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GEOMETRIC ANALYSIS ON SMALL UNITARY REPRESENTATIONS OF $GL(N,\mathbb{R})$

TOSHIYUKI KOBAYASHI, BENT ØRSTED, MICHAEL PEVZNER

Abstract. The most degenerate unitary principal series representations $\pi_{i\lambda,\delta}$ ($\lambda \in \mathbb{R}$, $\delta \in \mathbb{Z}/2\mathbb{Z}$) of $G = GL(N,\mathbb{R})$ attain the minimum of the Gelfand–Kirillov dimension among all irreducible unitary representations of G. This article gives an explicit formula of the irreducible decomposition of the restriction $\pi_{i\lambda,\delta}|_H$ (*branching law*) with respect to all symmetric pairs (G,H). For $N = 2n$ with $n \geq 2$, the restriction $\pi_{i\lambda,\delta}|_H$ remains irreducible for $H = Sp(n,\mathbb{R})$ if $\lambda \neq 0$ and splits into two irreducible representations if $\lambda = 0$. The branching law of the restriction $\pi_{i\lambda,\delta}|_H$ is purely discrete for $H = GL(n,\mathbb{C})$, consists only of continuous spectrum for $H = GL(p,\mathbb{R}) \times GL(q,\mathbb{R})$ ($p+q=N$), and contains both discrete and continuous spectra for $H = O(p,q)$ ($p > q \geq 1$). Our emphasis is laid on geometric analysis, which arises from the restriction of 'small representations' to various subgroups.

Key words and phrases: *small representation, branching law, symmetric pair, reductive group, phase space representation, symplectic group, degenerate principal series representations.*

1. Introduction

The subject of our study is geometric analysis on 'small representations' of $GL(N,\mathbb{R})$ through branching problems to non-compact subgroups.

Here, by a branching problem, we mean a general question on the understanding how irreducible representations of a group decompose when restricted to a subgroup. A classic example is studying the irreducible decomposition of the tensor product of two representations. Branching problems are one of the most basic problems in representation theory, however, it is hard in general to find explicit branching laws for unitary representations of non-compact reductive groups. For reductive symmetric spaces G/H, the multiplicities in the Plancherel formula of $L^2(G/H)$ are finite [1, 5], whereas the multiplicities in the branching laws for the restriction $G \downarrow H$ are often infinite even when

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(G, H) are symmetric pairs (see *e.g.* [17] for recent developments and open problems in this area).

Our standing point is that '*small representations' of a group should have 'large symmetries' in the representation spaces*, as was advocated by one of the authors from the perspectives in global analysis [18]. In particular, considering the restrictions of 'small representations' to reasonable subgroups, we expect that their breaking symmetries should have still fairly large symmetries, for which geometric analysis would deserve finer study.

Then, what are 'small representations'? For this, the Gelfand–Kirillov dimension serves as a coarse measure of the 'size' of infinite dimensional representations. We recall that for an irreducible unitary representation π of a real reductive Lie group G the Gelfand–Kirillov dimension $\mathrm{DIM}(\pi)$ takes the value in the set of half the dimensions of nilpotent orbits in the Lie algebra $\mathfrak{g}$. We may think of π as one of the 'smallest' infinite dimensional representations of G, if $\mathrm{DIM}(\pi)$ equals $n(G)$, half the dimension of the minimal nilpotent orbit.

For the metaplectic group $G = Mp(m, \mathbb{R})$, the connected two-fold covering group of the symplectic group $Sp(m, \mathbb{R})$ of rank m, the Gelfand–Kirillov dimension attains its minimum $n(G) = m$ at the Segal–Shale–Weil representation. For the indefinite orthogonal group $G = O(p, q)$ $(p, q > 3)$, there exists π such that $\mathrm{DIM}(\pi) = n(G)$ $(= p + q - 3)$ if and only if $p + q$ is even according to an algebraic result of Howe and Vogan. See *e.g.* a survey paper [12] for the algebraic theory of 'minimal representations', and [11, 18, 19, 20] for their analytic aspects.

In general, a real reductive Lie group G admits at most finitely many irreducible unitary representations π with $\mathrm{DIM}(\pi) = n(G)$ if the complexified Lie algebra $\mathfrak{g}_{\mathbb{C}}$ does not contain a simple factor of type A (see [12]). In contrast, for $G = GL(N, \mathbb{R})$, there exist infinitely many irreducible unitary representations π with $\mathrm{DIM}(\pi) = n(G)$ $(= N - 1)$. For example, the unitarily induced representations

$$\pi_{i\lambda,\delta}^{GL(N,\mathbb{R})} := \mathrm{Ind}_{P_N}^{GL(N,\mathbb{R})}(\chi_{i\lambda,\delta}) \tag{1.1}$$

from a unitary character $\chi_{i\lambda,\delta}$ of a maximal parabolic subgroup

$$P_N := (GL(1, \mathbb{R}) \times GL(N-1, \mathbb{R})) \ltimes \mathbb{R}^{N-1}, \tag{1.2}$$

are such representations with parameter $\lambda \in \mathbb{R}$ and $\delta \in \mathbb{Z}/2\mathbb{Z}$.

In this paper, we find the irreducible decomposition of these 'small representations' $\pi_{i\lambda,\delta}^{GL(N,\mathbb{R})}$ with respect to *all symmetric pairs*.

We recall that a pair of Lie groups (G, H) is said to be a symmetric pair if there exists an involutive automorphism σ of G such that H is an

open subgroup of $G^\sigma := \{g \in G : \sigma g = g\}$. According to M. Berger's classification [4], the following subgroups $H = K$, $G_j\,(1 \le j \le 4)$ and G exhaust all symmetric pairs (G, H) for $G = GL(N,\mathbb{R})$ up to local isomorphisms and the center of G:

$$\begin{aligned}
K &:= O(N) && \text{(maximal compact subgroup)},\\
G_1 &:= Sp(n,\mathbb{R}) && (N = 2n),\\
G_2 &:= GL(n,\mathbb{C}) && (N = 2n),\\
G_3 &:= GL(p,\mathbb{R}) \times GL(q,\mathbb{R}) && (N = p+q),\\
G_4 &:= O(p,q) && (N = p+q).
\end{aligned}$$

It turns out that the branching laws for the restrictions of $\pi^{GL(N,\mathbb{R})}_{i\lambda,\delta}$ with respect to these subgroups behave nicely in all the cases, and in particular, the multiplicities of irreducible representations in the branching laws are uniformly bounded.

To be more specific, the restriction of $\pi^{GL(N,\mathbb{R})}_{i\lambda,\delta}$ to K splits discretely into the space of spherical harmonics on $\mathbb{R}^N$, and the resulting K-type formula is multiplicity-free and so called of ladder type. For the non-compact subgroups $G_j\,(1 \le j \le 4)$, we prove the following irreducible decompositions in Theorems 8.1, 9.1, 10.1 and 11.1:

Theorem 1.1. *For $\lambda \in \mathbb{R}$ and $\delta \in \mathbb{Z}/2\mathbb{Z}$, the irriducible unitary representation $\pi^{GL(N,\mathbb{R})}_{i\lambda,\delta}$ decomposes when restricted to symmetric pairs as follows:*

1) $GL(2n,\mathbb{R}) \downarrow Sp(n,\mathbb{R})$, $(n \ge 2)$*:*

$$\pi^{GL(2n,\mathbb{R})}_{i\lambda,\delta}\Big|_{G_1} \simeq \begin{cases} \text{Irreducible} & (\lambda \neq 0),\\ \left(\pi^{Sp(n,\mathbb{R})}_{0,\delta}\right)^+ \oplus \left(\pi^{Sp(n,\mathbb{R})}_{0,\delta}\right)^- & (\lambda = 0). \end{cases}$$

2) $GL(2n,\mathbb{R}) \downarrow GL(n,\mathbb{C})$*:*

$$\pi^{GL(2n,\mathbb{R})}_{i\lambda,\delta}\Big|_{G_2} \simeq \sum_{m \in 2\mathbb{Z}+\delta}^{\oplus} \pi^{GL(n,\mathbb{C})}_{i\lambda,m}.$$

3) $GL(p+q,\mathbb{R}) \downarrow GL(p,\mathbb{R}) \times GL(q,\mathbb{R})$:

$$\pi^{GL(p+q,\mathbb{R})}_{i\lambda,\delta}\Big|_{G_3} \simeq \sum_{\delta' \in \mathbb{Z}/2\mathbb{Z}} \int_{\mathbb{R}}^{\oplus} \pi^{GL(p,\mathbb{R})}_{i\lambda',\delta'} \boxtimes \pi^{GL(q,\mathbb{R})}_{i(\lambda-\lambda'),\delta-\delta'} d\lambda'.$$

4) $GL(p+q,\mathbb{R}) \downarrow O(p,q)$:

$$\pi^{GL(p+q,\mathbb{R})}_{i\lambda,\delta}\Big|_{G_4} \simeq \sum_{\nu \in A^\delta_+(p,q)}^{\oplus} \pi^{O(p,q)}_{+,\nu} \oplus \sum_{\nu \in A^\delta_+(q,p)}^{\oplus} \pi^{O(p,q)}_{-,\nu} \oplus 2\int_{\mathbb{R}_+}^{\oplus} \pi^{O(p,q)}_{i\nu,\delta} d\nu.$$

Here, each summand in the right-hand side stands for (pairwise inequivalent) irreducible representations of the corresponding subgroups which will be defined explicitly in Sections 8, 9, 10 and 11.

As indicated above, we see that the representation $\pi_{i\lambda,\delta}^{GL(2n,\mathbb{R})}$ remains generically irreducible when restricted to the subgroup $G_1 = Sp(n,\mathbb{R})$ and splits into a direct sum of two irreducible subrepresentations for $\lambda = 0$ and $n > 1$. The case $n = 1$ is well known (*cf.* [3]): the group $Sp(1,\mathbb{R})$ is isomorphic to $SL(2,\mathbb{R})$, and $\pi_{i\lambda,\delta}$ are irreducible except for $(\lambda,\delta) = (0,1)$, while $\pi_{0,1}$ splits into the direct sum of two irreducible unitary representations i.e. the (classical) Hardy space and its dual.

The representation $\pi_{i\lambda,\delta}^{GL(2n,\mathbb{R})}$ is discretely decomposable in the sense of [16] when restricted to the subgroup $G_2 = GL(n,\mathbb{C})$. In other words, the non-compact group G_2 behaves in the representation space of $\pi_{i\lambda,\delta}^{GL(2n,\mathbb{R})}$ as if it were a compact subgroup. In contrast, the restriction of $\pi_{i\lambda,\delta}^{GL(p+q,\mathbb{R})}$ to another subgroup $G_3 = GL(p,\mathbb{R}) \times GL(q,\mathbb{R})$ decomposes without discrete spectrum, while both discrete and continuous spectra appear for the restriction of $\pi_{i\lambda,\delta}^{GL(p+q,\mathbb{R})}$ to $G_4 = O(p,q)$ if $p,q \geq 1$ and $(p,q) \neq (1,1)$. Finally, in Theorem 12.1 we give an irreducible decomposition of the tensor product of the Segal–Shale–Weil representation with its dual, giving another example of explicit branching laws of small representations with respect to symmetric pairs.

We have stated Theorem 1.1 from representation theoretic viewpoint. However, our emphasis is not only on results of this nature but also on geometric analysis of concrete models via branching laws of small representations, which we find surprisingly rich in its interaction with various domains of classical analysis and their new aspects. It includes the theory of Hilbert-space valued Hardy spaces (Section 2), the Weyl operator calculus (Section 3), representation theory of Jacobi and Heisenberg groups, the Segal–Shale–Weil representation of the metaplectic group (Section 4), (complex) spherical harmonics (Section 5), the K-Bessel functions (Section 7), and global analysis on space forms of indefinite-Riemannian manifolds (Section 11).

Further, we introduce a *non-standard* L^2-model for the degenerate principal series representations of $Sp(n,\mathbb{R})$ where the Knapp–Stein intertwining operator becomes an *algebraic operator* (Theorem 6.1). In this model the minimal K-types are given in terms of Bessel functions (Proposition 7.1). The two irreducible components $\pi_{0,\delta}^{\pm}$ at $\lambda = 0$ in Theorem 1.1 1) will be presented in three ways, that is, in terms of Hardy spaces based on the Weyl operator calculus as giving the P-module

structure, complex spherical harmonics as giving the K-module structure, and the eigenspaces of the Knapp–Stein intertwining operators (see Theorem 8.3).

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Notation: $\mathbb{N} = \{0, 1, 2, \dots\}$, $\mathbb{N}_+ = \{1, 2, 3, \dots\}$, $\mathbb{R}_\pm = \{\rho \in \mathbb{R} : \pm\rho \geq 0\}$, $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$, and $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$.

2. Hilbert space valued Hardy space

Let W be a (separable) Hilbert space. Then, we can define the Bochner integrals of weakly measurable functions on $\mathbb{R}$ with values in W. For a measurable set E in $\mathbb{R}$, we denote by $L^2(E, W)$ the Hilbert space consisting of W-valued square integrable functions on E. Clearly, it is a closed subspace of $L^2(\mathbb{R}, W)$.

Suppose F is a W-valued function defined on an open subset in $\mathbb{C}$. We say F is holomorphic if the scalar product $(F, w)_W$ is a holomorphic function for any $w \in W$.

Let Π_+ be the upper half plane $\{z = t + iu \in \mathbb{C} : u = \operatorname{Im} z > 0\}$. Then, the W-valued Hardy space is defined as

(2.1)
$$\mathcal{H}^2_+(W) := \{F : \Pi_+ \to W : F \text{ is holomorphic and } ||F||_{\mathcal{H}^2_+(W)} < \infty\},$$

where the norm $\|F\|_{\mathcal{H}^2_+}(W)$ is given by

$$||F||_{\mathcal{H}^2_+(W)} := \left(\sup_{u>0} \int_{\mathbb{R}} ||F(t+iu)||^2_W dt\right)^{\frac{1}{2}}.$$

Similarly, $\mathcal{H}^2_-(W)$ is defined by replacing Π_+ with the lower half plane Π_-. Notice that $\mathcal{H}^2_+(W)$ is the classical Hardy space, if $W = \mathbb{C}$.

Next, we define the W-valued Fourier transform $\mathcal{F}$ as

$$\mathcal{F} : L^2(\mathbb{R}, W) \to L^2(\mathbb{R}, W), \quad f(t) \mapsto (\mathcal{F}f)(\rho) := \int_{\mathbb{R}} f(t) e^{-2\pi i \rho t}\, dt.$$

Here, the Bochner integral converges for $f \in (L^1 \cap L^2)(\mathbb{R}, W)$ with obvious notation. Then, $\mathcal{F}$ extends to the Hilbert space $L^2(\mathbb{R}, W)$ as a unitary isomorphism.

Example 2.1. *Suppose $W = L^2(\mathbb{R}^k)$ for some k. Then, we have a natural unitary isomorphism $L^2(\mathbb{R}, W) \simeq L^2(\mathbb{R}^{k+1})$. Via this isomorphism, the $L^2(\mathbb{R}^k)$-valued Fourier transform $\mathcal{F}$ is identified with the*

partial Fourier transform $\mathcal{F}_t$ with respect to the first variable t as follows:

$$\begin{array}{ccc} L^2(\mathbb{R}, L^2(\mathbb{R}^k)) & \overset{\sim}{\underset{\mathcal{F}}{\to}} & L^2(\mathbb{R}, L^2(\mathbb{R}^k)) \\ \wr & & \wr \\ L^2(\mathbb{R}^{k+1}) & \overset{\sim}{\underset{\mathcal{F}_t}{\to}} & L^2(\mathbb{R}^{k+1}). \end{array} \tag{2.2}$$

As in the case of the classical theory on the (scalar-valued) Hardy space $\mathcal{H}^2_+ \equiv \mathcal{H}^2_+(\mathbb{C})$, we can characterize $\mathcal{H}^2_\pm(W)$ by means of the Fourier transform:

Lemma 2.2. *Let W be a separable Hilbert space, and $\mathcal{H}^2_\pm(W)$ the W-valued Hardy spaces (see (2.1)).*

1) *For $F \in \mathcal{H}^2_\pm(W)$, the boundary value*
$$F(t \pm i0) := \lim_{u \downarrow 0} F(t \pm iu)$$
exists as a weak limit in the Hilbert space $L^2(\mathbb{R}, W)$, and defines an isometric embedding:
$$\mathcal{H}^2_\pm(W) \hookrightarrow L^2(\mathbb{R}, W). \tag{2.3}$$
From now, we regard $\mathcal{H}^2_\pm(W)$ as a closed subspace of $L^2(\mathbb{R}, W)$.
2) *The W-valued Fourier transform $\mathcal{F}$ induces the unitary isomorphism:*
$$\mathcal{F} : \mathcal{H}^2_\pm(W) \overset{\sim}{\to} L^2(\mathbb{R}_\pm, W).$$
3) *$L^2(\mathbb{R}, W) = \mathcal{H}^2_+(W) \oplus \mathcal{H}^2_-(W)$ (direct sum).*
4) *If a function $F \in \mathcal{H}^2_+(W)$ satisfies $F(t + i0) = F(-t + i0)$ then $F \equiv 0$.*

Proof. The idea is to reduce the general case to the classical one by using a uniform estimate on norms as the imaginary part u tends to zero.

