A Nonlinear Adiabatic Theorem for Coherent States

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RÉMI CARLES AND CLOTILDE FERMANIAN-KAMMERER

Abstract. We consider the propagation of wave packets for a one-dimensional nonlinear Schrödinger equation with a matrix-valued potential, in the semi-classical limit. For an initial coherent state polarized along some eigenvector, we prove that the nonlinear evolution preserves the separation of modes, in a scaling such that nonlinear effects are critical (the envelope equation is nonlinear). The proof relies on a fine geometric analysis of the role of spectral projectors, which is compatible with the treatment of nonlinearities. We also prove a nonlinear superposition principle for these adiabatic wave packets.

1. Introduction

We consider the semi-classical limit $\varepsilon \to 0$ for the nonlinear Schrödinger equation

$$\begin{cases} i\varepsilon\partial_t\psi^\varepsilon + \dfrac{\varepsilon^2}{2}\partial_x^2\psi^\varepsilon = V(x)\psi^\varepsilon + \Lambda|\psi^\varepsilon|^2_{\mathbf{C}^N}\psi^\varepsilon, \quad (t,x)\in\mathbf{R}\times\mathbf{R}, \\ \psi^\varepsilon_{|t=0} = \psi^\varepsilon_0 \end{cases} \tag{1.1}$$

where $\Lambda \in \mathbf{R}$. The data ψ_0^ε and the solution $\psi^\varepsilon(t)$ are vectors of $\mathbf{C}^N$, $N \geqslant 1$. The quantity $|\psi^\varepsilon|^2_{\mathbf{C}^N}$ denotes the square of the Hermitian norm in $\mathbf{C}^N$ of the vector ψ^ε. Finally, the potential V is smooth and valued in the set of N by N Hermitian matrices. Such systems appear in the modelling of Bose-Einstein condensate (see [1] and references therein).

Definition 1.1. *We say that a function f is* at most quadratic *if $f \in \mathcal{C}^\infty(\mathbf{R})$ and for all $k \geqslant 2$, $f^{(k)} \in L^\infty(\mathbf{R})$.*

We make the following assumptions on the potential V:

Assumption 1.2. (1) *We have $V(x) = D(x) + W(x)$ with $D, W \in \mathcal{C}^\infty(\mathbf{R}, \mathbf{R}^{N\times N})$, D diagonal with at most quadratic coefficients, and W symmetric and bounded as well as its derivatives, $W \in W^{\infty,\infty}(\mathbf{R})$.*
(2) *The matrix V has P distinct, at most quadratic, eigenvalues $\lambda_1, \dots, \lambda_P$ and*

$$\exists c_0, n_0 \in \mathbf{R}^+, \ \ \forall j \neq k, \ \ \forall x \in \mathbf{R}, \ \ |\lambda_j(x) - \lambda_k(x)| \geqslant c_0 \langle x\rangle^{-n_0}. \tag{1.2}$$

Under these assumptions (the first point suffices), we can prove global existence of the solution ψ^ε for fixed $\varepsilon > 0$:

Lemma 1.3. *If V satisfies Assumption 1.2 and $\psi_0^\varepsilon \in L^2(\mathbf{R})$, there exists a unique, global, solution to* (1.1)

$$\psi^\varepsilon \in \mathcal{C}\left(\mathbf{R}; L^2(\mathbf{R})\right) \cap L^8_{\text{loc}}\left(\mathbf{R}; L^4(\mathbf{R})\right).$$

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The L^2-norm of ψ^ε does not depend on time: $\|\psi^\varepsilon(t)\|_{L^2(\mathbf{R})} = \|\psi_0^\varepsilon\|_{L^2(\mathbf{R})}, \ \ \forall t \in \mathbf{R}$.

The proof of this lemma is sketched in Appendix A.

In this nonlinear setting, the size of the initial data is crucial. As in [4], we choose to consider initial data of order 1 (in L^2), and to introduce a dependence upon ε in the coupling constant (note that the nonlinearity is homogeneous). This leads to the equation

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\partial_x^2\psi^\varepsilon = V(x)\psi^\varepsilon + \Lambda\varepsilon^{2\beta}|\psi^\varepsilon|^2_{\mathbf{C}^N}\psi^\varepsilon,$$

and we choose the exponent $\beta = 3/4$, which is critical for the type of initial data we want to consider (coherent state) when the potential V is *scalar* (see [4]). We are left with the nonlinear semi-classical Schrödinger equation

$$i\varepsilon\partial_t\psi^\varepsilon + \frac{\varepsilon^2}{2}\partial_x^2\psi^\varepsilon = V(x)\psi^\varepsilon + \Lambda\varepsilon^{3/2}|\psi^\varepsilon|^2_{\mathbf{C}^N}\psi^\varepsilon \quad ; \quad \psi^\varepsilon_{|t=0} = \psi_0^\varepsilon. \tag{1.3}$$

