ is $3-$edge colorable and $\rho(G) \leq |X| \leq s(G)$. □

The following theorem was first proved by Payan in [6].

**Theorem 2.3.** [2] [6] *Let $G$ be a graph with maximum degree at most 3. Then $G$ has a $\delta-$minimum edge-coloring $\phi$ where $E_{\phi}(\delta)$ is a strong matching and, moreover, any edge in $E_{\phi}(\delta)$ has its two ends of degree 3 in $G$.*

In [7] Steffen showed that it is always possible to find an independent set of size at most $s(G)$ in a graph with maximum degree 3 whose deletion leaves a $3-$edge colorable graph. It can be noticed that this is an immediate corollary of Theorem 2.3. Denote by $\rho_{\alpha}(G)$ the minimum number of vertices of an independent set that must be deleted from $G$ so that the resulting graph is $3-$edge colorable.

**Theorem 2.4.** *Let $G$ be a graph with maximum degree 3 then $s(G) = \rho_{\alpha}(G)$*

**Proof** Obviously we have $\rho(G) \leq \rho_{\alpha}(G)$ and hence $s(G) \leq \rho_{\alpha}(G)$ by Theorem 2.2. Conversely let $\phi$ be a $\delta-$minimum edge-coloring of $G$ such that $E_{\phi}(\delta)$ is a strong matching (Theorem 2.3) and let $X \subseteq V(G)$ be a minimal set of vertices intersecting each edge of $E_{\phi}(\delta)$. Then $X$ is an independent set and $G - X$ is $3-$edge colorable. Hence $\rho_{\alpha}(G) \leq |X| \leq s(G)$. □

By Theorem 2.3 an optimal set satisfying Theorem 2.4 can be found with all vertices of degree 3. In fact Theorem 2.4 says that we can find at least $2^{s(G)}$ such independent sets (when $s(G) > 0$).

In [8] Steffen showed that $s(G) \leq \frac{n}{8}$ when $G$ is a bridgeless cubic graph with at least 16 vertices. In fact this result can be easily extended to a subclass of graphs with maximum degree 3.

**Proposition 2.5.** *Let $G$ be a bridgeless graph with maximum degree 3 and $k \neq 1$ vertices of degree 2 with at least 16 vertices. Then $s(G) \leq \frac{n}{8}$.*

**Proof** By the result of Steffen [8], we can assume that $G$ has $k \geq 2$ vertices of degree 2. Let $v_1 \dots v_k$ be these vertices. Let $G^{'}$ be a copy of $G$ and $v_1^{'} \dots v_k^{'}$ be the corresponding vertices of degree 2. Let $H$ be the graph obtained from these two copies by joining the vertices $v_i$ and $v_i^{'}$ $(i = 1 \dots k)$ by an edge. It is an easy matter to see that $H$ is a bridgeless cubic graph. By the result of Steffen [8], we have $s(H) \leq \frac{2n}{8}$.

Let $p$ ($q$ respectively) be the number of edges colored with $\delta$ in the copy of $G$ ($G^{'}$ respectively). We obviously have $p \leq q$ and thus $p \leq \frac{2n}{16}$. That is $s(G) \leq \frac{n}{8}$ as claimed.

□

**Corollary 2.6.** *Let $G$ be a bridgeless graph with maximum degree 3 and at least 16 vertices. Assume that $G$ has $k \neq 1$ vertices of degree 2. Then $s(G) = \rho(G) = \rho_\alpha(G) \leq \frac{n}{8}$*

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