Let $\{e_j\}$ be an orthonormal basis of W. Suppose $F \in \mathcal{H}^2_+(W)$. Then we have

$$\begin{aligned} \|F\|^2_{\mathcal{H}^2_+(W)} &= \sup_{u>0} \int_{\mathbb{R}} \|F(t+iu)\|^2_W dt \\ &= \sup_{u>0} \sum_j I_j(u), \end{aligned} \tag{2.4}$$

where we set

$$I_j(u) := \int_{\mathbb{R}} |(F(t+iu), e_j)_W|^2 dt.$$

Then, it follows from (2.4) that for any j $\sup_{u>0} I_j(u) < \infty$ and therefore

$$F_j(z) := (F_j(z), e_j)_W, \quad (z = t + iu \in \Pi_+)$$

belongs to the (scalar-valued) Hardy space $\mathcal{H}^2_+$. By the classical Paley–Wiener theorem for the (scalar-valued) Hardy space $\mathcal{H}^2_+$, we have:

$$F_j(t + i0) := \lim_{u \downarrow 0} F_j(t + iu) \text{ (weak limit in } L^2(\mathbb{R})), \tag{2.5}$$

$$\mathcal{F}F_j(t + i0) \in L^2(\mathbb{R}_+), \tag{2.6}$$

$$(\mathcal{F}F_j(t + iu))(\rho) = e^{-2\pi u\rho}(\mathcal{F}F_j(t + i0))(\rho) \text{ for } u > 0, \tag{2.7}$$

$$I_j(u) \text{ is a monotonely decreasing function of } u > 0, \tag{2.8}$$

$$\lim_{u \downarrow 0} I_j(u) = \|F_j(t + iu)\|^2_{\mathcal{H}^2_+} = \|F_j(t + i0)\|^2_{L^2(\mathbb{R})}. \tag{2.9}$$

The formula (2.7) shows (2.8), which is crucial in the uniform estimate as below. In fact by (2.8) we can exchange $\sup_{u>0}$ and $\sum_j$ in (2.4). Thus, we get

$$\|F\|^2_{\mathcal{H}^2_+(W)} = \sum_j \lim_{u \downarrow 0} I_j(u) = \sum_j \|F_j(t + i0)\|^2_{L^2(\mathbb{R})}.$$

Hence we can define an element of $L^2(\mathbb{R}, W)$ as the following weak limit:

$$F(t + i0) := \sum_j F_j(t + i0)e_j.$$

Equivalently, $F(t + i0)$ is the weak limit of $F(t + iu)$ in $L^2(\mathbb{R}, W)$ as $u \to 0$. Further, (2.6) implies $\operatorname{supp} \mathcal{F}F(t + i0) \subset \mathbb{R}_+$ because

$$\mathcal{F}F(t + i0) = \sum_j \mathcal{F}F_j(t + i0)e_j \quad \text{(weak limit)}.$$

In summary we have shown that $F(t + i0) \in L^2(\mathbb{R}, W)$, $\mathcal{F}F(t + i0) \in L^2(\mathbb{R}_+, W)$, and

$$\|F\|_{\mathcal{H}^2_+(W)} = \|F(t + i0)\|_{L^2(\mathbb{R},W)} = \|\mathcal{F}F(t + i0)\|_{L^2(\mathbb{R}_+,W)}$$

for any $F \in \mathcal{H}^2_+(W)$. Thus, we have proved that the map

$$\mathcal{F} : \mathcal{H}^2_+(W) \to L^2(\mathbb{R}_+, W)$$

is well-defined and isometric.

Conversely, the opposite inclusion $\mathcal{F}^{-1}(L^2(\mathbb{R}_+, W)) \subset \mathcal{H}^2_+(W)$ is proved in a similar way. Hence the statements 1), 2) and 3) follow.

The last statement is now immediate from 2) because $\mathcal{F}F(t+i0)(\rho) = \mathcal{F}F(-t + i0)(-\rho)$. □

3. Weyl Operator Calculus

In this section, based on the well-known construction of the Schrödinger representation and the Segal–Shale–Weil representation, we introduce the action of the outer automorphisms of the Heisenberg group on the Weyl operator calculus (see (3.11), (3.13), and (3.14)), and discuss carefully its basic properties, see Proposition 3.2 and Lemma 3.4. In particular, the results of this section will be used in analyzing of the 'small representation' $\pi^{GL(2n,\mathbb{R})}_{i\lambda,\delta}$, when restricted to a certain maximal parabolic subgroup of $Sp(n,\mathbb{R})$, see e.g. the identity (4.12).

Let $\mathbb{R}^{2m}$ be the $2m$-dimensional Euclidean vector space endowed with the standard symplectic form

$$\omega(X,Y) \equiv \omega((x,\xi),(y,\eta)) := \langle \xi, y\rangle - \langle x, \eta\rangle. \tag{3.1}$$

The choice of this non-degenerate closed 2-form gives a standard realization of the symplectic group $Sp(m,\mathbb{R})$ and the Heisenberg group H^{2m+1}. Namely,

$$Sp(m,\mathbb{R}) := \{T \in GL(2m,\mathbb{R}) : \omega(TX,TY) = \omega(X,Y)\}$$

and

$$H^{2m+1} := \{g = (s,A) \in \mathbb{R}\times\mathbb{R}^{2m}\}$$

equipped with the product

$$g\cdot g' \equiv (s,A)\cdot(s',A') := (s+s'+\frac{1}{2}\omega(A,A'), A+A').$$

Accordingly, the Heisenberg Lie algebra $\mathfrak{h}^{2m+1}$ is then defined by

$$[(s,X),(t,Y)] = (\omega(X,Y),0).$$

Finally we denote by Z the center $\{(s,0) : s\in\mathbb{R}\}$ of H^{2m+1}.

The Heisenberg group H^{2m+1} admits a unitary representation, denoted by ϑ, on the *configuration space* $L^2(\mathbb{R}^m)$ by the formula

$$\vartheta(g)\varphi(x) = e^{2\pi i(s+\langle x,\alpha\rangle - \frac{1}{2}\langle a,\alpha\rangle)}\varphi(x-a), \quad g=(s,a,\alpha). \tag{3.2}$$

This representation, referred to as the *Schrödinger representation*, is irreducible and unitary [23]. The symplectic group, or more precisely its double covering, also acts on the same Hilbert space $L^2(\mathbb{R}^m)$.

In order to track the effect of $\mathrm{Aut}(H^{2m+1})$, we recall briefly its construction. The group $Sp(m,\mathbb{R})$ acts by automorphisms of H^{2m+1} preserving the center Z pointwise. Composing ϑ with such automorphisms $T\in Sp(m,\mathbb{R})$ one gets a new representation $\vartheta\circ T$ of H^{2m+1} on $L^2(\mathbb{R}^m)$. Notice that these representations have the same central character, namely $\vartheta\circ T(s,0,0) = e^{2\pi i s}\,\mathrm{id} = \vartheta(s,0,0)$. According to

the Stone–von Neumann theorem (see Fact 3.3 below) the representations ϑ and $\vartheta\circ T$ are equivalent as irreducible unitary representations of H^{2m+1}. Thus, there exists a unitary operator $\mathrm{Met}(T)$ acting on $L^2(\mathbb{R}^m)$ in such a way that

$$(\vartheta\circ T)(g) = \mathrm{Met}(T)\vartheta(g)\mathrm{Met}(T)^{-1}, \quad g\in H^{2m+1}. \tag{3.3}$$

Because ϑ is irreducible, Met is defined up to a scalar and gives rise to a projective unitary representation of $Sp(m,\mathbb{R})$. It is known that this scalar factor may be chosen in one and only one way, up to a sign, so that Met becomes a double-valued representation of $Sp(m,\mathbb{R})$. The resulting unitary representation of the metaplectic group, that we keep denoting Met, is referred to as the *Segal–Shale–Weil representation* and it is a lowest weight module with respect to a fixed Borel subalgebra. Notice that choosing the opposite sign of the scalar factor in the definition of Met one gets a highest weight module which is isomorphic to the contragredient representation $\mathrm{Met}^\vee$.

The unitary representation Met splits into two irreducible and inequivalent subrepresentations Met_0 and Met_1 according to the decomposition of the Hilbert space $L^2(\mathbb{R}^m) = L^2(\mathbb{R}^m)_{\mathrm{even}}\oplus L^2(\mathbb{R}^m)_{\mathrm{odd}}$.

The *Weyl quantization*, or the *Weyl operator calculus*, is a way to associate to a function $\mathfrak{S}(x,\,\xi)$ the operator $\mathrm{Op}(\mathfrak{S})$ on $L^2(\mathbb{R}^m)$ defined by the equation

$$(\mathrm{Op}(\mathfrak{S})\,u)(x) = \int_{\mathbb{R}^m\times\mathbb{R}^m} \mathfrak{S}\left(\frac{x+y}{2},\,\eta\right) e^{2\pi i\langle x-y,\,\eta\rangle}\,u(y)\;dy\,d\eta\,. \tag{3.4}$$

Such a linear operator sets up an isometry

$$\mathrm{Op} : L^2(\mathbb{R}^{2m}) \xrightarrow{\sim} \mathrm{HS}(L^2(\mathbb{R}^m), L^2(\mathbb{R}^m)). \tag{3.5}$$

from the *phase space* $L^2(\mathbb{R}^m\times\mathbb{R}^m)$ onto the Hilbert space consisting of all Hilbert–Schmidt operators on the configuration space $L^2(\mathbb{R}^m)$. Introducing the *symplectic* Fourier transformation $\mathcal{F}_{\mathrm{symp}}$ by:

$$(\mathcal{F}_{\mathrm{symp}}\,\mathfrak{S})(X) := \int_{\mathbb{R}^m\times\mathbb{R}^m} \mathfrak{S}(Y)\,e^{-2i\pi\omega(X,Y)}\,dY\,, \tag{3.6}$$

one may give another, fully equivalent, definition of the Weyl operator by means of the equation

$$\mathrm{Op}(\mathfrak{S}) = \int_{\mathbb{R}^{2m}} (\mathcal{F}_{\mathrm{symp}}\,\mathfrak{S})(Y)\,\vartheta(0,Y)\,dY, \tag{3.7}$$

where the right-hand side is a Bochner operator-valued integral.

The Heisenberg group H^{2m+1} acts on $\mathbb{R}^{2m}\simeq H^{2m+1}/Z$, by

$$\mathbb{R}^{2m}\to\mathbb{R}^{2m},\ X\mapsto X+A \quad \text{for } g=(s,A),$$

and consequently it acts on the phase space $L^2(\mathbb{R}^{2m})$ by left translations. The symplectic group $Sp(m,\mathbb{R})$ also acts on the same Hilbert space $L^2(\mathbb{R}^{2m})$ by left translations. (This representation is reducible. See Section 12 for its irreducible decomposition.) In fact, both representations come from an action on $L^2(\mathbb{R}^{2m})$ of the semidirect product group $G^J := Sp(m,\mathbb{R}) \ltimes H^{2m+1}$ which is referred to as the *Jacobi group*.

Let us recall some classical facts in a way that we shall use them in the sequel:

Fact 3.1.

1) *The representations ϑ and Met form a unitary representation of the double covering $Mp(m,\mathbb{R}) \ltimes H^{2m+1}$ of G^J on the configuration space $L^2(\mathbb{R}^m)$. This action induces a representation of the Jacobi group G^J on the Hilbert space of Hilbert–Schmidt operators* $\mathrm{HS}(L^2(\mathbb{R}^m), L^2(\mathbb{R}^m))$ *by conjugations.*

2) *The Weyl quantization map Op intertwines the action of G^J on $L^2(\mathbb{R}^{2m})$ with the representation* $\mathrm{Met} \ltimes \vartheta$ *on the Hilbert space* $\mathrm{HS}(L^2(\mathbb{R}^m), L^2(\mathbb{R}^m))$ *defined in 2). Namely,*

$$\vartheta(g)\,\mathrm{Op}(\mathfrak{S})\,\vartheta(g^{-1}) = \mathrm{Op}(\mathfrak{S} \circ g^{-1}), \quad g \in H^{2m+1}. \tag{3.8}$$

$$\mathrm{Met}(g)\,\mathrm{Op}(\mathfrak{S})\,\mathrm{Met}^{-1}(g) = \mathrm{Op}(\mathfrak{S} \circ g^{-1}), \quad g \in Sp(m,\mathbb{R}). \tag{3.9}$$

3) *Any unitary operator satisfying (3.8) and (3.9) is a scalar multiple of the Weyl quantization map Op.*

Proof. Most of these statements may be found in the literature (*e.g.* [11, Chapter 2] for the second statement), but we give a brief explanation of some of them for the convenience of the reader. Namely, the first statement follows from (3.3). Consequently, the semi-direct product $Mp(m,\mathbb{R}) \ltimes H^{2m+1}$ also acts by conjugations on the space $\mathrm{HS}(L^2(\mathbb{R}^m), L^2(\mathbb{R}^m))$, and this action is well defined for the Jacobi group $G^J = Sp(m,\mathbb{R}) \ltimes H^{2m+1}$ because the kernel of the metaplectic cover $Mp(m,\mathbb{R}) \to Sp(m,\mathbb{R})$ acts trivially on $\mathrm{HS}(L^2(\mathbb{R}^m), L^2(\mathbb{R}^m))$.

The third statement follows from the fact that $L^2(\mathbb{R}^{2m})$ is already irreducible by the codimension one subgroup $Sp(m,\mathbb{R}) \ltimes \mathbb{R}^{2m}$ of G^J. Indeed, any translation–invariant closed subspace of $L^2(\mathbb{R}^{2m})$ is a Wiener space, *i.e.* the pre-image by the Fourier transform of $L^2(E)$ for some measurable set E in $\mathbb{R}^{2m}$. On the other hand, the symplectic group acts ergodically on $\mathbb{R}^{2m}$, in the sense that the only $Sp(m,\mathbb{R})$–invariant measurable subsets of $\mathbb{R}^{2m}$ are either null or conull with respect to the Lebesgue measure. Hence, the whole group $Sp(m,\mathbb{R}) \ltimes \mathbb{R}^{2m+1}$ acts irreducibly on $L^2(\mathbb{R}^{2m})$. □

Now we consider the 'twist' of the metaplectic representation by automorphisms of the Heisenberg group.

The group of automorphisms of the Heisenberg group H^{2m+1}, to be denoted by $\mathrm{Aut}(H^{2m+1})$, is generated by

- symplectic maps : $(s,A) \mapsto (s, T(A))$, where $T \in Sp(m,\mathbb{R})$;
- inner automorphisms $(s,A) \mapsto I_{(t,B)}(s,A) := (t,B)(s,A)(t,B)^{-1}$ $= (s - \omega(A,B), A)$, where $(t,B) \in H^{2m+1}$;
- dilations $(s,A) \mapsto d(r)(s,A) := (r^2 s, rA)$, where $r > 0$;
- inversion: $(s,A) \mapsto i(s,A) := (-s, \alpha, a)$, where $A = (a,\alpha)$.

In the sequel we shall pay a particular attention to the *rescaling map* τ_ρ which is defined for every $\rho \neq 0$ by

$$\tau_\rho : H^{2m+1} \to H^{2m+1}, \quad (s,a,\alpha) \mapsto \left(\frac{\rho}{4}\, s, a, \frac{\rho}{4}\, \alpha\right). \tag{3.10}$$

Here we have adopted the parametrization of τ_ρ in a way that it fits well into Lemma 4.2. We note that $(\tau_{-4})^2 = \mathrm{id}$ and $\tau_4 = \mathrm{id}$.

The whole group $\mathrm{Aut}(H^{2m+1})$ of automorphisms is generated by G^J and $\{\tau_\rho : \rho \in \mathbb{R}^\times\}$. We denote by $\mathrm{Aut}(H^{2m+1})_o$ the identity component of $\mathrm{Aut}(H^{2m+1})$. Then we have

$$\mathrm{Aut}(H^{2m+1}) = \{1, \tau_{-4}\} \cdot \mathrm{Aut}(H^{2m+1})_o.$$

For any given automorphism $\tau \in \mathrm{Aut}(H^{2m+1})$, we denote by $\overline{\tau}$ the induced linear operator on $H^{2m+1}/Z \simeq \mathbb{R}^{2m}$ and by $\pi(\tau)$ its pull-back $\pi(\tau)f := f \circ (\overline{\tau})^{-1}$. We notice that $\pi(\tau)$ is a unitary operator on $L^2(\mathbb{R}^{2m})$ if $\tau \in G^J$.

Further, we define the τ-*twist* Op_τ of the Weyl quantization map Op by

$$\mathrm{Op}_\tau := \mathrm{Op} \circ \pi(\tau). \tag{3.11}$$

In particular, it follows from (3.4) and (3.10) that

(3.12)
$$(\mathrm{Op}_{\tau_\rho}(\mathfrak{S})\, u)(x) = \int_{\mathbb{R}^m \times \mathbb{R}^m} \mathfrak{S}\left(\frac{x+y}{2}, \frac{4}{\rho}\xi\right) e^{2\pi i \langle x-y, \xi\rangle}\, u(y)\, dy\, d\eta\,.$$

Similarly, we define the τ-twist ϑ_τ of the Schrödinger representation ϑ by

$$\vartheta_\tau := \vartheta \circ \tau^{-1}. \tag{3.13}$$

Finally, we define the τ-twist Met_τ of the Segal–Shale–Weil representation Met. For this, we begin with the identity component $\mathrm{Aut}(H^{2m+1})_o$.

We set

$$\mathrm{Met}_\tau := A^{-1} \circ \mathrm{Met} \circ A, \qquad \text{where} \tag{3.14}$$

$$A = \begin{cases} \mathrm{Met}(\tau), & \text{for } \tau \in Sp(m,\mathbb{R}), \\ \vartheta(\tau), & \text{for } \tau \in H^{2m+1}, \\ \mathrm{Id}, & \text{for } \tau = d(r). \end{cases}$$

It follows from Fact 3.1 1) that Met_τ is well-defined for $\tau \in \mathrm{Aut}(H^{2m+1})_o$. For the connected component containing τ_{-4}, we set

$$\mathrm{Met}_\tau := (\mathrm{Met}_{\tau'})^\vee \tag{3.15}$$

for $\tau = \tau_{-4}\tau'$, $\tau' \in \mathrm{Aut}(H^{2m+1})_o$.

Thereby, Met_τ is a unitary representation of $Mp(m,\mathbb{R})$ on $L^2(\mathbb{R}^m)$ characterized for every $T \in Sp(m,\mathbb{R})$ by

$$Met_\tau(T)\vartheta_\tau(g)Met_\tau(T)^{-1} = \vartheta_\tau(T(g)).$$

Hence, the group $\mathrm{Aut}(H^{2m+1})$ acts on $L^2(\mathbb{R}^{2m})$ in such a way that the following proposition holds.

Proposition 3.2.