We focus on initial data which are perturbation of wave packets

$$\psi_0^\varepsilon(x) = \varepsilon^{-1/4}e^{i\xi_0(x-x_0)/\varepsilon}a\left(\frac{x-x_0}{\sqrt{\varepsilon}}\right)\chi(x) + r_0^\varepsilon(x), \tag{1.4}$$

where the initial error satisfies

$$\|r_0^\varepsilon\|_{L^2(\mathbf{R})} + \|xr_0^\varepsilon\|_{L^2(\mathbf{R})} + \|\varepsilon\partial_x r_0^\varepsilon\|_{L^2(\mathbf{R})} = \mathcal{O}(\varepsilon^\kappa) \text{ for some } \kappa > \frac{1}{4}. \tag{1.5}$$

The profile a belongs to the Schwartz class, $a \in \mathcal{S}(\mathbf{R};\mathbf{C})$, and the initial datum is polarized along an eigenvector $\chi(x) \in \mathcal{C}^\infty(\mathbf{R};\mathbf{C}^N)$:

$$V(x)\chi(x) = \lambda_1(x)\chi(x), \quad \text{ with } |\chi(x)|_{\mathbf{C}^N} = 1.$$

Note that λ_1 is simply a notation for *some* eigenvalue, up to a renumbering of eigenvalues. The L^2-norm of ψ_0^ε is independent of ε, $\|\psi_0^\varepsilon\|_{L^2(\mathbf{R})} = \|a\|_{L^2(\mathbf{R})}$. As pointed out above, this is equivalent to considering (1.1) with initial data of the same form (1.4), but of order $\varepsilon^{3/4}$ in $L^2(\mathbf{R})$. The evolution of such data when a is a Gaussian has been extensively studied by G. Hagedorn on the one hand, and by G. Hagedorn and A. Joye on the other hand, in the linear context $\Lambda = 0$ (see [8, 9]). These data are also particularly interesting for numerics (see [12] and the references therein).

Because of the gap condition, the matrix V has smooth eigenvalues and eigenprojectors (see [11]). Besides, the gap condition (1.2) also implies that we control the growth of the eigenprojectors (see Lemma C.2). Note however than in dimension 1 ($x \in \mathbf{R}$), one can have smooth eigenprojectors without any gap condition. We explain this fact below and give an example of projectors that we can consider; we also illustrate why things may be more complicated in higher dimensions ($d \geqslant 2$).

Example 1.4. For $N = 2$ and $x \in \mathbf{R}$, consider

$$V(x) = (ax^2 + b)\mathrm{Id} + \begin{pmatrix} u(x) & v(x) \\ v(x) & -u(x) \end{pmatrix}, \tag{1.6}$$

for $a, b \in \mathbf{R}$, and u and v smooth and bounded with bounded derivatives. Such a potential satisfies Assumption 1.2. Its eigenvalues are the two functions

$$\lambda^\pm(x) = ax^2 + b \pm \sqrt{u(x)^2 + v(x)^2}.$$

These functions are clearly smooth outside the set of points x_0 such that $u(x_0)^2 + v(x_0)^2 = 0$. Besides, for such points, one can renumber the modes in order to build smooth eigenvalues. More precisely, observe first that if $u(x)^2 + v(x)^2 = O((x - x_0)^\infty)$ close to x_0, the functions $\lambda^\pm$ are smooth close to x_0. Moreover, if $u(x)^2 + v(x)^2 = (x - x_0)^k f(x)$ with $f(x_0) \neq 0$, necessarily $f(x_0) > 0$ and $k = 2p$, so we have

$$\lambda^\pm(x) = ax^2 + b \pm |x - x_0|^p \sqrt{f(x)}.$$

For p even these functions are again smooth. However, when p is odd, they are no longer smooth and we perform a renumbering of the eigenfunctions, observing that

$$x \mapsto ax^2 + b + (x - x_0)^p \sqrt{f(x)}$$

are smooth eigenvalues of V close to x_0.

Example 1.5. Resume the above example, with now $x \in \mathbf{R}^d$, $d \geqslant 2$. The smoothness of the eigenvalues is no longer guaranteed: suppose $u(x) = x_1$ and $v(x) = x_2$, then the functions $\lambda_\pm$ are not smooth and one cannot find any renumbering which makes them smooth.

Example 1.6. For an example of a potential which satisfies 1.2, we simply choose V as in (1.6) with

$$c_u u(x) = c_v v(x) = \langle x \rangle^{-n_0}, \quad c_u^2 + c_v^2 \neq 0.$$

1.1. **The ansatz.** We consider the classical trajectories $(x(t), \xi(t))$ solutions to

$$\dot{x}(t) = \xi(t), \;\; \dot{\xi}(t) = -\nabla\lambda_1(x(t)), \;\; x(0) = x_0, \; \xi(0) = \xi_0. \tag{1.7}$$

Because λ_1 is at most quadratic, the classical trajectories grow at most exponentially in time (see e.g. [4]):

$$\exists C > 0, \;\; |\xi(t)| + |x(t)| \lesssim \mathrm{e}^{Ct}. \tag{1.8}$$

We denote by S the action associated with $(x(t), \xi(t))$

$$S(t) = \int_0^t \left(\frac{1}{2}|\xi(s)|^2 - \lambda_1(x(s)) \right) ds. \tag{1.9}$$

We consider the function $u = u(t, y)$ solution to

$$i\partial_t u + \frac{1}{2}\partial_y^2 u = \frac{1}{2}\lambda_1''(x(t))\, y^2 u + \Lambda |u|^2 u \quad ; \quad u(0, y) = a(y), \tag{1.10}$$

and we denote by φ^ε the function associated with u, x, ξ, S by:

$$\varphi^\varepsilon(t, x) = \varepsilon^{-1/4} u\left(t, \frac{x - x(t)}{\sqrt{\varepsilon}}\right) e^{i(S(t) + \xi(t)(x - x(t)))/\varepsilon}. \tag{1.11}$$

Global existence of u and control of its derivatives and momenta are proved in [3]. More precisely, we have the following result.