1) *The τ-twisted Weyl calculus is covariant with respect to the Jacobi group:*

$$\vartheta_\tau(g)\,\mathrm{Op}_\tau(\mathfrak{S})\,\vartheta_\tau(g^{-1}) = \mathrm{Op}_\tau(\mathfrak{S} \circ g^{-1}), \quad g \in H^{2m+1}, \tag{3.16}$$

$$\mathrm{Met}_\tau(g)\,\mathrm{Op}_\tau(\mathfrak{S})\,\mathrm{Met}_\tau^{-1}(g) = \mathrm{Op}_\tau(\mathfrak{S} \circ g^{-1}), \quad g \in Sp(m,\mathbb{R}). \tag{3.17}$$

2) *For any $\tau \in \mathrm{Aut}(H^{2m+1})$ the representation Met_τ is equivalent either to Met or to its contragredient $Met^\vee$.*

The special case of the τ-twist, namely, the τ-twist associated with the rescaling map τ_ρ (3.10) deserves our attention for at least the following two reasons. First, the parameter $\frac{\rho}{4}$ has a concrete physical meaning - this is the inverse of the Planck constant h (see [11, Theorem 4.57], where a slightly different notation was used. Namely, the Schrödinger representations that we denote by ϑ_{τ_ρ} correspond therein to ρ_h with $h = \frac{4}{\rho}$). Secondly, dilations do not preserve the center Z of the Heisenberg while the symplectic automorphisms of H^{2m+1} do. More precisely, the whole Jacobi group G^J fixes Z pointwise. The last observation together with the Stone – von Neumann theorem (see below) shows that the action of $\mathrm{Aut}(H^{2m+1})/G^J \simeq \{\tau_\rho : \rho \in \mathbb{R}^\times\} (\simeq \mathbb{R}^\times)$ is sufficient in order to obtain all infinite dimensional irreducible unitary representations of the Heisenberg group.

We set

$$\vartheta_\rho := \vartheta_{\tau_\rho}, \tag{3.18}$$

to which we refer as the *Schrödinger representations* with central character ρ.

Fact 3.3 (Stone–von Neumann Theorem, [13, 23])**.** *The representations ϑ_ρ constitute a family of irreducible pairwise inequivalent unitary representations with real parameter ρ. Any infinite dimensional irreducible unitary representation of H^{2m+1} is uniquely determined by its central character and thus equivalent to one of the ϑ_ρ's.*

To end this section, we give yet another algebraic property of the Weyl operator calculus. We shall see in Lemma 4.5 that the irreducible decomposition of $\pi_{i\lambda,\delta}^{GL(2n,\mathbb{R})}$, when restricted to a maximal parabolic subgroup of $Sp(n,\mathbb{R})$, is based on an involution of the phase space coming from the parity preserving involution on the configuration space.

Consider on $L^2(\mathbb{R}^m)$ an involution defined by $\check{u}(x) := u(-x)$ and induce through the map $\mathrm{Op}_{\tau_\rho} : L^2(\mathbb{R}^{2m}) \to \mathrm{HS}(L^2(\mathbb{R}^m), L^2(\mathbb{R}^m))$ two involutions on $L^2(\mathbb{R}^{2m})$, denoted by $\mathfrak{S} \mapsto {}^{\dagger_\rho}\mathfrak{S}$ and $\mathfrak{S} \mapsto \mathfrak{S}^{\dagger_\rho}$, by the following identities:

$$\mathrm{Op}_{\tau_\rho}({}^{\dagger_\rho}\mathfrak{S})(u) = \mathrm{Op}_{\tau_\rho}(\mathfrak{S})(\check{u}), \tag{3.19}$$

$$\mathrm{Op}_{\tau_\rho}(\mathfrak{S}^{\dagger_\rho})(u) = (\mathrm{Op}_{\tau_\rho}(\mathfrak{S})(u))\check{}. \tag{3.20}$$

Then ${}^{\dagger_\rho}\mathfrak{S}$ and $\mathfrak{S}^{\dagger_\rho}$ are characterized by their partial Fourier transforms defined by

$$(\mathcal{F}_\xi\mathfrak{S})(x,\eta) := \int_{\mathbb{R}^m} \mathfrak{S}(x,\xi)e^{-2\pi i\langle\xi,\eta\rangle}d\xi \quad \text{for } \mathfrak{S} \in L^2(\mathbb{R}^{2m}).$$

Lemma 3.4.

$$\left(\mathcal{F}_\xi {}^{\dagger_\rho}\mathfrak{S}\right)(x,\eta) = (\mathcal{F}_\xi\mathfrak{S})\left(-\frac{2}{\rho}\eta, -\frac{\rho}{2}x\right),$$

$$\left(\mathcal{F}_\xi \mathfrak{S}^{\dagger_\rho}\right)(x,\eta) = (\mathcal{F}_\xi\mathfrak{S})\left(\frac{2}{\rho}\eta, \frac{\rho}{2}x\right).$$

Proof. By (3.12) the first equality (3.19) amounts to

$$\int_{\mathbb{R}^m\times\mathbb{R}^m} {}^{\dagger_\rho}\mathfrak{S}\left(\frac{x+y}{2}, \frac{4}{\rho}\xi\right) e^{2i\pi\langle x-y,\xi\rangle}u(y)dyd\xi$$
$$= \int_{\mathbb{R}^m\times\mathbb{R}^m} \mathfrak{S}\left(\frac{x+y}{2}, \frac{4}{\rho}\xi\right) e^{2i\pi\langle x-y,\xi\rangle}u(-y)dyd\xi.$$

The right-hand side equals

$$\left(\frac{|\rho|}{4}\right)^n \int_{\mathbb{R}^m\times\mathbb{R}^m} \mathfrak{S}\left(\frac{x-y}{2}, \xi\right) e^{2i\pi\langle\frac{\rho}{4}(x+y),\xi\rangle}u(y)dyd\xi.$$

This equality holds for all $u \in L^2(\mathbb{R}^m)$, and therefore,

$$\int_{\mathbb{R}^m} {}^{\dagger_\rho}\mathfrak{S}\left(\frac{x+y}{2}, \frac{4}{\rho}\xi\right) e^{2i\pi\langle x-y,\xi\rangle} d\xi = \left(\frac{|\rho|}{4}\right)^n \int_{\mathbb{R}^m} \mathfrak{S}\left(\frac{x-y}{2}, \xi\right) e^{2i\pi\langle \frac{\rho}{4}(x+y),\xi\rangle} d\xi.$$

Namely,

$$\left(\mathcal{F}_\xi {}^{\dagger_\rho}\mathfrak{S}\right)\left(\frac{x+y}{2}, \frac{\rho}{4}(y-x)\right) = \left(\mathcal{F}_\xi \mathfrak{S}\right)\left(\frac{x-y}{2}, -\frac{\rho}{4}(x+y)\right).$$

Thus the first statement follows and the second may be proved in the same way. □

4. Restriction of $\pi_{i\lambda,\delta}$ to a maximal parabolic subgroup

Let $n = m + 1$. Consider the space of homogeneous functions

(4.1)
$$V^\infty_{\mu,\delta} := \{f \in C^\infty(\mathbb{R}^{2n} \setminus \{0\}) : f(r\,\cdot) = (\operatorname{sgn} r)^\delta |r|^{-n-\mu} f(\cdot), r \in \mathbb{R}^\times\},$$

for $\delta = 0, 1$ and $\mu \in \mathbb{C}$. It may be seen as the space of even or odd smooth functions on the unit sphere S^{2n-1} according to $\delta = 0$ or 1, since homogeneous functions are determined by their restriction to S^{2n-1}. Let $V_{\mu,\delta}$ denote its completion with respect to the L^2-norm over S^{2n-1}. Likewise, by restricting to the hyperplane defined by the first coordinate to be 1, we can identify the space $V_{\mu,\delta}$ with the Hilbert space $L^2(\mathbb{R}^{2n-1})$ up to a scalar multiple on the inner product.

The normalized degenerate principal series representations $\pi^{GL(2n,\mathbb{R})}_{\mu,\delta}$ induced from the character $\chi_{\mu,\delta}$ of a maximal parabolic subgroup P_{2n} of $GL(2n,\mathbb{R})$ corresponding to the partition $2n = 1 + (2n-1)$ may be realized on these functional spaces. The realization of the same representation on $V_{\mu,\delta}$ will be referred to as the *K-picture*, and on $L^2(\mathbb{R}^{2n-1})$ as the *N-picture*.

In addition to these standard models of $\pi^{GL(2n,\mathbb{R})}_{\mu,\delta}$, we shall use another model $L^2(\mathbb{R}, \mathrm{HS}(L^2(\mathbb{R}^m), L^2(\mathbb{R}^m)))$, which we call the *operator calculus model*. It gives a strong machinery for investigating the restriction to the maximal parabolic subgroup of $Sp(n,\mathbb{R})$ (see (4.3) below).

Let us denote by

$$\mathcal{F}_t(f)(\rho, X) = \int_{\mathbb{R}} f(t, X)\, e^{-2i\pi t\rho}\, dt,$$

the partial Fourier transform of $f(t, X) \in L^2(\mathbb{R}^{1+2m})$ with respect to the first variable. Applying the direct integral of the operators Op_{τ_ρ}

and using (2.2), we obtain the unitary isomorphisms

$$V_{\mu,\delta} \simeq L^2(\mathbb{R}^{1+2m}) \simeq L^2(\mathbb{R}, L^2(\mathbb{R}^{2m})) \underset{\mathcal{F}_t}{\simeq} L^2(\mathbb{R}, L^2(\mathbb{R}^{2m})) \xrightarrow[\int \mathrm{Op}_{\tau_\rho} d\rho]{\sim} L^2(\mathbb{R}, \mathrm{HS}(L^2(\mathbb{R}^m), L^2(\mathbb{R}^m))). \tag{4.2}$$

According to situations we shall use following geometric models for the induced representations:

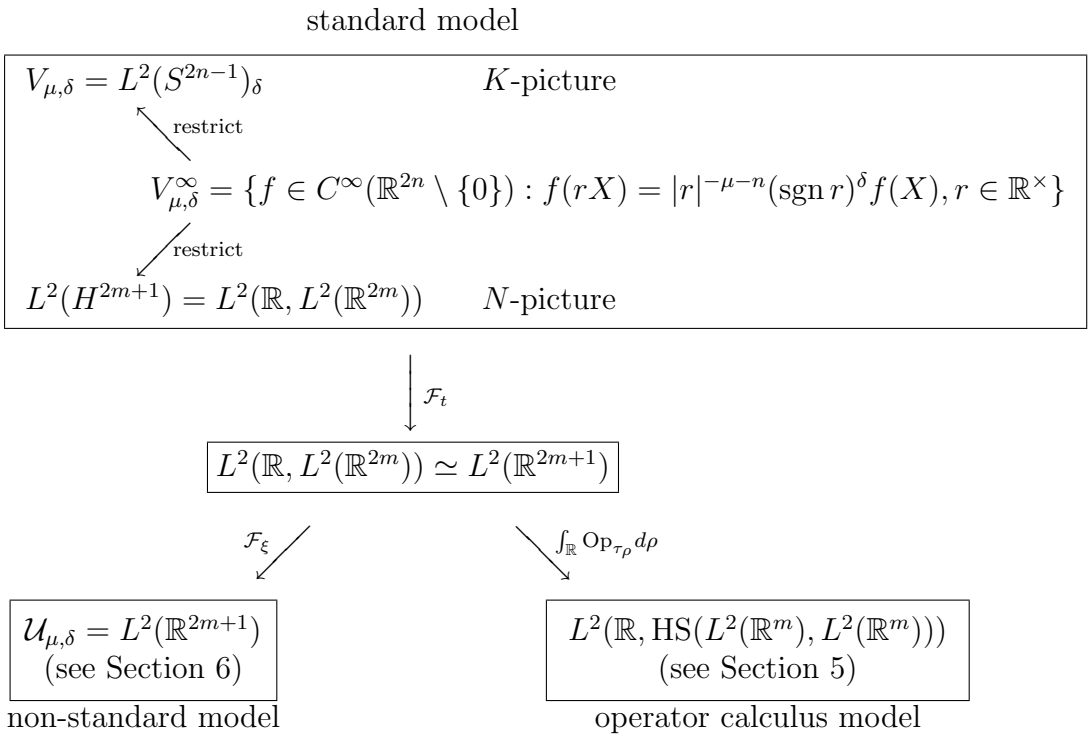


FIGURE 4.1.

The group $G_1 = Sp(n,\mathbb{R})(= Sp(m+1,\mathbb{R}))$ acts by linear symplectomorphisms on $\mathbb{R}^{2n}$ and thus it also acts on the real projective space $\mathbb{P}^{2m+1}\mathbb{R}$. Fix a point in $\mathbb{P}^{2m+1}\mathbb{R}$ and denote by P its stabilizer in G_1. This is a maximal parabolic subgroup of G_1 with Langlands decomposition

$$P = MA\overline{N} \simeq (\mathbb{R}^\times \cdot Sp(m,\mathbb{R})) \ltimes H^{2m+1}. \tag{4.3}$$

Let $\mathfrak{g}_1 = \mathfrak{n} + \mathfrak{m} + \mathfrak{a} + \overline{\mathfrak{n}}$ be the Gelfand–Naimark decomposition for the Lie algebra $\mathfrak{g}_1 = Lie(G_1)$.

We identify the standard Heisenberg Lie group H^{2m+1} with the subgroup $N = \exp \mathfrak{n}$ through the following Lie groups isomorphism:

$$(s, x, \xi) \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 \\ x & I_m & 0 & 0 \\ 2s & {}^t\xi & 1 & -{}^t x \\ \xi & 0 & 0 & I_m \end{pmatrix}. \tag{4.4}$$

Thus, in the coordinates $(t, x, \xi) \in H^{1+2m}$, the restriction map $V^\infty_{\mu,\delta} \to L^2(H^{2m+1})$ is given by

$$f \mapsto f(1, 2t, x, \xi). \tag{4.5}$$

The action of G_1 on $\mathbb{P}^{2n-1}\mathbb{R}$ is transitive, and all such isotropy subgroups are conjugate to each other. Therefore, we may assume that $P = Sp(n, \mathbb{R}) \cap P_{2n}$. Then, the natural inclusion $Sp(n, \mathbb{R}) \subset GL(2n, \mathbb{R})$ induces the following isomorphisms

$$Sp(n, \mathbb{R})/P \xrightarrow{\sim} GL(2n, \mathbb{R})/P_{2n} \simeq \mathbb{P}^{2n-1}\mathbb{R}.$$

Hence, the (normalized) induced representation $\pi_{\mu,\delta} \equiv \pi^{Sp(n,\mathbb{R})}_{\mu,\delta} := \mathrm{Ind}^{Sp(n,\mathbb{R})}_P \chi_{\mu,\delta}$ can (*cf.* Section 8) also be realized on the Hilbert space $V_{\mu,\delta}$. Therefore, $\pi_{\mu,\delta}$ is equivalent to the restriction of $\pi^{GL(2n,\mathbb{R})}_{\mu,\delta}$ with respect to $Sp(n, \mathbb{R})$. Notice that $\pi_{\mu,\delta}$ is unitary for $\mu = i\lambda$, $\lambda \in \mathbb{R}$.

It is noteworthy that the unipotent radical $\overline{N}$ of P is the Heisenberg group H^{2n-1} which is not abelian if $n \geq 2$, although the unipotent radical of P_{2n} clearly is. Notice also that the automorphism group $\mathrm{Aut}(H^{2n-1})$ contains $P/\{\pm 1\}$ as a subgroup of index 2.

Denote by $M_o \simeq Sp(m, \mathbb{R})$ the identity component of $M \simeq O(1) \times Sp(m, \mathbb{R})$. The subgroup $M_o \ltimes N$ is isomorphic to the Jacobi group G^J introduced in Section 3.

We have then the following inclusive relations for subgroups of symplectomorphisms:

$$\underset{\text{Symplectic group}}{G_1} \supset MAN \supset \underset{\text{Jacobi group}}{G^J = M_o N} \supset \underset{\text{Heisenberg group}}{N}.$$

Our strategy of analyzing the representations $\pi_{i\lambda,\delta}$ of G_1 (see Theorem 8.3) will be based on their restrictions to these subgroups (see Lemmas 4.1 and 4.5).

We recall from (3.18) that ϑ_ρ is the Schrödinger representation of the Heisenberg group H^{2m+1} with central character ρ. While the abstract Plancherel formula for the group $N \simeq H^{2m+1}$:

$$L^2(N) = \int_{\mathbb{R}} \vartheta_\rho \otimes \vartheta^\vee_\rho d\rho,$$

underlines the decomposition with respect to left and right regular actions of the group N, we shall consider the decomposition of this space with respect to the restriction of the principal series representation $\pi_{i\lambda,\delta}$ to the Jacobi group $G^J = Sp(m,\mathbb{R}) \ltimes H^{2m+1}$ (see Lemma 4.1).

Let us examine how the restriction $\pi_{i\lambda,\delta}|_{G^J}$ defined on the Hilbert space $V_{i\lambda,\delta}$ on the left-hand side of (4.2) is transferred to $L^2(\mathbb{R}, L^2(\mathbb{R}^{2m}))$ via the partial Fourier transform $\mathcal{F}_t$.

The restriction $\pi_{i\lambda,\delta}|_N$ coincides with the left regular representation of N on $L^2(\mathbb{R}^{1+2m})$ given by

$$\begin{aligned}(4.6)\qquad \pi_{i\lambda,\delta}(g)f(t,X) &= f(t-s-\frac{1}{2}\omega(A,X), X-A)\\ &= f(t-s+\frac{1}{2}(\langle\xi,a\rangle-\langle x,\alpha\rangle), x-a, \xi-\alpha),\end{aligned}$$

for $f(t,X)\in L^2(\mathbb{R}^{1+2m})$ and $g=(s,A)\equiv(s,a,\alpha)\in\mathbb{H}^{2m+1}$.

Taking the partial Fourier transform $\mathcal{F}_t$ of (4.6), we get

(4.7)
$$(\mathcal{F}_t\,(\pi_{i\lambda,\delta}(g)\,f))(\rho,x,\,\xi) = e^{-2\pi i\rho\,(s-\frac{1}{2}(\langle\xi,a\rangle-\langle x,\alpha\rangle))}\,(\mathcal{F}_t f)(\rho, x-a,\,\xi-\alpha).$$

Now, for each $\rho\in\mathbb{R}$, we define a representation ϖ_ρ of N on $L^2(\mathbb{R}^{2m})$ by

$$(4.8)\qquad \varpi_\rho(g)h(x,\xi) := e^{-2\pi i\rho\,(s-\frac{1}{2}(\langle\xi,a\rangle-\langle x,\alpha\rangle))}\,h(x-a,\xi-\alpha),$$

for $g=(s,a,\alpha)\in N$ and $h\in L^2(\mathbb{R}^{2m})$. Then, ϖ_ρ is a unitary representation of N for any ρ, and the formula (4.7) may be written as:

$$(4.9)\qquad (\mathcal{F}_t\pi_{i\lambda,\delta}(g)f)(\rho,x,\xi) = \varpi_\rho(g)(\mathcal{F}_t f)(\rho,x,\xi),$$

for $g\in N$. Here, we let $\varpi_\rho(g)$ act on $\mathcal{F}_t f$ seen as a function of (x,ξ).

For each $\rho\in\mathbb{R}$, we can extend the representation ϖ_ρ of N to a unitary representation of the Jacobi group G^J by letting M_o act on $L^2(\mathbb{R}^{2m})$ by

$$\varpi_\rho(g)h(x,\xi) = h(y,\eta),\quad \text{with}\,(y,\eta)=g^{-1}(x,\xi),\ g\in M_o\simeq Sp(m,\mathbb{R}).$$

Then, clearly the identity (4.9) holds also for $g\in M_o$. Thus, we have proved the following decomposition formula:

Lemma 4.1. *For any $(\lambda,\delta)\in\mathbb{R}\times\mathbb{Z}/2\mathbb{Z}$, the restriction of $\pi_{i\lambda,\delta}$ to the Jacobi group is unitarily equivalent to the direct integral of unitary representations ϖ_ρ via $\mathcal{F}_t$ (see (4.2)):*

$$(4.10)\qquad \pi_{i\lambda,\delta}|_{G^J} \underset{\mathcal{F}_t}{\simeq} \int_{\mathbb{R}}^{\oplus} \varpi_\rho d\rho.$$

Next we establish the link between the representations $(\varpi_\rho, L^2(\mathbb{R}^{2m}))$ and $(\vartheta_\rho, L^2(\mathbb{R}^m))$ of the Heisenberg group $N \simeq H^{2m+1}$. For this we note that the representation ϖ_ρ brings us to the changeover of one parameter families of automorphisms of H^{2m+1}, from $\{\tau_\rho : \rho \in \mathbb{R}^\times\}$ to $\{\psi_\rho : \rho \in \mathbb{R}^\times\}$ which defined by

$$\psi_\rho(s, a, \alpha) := \left(\frac{1}{\rho}s, \frac{1}{2}a, \frac{2}{\rho}\alpha\right). \tag{4.11}$$

Then we state the following covariance relation given by Op_{τ_ρ}:

Lemma 4.2. *For every $g \in H^{2m+1}$ the following identity in* $\mathrm{End}(L^2(\mathbb{R}^m))$ *holds for any $\mathfrak{S} \in L^2(\mathbb{R}^{2m})$:*

$$\mathrm{Op}_{\tau_\rho}(\varpi_\rho(g)\mathfrak{S}) = \mathrm{Op}_{\tau_\rho}(\mathfrak{S}) \circ \vartheta_{\psi_\rho}(g^{-1}). \tag{4.12}$$

Proof. Let $g = (s, a, \alpha) \in H^{2m+1}$ and take an arbitrary function $u \in L^2(\mathbb{R}^m)$. Using the integral formula (3.12) for Op_{τ_ρ}, we get

$$\begin{aligned}
&\mathrm{Op}_{\tau_\rho}(\varpi_\rho(g)\mathfrak{S})u(x)\\
=\ & \int_{\mathbb{R}^m\times\mathbb{R}^m} (\varpi_\rho(g)\mathfrak{S})\left(\frac{x+y}{2}, \frac{4}{\rho}\xi\right) e^{2\pi i\langle x-y,\xi\rangle} u(y)dyd\xi\\
=\ & \int_{\mathbb{R}^m\times\mathbb{R}^m} e^{-2\pi i\rho\left(s-\frac{1}{2}\left(\langle\frac{4}{\rho}\xi,a\rangle-\left\langle\frac{x+y}{2},\alpha\right\rangle\right)\right)}\mathfrak{S}\left(\frac{x+y}{2}-a, \frac{4}{\rho}\xi-\alpha\right) e^{2\pi i\langle x-y,\xi\rangle} u(y)dyd\xi\\
=\ & \int_{\mathbb{R}^m\times\mathbb{R}^m} e^{-2\pi iB}\mathfrak{S}\left(\frac{x+y}{2}, \frac{4}{\rho}\xi\right) u(y+2a)dyd\xi,
\end{aligned}$$

where

$$\begin{aligned}
B &= \rho s - \frac{\rho}{2}\left(\langle\frac{4}{\rho}\xi+\alpha, a\rangle - \left\langle\frac{x+y+2a}{2}, \alpha\right\rangle\right) - \left\langle x-y-2a, \xi+\frac{\rho}{4}\alpha\right\rangle\\
&= \rho s + \frac{\rho}{2}\langle a+y, \alpha\rangle - \langle x-y, \xi\rangle.
\end{aligned}$$

In view of the definitions (3.13) anf (4.11),

$$\vartheta_{\psi_\rho}(g^{-1}) = \vartheta(\psi_\rho^{-1}g^{-1}) = \vartheta(-\rho s, -2a, -\frac{\rho}{2}\alpha).$$

Thus, by the definition (3.2) of the Schrödinger representation ϑ, we have

$$(\vartheta_{\psi_\rho}(s^{-1})u)(y) = e^{-2\pi i(\rho s+\frac{\rho}{2}\langle a+y,\alpha\rangle)}u(y+2a).$$

Hence, the last integral equals

$$\begin{aligned}
&\int_{\mathbb{R}^m\times\mathbb{R}^m} \mathfrak{S}\left(\frac{x+y}{2}, \frac{4}{\rho}\xi\right)(\vartheta_{\psi_\rho}(g^{-1})u)(y)e^{2\pi i\langle x-y,\xi\rangle}dyd\xi\\
=\ & \left(\mathrm{Op}_{\tau_\rho}(\mathfrak{S})\vartheta_{\psi_\rho}(g^{-1})u\right)(x).
\end{aligned}$$

□

Then, it turns out that the decomposition (4.10) is not irreducible, but the following lemma holds:

Lemma 4.3. *For any $\rho \in \mathbb{R}^\times$, ϖ_ρ is a unitary representation of the Jacobi group G^J on $L^2(\mathbb{R}^{2m})$, which splits into a direct sum $\varpi_\rho^0 \oplus \varpi_\rho^1$ of two pairwise inequivalent unitary irreducible representations.*

Proof. Consider the rescaling map τ_ρ introduced by (3.10) and recall that the τ_ρ–twisted Weyl quantization map induces a G^J equivariant isomorphism

$$\mathrm{Op}_{\tau_\rho} : L^2(\mathbb{R}^{2m}) \xrightarrow{\sim} \mathrm{HS}(L^2(\mathbb{R}^m), L^2(\mathbb{R}^m)) \tag{4.13}$$

intertwining the ϖ_ρ and ϑ_{ψ_ρ} actions (4.12).

The irreducibility of the Schrödinger representation ϑ_ρ of the group N (Fact 3.3) implies therefore that any N-invariant closed subspace in $\mathrm{HS}(L^2(\mathbb{R}^m), L^2(\mathbb{R}^m))$ must be of the form $\mathrm{HS}(L^2(\mathbb{R}^m), U)$ for some closed subspace $U \subset L^2(\mathbb{R}^m)$.

In view of the covariance relation (3.17) of the Weyl quantization, the subspace $\mathrm{HS}(L^2(\mathbb{R}^m), U)$ is $Sp(m,\mathbb{R})$-invariant if and only if U itself is $Mp(m,\mathbb{R})$-invariant (see Proposition 3.2), and the latter happens only if U is one of $\{0\}$, $L^2(\mathbb{R}^m)_{\mathrm{even}}, L^2(\mathbb{R}^m)_{\mathrm{odd}}$ or $L^2(\mathbb{R}^m)$. Thus, we have the following irreducible decomposition of ϖ_ρ, seen as a representation of G^J on $L^2(\mathbb{R}^{2m})$:

$$\begin{aligned} L^2(\mathbb{R}^{2m}) \quad &= \quad W_+ \oplus W_- \\ &\underset{\mathrm{Op}_{\tau_\rho}}{\overset{\sim}{\longrightarrow}} \quad \mathrm{HS}(L^2(\mathbb{R}^m), L^2(\mathbb{R}^m)_{\mathrm{even}}) \oplus \mathrm{HS}(L^2(\mathbb{R}^m), L^2(\mathbb{R}^m)_{\mathrm{odd}}). \end{aligned} \tag{4.14}$$

From Proposition 3.2 2) we deduce that the corresponding representations, to be denoted by ϖ_ρ^δ, of G^J, where δ labels the parity, are pairwise inequivalent, i.e. $\varpi_\rho^\delta = \varpi_{\rho'}^{\delta'}$ if and only if $\rho = \rho'$ and $\delta = \delta'$ for all $\rho, \rho' \in \mathbb{R}$ and $\delta, \delta' \in \mathbb{Z}/2\mathbb{Z}$. □

The following lemma is straightforward from the definition of the involution $\mathfrak{S} \mapsto \mathfrak{S}^{\dagger_\rho}$ (see (3.19)).

Lemma 4.4. *The subspaces W_+ and W_- introduced above are the $+1$ and -1 eigenspaces of the involution $\mathfrak{S} \mapsto \mathfrak{S}^{\dagger_\rho}$, respectively.*

Eventually, we take the A-action into account, and give the branching law of the (degenerate) principal series representation $\pi_{i\lambda,\delta}$ of G_1 when restricted to the maximal parabolic subgroup MAN.

Lemma 4.5. (Branching law for $G_1 \downarrow MAN$). *For every $(\lambda, \delta) \in \mathbb{R} \times \mathbb{Z}/2\mathbb{Z}$ the space $V_{i\lambda,\delta}$ acted upon by the representation $\pi_{i\lambda,\delta}|_{MAN}$ splits into the direct sum of four irreducible representations:*

$$V_{i\lambda,\delta} \simeq \mathcal{H}^2_+(W_+) \oplus \mathcal{H}^2_+(W_-) \oplus \mathcal{H}^2_-(W_+) \oplus \mathcal{H}^2_-(W_-). \tag{4.15}$$

Proof. We shall prove first that each summand in (4.15) is already irreducible as a representation of $M_oAN \simeq G^JA$. Then we see that it is stable by the group MAN and thus irreducible because M is generated by M_o and $-I_{2n}$,which acts on $V_{i\lambda,\delta}$ by the scalar $(-1)^\delta$.

In light of the G^J-irreducible decomposition (4.10), any G^J-invariant closed subspace U of $V_{i\lambda,\delta}$ must be of the form

$$U = \mathcal{F}_t^{-1}(L^2(E_+, W_+)) \oplus \mathcal{F}_t^{-1}(L^2(E_-, W_-)),$$

for some measurable sets $E_\pm$ in $\mathbb{R}$.

Suppose furthermore that U is A-invariant. Notice that the group A acts on $V_{i\lambda,\delta} \simeq L^2(\mathbb{R}^{2m+1})$ by

$$\pi_{i\lambda,\delta}(a)f(t, X) = a^{-1-m-i\lambda} f(a^{-2}t, a^{-1}X).$$

In turn, their partial Fourier transforms with respect to the $t \in \mathbb{R}$ variable are given by

$$(\mathcal{F}_t \pi_{i\lambda,\delta}(a)f)(\rho, X) = a^{1-m-i\lambda}(\mathcal{F}_t f)(a^2\rho, a^{-1}X).$$

Therefore, $\mathcal{F}_t f$ is supported in $E_\pm$ if and only if $\mathcal{F}_t \pi_{i\lambda,\delta}(a)f$ is supported in $a^{-2}E_\pm$ as a $W_\pm$-valued function on $\mathbb{R}$. In particular, U is an A-invariant subspace if and only if $E_\pm$ is an invariant measurable set under the dilation $\rho \mapsto a^2\rho$ $(a > 0)$, namely, $E_\pm = \{0\}$, $\mathbb{R}_-$, $\mathbb{R}_+$, or $\mathbb{R}$ (up to measure zero sets).

Since $M_oAN \simeq G^JA$, M_oAN-invariant proper closed subspaces must be of the form $\mathcal{F}_t^{-1}(L^2(\mathbb{R}_\pm, W_\varepsilon))$ with $\varepsilon = +$ or $-$.

We recall from Lemma 2.2 that the Hilbert space $L^2(\mathbb{R}, W_\varepsilon)$ is a sum of W_ε-valued Hardy spaces:

$$L^2(\mathbb{R}, W_\varepsilon) = \mathcal{H}^2_+(W_\varepsilon) \oplus \mathcal{H}^2_-(W_\varepsilon) \underset{\mathcal{F}_t}{\tilde{\to}} L^2(\mathbb{R}_+, W_\varepsilon) \oplus L^2(\mathbb{R}_-, W_\varepsilon). \tag{4.16}$$

Now Lemma 4.5 has been proved. □

Lemma 4.5 implies that the representation $\pi_{i\lambda,\delta}$ of G_1 has at most four irreducible subrepresentations. The precise statement for this will be given in Theorem 8.3 .

5. Restriction of $\pi_{i\lambda,\delta}$ to a maximal compact subgroup

As the operator calculus model $L^2(\mathbb{R}, \mathrm{HS}(L^2(\mathbb{R}^m), L^2(\mathbb{R}^m)))$ was appropriate for studying the P-structure of $\pi_{i\lambda,\delta}$, we use complex spherical harmonics for the analysis of the K-structure of these representations.

We retain the convention $n = m + 1$. Identifying the symplectic form ω on $\mathbb{R}^{2n}$ with the imaginary part of the Hermitian inner product on $\mathbb{C}^n$ we realize the group of unitary transformations $K = U(n)$ as a subgroup of $G_1 = Sp(n, \mathbb{R})$. Then the group K is a maximal compact subgroup of G_1.

Analogously to the classical spherical harmonics on $\mathbb{R}^n$, consider harmonic polynomials on $\mathbb{C}^n$ as follows. For $\alpha, \beta \in \mathbb{N}$, let $\mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)$ denote the vector space of polynomials $p(z_0, \dots, z_m, \bar{z}_0, \dots, \bar{z}_m)$ on $\mathbb{C}^n$ which

(1) are homogeneous of degree α in $(z_0, \dots, z_m)$ and of degree β in $(\bar{z}_0, \dots, \bar{z}_m)$;

(2) belong to the kernel of the differential operator $\displaystyle\sum_{i=0}^{m} \frac{\partial^2}{\partial z_i \partial \bar{z}_i}$.

Then, $\mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)$ is a finite dimensional vector space. It is non-zero except for the case where $n = 1$ and $\alpha, \beta \geq 1$. The natural action of K on polynomials,

$$p(z_0, \dots, z_m, \bar{z}_0, \dots, \bar{z}_m) \mapsto p(g^{-1}(z_0, \dots, z_m), \overline{g^{-1}(z_0, \dots, z_m)}) \quad (g \in K),$$

leaves $\mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)$ invariant. The resulting representations of K on $\mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)$, which we denote by the same symbol $\mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)$, are irreducible and pairwise inequivalent for any such α, β.

The restriction of $\mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)$ to the unit sphere $S^{2m+1} = \{(z_0, \dots, z_m) \in \mathbb{C}^n : \sum_{j=0}^{m} |z_j|^2 = 1\}$ is injective and gives a complete orthogonal basis of $L^2(S^{2m+1})$, and we have a discrete sum decomposition

$$L^2(S^{2m+1}) \simeq \sum_{\alpha,\beta \in \mathbb{N}}^{\oplus} \mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)|_{S^{2m+1}} \quad (m \geq 1). \tag{5.1}$$

The case $m = 0$ collapses to

$$L^2(S^1) \simeq \sum_{\alpha \in \mathbb{N}}^{\oplus} \mathcal{H}^{\alpha,0}(\mathbb{C}^1)|_{S^1} \oplus \sum_{\beta \in \mathbb{N}_+}^{\oplus} \mathcal{H}^{0,\beta}(\mathbb{C}^1)|_{S^1}.$$

Fixing a $\mu \in \mathbb{C}$ we may extend functions on S^{2m+1} to homogeneous functions of degree $-(m + 1 + \mu)$. The decomposition (5.1) gives rise to the branching law (*K-type formula*) with respect to the maximal compact subgroup.

Lemma 5.1. (Branching law for $G_1 \downarrow K$). *The restriction of $\pi_{\mu,\delta}$ to the subgroup K of G_1 is decomposed into a discrete direct sum of pairwise inequivalent representations:*

$$\pi_{\mu,\delta}|_K \simeq \sum_{\substack{\alpha,\beta\in\mathbb{N}\\ \alpha+\beta\equiv\delta \bmod 2}}^{\oplus} \mathcal{H}^{\alpha,\beta}(\mathbb{C}^n) \qquad (m \geq 1),$$

$$\pi_{\mu,\delta}|_K \simeq \sum_{\substack{\alpha\in\mathbb{N}\\ \alpha\equiv\delta \bmod 2}}^{\oplus} \mathcal{H}^{\alpha,0}(\mathbb{C}) \oplus \sum_{\substack{\beta\in\mathbb{N}_+\\ \beta\equiv\delta \bmod 2}}^{\oplus} \mathcal{H}^{0,\beta}(\mathbb{C}) \quad (m = 0).$$

We shall refer to $\mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)$ as a K-type of the representation $\pi_{\mu,\delta}$.