Theorem 1.7 (From [3])**.** *Suppose $a \in \mathcal{S}(\mathbf{R})$. There exists a unique, global solution $u \in \mathcal{C}(\mathbf{R}; L^2(\mathbf{R})) \cap L^8_{\mathrm{loc}}(\mathbf{R}; L^4(\mathbf{R}))$ to (1.10). In addition, for all $k, p \in \mathbf{N}$, $\langle y \rangle^k \partial_y^p u \in \mathcal{C}(\mathbf{R}; L^2(\mathbf{R}))$ and*

$$\forall k, p \in \mathbf{N}, \;\; \exists C > 0, \;\; \forall t \in \mathbf{R}^+, \;\; \| \langle y \rangle^k \partial_y^p u(t, \cdot) \|_{L^2(\mathbf{R})} \lesssim \mathrm{e}^{Ct}. \tag{1.12}$$

In particular, note that $\partial_y^p u(t, \cdot)$ is in L^∞ for all $p \in \mathbf{N}$. These results have consequences on φ^ε. As far as the L^∞ norm is concerned, we infer, using (1.8),

$$\forall p \in \mathbf{N}, \;\; \|(\varepsilon\partial_x)^p \varphi^\varepsilon(t)\|_{L^\infty} \lesssim \varepsilon^{-1/4} \mathrm{e}^{C_p t}. \tag{1.13}$$

We use the time-dependent eigenvectors constructed in [8] (see also [9] and [14]). To make the notations precise, we denote by d_j the multiplicity of the eigenvalue λ_j, $1 \leqslant j \leqslant P$ (note that $\sum_{1\leqslant j\leqslant P} d_j = N$).

Proposition 1.8. *There exists a smooth orthonormal family* $\left(\chi^\ell(t,x)\right)_{1\leqslant \ell\leqslant d_1}$ *such that for all t,* $\left(\chi^\ell(t,x)\right)_{1\leqslant \ell\leqslant d_1}$ *spans the eigenspace associated to* λ_1*,* $\chi^1(0,x)=\chi(x)$ *and for* $m\in\{1,\cdots,d_1\}$*,*

$$\left(\chi^m(t,x), \partial_t\chi^\ell(t,x)+\xi(t)\partial_x\chi^\ell(t,x)\right)_{\mathbf{C}^N}=0. \tag{1.14}$$

Moreover, for $\ell\in\{1,\dots,d_1\}$*,* $k,p\in\mathbf{N}$*, there exists a constant* $C=C(p,k)$ *such that*

$$\left|\partial_t^p\partial_x^k\chi^\ell(t,x)\right|_{\mathbf{C}^N}\leqslant C\,\mathrm{e}^{Ct}\left\langle x\right\rangle^{(k+p)(1+n_0)},$$

where n_0 *appears in* (1.2)*.*

Note that equation (1.14) for $m=\ell$ is true as soon as the eigenvector χ^ℓ is normalized and real-valued.

Equation (1.14) is often referred to as *parallel transport*. These time-dependent eigenvectors are commonly used in adiabatic theory and are connected with the Berry phase (see [14]). Their construction is recalled in Section 2, where the control of their growth is also established.

Notation. In the case of a single coherent state, we complete the family $\left(\chi^\ell(t,x)\right)_{1\leqslant\ell\leqslant d_1}$ as an orthonormal basis $\left(\chi_j^\ell\right)_{\substack{1\leqslant j\leqslant P\\ 1\leqslant \ell\leqslant d_j}}$ of $\mathbf{C}^N$ as follows:

- $\chi_1^\ell=\chi^\ell$,
- For $j\geqslant 2$ and $1\leqslant\ell\leqslant d_j$, $\chi_j^\ell=\chi_j^\ell(x)$ does not depend on time,
- For $j\geqslant 2$, $(\chi_j^\ell)_{1\leqslant j\leqslant d_j}$ spans the eigenspace associated to λ_j.

1.2. **The results.** We prove that there is adiabatic decoupling for the solution of (1.3) with initial data which are coherent states of the form (1.4): the solution keeps the same form and remains in the same eigenspace.