The restriction $G_1 \downarrow K$ is multiplicity free. Therefore any K-intertwining operator (in particular, any G_1-intertwining operator) acts as a scalar on every K-type by Schur's lemma. We give an explicit formula of this scalar for the Knapp–Stein intertwining operator:

$$\mathcal{T}_{\mu,\delta} : V_{-\mu,\delta} \to V_{\mu,\delta},$$

which is defined as the meromorphic continuation of the following integral operator

$$(\mathcal{T}_{\mu,\delta} f)(\eta) := \int_{S^{2n-1}} f(\xi)\, |\omega(\xi,\eta)|\, |^{-\mu-n} \left(\operatorname{sgn}\omega(\xi,\eta)\right)^{\delta} d\sigma(\xi).$$

Here $d\sigma$ is the Euclidean measure on the unit sphere. Further, we normalize it by

$$\widetilde{\mathcal{T}}_{\mu,\delta} := \frac{1}{C_{2n}(\mu,\delta)} \mathcal{T}_{\mu,\delta}, \tag{5.2}$$

where

$$C_{2n}(\mu,\delta) := 2\pi^{\mu+n-\frac{1}{2}} \times \begin{cases} \frac{\Gamma\left(\frac{1-\mu-n}{2}\right)}{\Gamma\left(\frac{\mu+n}{2}\right)} & (\delta = 0),\\ -i\frac{\Gamma\left(\frac{2-\mu-n}{2}\right)}{\Gamma\left(\frac{\mu+n+1}{2}\right)} & (\delta = 1). \end{cases}$$

Proposition 5.2. *For $\alpha, \beta \in \mathbb{N}$, we set $\delta \equiv \alpha + \beta \bmod 2$. The normalized Knapp–Stein intertwining operator $\widetilde{\mathcal{T}}_{\mu,\delta}$ acts on $\mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)$ as the following scalar*

$$(-1)^{\beta}\pi^{-\mu}\frac{\Gamma\left(\frac{\alpha+\beta+\mu+n}{2}\right)}{\Gamma\left(\frac{\alpha+\beta-\mu+n}{2}\right)}.$$

Proof. See [6, Theorem 2.1] for $\delta = 0$. The proof for $\delta = 1$ works as well by using Lemma 5.4. □

Remark 5.3. Without normalization, the Knapp–Stein intertwining operator $\mathcal{T}_{\mu,\delta}$ acts on $\mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)$ as

$$\mathcal{T}_{\mu,\delta}\Big|_{\mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)} = (-1)^{\beta} A_{\alpha+\beta}(\mu)\,\mathrm{id},$$

where $\delta \equiv \alpha+\beta \bmod 2$ and

$$A_k(\mu) := 2\pi^{n-\frac{1}{2}} \frac{\Gamma\left(\frac{k+\mu+n}{2}\right)}{\Gamma\left(\frac{k-\mu+n}{2}\right)} \times \begin{cases} \frac{\Gamma\left(\frac{1-\mu-n}{2}\right)}{\Gamma\left(\frac{\mu+n}{2}\right)} & (k \in 2\mathbb{N}), \\ -i\frac{\Gamma\left(\frac{2-\mu-n}{2}\right)}{\Gamma\left(\frac{\mu+n+1}{2}\right)} & (k \in 2\mathbb{N}+1), \end{cases}$$

The symplectic Fourier transform $\mathcal{F}_{\mathrm{symp}}$, defined by (3.6), may be written as:

$$(\mathcal{F}_{\mathrm{symp}} f)(Y) = \int_{\mathbb{R}^{2n}} f(X) e^{-2\pi i \omega(X,Y)} dX = \left(\mathcal{F}_{\mathbb{R}^{2n}} f\right)(JY),$$

where $J : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is given by $J(x,\xi) := (-\xi, x)$.

For generic complex parameter μ (*e.g.* $\mu \neq n, n+2, \dots$ for $\delta = 0$), the space $V^{\infty}_{\mu,\delta}$ of homogeneous functions on $\mathbb{R}^{2n} \setminus \{0\}$ may be regarded as a subspace of the space $\mathcal{S}'(\mathbb{R}^{2n})$ of tempered distributions, and we have the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{F}_{\mathrm{symp}} & : & \mathcal{S}'(\mathbb{R}^{2n}) & \xrightarrow{\sim} & \mathcal{S}'(\mathbb{R}^{2n}) \\ & & \cup & \circlearrowleft & \cup \\ & & V_{-\mu,\delta} & \xrightarrow{\sim} & V_{\mu,\delta} \end{array}$$

Lemma 5.4. *As operators that depend meromorphically on μ, $\widetilde{\mathcal{T}}_{\mu,\delta}$ satisfy the following identity:*

$$\widetilde{\mathcal{T}}_{\mu,\delta} = \mathcal{F}_{\mathrm{symp}}|_{V_{-\mu,\delta}}.$$

Proof. The proof parallels that of [6, Proposition 2.3]. For $h \in C^{\infty}(S^{2n-1})_{\delta}$, we define a homogeneous function $h_{\mu-n} \in V^{\infty}_{-\mu,\delta}$ by

$$h_{\mu-n}(r\xi) := r^{\mu-n} h(\xi) \quad (r > 0,\ \xi \in S^{2n-1}).$$

Then we recall from [6, Proposition 2.2] the following formula:

$$\mathcal{F}_{\mathbb{R}^{2n}} h_{\mu-n}(s\eta) = \frac{\Gamma(\mu+n) e^{-\frac{\pi i}{2}(\mu+n)}}{(2\pi)^{\mu+n} s^{\mu+n}} \int_{S^{2n-1}} (\langle \xi, \eta\rangle - i0)^{-\mu-n} h(\xi) d\sigma(\xi),$$

where $(\langle \xi,\eta\rangle - i0)^{\lambda}$ is a distribution of ξ, η, obtained by the substitution of $t = \langle \xi, \eta\rangle$ into the distribution $(t-i0)^{\lambda}$ of one variable t.

To conclude, we use

$$\begin{aligned}
&(t-i0)^{-\mu-n}\\
=\ & e^{\frac{\pi}{2}i(\mu+n)}\left(\cos\frac{\pi(\mu+n)}{2}|t|^{-\mu-n}-i\sin\frac{\pi(\mu+n)}{2}|t|^{-\mu-n}\mathrm{sgn}\,t\right)\\
=\ & \pi e^{\frac{\pi}{2}i(\mu+n)}\left(\frac{|t|^{-\mu-n}}{\Gamma\left(\frac{1+\mu+n}{2}\right)\Gamma\left(\frac{1-\mu-n}{2}\right)}-i\frac{|t|^{-\mu-n}\mathrm{sgn}\,t}{\Gamma\left(\frac{\mu+n}{2}\right)\Gamma\left(\frac{2-\mu-n}{2}\right)}\right).
\end{aligned}$$

□

We note that the Knapp–Stein intertwining operator induces a unitary equivalence of representations $\pi_{i\lambda,\delta}$ and $\pi_{-i\lambda,\delta}$ of $G_1 = Sp(n,\mathbb{R})$:

$$\pi_{i\lambda,\delta} \simeq \pi_{-i\lambda,\delta}, \quad \text{for any } \lambda \in \mathbb{R} \text{ and } \delta \in \mathbb{Z}/2\mathbb{Z}. \tag{5.3}$$

6. Algebraic Knapp–Stein intertwining operator

We introduce yet another model $\mathcal{U}_{\mu,\delta} \simeq L^2(\mathbb{R}^{2m+1})$, referred to as the *non-standard model*, of the representation $\pi_{\mu,\delta}$ as the image of the partial Fourier transform

$$\mathcal{F}_\xi : L^2(\mathbb{R}^{1+m+m}) \xrightarrow{\sim} L^2(\mathbb{R}^{1+m+m}),$$

where ξ denotes the last variable in $\mathbb{R}^m$. Then the space $\mathcal{U}_{\mu,\delta}$ inherits a G_1-module structure from $(\pi_{\mu,\delta}, V_{\mu,\delta})$ through $\mathcal{F}_\xi \circ \mathcal{F}_t$ (see Figure 4.1).

The advantage of this model is that the Knapp–Stein intertwining operator becomes an algebraic operator (see Theorem 6.1 below). The price to pay is that the Lie algebra $\mathfrak{k}$ acts on $\mathcal{U}_{\mu,\delta}$ by second order differential operators. We can still give an explicit form of minimal K-types on the model $\mathcal{U}_{\mu,\delta}$ when it splits into two irreducible components ($\mu = 0, \delta = 0, 1$) by means of K-Bessel functions (Section7).

We define an endomorphism of $L^2(\mathbb{R}^{2m+1})$ by

$$(T_{\mu,\delta}H)(\rho,x,\eta) := \left|\frac{\rho}{2}\right|^{-\mu}(\mathrm{sgn}\rho)^\delta H\left(\rho, \frac{2}{\rho}\eta, \frac{\rho}{2}x\right). \tag{6.1}$$

Regarding $\widetilde{\mathcal{T}}_{\mu,\delta}$ as an operator on the N-picture, we have

Theorem 6.1 (algebraic Knapp–Stein intertwining operator)**.** *For any $\mu \in \mathbb{C}$ and $\delta \in \mathbb{Z}/2\mathbb{Z}$, the following diagram commutes:*

$$\begin{array}{ccc}
V_{-\mu,\delta} & \xrightarrow{\widetilde{\mathcal{T}}_{\mu,\delta}} & V_{\mu,\delta}\\
{\scriptstyle \mathcal{F}_\xi\mathcal{F}_t}\downarrow & & \downarrow{\scriptstyle \mathcal{F}_\xi\mathcal{F}_t}\\
\mathcal{U}_{-\mu,\delta} & \xrightarrow{T_{\mu,\delta}} & \mathcal{U}_{\mu,\delta}
\end{array}$$

To prove Theorem 6.1, we work on the ambient space $\mathbb{R}^{2n}(=\mathbb{R}^{2m+2})$. Let $\mathcal{F}_{\mathbb{R}^n}$ denote the partial Fourier transform of the last n coordinates in $\mathbb{R}^{2n}$.

Lemma 6.2. 1) *For $f \in V_{-\mu,\delta}$, the function $\mathcal{F}_{\mathbb{R}^n} f$ satisfies*

$$(\mathcal{F}_{\mathbb{R}^n} f)(rx, r^{-1}\eta) = |r|^{\mu}(\operatorname{sgn} r)^{\delta}(\mathcal{F}_{\mathbb{R}^n} f)(x,\eta), \quad r \in \mathbb{R}^{\times},\ x,\eta \in \mathbb{R}^n.$$

2) *For $f \in \mathcal{S}'(\mathbb{R}^{2n}), x, \eta \in \mathbb{R}^n$, we have*

$$(\mathcal{F}_{\mathbb{R}^n} \circ \mathcal{F}_{\mathrm{symp}} \circ \mathcal{F}_{\mathbb{R}^n}^{-1}) f(x,\xi) = f(\xi, x).$$

Proof of Lemma 6.2. 1) This is a straightforward computation.

2) For $f(x,\xi') \in \mathcal{S}(\mathbb{R}^{2n})$,

$$\begin{aligned}
&(\mathcal{F}_{\mathbb{R}^n} \circ \mathcal{F}_{\mathrm{symp}} \circ \mathcal{F}_{\mathbb{R}^n}^{-1} f)(y,\eta') \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^{2n}}\int_{\mathbb{R}^n} f(x,\xi') e^{2\pi i\langle \xi,\xi'\rangle} e^{-2\pi i(\langle \xi, y\rangle - \langle x,\eta\rangle)} e^{-2\pi i\langle \eta,\eta'\rangle} d\xi' dx d\xi d\eta \\
&= \int_{\mathbb{R}^n\times\mathbb{R}^{2n}\times\mathbb{R}^n} f(x,\xi') e^{2\pi i\langle \xi'-y,\xi\rangle} e^{-2\pi i\langle \eta'-x,\eta\rangle} d\xi' dx d\xi d\eta \\
&= \int_{\mathbb{R}^n} f(x,\xi')\delta(\xi'-y)\delta(\eta'-x) dx d\xi' \\
&= f(\eta', y).
\end{aligned}$$

□

From now x, ξ, η will stand again for elements of $\mathbb{R}^m$, where $m = n-1$.

Proof of Theorem 6.1. According to the choice of the isomorphism (4.4) between the Lie group N and the standard Heisenberg Lie group, for $f \in V_{-\mu,\delta}$, we set

$$\begin{aligned}
F(t,x,\xi) &:= f(1,x,2t,\xi), \\
H(\rho,x,\eta) &:= (\mathcal{F}_t\mathcal{F}_\xi F)(\rho,x,\eta),
\end{aligned}$$

where $t, \rho \in \mathbb{R}$ and $x, \xi \in \mathbb{R}^m$. Then $H(\rho,x,\eta) = \frac{1}{2}(\mathcal{F}_{\mathbb{R}^n} f)(1,x,\frac{\rho}{2},\eta)$. Thus, according to Lemma 6.2,

$$\begin{aligned}
\mathcal{F}_t\mathcal{F}_\xi(\mathcal{F}_{\mathrm{symp}} f)(\rho,x,\eta) &= \frac{1}{2}\mathcal{F}_{\mathbb{R}^n}(\mathcal{F}_{\mathrm{symp}} f)(1,x,\frac{\rho}{2},\eta) \\
&= \frac{1}{2}(\mathcal{F}_{\mathbb{R}^n} f)(\frac{\rho}{2},\eta,1,x) \\
&= \frac{1}{2}\left|\frac{\rho}{2}\right|^{-\mu}(\operatorname{sgn}\rho)^{\delta}(\mathcal{F}_{\mathbb{R}^n} f)(1,\frac{2}{\rho}\eta,\frac{\rho}{2},\frac{\rho}{2}x) \\
&= \left|\frac{\rho}{2}\right|^{-\mu}(\operatorname{sgn}\rho)^{\delta} H(\rho,\frac{2}{\rho}\eta,\frac{\rho}{2}x).
\end{aligned}$$

Now Theorem follows from Lemma 5.4. □

7. Minimal K-type in a non-standard model

We give an explicit formula for two particular K-finite vectors of $\pi_{0,\delta}$ (in fact, minimal K-types of irreducible components $\pi^{\pm}_{0,\delta}$ of $\pi_{0,\delta}$; see Theorem 8.3 1)) in the non-standard L^2-model $\mathcal{U}_{0,\delta}(\simeq L^2(\mathbb{R}^{2m+1}))$. The main results (see Proposition 7.1) show that minimal K-types are represented in terms of K-Bessel functions in this model. Although we do not use these results in the proof of Theorem 8.3, we think they are interesting of their own from the view point of geometric analysis of small representations. It is noteworthy that similar feature to Proposition 7.1 has been observed in the L^2-model of minimal representations of some other reductive groups (see e.g. [19, Part III]).

We begin with the identification

$$\mathbb{C} \xrightarrow{\sim} \mathcal{H}^{0,0}(\mathbb{C}^{m+1}), \quad 1 \mapsto \mathbf{1} \quad \text{(constant function)},$$

and extend it to a homogeneous function on $\mathbb{R}^{2n}$ belonging to $V_{0,0}$ (see (4.1)). Using the formula (4.5) in the N-picture, we set

$$h^+(t,x,\xi) := (1+4t^2+|x|^2+|\xi|^2)^{-\frac{m+1}{2}}.$$

Notice that $h^+(t,x,\xi) \in V_{0,0} \cap \mathcal{H}^{0,0}(\mathbb{C}^{m+1})$ in the K-type formula of $\pi_{0,0}$ (see Lemma 5.1).

Let

$$\psi(\rho,x,\eta) := \left(1+|x|^2\right)^{\frac{1}{2}} \left(\frac{\rho^2}{4}+|\eta|^2\right)^{\frac{1}{2}}. \tag{7.1}$$

Likewise we identify

$$\mathbb{C}^{m+1} \xrightarrow{\sim} \mathcal{H}^{0,1}(\mathbb{C}^{m+1}),\ b \mapsto \sum_{j=0}^{m} b_j \overline{z}_j,$$

and set

(7.2)

$$h_b^-(t,x,\xi) := (1+4t^2+|x|^2+|\xi|^2)^{-\frac{m+2}{2}} (b_0(1-2it) + \sum_{j=1}^{m} b_j(x_j - i\xi_j)),$$

$$\varphi_b(\rho,x,\eta) := \omega(\begin{pmatrix} (1+|x|^2)^{\frac{1}{2}} \\ (\frac{\rho^2}{4}+|\eta|^2)^{\frac{1}{2}} \end{pmatrix}, b_0 \begin{pmatrix} 1 \\ \frac{\rho}{2} \end{pmatrix} + \sum_{j=1}^{m} b_j \begin{pmatrix} x_j \\ \eta_j \end{pmatrix}), \tag{7.3}$$

where ω denotes the standard symplectic form on $\mathbb{C}^2$ defined as in (3.1). Then $h_b^- \in V_{0,1} \cap \mathcal{H}^{0,1}(\mathbb{C}^{m+1})$ in the K-type formula of $\pi_{0,1}$ (see Lemma 5.1).

Let $K_\nu(z)$ denote the modified Bessel function of the second kind (K-Bessel function for short). Then the K-finite vectors h^+ and h_b^- ($b \in \mathbb{C}^{m+1}$) in the standard model (N-picture) are of the following form in the non-standard model $\mathcal{U}_{0,\delta}$.

Proposition 7.1.

$$1)\ (\mathcal{F}_t\mathcal{F}_\xi h^+)(\rho,x,\eta) = \frac{\pi^{\frac{m+2}{2}}}{\Gamma(\frac{m+1}{2})}\, K_0\left(2\pi\psi(\rho,x,\eta)\right).$$

$$2)\ (\mathcal{F}_t\mathcal{F}_\xi h_b^-)(\rho,x,\eta) = \frac{\pi^{\frac{m+2}{2}}}{2\Gamma(\frac{m+2}{2})}\, \frac{\varphi_b(\rho,x,\eta)}{\psi(\rho,x,\eta)}\, \exp\left(-2\pi\psi(\rho,x,\eta)\right).$$

The rest of this section is devoted to the proof of Proposition 7.1. In order to get simpler formulas we also use the following normalization $\widetilde{K}_\nu(z) := \left(\frac{z}{2}\right)^{-\nu} K_\nu(z)$ [20, Section 7.2].