Theorem 1.9. *Let* $a\in\mathcal{S}(\mathbf{R})$ *and* r_0^ε *satisfying* (1.5)*. Under Assumption 1.2, consider* ψ^ε *solution to the Cauchy problem* (1.3)–(1.4)*, and the approximate solution* φ^ε *given by* (1.11)*. There exists a constant* $C>0$ *such that the function*

$$w^\varepsilon(t,x)=\psi^\varepsilon(t,x)-\varphi^\varepsilon(t,x)\chi^1(t,x),$$

where χ^1 *is given by Proposition 1.8, satisfies*

$$\sup_{|t|\leqslant C\log\log\frac{1}{\varepsilon}}\left(\|w^\varepsilon(t)\|_{L^2}+\|xw^\varepsilon(t)\|_{L^2}+\|\varepsilon\partial_x w^\varepsilon(t)\|_{L^2}\right)\underset{\varepsilon\to 0}{\longrightarrow}0.$$

This adiabatic decoupling between the modes is well-known in the linear setting and is at the basis of numerous results on semi-classical Schrödinger operator with matrix-valued potential in the framework of Born-Oppenheimer approximation for molecular dynamics. On this subject, the reader can consult the article of H. Spohn and S. Teufel [13] or the book of S. Teufel [14] for a review on the topic (see also [2] for an adiabatic result in a nonlinear context and [10] for application of adiabatic theory to the obtention of resolvent estimates).

Remark 1.10. Suppose that V depends on ε with $V^\varepsilon=D+\varepsilon W$, where D and W are as in Assumption 1.2; this is so in the model presented in [1]. Then the above result remains true for $|t|\leqslant C\log\left(\frac{1}{\varepsilon}\right)$: we gain one logarithm. See Remark 4.1 for the key arguments. Also, the assumption on the initial error can be relaxed: to prove the analogue of Theorem 1.9 with an approximation in L^2 up to $C\log 1/\varepsilon$, (1.5) can be replaced with

$$\|r_0^\varepsilon\|_{L^2(\mathbf{R})} \to 0 \text{ as } \varepsilon \to 0.$$

In contrast with the general framework of this paper, no rate is needed: the rate in (1.5) is due to the fact that we cannot use Strichartz estimates here.

It is also interesting to analyze the evolution of solution associated with data which are the superposition of two data of the studied form. We suppose

$$\psi_0^\varepsilon(x) = \varphi_1^\varepsilon(0,x)\chi_1(x) + \varphi_2^\varepsilon(0,x)\chi_2(x),$$

where both functions φ_1^ε and φ_2^ε have the form (1.11), for two eigenvectors of V, χ_1 and χ_2, and phase space points (x_1,ξ_1) and (x_2,ξ_2). We assume

$$(\chi_1, x_1, \xi_1) \neq (\chi_2, x_2, \xi_2)\,.$$

We associate with the phase space points (x_j,ξ_j), $j\in\{1,2\}$ the classical trajectories $(x_j(t),\xi_j(t))$, and the action $S_j(t)$ associated with $\tilde\lambda_j$ such that

$$V(x)\chi_j(x) = \tilde\lambda_j(x)\chi_j(x).$$

Note that we may have $\tilde\lambda_1=\tilde\lambda_2$. Let us denote by $\chi_j^\ell(t)_{\substack{1\leqslant j\leqslant P\\ 1\leqslant \ell\leqslant d_j}}$ a time-dependent orthonormal basis of eigenvectors defined according to Proposition 2.1 (see also Proposition 1.8 above) with $\chi_1^1(0,x)=\chi_1(x)$, $\chi_2(x)=\chi_1^2(0,x)$ if $\tilde\lambda_1=\tilde\lambda_2$, $\chi_2(x)=\chi_2^1(0,x)$ otherwise, and by φ_j^ε the ansatz defined by (1.11). To unify the presentation, we write

$$\chi^1=\chi_1^1 \quad ; \quad \chi^2=\begin{cases}\chi_1^2 \text{ if } \tilde\lambda_1=\tilde\lambda_2,\\ \chi_2^1 \text{ otherwise.}\end{cases}$$

Theorem 1.11. *Set* $E_j=\frac{\xi_j^2}{2}+\tilde\lambda_j(x_j)$ *for* $j\in\{1,2\}$ *and suppose*

$$\Gamma=\inf_{x\in\mathbf{R}}\left|\tilde\lambda_1(x)-\tilde\lambda_2(x)-(E_1-E_2)\right|>0.$$

There exists $C>0$ *such that the function*

$$w^\varepsilon(t)=\psi^\varepsilon(t)-\varphi_1^\varepsilon\chi^1(t,x)-\varphi_2^\varepsilon\chi^2(t,x).$$

satisfies

$$\sup_{t\leqslant C\log\log\frac{1}{\varepsilon}}\left(\|w^\varepsilon(t)\|_{L^2}+\|xw^\varepsilon(t)\|_{L^2}+\|\varepsilon\partial_x w^\varepsilon(t)\|_{L^2}\right)\underset{\varepsilon\to 0}{\longrightarrow}0.$$

Note that if $\tilde\lambda_1=\tilde\lambda_2$, one recovers the condition $E_1\neq E_2$ of [4]. The proof of Theorem 1.11 follows the same lines as in [4, Section 6]. The constant Γ controls the frequencies of time interval where trajectories cross.

Remark 1.12. In finite time, the situation is different whether $\tilde\lambda_1=\tilde\lambda_2$ or not. If $\tilde\lambda_1=\tilde\lambda_2$, the superposition holds in finite time without any condition on Γ; this comes from the fact that the trajectories $x_1(t)$ and $x_2(t)$ only cross on isolated points (see [4]). However, if $\tilde\lambda_1\neq\tilde\lambda_2$ one may have $x_1(t)=x_2(t)$ on intervals of non-empty interior: the condition $\Gamma\neq 0$ prevents this situation from happening. For example, if

$$V(x)=\begin{pmatrix}\cos x & \sin x\\ \sin x & -\cos x\end{pmatrix}+v(x)\mathrm{Id}$$

with v smooth and at most quadratic, we have $\lambda_1(x) = v(x)-1$ and $\lambda_2(x) = v(x)+1$: classical trajectories for both modes, issued from the same point of the phase space, are equal.

1.3. **Strategy of the proof of Theorem 1.9.** The proof is more complicated than in the scalar case [4], due to the fact that the spectral projectors do not commute with the Laplace operator. From this perspective, a much finer geometric understanding is needed and we revisit [8, 7, 9, 13, 14] by adapting to our nonlinear context ideas contained therein.

Observe first that the function φ^ε satisfies

$$i\varepsilon\partial_t\varphi^\varepsilon + \frac{\varepsilon^2}{2}\partial_x^2\varphi^\varepsilon = \mathcal{T}_\varepsilon(t,x)\varphi^\varepsilon + \Lambda\,\varepsilon^{3/2}|\varphi^\varepsilon|^2_{\mathbf{C}^N}\varphi^\varepsilon, \tag{1.15}$$

where

$$\mathcal{T}_\varepsilon(t,x) = \lambda_1(x(t)) + \lambda_1'(x(t))(x-x(t)) + \frac{1}{2}\lambda''(x(t))(x-x(t))^2.$$

This term corresponds to the beginning of the Taylor expansion of λ_1 about $x(t)$. Therefore, the function $w^\varepsilon(t,x) = \psi^\varepsilon(t,x) - \varphi^\varepsilon(t,x)\chi^1(t,x)$ satisfies $w^\varepsilon_{|t=0} = r_0^\varepsilon$ and

$$i\varepsilon\partial_t w_\varepsilon(t,x) + \frac{\varepsilon^2}{2}\partial_x^2 w^\varepsilon(t,x) - V(x)w^\varepsilon(t,x) = \varepsilon\widetilde{NL}^\varepsilon(t,x) + \varepsilon\widetilde{L}^\varepsilon(t,x)$$

where

$$\begin{aligned}\widetilde{NL}^\varepsilon &= \Lambda\,\varepsilon^{1/2}\left(\left|\varphi^\varepsilon\chi^1 + w^\varepsilon\right|^2_{\mathbf{C}^N}(\varphi^\varepsilon\chi^1 + w^\varepsilon) - |\varphi^\varepsilon|^2\varphi^\varepsilon\chi^1\right),\\ \widetilde{L}^\varepsilon &= i\partial_t\chi^1\varphi^\varepsilon + \varepsilon\partial_x\chi^1\partial_x\varphi^\varepsilon + \frac{\varepsilon}{2}\varphi^\varepsilon\partial_x^2\chi^1 + \varepsilon^{-1}\left(\lambda_1(x) - \mathcal{T}_\varepsilon\right)\varphi^\varepsilon\chi^1.\end{aligned}$$

Since φ^ε is concentrated near $x = x(t)$ at scale $\sqrt{\varepsilon}$, we have

$$(\lambda_1(x) - \mathcal{T}_\varepsilon)\varphi^\varepsilon = \mathcal{O}\left(\varepsilon^{3/2}\mathrm{e}^{Ct}\right) \text{ in } L^2(\mathbf{R}),$$

where we have used Theorem 1.7. The term $\widetilde{L}^\varepsilon$ *a priori* presents an $\mathcal{O}(1)$ contribution, which is an obstruction to infer that w^ε is small by applying Gronwall Lemma. Observing that in view of the estimates on the classical flow (see (1.8))

$$\varepsilon\partial_x\varphi^\varepsilon = i\xi(t)\varphi^\varepsilon + \mathcal{O}\left(\sqrt{\varepsilon}\,\mathrm{e}^{Ct}\right) \quad \text{in } L^2(\mathbf{R}),$$

we write,

$$\widetilde{L}^\varepsilon = i\left(\partial_t\chi^1 + \xi(t)\partial_x\chi^1\right)\varphi^\varepsilon + \mathcal{O}\left(\sqrt{\varepsilon}\,\mathrm{e}^{Ct}\right) \text{ in } L^2(\mathbf{R}).$$

The choice of the time-dependent eigenvectors ensures that for all time, the $\mathcal{O}(1)$ contribution of $\widetilde{L}^\varepsilon$ is orthogonal to the first mode (the eigenspace associated with λ^1). Then, to get rid of these terms, we introduce a correction term to w^ε. We set

$$\theta^\varepsilon(t,x) = w^\varepsilon(t,x) + \varepsilon g^\varepsilon(t,x), \quad g^\varepsilon(t,x) = \sum_{2\leqslant j\leqslant P}\;\sum_{1\leqslant\ell\leqslant d_j} g^\varepsilon_{j,\ell}(t,x)\chi_j^\ell(x),$$

where for $j \geqslant 2$ and for $1 \leqslant \ell \leqslant d_j$, the function $g^\varepsilon_{j,\ell}(t,x)$ solves the scalar Schrödinger equation

$$i\varepsilon\partial_t g^\varepsilon_{j,\ell} + \frac{\varepsilon^2}{2}\partial_x^2 g^\varepsilon_{j,\ell} - \lambda_j(x)g^\varepsilon_{j,\ell} = \varphi^\varepsilon r_{j,\ell} \quad ; \quad g^\varepsilon_{j,\ell|t=0} = 0, \tag{1.16}$$

where

$$r_{j,\ell}(t,x) = -i\left(\partial_t\chi^1(t,x) + \xi(t)\partial_x\chi^1(t,x)\,,\;\chi_j^\ell(x)\right)_{\mathbf{C}^N}. \tag{1.17}$$