Lemma 7.2. *For every $\mu \in \mathbb{R}$ let us define the following function on $\mathbb{R}\times\mathbb{R}^m$:*

$$I_\mu \equiv I_\mu(a,\eta) := \int_{\mathbb{R}^m} (a^2+|\xi|^2)^{-\mu} e^{-2i\pi\langle\xi,\eta\rangle} d\xi.$$

Then,

$$I_\mu(a,\eta) = \frac{2\pi^{\frac{m}{2}}}{\Gamma(\mu)} a^{m-2\mu} \widetilde{K}_{\frac{m}{2}-\mu}(2\pi a|\eta|). \tag{7.4}$$

Proof. Recall the classical Bochner formula

$$\int_{S^{m-1}} e^{-2i\pi s\langle\xi,\xi'\rangle} d\sigma(\xi) = 2\pi s^{1-\frac{m}{2}} J_{\frac{m}{2}-1}(2\pi s), \quad \text{for } \xi' \in S^{m-1},$$

where $J_\nu(z)$ denotes the Bessel function of the first kind. Then,

$$\begin{aligned} I_\mu(a,\eta) &= \int_0^\infty \int_{S^{m-1}} (a^2+r^2)^{-\mu} e^{-2i\pi r|\eta|\langle\xi,\frac{\eta}{|\eta|}\rangle} r^{m-1} dr d\sigma(\xi) \\ &= 2\pi|\eta|^{1-\frac{m}{2}} \int_0^\infty r^{\frac{m}{2}} J_{\frac{m}{2}-1}(2\pi r|\eta|)(r^2+a^2)^{-\mu} dr. \end{aligned}$$

According to [8, 8.5 (20)] we have

$$\int_0^\infty x^{\nu+\frac{1}{2}} (x^2+a^2)^{-\mu-1} J_\nu(xy)(xy)^{\frac{1}{2}} dx = \frac{a^{\nu-\mu}\, y^{\mu+\frac{1}{2}}\, K_{\nu-\mu}(ay)}{2^\mu\, \Gamma(\mu+1)},$$

for $\operatorname{Re} a > 0$ and $-1 < \operatorname{Re}\nu < 2\operatorname{Re}\mu + \frac{3}{2}$, which implies

$$\begin{aligned} I_\mu(a,\eta) &= \frac{2\pi^\mu}{\Gamma(\mu)} \left(\frac{a}{|\eta|}\right)^{\frac{m}{2}-\mu} K_{\frac{m}{2}-\mu}(2\pi a|\eta|) \\ &= \frac{2\pi^{\frac{m}{2}}}{\Gamma(\mu)} a^{m-2\mu} \widetilde{K}_{\frac{m}{2}-\mu}(2\pi a|\eta|). \end{aligned}$$

□

In particular, we have

$$I_{\frac{m+1}{2}}(a,\eta) = \frac{\pi^{\frac{m+2}{2}}}{\Gamma(\frac{m+1}{2})}\,\frac{\exp(-2\pi a|\eta|)}{a},$$
$$I_{\frac{m+2}{2}}(a,\eta) = \frac{2\pi^{\frac{m+2}{2}}}{\Gamma(\frac{m+2}{2})}\,\frac{|\eta|}{a}\,K_1(2\pi a|\eta|).$$

Here we used $\widetilde{K}_{-\frac{1}{2}}(z) = \frac{\sqrt{\pi}}{2}e^{-z}$ in the first identity. By a little abuse of notation, we write $h^-_{(0)}$ and $h^-_{(1)}$ for $h^-_{(1,0,\dots,0)}$ and $h^-_{(0,1,0,\dots,0)}$, respectively.

Lemma 7.3. *For $(t,x) \in \mathbb{R}\times\mathbb{R}^m$, we set*

$$a \equiv a(t,x) := \sqrt{1+4t^2+|x|^2}.$$

Then,

$$(\mathcal{F}_\xi h^+)(t,x,\eta) = I_{\frac{m+1}{2}}(a(t,x),\eta),$$
$$(\mathcal{F}_\xi h^-_{(1)})(t,x,\eta) = \Big(x_1 + \frac{1}{2\pi}\frac{\partial}{\partial\eta_1}\Big) I_{\frac{m+2}{2}}(a(t,x),\eta).$$

Proof. By definition

$$(\mathcal{F}_\xi h^+)(t,x,\eta) = \int_{\mathbb{R}^m} (1+4t^2+|x|^2+|\xi|^2)^{-\frac{m+1}{2}} e^{-2\pi i\langle\xi,\eta\rangle} d\xi$$

$$\begin{aligned}(\mathcal{F}_\xi h^-_{(1)})(t,x,\eta) &= \int_{\mathbb{R}^m} (1+4t^2+|x|^2+|\xi|^2)^{-\frac{m+2}{2}} (x_1 - i\xi_1) e^{-2\pi i\langle\xi,\eta\rangle} d\xi \\ &= \Big(x_1 + \frac{1}{2\pi}\frac{\partial}{\partial\eta_1}\Big) I_{\frac{m+2}{2}}\left(\sqrt{1+4t^2+|x|^2},\eta\right).\end{aligned}$$

Hence Lemma 7.3 is proved. □

Proof of Proposition 7.1. We recall from [8, vol. I, 1.4 (27); 1.13 (45); 2.13 (43)] the following formulas: For $\operatorname{Re} d > 0$, $\operatorname{Re} c > 0$ and $s > 0$,

$$\int_0^\infty \frac{\exp(-d(t^2+c^2)^{\frac{1}{2}})}{(t^2+c^2)^{\frac{1}{2}}} \cos(st)dt = K_0(c(s^2+d^2)^{\frac{1}{2}}). \tag{7.5}$$

$$\begin{aligned}\int_0^\infty \frac{K_\nu(d(t^2+c^2)^{\frac{1}{2}})}{(t^2+c^2)^{\frac{\nu}{2}}} \cos(st)dt &= \sqrt{\frac{\pi}{2}}\, \frac{K_{\nu-\frac{1}{2}}(c(s^2+d^2)^{\frac{1}{2}})}{d^\nu c^{\nu-\frac{1}{2}}(s^2+d^2)^{\frac{1}{4}-\frac{1}{2}\nu}} \\ &= 2^{\nu-1}\sqrt{\pi}\, d^{-\nu}\, c^{1-2\nu}\, \widetilde{K}_{-\nu+\frac{1}{2}}(c(s^2+d^2)^{\frac{1}{2}}).\end{aligned} \tag{7.6}$$

$$\int_0^\infty \frac{tK_1(d(t^2+c^2)^{\frac{1}{2}})}{(t^2+c^2)^{\frac{1}{2}}} \sin(st)dt = \frac{\pi s}{2d} \frac{\exp(-c(s^2+d^2)^{\frac{1}{2}})}{(s^2+d^2)^{\frac{1}{2}}}. \tag{7.7}$$

We apply the formulas (7.5) and (7.6) with $d = 4\pi|\eta|$, $c = \frac{1}{2}(1+|x|^2)^{\frac{1}{2}}$ and $s = 2\pi\rho$. In view that $a \equiv a(t,x) = 2(t^2+c^2)^{\frac{1}{2}}$ and $2\pi\psi(\rho,x,\eta) = c(s^2+d^2)^{\frac{1}{2}}$, we get

$$\begin{aligned}&\int_{-\infty}^\infty \frac{\exp(-2\pi a|\eta|)}{a}\, e^{-2\pi i t\rho} dt = K_0(2\pi\psi(\rho,x,\eta)), \\ &\int_{-\infty}^\infty \frac{K_1(2\pi a|\eta|)}{a}\, e^{-2\pi i t\rho} dt = \frac{1}{4|\eta|(1+|x|^2)^{\frac{1}{2}}} \exp(-2\pi\psi(\rho,x,\eta)).\end{aligned}$$

Here, we have used again $\widetilde{K}_{-\frac{1}{2}}(z) = \frac{\sqrt{\pi}}{2}\, e^{-z}$ for the second equation. Thus the first statement has been proved.

To see the second statement, it is sufficient to treat the following two cases: $b = (1,0,\dots,0)$ and $b = (0,1,0,\dots,0)$. We use

$$\begin{aligned}\mathcal{F}_t\mathcal{F}_\xi h^-_{(1)} &= \mathcal{F}_t\left(\Big(x_1 + \frac{1}{2\pi}\frac{\partial}{\partial\eta_1}\Big) I_{\frac{m+2}{2}}(a(t,x),\eta)\right) \\ &= \Big(x_1 + \frac{1}{2\pi}\frac{\partial}{\partial\eta_1}\Big)\mathcal{F}_t\Big(I_{\frac{m+2}{2}}(a(t,x),\eta)\Big).\end{aligned}$$

Now use

$$\Big(x_1 + \frac{1}{2\pi}\frac{\partial}{\partial\eta_1}\Big)\frac{\exp(-2\pi\psi(\rho,x,\eta))}{(1+|x|^2)^{\frac{1}{2}}} = \frac{\varphi_{(1)}(\rho,x,\eta)\exp(-2\pi\psi(\rho,x,\eta))}{\psi(\rho,x,\eta)}.$$

The case $b = (1,0,\dots,0)$ goes similarly by using the formula (7.7). $\square$

8. Branching law for $GL(2n,\mathbb{R}) \downarrow Sp(n,\mathbb{R})$

From now we give a proof of Theorem 1.1 with emphasis on geometric analysis involved.

Our strategy is the following. Suppose P is a closed subgroup of a Lie group G, $\chi : P \to \mathbb{C}^\times$ a unitary character, and $\mathcal{L} := G \times_P \chi$ a G-equivariant line bundle over G/P. We write $L^2(G/P, \mathcal{L})$ for the Hilbert space consisting of L^2-sections for the line bundle $\mathcal{L} \otimes (\Lambda^{top} T^*(G/P))^{\frac{1}{2}}$. Then the group G acts on $L^2(G/P, \mathcal{L})$ as a unitary representation, to be denoted by π^G_χ, by translations.

If (G, H) is a reductive symmetric pair and P is a parabolic subgroup of G, then there exist finitely many open H-orbits $\mathcal{O}^{(j)}$ on the real flag variety G/P such that $\cup_j \mathcal{O}^{(j)}$ is open dense in G/P. (In our cases below, the number of open H-orbits is at most two.) Applying the Mackey theory, we see that the restriction of the unitary representation π^G_χ to the subgroup H is unitarily equivalent to a finite direct sum:

$$\pi^G_\chi|_H \simeq \bigoplus_j L^2(\mathcal{O}^{(j)}, \mathcal{L}|_{\mathcal{O}^{(j)}}).$$

Thus the branching problem is reduced to the irreducible decomposition of $L^2(\mathcal{O}^{(j)}, \mathcal{L}|_{\mathcal{O}^{(j)}})$, equivalently, the Plancherel formula for the homogeneous line bundle $\mathcal{L}|_{\mathcal{O}^{(j)}}$ over open H-orbits $\mathcal{O}^{(j)}$.

In our specific setting, where $G = GL(N,\mathbb{R})$ and $P = P_N$ (see (1.2)), the base space G/P is the real projective space $\mathbb{P}^{N-1}\mathbb{R}$. For $(\lambda, \delta) \in \mathbb{R} \times \mathbb{Z}/2\mathbb{Z}$, we define a unitary character $\chi_{i\lambda,\delta}$ of P_N by

$$\chi_{i\lambda,\delta}\begin{pmatrix} a & {}^t b \\ 0 & C \end{pmatrix} := |a|^\lambda (\operatorname{sgn} a)^\delta,\ a \in GL(1,\mathbb{R}),\ C \in GL(N-1,\mathbb{R}),\ b \in \mathbb{R}^{N-1},$$

in the matrix realization of P_N. Then $\pi^G_{\chi_{i\lambda,\delta}}$ coincides with $\pi^G_{i\lambda,\delta}$ in previous notation. In this and the next three sections, we find the explicit irreducible decomposition of $L^2(\mathcal{O}^{(j)}, \mathcal{L}|_{\mathcal{O}^{(j)}})$ with respect to $\pi^G_{i\lambda,\delta}$.

We begin with the case $H = G_1$, *i.e.*

$$(G, H) \equiv (GL(2n,\mathbb{R}),\ Sp(n,\mathbb{R}))\,.$$

As we have already seen in Section 4 the group G_1 acts transitively on G/P_N, and we have the following unitary equivalence of unitary representations of $G_1 = Sp(n,\mathbb{R})$:

$$\pi^G_{i\lambda,\delta}|_{G_1} \simeq \pi^{G_1}_{i\lambda,\delta}.$$

Here $\pi_{i\lambda,\delta}^{Sp(n,\mathbb{R})}$ is a unitary representation of $Sp(n,\mathbb{R})$ induced from the maximal parabolic subgroup $P = G_1 \cap P_N \simeq (GL(1,\mathbb{R}) \times Sp(n-1,\mathbb{R})) \ltimes H^{2n-1}$.

Thus the following two statements are equivalent.

Theorem 8.1. *The restriction of $\pi_{i\lambda,\delta}^{GL(2n,\mathbb{R})}$ from $GL(2n,\mathbb{R})$ to $Sp(n,\mathbb{R})$ stays irreducible for any $\lambda \in \mathbb{R}^\times$ and $\delta \in \{0,1\}$. It splits into two irreducible components for $\lambda = 0, \delta = 0, 1$ and $n \geq 2$.*

Theorem 8.2. *Let P be a maximal parabolic subgroup of G_1 whose Levi part is isomorphic to $GL(1,\mathbb{R}) \times Sp(n-1,\mathbb{R})$, and denote by $\pi_{i\lambda,\delta}$ ($\lambda \in \mathbb{R}, \delta = 0, 1$) the corresponding unitary (degenerate) principal series representation of G_1. Then for $n \geq 2$, $\pi_{i\lambda,\delta}$ are irreducible for any $(\lambda,\delta) \in \mathbb{R}^\times \times \mathbb{Z}/2\mathbb{Z}$, and splits into a direct sum of two irreducible components for $\lambda = 0, \delta = 0, 1$.*

Theorem 8.2 itself was proved in [21, Theorem 7.3]. The case of $\delta = 0$ was studied by different methods earlier in [10] and also very recently in [2] ($\lambda = 0$ and $\delta = 0$) in the context of special unipotent representations of the split group $Sp(n,\mathbb{R})$. We give yet another proof of Theorem 8.2 in the most interesting case, i.e. in the case $\lambda = 0$ and $\delta = 0, 1$ below.

Theorem 8.3 describes a finer structure of the irreducible summands. The novelty here (even for the $\delta = 0$ case) is that we characterize explicitly the two irreducible summands by their K-module structure, and also by their P-module structure. The former is given in terms of complex spherical harmonics (*cf.* Lemma 5.1) and the latter in terms of Hardy spaces (*cf.* Lemma 4.5), as follows:

Theorem 8.3. *Let $n \geq 2$ and $\delta \in \mathbb{Z}/2\mathbb{Z}$. The unitary representation $\pi_{0,\delta}$ of $G_1 = Sp(n,\mathbb{R})$ splits into the direct sum of two irreducible representations of G_1:*

$$\pi_{0,\delta} = \pi_{0,\delta}^{+} \oplus \pi_{0,\delta}^{-}. \tag{8.1}$$

1) (Characterization by K-type). Each irreducible summand in (8.1) has the following K-type formula:

$$\pi_{0,\delta}^{+} \simeq \sum_{\substack{\beta \in 2\mathbb{N} \\ \alpha \equiv \beta + \delta \bmod 2}}^{\oplus} \mathcal{H}^{\alpha,\beta}(\mathbb{C}^n),$$

$$\pi_{0,\delta}^{-} \simeq \sum_{\substack{\beta \in 2\mathbb{N}+1 \\ \alpha \equiv \beta + \delta \bmod 2}}^{\oplus} \mathcal{H}^{\alpha,\beta}(\mathbb{C}^n),$$

where $\sum^{\oplus}$ denotes the Hilbert completion of the algebraic direct sum.

2) (Characterization by Hardy spaces). The irreducible summands $\pi^{\pm}_{0,\delta}$ consist of two Hardy spaces via the isomorphism (4.15):

$$\pi^{+}_{0,0} \simeq \mathcal{H}^2_{+}(W_{+}) \oplus \mathcal{H}^2_{-}(W_{+}), \qquad \pi^{-}_{0,0} \simeq \mathcal{H}^2_{+}(W_{-}) \oplus \mathcal{H}^2_{-}(W_{-}),$$
$$\pi^{+}_{0,1} \simeq \mathcal{H}^2_{+}(W_{+}) \oplus \mathcal{H}^2_{-}(W_{-}), \qquad \pi^{-}_{0,1} \simeq \mathcal{H}^2_{+}(W_{-}) \oplus \mathcal{H}^2_{-}(W_{+}).$$

Here, $W_{\pm}$ are the subspaces of $L^2(\mathbb{R}^{2m})$ defined in (4.14)*, and $\mathcal{H}^2_{\pm}(W_{\varepsilon})$ are the W_{ε}-valued Hardy spaces.*

3) (Characterization by the Knapp–Stein intertwining operator). The irreducible summands $\pi^{\pm}_{0,\delta}$ are the ± 1 eigenspaces of the normalized Knapp–Stein intertwining operator $\widetilde{\mathcal{T}_{0,\delta}}$ (see (5.2)).

Proof. 1) and 3) The normalized Knapp–Stein intertwining operator $\widetilde{\mathcal{T}}_{0,\delta}$ has eigenvalues either 1 or -1 according to the parity of the K-type $\mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)$, namely $\beta \equiv 0$ or $\beta \equiv 1 \bmod 2$ by Proposition 5.2. Hence the statements 1) and 3) are proved.

2) In the model $\mathcal{U}_{0,\delta} \simeq L^2(\mathbb{R}^{2m+1})$ (see Section 6), the Knapp–Stein intertwining operator $\widetilde{\mathcal{T}}_{0,\delta}$ is equivalent to the algebraic operator

$$T_{0,\delta} : H(\rho, x, \eta) \to (\operatorname{sgn}\rho)^{\delta} H\left(\rho, \frac{2}{\rho}\eta, \frac{\rho}{2}x\right),$$

by Theorem 6.1.

In turn, it follows from Lemma 3.4 that $T_{0,\delta}$ is transfered to the operator

$$\mathfrak{S}(\rho, *) \mapsto (\operatorname{sgn}\rho)^{\delta} \mathfrak{S}^{\dagger_\rho}(\rho, *) \tag{8.2}$$

in the operator calculus model $L^2(\mathbb{R}, \mathrm{HS}(L^2(\mathbb{R}^m), L^2(\mathbb{R}^m)))$ (see Figure 4.1). In view of the ± 1 eigenspaces of the transform (8.2), we see that the statement 2) follows from the characterization of $W_{\pm}$ (see Lemma 4.4) and the isomorphism $\mathcal{F}_t : \mathcal{H}^2_{\pm}(W_{\varepsilon}) \xrightarrow{\sim} L^2(\mathbb{R}_{\pm}, W_{\varepsilon})$ given in Lemma 2.2.

Finally, we need to prove that the summands $\pi^{\pm}_{0,\delta}$ are irreducible G_1-modules. This is deduced from the decomposition of $\pi^{\pm}_{0,\delta}$ by means of Hardy spaces in 2) and from the following lemma. □

Lemma 8.4. *For any $\delta \in \mathbb{Z}/2\mathbb{Z}$, none of the Hardy spaces $\mathcal{H}^2_{\pm}(W_{\varepsilon})$ ($\varepsilon = \pm$) is G_1-stable with respect to $\pi_{0,\delta}$.*

Proof. For $Z := (z_1, \dots, z_m) = x + i\xi \in \mathbb{C}^m \simeq \mathbb{R}^{2m}$ (see (4.5)), we set

$$\begin{aligned}
f_{0,0}(t,x,\xi) &:= (1+4t^2+|x|^2+|\xi|^2)^{-\frac{m+1}{2}},\\
f_{0,1}(t,x,\xi) &:= (1+4t^2+|x|^2+|\xi|^2)^{-\frac{m+2}{2}}\,(x_1 - i\xi_1),\\
f_{1,0}(t,x,\xi) &:= (1+4t^2+|x|^2+|\xi|^2)^{-\frac{m+2}{2}}\,(x_1 + i\xi_1),\\
f_{1,1}(t,x,\xi) &:= (1+4t^2+|x|^2+|\xi|^2)^{-\frac{m+3}{2}}\,(1+4t^2-x_1^2-\xi_1^2).
\end{aligned}$$

We note that $f_{0,0} = h^+$ and $f_{0,1} = h^-_{(0,1,0,\dots,0)} = h^-_{(1)}$ in the notation of Section 7. Then we have $f_{\alpha,\beta} \in \mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)$ for any $\alpha, \beta \in \{0,1\}$. In view of Theorem 8.3 1), we get

$$\begin{aligned}
f_{0,0}(t,x,\xi) &\in \mathcal{H}^{0,0}(\mathbb{C}^n) \subset V^+_{0,0},\\
f_{0,1}(t,x,\xi) &\in \mathcal{H}^{0,1}(\mathbb{C}^n) \subset V^-_{0,1},\\
f_{1,0}(t,x,\xi) &\in \mathcal{H}^{1,0}(\mathbb{C}^n) \subset V^+_{0,1},\\
f_{1,1}(t,x,\xi) &\in \mathcal{H}^{1,1}(\mathbb{C}^n) \subset V^-_{0,0},
\end{aligned}$$

where $V^\pm_{0,\delta}$ stands for the representaion space in the N-picture corresponding to $\pi^\pm_{0,\delta}$ in Theorem 8.3. Suppose now that one of the Hardy spaces $\mathcal{H}^2_\pm(W_\varepsilon)$ were G_1-stable with respect to $\pi_{0,\delta}$. Then its orthogonal complementary subspace for the decomposition in Theorem 8.3 2) would be also G_1-stable. Since K-type is multiplicity-free in $\pi_{0,\delta}$ by Lemma 5.1, either $\mathcal{H}^2_\pm(W_\varepsilon)$ or its complementary subspace should contain the K-type $\mathcal{H}^{\alpha,\beta}(\mathbb{C}^n)$ for some $\alpha, \beta = 0$ or 1. But this never happens because $f_{\alpha,\beta}(t,x,\xi) = f_{\alpha,\beta}(-t,x,\xi)$ and thus $\operatorname{supp} \mathcal{F}_t f_{\alpha,\beta} \not\subseteq \mathbb{R}_\pm$ (see Lemma 2.2 4)). Thus lemma is proved. □

Remark 8.5. The case $n = 1$ is well known. Here the group $Sp(1,\mathbb{R})$ is isomorphic to $SL(2,\mathbb{R})$, and $\pi_{i\lambda,\delta}$ are irreducible except for $(\lambda,\delta) = (0,1)$, while $\pi_{0,1}$ splits into the direct sum of two irreducible unitary representations:

$$\begin{aligned}
\pi^{Sp(1,\mathbb{R})}_{0,1} &\simeq \mathcal{H}^2_+(\mathbb{C}) \oplus \mathcal{H}^2_-(\mathbb{C})\\
&\simeq \left(\sum_{\alpha \in 2\mathbb{N}+1}^{\oplus} \mathcal{H}^{\alpha,0}(\mathbb{C}) \right) \oplus \left(\sum_{\beta \in 2\mathbb{N}+1}^{\oplus} \mathcal{H}^{0,\beta}(\mathbb{C}) \right).
\end{aligned}$$

The spaces $\mathcal{H}^{\alpha,0}(\mathbb{C})$ and $\mathcal{H}^{0,\beta}(\mathbb{C})$ are one dimensional, and

$$\begin{aligned}
(t+i)^\alpha (t^2+1)^{-\frac{\alpha+1}{2}} &\in \mathcal{H}^{\alpha,0}(\mathbb{C}) \cap V_{0,1},\\
(t-i)^\beta (t^2+1)^{-\frac{\beta+1}{2}} &\in \mathcal{H}^{0,\beta}(\mathbb{C}) \cap V_{0,1}.
\end{aligned}$$

The former function extends holomorphically to the upper half plane Π_+, and the latter one extends holomorphically to Π_- if $\alpha, \beta \equiv 1 \bmod 2$, namely, if $\delta \equiv 1$.

As formulated in Theorem 8.2, our result may be compared with general theory on (degenerate) principal series representations of real reductive groups. For instance, according to Harish-Chandra and Vogan–Wallach [25], such representations are at most a finite sum of irreducible representations and are 'generically' irreducible. A theorem of Kostant [22] asserts that spherical unitary principal series representations (induced from minimal parabolic subgroups) are irreducible.

There has been also extensive research on the structure of (degenerate) principal series representations in specific cases, in particular, in the case where the unipotent radical of P is abelian by A. U. Klimyk, B. Gruber, R. Howe, E.–T. Tan, S.–T. Lee, S. Sahi and others by algebraic and combinatorial methods (see *e.g.* [14] and references therein).

We have not adopted here the aforementioned methods, but have used the idea of branching laws to *non-compact subgroups* (see [17]) primarily because of the belief that the latter approach to very small representations will open new aspects of the theory of geometric analysis.

9. Branching law for $GL(2n, \mathbb{R}) \downarrow GL(n, \mathbb{C})$

Let $P_n^{\mathbb{C}} = L_n^{\mathbb{C}} N_n^{\mathbb{C}}$ be the standard maximal parabolic subgroup of $GL(n, \mathbb{C})$ corresponding to the partition $n = 1 + (n-1)$, namely, the Levi subgroup $L_n^{\mathbb{C}}$ of $P_n^{\mathbb{C}}$ is isomorphic to $GL(1, \mathbb{C}) \times GL(n-1, \mathbb{C})$ and the unipotent radical $N_n^{\mathbb{C}}$ is the complex abelian group $\mathbb{C}^{n-1}$. Inducing from a unitary character $(\nu, m) \in \mathbb{R} \times \mathbb{Z}$ of the first factor of $L_n^{\mathbb{C}}$, $GL(1, \mathbb{C}) \simeq \mathbb{R}_+ \times S^1$ we define a degenerate principal series representation $\pi_{i\nu,m}^{GL(n,\mathbb{C})}$ of $GL(n, \mathbb{C})$. They are pairwise inequivalent, irreducible unitary representations of $GL(n, \mathbb{C})$ (see [14, Corollary 2.4.3]).

We identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, and regard

$$G_2 := GL(n, \mathbb{C})$$

as a subgroup of $G = GL(2n, \mathbb{R})$.

Theorem 9.1 (Branching law $GL(2n, \mathbb{R}) \downarrow GL(n, \mathbb{C})$)**.**

$$\pi_{i\lambda,\delta}^{GL(2n,\mathbb{R})}|_{GL(n,\mathbb{C})} \simeq \sum_{m \in 2\mathbb{Z}+\delta}^{\oplus} \pi_{i\lambda,m}^{GL(n,\mathbb{C})}. \tag{9.1}$$

Proof. The group $G_2 = GL(n, \mathbb{C})$ acts transitively on the real projective space $\mathbb{P}^{2n-1}\mathbb{R}$, and the unique (open) orbit $\mathcal{O}_2 := \mathbb{P}^{2n-1}\mathbb{R}$ is represented

as a homogeneous space G_2/H_2 where the isotropy group H_2 is of the form

$$H_2 \simeq (O(1) \times GL(n-1,\mathbb{C}))N_n^{\mathbb{C}}.$$

Since $P_n^{\mathbb{C}}/H_2 \simeq S^1/\{\pm 1\}$, we have a G_2-equivariant fibration:

$$S^1/\{\pm 1\} \to \mathbb{P}^{2n-1}\mathbb{R} \to GL(n,\mathbb{C})/P_n^{\mathbb{C}}.$$

Further, if we denote by $\mathbb{C}_\delta$ the one-dimensional representation of H_2 obtained as the following compositions:

$$H_2 \to H_2/GL(n-1,\mathbb{C})N_n^{\mathbb{C}} \xrightarrow{\delta} \mathbb{C}^\times,$$

then the G-equivariant line bundle $\mathcal{L}_{i\lambda,\delta} = G \times_P \mathbb{C}_{i\lambda,\delta}$ is represented as a G_2-equivariant line bundle simply by

$$\mathcal{L}_\delta := \mathcal{L}_{i\lambda,\delta}|_{\mathcal{O}_2} \simeq GL(n,\mathbb{C}) \times_{H_2} \mathbb{C}_\delta.$$

Therefore, we have an isomorphism as unitary representations of G_2:

$$\mathcal{H}_{i\lambda,\delta}^{GL(2n,\mathbb{R})}|_{G_2} \simeq L^2(\mathcal{O}_2, \mathcal{L}_\delta).$$

Taking the Fourier series expansion of $L^2(\mathcal{O}_2, \mathcal{L}_\delta)$ along the fiber $S^1/\{\pm 1\}$, we get the irreducible decomposition (9.1). □

An interesting feature of Theorem 9.1 is that the degenerate principal series representation $\pi_{i\lambda,\delta}^{GL(2n,\mathbb{R})}$ is discretely decomposable with respect to the restriction $GL(2n,\mathbb{R}) \downarrow GL(n,\mathbb{C})$. We have seen this by finding explicit branching law, however, discrete decomposability of the restriction $\pi_{i\lambda,\delta}^{GL(2n,\mathbb{R})}|_{GL(n,\mathbb{C})}$ can be explained also by the general theory [16] as follows:

Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{o}(2n)$, and we take a standard basis $\{f_1, \dots, f_n\}$ in $i\mathfrak{t}^*$ such that the dominant Weyl chamber for the disconnected group $K = O(2n)$ is given as

$$i\mathfrak{t}_+^* = \{(\lambda_1, \dots, \lambda_n) : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0\}.$$

For $K_2 := G_2 \cap K \simeq U(n)$ the Hamiltonian action of K on the cotangent bundle $T^*(K/K_2)$ has the momentum map $T^*(K/K_2) \to i\mathfrak{k}^*$. The intersection of its image with the dominant Weyl chamber $i\mathfrak{t}_+^*$ is given by

$$\begin{aligned} &i\mathfrak{t}_+^* \cap \mathrm{Ad}^\vee(K)(i\mathfrak{k}_2^\perp) \\ &= \left\{(\lambda_1, \dots, \lambda_n) \in i\mathfrak{t}_+^* : \lambda_{2i-1} = \lambda_i \text{ for } 1 \leq i \leq \left\lceil \frac{n}{2} \right\rceil \right\}. \end{aligned}$$

On the other hand, it follows from Lemma 5.1 that the asymptotic K-support of $\pi_{i\lambda,\delta}$ amounts to

$$AS_K(\pi_{i\lambda,\delta}) = \mathbb{R}_+(1, 0, \dots, 0).$$

Hence, the triple $(G, G_2, \pi_{i\lambda,\delta})$ satisfies

$$AS_K(\pi_{i\lambda,\delta}) \cap \mathrm{Ad}^\vee(K)(i\mathfrak{k}_2^\perp) = \{0\}. \tag{9.2}$$

This is nothing but the criterion for discrete decomposability of the restriction of the unitary representation $\pi_{i\lambda,\delta}|_{G_2}$ ([16, Theorem 2,9]).

For $G_1 = Sp(n, \mathbb{R})$, we saw in Theorem 8.1 that the restriction $\pi^{GL(2n,\mathbb{R})}_{i\lambda,\delta}|_{G_1}$ stays irreducible. Thus, this is another (obvious) example of discretely decomposable branching law. We can see this fact directly from the observation that G_1 and G_2 have the same maximal compact subgroups,

$$(K_1 :=) K \cap G_1 = K \cap G_2 (=: K_2).$$

In fact, we get from (9.2)

$$AS_K(\pi_{i\lambda,\delta}) \cap \mathrm{Ad}^\vee(K)(i\mathfrak{k}_1^\perp) = \{0\}.$$

Therefore, the restriction $\pi_{i\lambda,\delta}|_{G_1}$ is discretely decomposable, too.

Remark 9.2. In contrast to the restriction of the quantization of elliptic orbits (equivalently, of Zuckerman's $A_{\mathfrak{q}}(\lambda)$-modules), it is rare that the restriction of the quantization of hyperbolic orbits (equivalently, unitarily induced representations from real parabolic subgroups) is discretely decomposable with respect to non-compact reductive subgroups. Another discretely decomposable case was found by Lee–Loke in their study of the Jordan–Hölder series of a certain degenerate principal series representations.

10. Branching law for $GL(N, \mathbb{R}) \downarrow GL(p, \mathbb{R}) \times GL(q, \mathbb{R})$

Let $N = p + q$ $(p, q \geq 1)$, and consider a subgroup $G_3 := GL(p, \mathbb{R}) \times GL(q, \mathbb{R})$ in $G := GL(N, \mathbb{R})$. The restriction of $\pi^{GL(N,\mathbb{R})}_{i\lambda,\delta}$ with respect to the symmetric pair

$$(G, G_3) = (GL(N, \mathbb{R}), GL(p, \mathbb{R}) \times GL(q, \mathbb{R}))$$

is decomposed into the same family of degenerate principal series representations of G_3:

Theorem 10.1 (Branching law $GL(p+q, \mathbb{R}) \downarrow GL(p, \mathbb{R}) \times GL(q, \mathbb{R})$)**.**

$$\pi^{GL(p+q,\mathbb{R})}_{i\lambda,\delta}|_{G_3} \simeq \sum_{\delta'=0,1} \int_{\mathbb{R}}^{\oplus} \pi^{GL(p,\mathbb{R})}_{i\lambda',\delta'} \boxtimes \pi^{GL(q,\mathbb{R})}_{i(\lambda-\lambda'),\delta-\delta'} d\lambda'.$$

Outline of Proof. The proof is similar to that of Theorem 9.1. The group $G_3 = GL(p,\mathbb{R}) \times GL(q,\mathbb{R})$ acts on $\mathbb{P}^{p+q-1}\mathbb{R}$ with an open dense orbit $\mathcal{O}_3$ which has a G_3-equivariant fibration

$$\mathbb{R}^\times \to \mathcal{O}_3 \to (GL(p,\mathbb{R})/P_p) \times (GL(q,\mathbb{R})/P_q)\,.$$

Hence, taking the Mellin transform by the $\mathbb{R}^\times$-action along the fiber, we get Theorem 10.1. □

11. Branching law for $GL(N,\mathbb{R}) \downarrow O(p,q)$

For $N = p+q$, we introduce the standard quadratic form of signature (p,q) by

$$Q(x) := x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 \quad \text{for } x \in \mathbb{R}^{p+q}.$$

Let G_4 be the indefinite orthogonal group defined by

$$O(p,q) := \{g \in GL(N,\mathbb{R}) : Q(gx) = Q(x) \quad \text{for any } x \in \mathbb{R}^{p+q}\}.$$

For $q = 0$, G_4 is nothing but a maximal compact subgroup $K = O(N)$ of G, and the branching law $\pi^{GL(N,\mathbb{R})}_{i\lambda,\delta}|_{G_4}$ is so called the K-type formula.

In order to describe the branching law $G \downarrow G_4$ for general p and q, we introduce a family of irreducible unitary representations of G_4, to be denoted by $\pi^{O(p,q)}_{+,\nu}$ ($\nu \in A_+(p,q)$ below), $\pi^{O(p,q)}_{-,\nu}$ ($\nu \in A_+(q,p)$), and $\pi^{O(p,q)}_{i\nu,\delta}$ ($\nu \in \mathbb{R}$) as follows. Let $\mathfrak{t}$ be a compact Cartan subalgebra of $\mathfrak{g}_4$, and we take a standard dual basis $\{e_j\}$ of $\mathfrak{t}$ such that the set of roots for $\mathfrak{k}_4 := \mathfrak{o}(p) \oplus \mathfrak{o}(q)$ is given by

$$\begin{aligned}\Delta(\mathfrak{k}_4,\mathfrak{t}_4) = \{&\pm(e_i \pm e_j) : 1 \le i < j \le [\frac{p}{2}] \ \text{ or } \ [\frac{p}{2}]+1 \le i < j \le [\frac{p}{2}]+[\frac{q}{2}]\}\\ &\cup \{\pm e_i : 1 \le i \le [\frac{p}{2}]\} \ (p : \text{odd})\\ &\cup \{\pm e_i : [\frac{p}{2}]+1 \le i \le [\frac{p}{2}]+[\frac{q}{2}]\} \ (q : \text{odd}).\end{aligned}$$

Then, attached to the coadjoint orbits $\mathrm{Ad}^\vee(G_4)(\nu e_i)$ for $\nu \in A_+(p,q)$ and $\mathrm{Ad}^\vee(G_4)(\nu e_{[\frac{p}{2}]+1})$ for $\nu \in A_+(q,p)$, we can define unitary representations of G_4, to be denoted by $\pi^{O(p,q)}_{+,\nu}$ and $\pi^{O(p,q)}_{-,\nu}$ as their geometric quantizations. These representations are realized in Dolbeault cohomologies over the corresponding coadjoint orbits endowed with G_4-invariant complex structures, and their underlying $(\mathfrak{g}_{\mathbb{C}}, K)$-modules are obtained also as cohomologically induced representations from characters of certain θ-stable parabolic subalgebras (see [19, §5] for details).