The function $\theta^\varepsilon(t)$ then solves

$$(1.18)\qquad \begin{cases} i\varepsilon\partial_t\theta^\varepsilon(t,x)+\dfrac{\varepsilon^2}{2}\partial_x^2\theta^\varepsilon(t,x)=V(x)\theta^\varepsilon(t,x)+\varepsilon NL^\varepsilon(t,x)+\varepsilon L^\varepsilon(t,x),\\ \theta^\varepsilon_{|t=0}=r_0^\varepsilon,\end{cases}$$

with

$$(1.19)\qquad NL^\varepsilon=\Lambda\,\varepsilon^{1/2}\left(|\varphi^\varepsilon\chi^1+\theta^\varepsilon-\varepsilon g^\varepsilon|^2_{\mathbf{C}^N}(\varphi^\varepsilon\chi^1+\theta^\varepsilon-\varepsilon g^\varepsilon)-|\varphi^\varepsilon|^2\varphi^\varepsilon\chi^1\right),$$

$$(1.20)\qquad L^\varepsilon=\widetilde{L}^\varepsilon+\left(i\varepsilon\partial_t+\frac{\varepsilon^2}{2}\partial_x^2-V(x)\right)g^\varepsilon(t,x)$$

$$=\mathcal{O}(\sqrt{\varepsilon}\mathrm{e}^{Ct})+\sum_{2\leqslant j\leqslant P}\ \sum_{1\leqslant\ell\leqslant d_j}\left[\frac{\varepsilon^2}{2}\partial_x^2,\chi_j^\ell\right]g^\varepsilon_{j,\ell}$$

where the $\mathcal{O}(\sqrt{\varepsilon}\mathrm{e}^{Ct})$ holds in L^2. The proof of the theorem then follows from a precise control of the functions χ_j^ℓ and $g^\varepsilon_{j,\ell}$, which is achieved in Sections 2 and 3, respectively. Then, the analysis of θ^ε as ε goes to zero by an energy method is presented in Section 4.

2. The family of time-dependent eigenvectors

In this section we prove Proposition 1.8, recalling the construction of the eigenvectors satisfying (1.14), and analyzing the behavior of their derivatives for large time. We follow the proof of [8]. More generally, we prove the following result which implies Proposition 1.8. We consider the Hamiltonian curves of $\frac{1}{2}|\xi|^2+\lambda_j(x)$, that we denote by $(x_j(t),\xi_j(t))$.

Proposition 2.1. *There exists a smooth orthonormal basis of* $\mathbf{C}^N$ $\left(\chi_j^\ell(t,x)\right)_{\substack{1\leqslant\ell\leqslant d_j\\1\leqslant j\leqslant P}}$ *such that for all* t, $\left(\chi_j^\ell(t,x)\right)_{1\leqslant\ell\leqslant d_j}$ *spans the eigenspace associated to* λ_j, *with* $\chi^1(0,x)=\chi(x)$ *and for* $m\in\{1,\cdots,d_j\}$,

$$\left(\chi_j^m(t,x),\partial_t\chi_j^\ell(t,x)+\xi_j(t)\partial_x\chi_j^\ell(t,x)\right)_{\mathbf{C}^N}=0.$$

Moreover, for $\ell\in\{1,\dots,d_j\}$, $k,p\in\mathbf{N}$, *there exists a constant* $C=C(p,k)$ *such that*

$$\left|\partial_t^p\partial_x^k\chi_j^\ell(t,x)\right|_{\mathbf{C}^N}\leqslant C\,\mathrm{e}^{Ct}\,\langle x\rangle^{(k+p)(1+n_0)},$$

where n_0 *appears in* (1.2).

Proof of Proposition 2.1. We consider a smooth basis of eigenvectors $(\chi_j^\ell(0))_{\substack{1\leqslant\ell\leqslant d_j\\1\leqslant j\leqslant P}}$ such that $\chi_1^1(0)=\chi$. Then, we denote by $\Pi_j(x)$ the smooth eigenprojector associated with the eigenvalue $\lambda_j(x)$ and define

$$K_j(x)=-i\left[\Pi_j(x),\partial_x\Pi_j(x)\right].$$

We set $z=x-x_j(t)$ and we consider the Schrödinger type equation

$$(2.1)\qquad i\partial_tY_j^\ell(t,z)=\xi_j(t)K_j(z+x_j(t))Y_j^\ell(t,z)\quad;\quad Y_j^\ell(0,z)=\chi_j^\ell(x_j(0)+z).$$

Let us prove that the vector $Y_j^\ell(t,z)$ is in the eigenspace of $\lambda_j(x_j(t)+z)$. Indeed, the evolution of $Z_j^\ell(t,z) = (\mathrm{Id} - \Pi_j(x_j(t)+z))\, Y_j^\ell(t,z)$ obeys to $Z_j^\ell(0,z)=0$ and

$$\begin{aligned}\partial_t Z_j^\ell(t,z) &= -\xi_j(t)\partial_x \Pi_j(x_j(t)+z) Y_j^\ell \\ &\qquad - \xi_j(t)(\mathrm{Id}-\Pi_j(x_j(t)+z))[\Pi_j(x_j(t)+z), \partial_x\Pi_j(x_j(t)+z)]Y_j^\ell \\ &= -\xi_j(t)\partial_x\Pi_j(x_j(t)+z)(\mathrm{Id}-\Pi_j(x_j(t)+z))Y_j^\ell \\ &= -\xi_j(t)\partial_x\Pi_j(x_j(t)+z)Z_j^\ell\end{aligned}$$

where we have used $\partial_x \Pi_j = \partial_x(\Pi_j^2) = \Pi_j\partial_x\Pi_j + (\partial_x\Pi_j)\,\Pi_j$, whence

$$\Pi_j(\partial_x\Pi_j)\Pi_j = \Pi_j\left(\Pi_j\partial_x\Pi_j + (\partial_x\Pi_j)\,\Pi_j\right)\Pi_j = 2\Pi_j(\partial_x\Pi_j)\Pi_j = 0.$$

Therefore, $Z_j^\ell(t)$ satisfies an equation of the form $\partial_t Z_j^\ell = A(t,z)Z_j^\ell$, which combined with $Z_j^\ell(0)=0$, implies $Z_j^\ell(t)=0$ for all $t\in\mathbf{R}$: the vectors $Y_j^\ell(t,z)$ are eigenvectors of $V(x_j(t)+z)$ for the eigenvalue $\lambda_j(x_j(t)+z)$.
Besides, since $\xi_j(t)K_j(z+x_j(t))$ is self-adjoint, $Y_j^\ell(t,z)$ is normalized for all t, and the family $(Y_j^\ell)_{1\leqslant \ell\leqslant d_j}$ is orthonormal. We define $\chi_j^\ell(t,x)$ by

$$\chi_j^\ell(t,x) = Y_j^\ell(t, x-x_j(t)) \tag{2.2}$$

and we obtain an orthonormal basis of eigenvectors of $V(x)$.
It remains to check that (1.14) holds. We have

$$\begin{aligned}\partial_t\chi_j^\ell + \xi_j(t)\partial_x\chi_j^\ell &= \partial_t Y_j^\ell(t,x-x_j(t)) \\ &= i\xi_j(t)K_j(x)\chi_j^\ell \\ &= -\xi_j(t)[\Pi_j(x),\partial_x\Pi_j(x)]\chi_j^\ell,\end{aligned}$$

whence

$$\begin{aligned}\left(\partial_t\chi_j^\ell + \xi_j(t)\partial_x\chi_j^\ell\ ,\ \chi_j^k\right)_{\mathbf{C}^N} &= -\xi_j(t)\left([\Pi_j,\partial_x\Pi_j]\chi_j^\ell,\chi_j^k\right)_{\mathbf{C}^N} \\ &= -\xi_j(t)\left(\Pi_j[\Pi_j,\partial_x\Pi_j]\Pi_j\chi_j^\ell,\chi_j^k\right)_{\mathbf{C}^N}\end{aligned}$$

since $\chi_j^{\ell/k} = \Pi_j\chi_j^{\ell/k}$. We then observe that $\Pi_j^2=\Pi_j$ implies

$$\Pi_j[\Pi_j,\partial_x\Pi_j]\Pi_j = \Pi_j^2\partial_x\Pi_j\Pi_j - \Pi_j\partial_x\Pi_j\Pi_j^2 = 0.$$

This concludes the first part of Proposition 1.8. It remains to study the behavior at infinity of the vectors $\chi_j^\ell(t,x)$ and of their derivatives.

By the definition of $\chi_j^\ell(t,x)$ in (2.2), it is enough to prove

$$|\partial_t^p\partial_x^k Y_j^\ell(t,z)|_{\mathbf{C}^N} \quad\lesssim\quad \mathrm{e}^{Ct}\left\langle x_j(t)+z\right\rangle^{(p+k)(1+n_0)}.$$

For this, we crucially use the estimates of Lemma C.2 and we argue by induction. Let us first consider the case $p=1$ and $k=0$. By Lemma C.2, we have $|K_j(x)|\lesssim \langle x\rangle^{1+n_0}$, whence (2.1) gives

$$|\partial_t Y_j^\ell(t,x-x_j(t))| \lesssim |\xi_j(t)||K_j(x)| \lesssim \mathrm{e}^{Ct}\langle x\rangle^{1+n_0}.$$