We normalize $\pi^{O(p,q)}_{+,\nu}$ such that its infinitesimal character is given by

$$(\nu, \frac{p+q}{2} - 2, \frac{p+q}{2} - 3, \dots, \frac{p+q}{2} - [\frac{p+q}{2}])$$

in the Harish-Chandra parametrization. The parameter set that we need for $\pi^{O(p,q)}_{+,\nu}$ is $A_+(p,q) := A^0_+(p,q) \cup A^1_+(p,q)$ where

$$A^\delta_+(p,q) := \begin{cases} \{\nu \in 2\mathbb{Z} + \frac{p-q}{2} + 1 + \delta : \nu > 0\}, & (p > 1, q \neq 0); \\ \{\nu \in 2\mathbb{Z} + \frac{p-q}{2} + 1 + \delta : \nu > \frac{p}{2} - 1\}, & (p > 1, q = 0); \\ \varnothing, & (p = 1, (q,\delta) \neq (0,1)) \\ & \text{or } (p = 0); \\ \{\frac{1}{2}\}, & (p = 1, (q,\delta) = (0,1)). \end{cases}$$

Notice that the identification $O(p,q) \simeq O(q,p)$ induces the equivalence $\pi^{O(p,q)}_{-,\nu} \simeq \pi^{O(q,p)}_{+,\nu}$.

For $p, q > 0$ the group $G_4 = O(p,q)$ is non-compact ant there are continuously many hyperbolic coadjoint orbits. Attached to (minimal) hyperbolic coadjoint orbits, we can define another family of irreducible unitary representations of G_4, to be denoted by $\pi^{O(p,q)}_{i\nu,\delta}$ for $\nu \in \mathbb{R}$ and $\delta \in \{0,1\}$. Namely, let $\pi^{O(p,q)}_{i\nu,\delta}$ be the unitary representation of G_4 induced from a unitary character $(i\nu, \delta)$ of a maximal parabolic subgroup of G_4 whose Levi part is $O(1,1) \times O(p-1, q-1)$.

We note that the Knapp–Stein intertwining operator gives a unitary isomorphism

$$\pi^{O(p,q)}_{i\nu,\delta} \simeq \pi^{O(p,q)}_{-i\nu,\delta} \quad (\nu \in \mathbb{R},\ \delta = 0, 1).$$

Theorem 11.1 (Branching law $GL(p+q,\mathbb{R}) \downarrow O(p,q)$)**.**

$$\pi^{GL(p+q,\mathbb{R})}_{i\lambda,\delta}|_{O(p,q)} \simeq \sum_{\nu \in A^\delta_+(p,q)}{}^{\oplus} \pi^{O(p,q)}_{+,\nu} \oplus \sum_{\nu \in A^\delta_+(q,p)}{}^{\oplus} \pi^{O(p,q)}_{-,\nu} \oplus 2 \int_{\mathbb{R}_+}{}^{\oplus} \pi^{O(p,q)}_{i\nu,\delta} d\nu.$$

Notice that in case when $q = 0$ the latter two components of the above decomposition do not occur and one gets the K-type formula $GL(n,\mathbb{R}) \downarrow O(n)$.

As a preparation of the proof, we formalize the Plancherel formula on the hyperboloid from a modern viewpoint of representation theory.

Let $X(p,q)_\pm$ be a hypersurface in $\mathbb{R}^{p+q}$ defined by

$$X(p,q)_\pm := \{x = (x', x'') \in \mathbb{R}^{p+q} \,:\, |x'|^2 - |x''|^2 = \pm 1\}.$$

We endow $X(p,q)_\pm$ with pseudo-Riemannian structures by restricting $ds^2 = dx_1^2 + \cdots + dx_p^2 - dx_{p+1}^2 - \cdots - dx_{p+q}^2$ on $\mathbb{R}^{p+q}$. Then, $X(p,q)_\pm$ becomes a *space form* of pseudo-Riemannian manifolds in the sense that its sectional curvature κ is constant. To be explicit, $X(p,q)_+$ has a pseudo-Riemannian structure of signature $(p-1, q)$ with sectional

curvature $\kappa \equiv 1$, whereas $X(p,q)_-$ has a signature $(p, q-1)$ with $\kappa \equiv -1$. Clearly, G_4 acts on $X(p,q)_\pm$ as isometries.

We denote by $L^2(X(p,q)_\pm)$ the Hilbert space consisting of square integrable functions on $X(p,q)_\pm$ with respect to the induced measure from $ds^2|_{X(p,q)}$.

The irreducible decomposition of the unitary representation of G_4 on $L^2(X(p,q)_\pm)$ is equivalent to the spectral decomposition of the Laplace–Beltrami operator on $X(p,q)_\pm$ with respect to the G_4-invariant pseudo-Riemannian structures. The latter viewpoint was established by Faraut [9] and Strichartz [24].

As we saw in [19, §5], the discrete series representations on hyperboloids $X(p,q)_\pm$ are isomorphic to $\pi_{\pm,\nu}^{O(p,q)}$ with parameter set $A_\pm(p,q)$.

$$L^2(X(p,q)_+)_\delta = \sum_{\nu \in A_+^\delta(p,q)} \pi_{+,\nu}^{O(p,q)} \oplus \int_{\mathbb{R}_+}^{\oplus} \pi_{i\nu,\delta}^{O(p,q)} d\nu, \tag{11.1}$$

$$L^2(X(p,q)_-)_\delta = \sum_{\nu \in A_+^\delta(q,p)} \pi_{-,\nu}^{O(p,q)} \oplus \int_{\mathbb{R}_+}^{\oplus} \pi_{i\nu,\delta}^{O(p,q)} d\nu. \tag{11.2}$$

Here we note that each irreducible decomposition is multiplicity free, the continuous spectra in both decompositions are the same and the discrete ones are distinct.

Proof of Theorem 11.1. According to the decomposition

$$\mathbb{R}^{p+q} \underset{\text{dense}}{\supset} \{x \in \mathbb{R}^{p+q} : Q(x) > 0\} \cup \{x \in \mathbb{R}^{p+q} : Q(x) < 0\},$$

the group $G_4 = O(p,q)$ acts on $\mathbb{P}^{p+q-1}\mathbb{R}$ with two open orbits, denoted by $\mathcal{O}_4^+$ and $\mathcal{O}_4^-$. A distinguishing feature for G_4 is that these open G_4-orbits are reductive homogeneous spaces. To be explicit, let H_4^+ and H_4^- be the isotropy subgroups of G_4 at $[e_1] \in \mathcal{O}_4^+$ and $[e_{p+q}] \in \mathcal{O}_4^-$, respectively, where $\{e_j\}$ denotes the standard basis of $\mathbb{R}^{p+q}$. Then we have

$$\mathcal{O}_4^+ \simeq G_4/H_4^+ = O(p,q)/(O(1) \times O(p-1,q)),$$
$$\mathcal{O}_4^- \simeq G_4/H_4^- = O(p,q)/(O(p,q-1) \times O(1)).$$

Correspondingly, the restriction of the line bundle $\mathcal{L}_{i\lambda,\delta} = G \times_P \chi_{i\lambda,\delta}$ to the open sets $\mathcal{O}_4^\pm$ of the base space G/P is given by

$$G_4 \times_{H_4^\pm} \mathbb{C}_\delta,$$

where $\mathbb{C}_\delta$ is a one-dimensional representation of $H_4^\pm$ defined by

$$O(1) \times O(p-1,q) \to \mathbb{C}^\times,\ (a,A) \mapsto a^\delta,$$
$$O(p,q-1) \times O(1) \to \mathbb{C}^\times,\ (B,b) \mapsto b^\delta,$$

respectively. It is noteworthy that unlike the cases $G_2 = GL(n,\mathbb{C})$ and $G_3 = GL(p,\mathbb{R}) \times GL(q,\mathbb{R})$, the continuous parameter λ is not involved in (11.1).

Since the union $\mathcal{O}_4^+ \cup \mathcal{O}_4^-$ is open dense in $\mathbb{P}^{p+q-1}\mathbb{R}$, we have a G_4-unitary equivalence (independent of λ):

$$\mathcal{H}^{GL(p+q,\mathbb{R})}_{i\lambda,\delta}|_{G_4} \simeq L^2(G_4 \times_{H_4} \mathbb{C}_\delta, \mathcal{O}_4^+) \oplus L^2(G_4 \times_{H_4} \mathbb{C}_\delta, \mathcal{O}_4^-).$$

Sections for the line bundle $G_4 \times_{H_4^\pm} \mathbb{C}_\delta$ over $\mathcal{O}_4^\pm$ are identified with even functions ($\delta = 0$) or odd functions ($\delta = 1$) on hyperboloids $X(p,q)_\pm$ because $X(p,q)_\pm$ are double covering manifolds of $\mathcal{O}_4^\pm$.

According to the parity of functions on the hyperboloid $X(p,q)_\pm$, we decompose

$$L^2(X(p,q)_\pm) = L^2(X(p,q)_\pm)_0 \oplus L^2(X(p,q)_\pm)_1.$$

Hence, we get Theorem 11.1. □

12. Tensor products $\mathrm{Met}^\vee \otimes \mathrm{Met}$

The irreducible decomposition of the tensor product of two representations is a special example of branching laws. It is well-understood that the tensor product of the same Segal–Shale–Weil representation (e.g. $\mathrm{Met} \otimes \mathrm{Met}$) decomposes into a *discrete* direct sum of lowest weight representations of $Sp(n,\mathbb{R})$ (see [15]). In this section, we prove:

Theorem 12.1. *Let* Met *be the Segal–Shale–Weil representation of the metaplectic group $Mp(n,\mathbb{R})$, and* $\mathrm{Met}^\vee$ *its contragredient representation. Then the tensor product representation* $\mathrm{Met}^\vee \otimes \mathrm{Met}$ *is well-defined as a representation of $Sp(n,\mathbb{R})$, and decomposes into the direct integral of irreducible unitary representations as follows:*

$$\mathrm{Met}^\vee \otimes \mathrm{Met} \simeq \sum_{\delta=0,1} \int_{\mathbb{R}_+}^{\oplus} 2\pi^{Sp(n,\mathbb{R})}_{i\lambda,\delta} d\lambda. \tag{12.1}$$

Remark 12.2. The branching formula in Theorem 12.1 may be regarded as the dual pair correspondence $O(1,1) \cdot Sp(n,\mathbb{R})$ with respect to the Segal–Shale–Weil representation of $Mp(2n,\mathbb{R})$. We note that the Lie group $O(1,1)$ is non-abelian, and its finite dimensional irreducible unitary representations are generically of dimension two, which corresponds the multiplicity two in the right-hand side of (12.1).

Proof. By Fact 3.2, the Weyl operatorcalculus

$$\mathrm{Op} : L^2(\mathbb{R}^{2n}) \xrightarrow{\sim} \mathrm{HS}(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n)) \tag{12.2}$$

gives an intertwining operator as unitary representations of $Mp(n,\mathbb{R})$. We write $L^2(\mathbb{R}^n)^\vee$ for the dual Hilbert space, and identify

$$\mathrm{HS}(L^2(\mathbb{R}^n), L^2(\mathbb{R}^n)) \simeq L^2(\mathbb{R}^n)^\vee \widehat{\otimes} L^2(\mathbb{R}^n), \tag{12.3}$$

where $\widehat{\otimes}$ denotes the completion of the tensor product of Hilbert spaces. Composing (12.2) and (12.3), we see that the tensor product representation $\mathrm{Met}^\vee \otimes \mathrm{Met}$ of $Mp(n,\mathbb{R})$ is unitarily equivalent to the regular representation on $L^2(\mathbb{R}^{2n})$. This representation on the phase space $L^2(\mathbb{R}^{2n})$ is well-defined as a representation of $Sp(n,\mathbb{R})$.

We consider the Mellin transform on $\mathbb{R}^{2n}$, which is defined as the Fourier transform along the radial direction:

$$f \to \frac{1}{4\pi} \int_{-\infty}^{\infty} |t|^{n-1+i\lambda} (\mathrm{sgn} t)^\delta f(tX) dt,$$

with $\lambda \in \mathbb{R}, \delta = 0, 1, X \in \mathbb{R}^{2n}$. Then, the Mellin transform gives a spectral decomposition of the Hilbert space $L^2(\mathbb{R}^{2n})$. Therefore, the phase space representation $L^2(\mathbb{R}^{2n})$ is decomposed as a direct integral of Hilbert spaces:

$$L^2(\mathbb{R}^{2n}) \simeq \sum_{\delta=0,1} \int_{\mathbb{R}}^{\oplus} V_{i\lambda,\delta}\, d\lambda. \tag{12.4}$$

Since $\pi^{Sp(n,\mathbb{R})}_{i\lambda,\delta} \simeq \pi^{Sp(n,\mathbb{R})}_{-i\lambda,\delta}$ (see (5.3)), we get Theorem 12.1. □

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References

[1] E. P. van den Ban and H. Schlichtkrull, The Plancherel decomposition for a reductive symmetric space. I, II *Invent. Math.* **161** (2005), 453-566; 567–628.
[2] D. Barbasch, The unitary spherical spectrum for split classical groups. *J. Inst. Math. Jussieu* **9** (2010), 265–356.
[3] V. Bargmann, Irreducible unitary representations of the Lorentz group, *Ann. of Math.* **48** (1947), pp. 568–640.
[4] M. Berger, Les espaces symétriques noncompacts, *Ann. Sci. École Norm. Sup.* (3) **74** (1957) 85–177.
[5] P. Delorme, Formule de Plancherel pour les espaces symétriques réductifs, *Ann. of Math.* (2) **147** (1998), no. 2, 417–452.

[6] J.–L. Clerc, T. Kobayashi, M. Pevzner, B. Ørsted, Generalized Bernstein–Reznikov integrals, *Math. Ann.*, Doi:/10.1007/s00208-010-0516-4, On line first, May 4, 2010.
[7] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi. *Higher transcendental functions*. Vol. II. McGraw-Hill, New York, 1953.
[8] A. Erdélyi, W. Magnus, F. Oberhettinger, F.G. Tricomi. *Tables of integral transforms*. Vol. I. II, McGraw-Hill, New York, 1954.
[9] J. Faraut, Distributions sphériques sur les espaces hyperboliques, *J. Math. Pures Appl.* **58** (1979), 369–444.
[10] T. A. Farmer, Irreducibility of certain degenerate principal series representations of Sp(n, R). *Proc. Amer. Math. Soc.* **83** (1981), 411–420.
[11] G. B. Folland, *Harmonic analysis in Phase space*, Princeton University Press, Princeton, 1989.
[12] W.-T. Gan and G. Savin, *On minimal representations definitions and properties*, Represent. Theory **9** (2005), 46–93.
[13] R. Howe, On the role of the Heisenberg group in harmonic analysis. *Bull. Amer. Math. Soc. (N.S.)* **3** (1980), pp. 821–843.
[14] R. Howe, S.-T. Lee, Degenerate principal series representations of $\mathrm{GL}_n(\mathbf{C})$ and $\mathrm{GL}_n(\mathbf{R})$. *J. Funct. Anal.* **166** (1999), pp. 244–309.
[15] M. Kashiwara, M. Vergne, On the Segal–Shale–Weil representations and harmonic polynomials, *Invent. Math.* **44**, (1978), pp. 1–47.
[16] T. Kobayashi, Discrete decomposability of the restriction of $A_{\mathfrak{q}}(\lambda)$ with respect to reductive subgroups II: Micro-local analysis and asymptotic K-support, *Ann. Math.,* **147** (1998), 709–729.
[17] T. Kobayashi, Branching problems of unitary representations, *Proc. of ICM 2002, Beijing,* vol. 2, 2002, pp. 615–627.
[18] T. Kobayashi, Algebraic analysis on minimal representations, to appear in *Publ. Res. Inst. Math. Sci.*, Special Issue in commemoration of the golden jubilee of algebraic analysis, E-preprint : arXiv:1001.0224.
[19] T. Kobayashi, B. Ørsted, Analysis on the Minimal Representation of $O(p, q)$-I, II, III., *Adv. Math.* **180**, (2003), pp. 486–512, 513–550, 551–595.
[20] T. Kobayashi and G. Mano, Integral formula of the unitary inversion operator for the minimal representation of $O(p, q)$, *Proc. Japan Acad. Ser. A* **83** (2007), 27–31; the full paper (to appear in the Mem. Amer. Math. Soc.) is available on arXiv:0712.1769.
[21] T. Kobayashi, B. Ørsted, M. Pevzner, and A. Unterberger, Composition formulas in the Weyl calculus, *J. Funct. Anal.* **257**, (2009), pp. 948–991.
[22] B. Kostant, On the existence and irreducibility of certain series of representations. *Bull. Amer. Math. Soc.* **75** (1969), pp. 627–642.
[23] J. v. Neumann, Die Eindeutigkeit der Schrödingerschen Operatoren. *Math. Ann.* **104** (1931), pp. 570–578.
[24] R. S. Strichartz, Harmonic analysis on hyperboloids, *J. Funct. Anal.* **12** (1973), 341–383.
[25] D. A. Vogan, Jr. and N. R. Wallach, Intertwining operators for real reductive groups. *Adv. Math.* **82** (1990), pp. 203–243.

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