Let us now suppose $k\geqslant 1$ and $p=0$. We observe that $\partial_z^k Y_j^\ell(t,z)$ solves

$$\begin{cases} i\partial_t\partial_z^k Y_j^\ell(t,z) = -i\xi_j(t)K_j(z+x_j(t))\partial_z^k Y_j^\ell(t,z) + f(t,z), \\ \quad \partial_z^k Y_j(0,z) = \partial_x^k\chi_j^\ell(0,z+x_j(0)),\end{cases} \tag{2.3}$$

where
$$f(t,z)=\sum_{0\leqslant\gamma\leqslant k-1} c_\gamma\,\xi_j(t)\partial_z^\gamma K_j(z+x_j(t))\partial_z^{k-\gamma}Y_j^\ell(t,z),$$
for some complex numbers c_γ independent of t and z. We obtain
$$\partial_z^k Y_j^\ell(t,z)=\mathcal{U}_j(t,0)\partial_x^k\chi_j^\ell(z+x_j(0))+\int_0^t \mathcal{U}_j(t,s)f(s,z)ds,$$
where $\mathcal{U}_j(t,s)$ denotes the unitary propagator associated to (2.1) (when the initial time is equal to s). We have by Lemma C.2
$$|\partial_z^k Y_j^\ell(0,z)|_{\mathbf{C}^N}=\left|\partial_x^k\chi_j^\ell(0,z+x_j(0))\right|_{\mathbf{C}^N}\lesssim\langle z+x_j(0)\rangle^{k(1+n_0)},$$
therefore the induction assumption
$$\forall\gamma\in\{0,\dots,k-1\},\ \ |\partial_x^\gamma Y_j^\ell(t,z)|_{\mathbf{C}^N}\lesssim \mathrm{e}^{Ct}\langle x_j(t)+z\rangle^{\gamma(1+n_0)}$$
implies, along with Lemma C.2,
$$|\partial_x^k Y_j^\ell(t,z)|_{\mathbf{C}^N}\lesssim \mathrm{e}^{Ct}\langle x_j(t)+z\rangle^{k(1+n_0)}.$$
We have obtained the estimate for $p=0$, $k\in\mathbf{N}$, and for $p=1$, $k=0$. Note that Equation (2.3) yields
$$\forall k\in\mathbf{N},\ \ |\partial_t\partial_x^k Y_j^\ell(t,z)|_{\mathbf{C}^N}\lesssim \mathrm{e}^{Ct}\langle x_j(t)+z\rangle^{(1+k)(1+n_0)},$$
and allows to prove the general estimate for time derivatives by an induction argument which crucially uses the fact that we have an exponential control of the derivatives in time of $\xi_j(t)$. This property follows by induction from (1.7), (1.8), and the fact that λ_j is at most quadratic. □

Before concluding this section, note that in view of the definition of the function $r_{j,\ell}$ in (1.17), Proposition 1.8 gives the following corollary.

Corollary 2.2. *For all $p\in\mathbf{N}$ and $k\in\mathbf{N}$, there exists a constant $C=C(p,k)$ such that, for $x\in\mathbf{R}$, $j\in\{1,\cdots,P\}$ and $\ell\in\{1,\cdots,d_j\}$,*
$$\left|\partial_t^p\partial_x^k r_{j,\ell}(t,x)\right|\lesssim \mathrm{e}^{Ct}\langle x\rangle^{(1+p+k)(1+n_0)}.$$

3. Analysis of the correction terms

In this section, we will make use of the following norms defined for $p\in\mathbf{N}$,
$$\|f\|_{\Sigma_\varepsilon^p}=\sup_{\alpha+\beta\leqslant p}\left\||x|^\alpha\varepsilon^\beta f^{(\beta)}(x)\right\|_{L^2}.$$
We associate with this norm the functional space Σ_ε^p defined by
$$\Sigma_\varepsilon^p=\{f\in L^2(\mathbf{R}^d),\quad \|f\|_{\Sigma_\varepsilon^p}<\infty\}.$$
In view of (1.8) and (1.13), for all $p\in\mathbf{N}$, there exists $c(p)$ such that
$$\|\varphi^\varepsilon(t)\|_{\Sigma_\varepsilon^p}\lesssim \mathrm{e}^{c(p)t},\quad \forall t\geqslant 0. \tag{3.1}$$
We can obviously take $c(0)=0$ by conservation of the L^2-norm, but in general, the norm of φ^ε in Σ_ε^1 potentially grows exponentially in time (see [3]). We denote by $U_k^\varepsilon(t)$ the semi-group associated with the operator $-\frac{\varepsilon^2}{2}\partial_x^2+\lambda_k(x)$ and we observe that for $p\in\mathbf{N}$, there exists a constant $C(p)$ such that
$$\|U_k^\varepsilon(t)\|_{\mathcal{L}(\Sigma_\varepsilon^p)}\leqslant C(p)\mathrm{e}^{C(p)|t|}. \tag{3.2}$$
The following averaging lemma shows an asymptotic orthogonality property.

Lemma 3.1. *For $T>0$ and $k\neq j$, there exists a constant C such that*

$$\forall t\in[0,T],\ \ \forall p\in \mathbf{N},\quad \left\|\frac{1}{i\varepsilon}\int_0^t U_k^\varepsilon(-s)U_j^\varepsilon(s)ds\right\|_{\mathcal{L}\left(\Sigma_\varepsilon^{(p+3)n_0+p+2},\Sigma_\varepsilon^p\right)}\leqslant C\mathrm{e}^{Ct}.$$

Proof. We first observe that

$$i\varepsilon\partial_t\left(U_k^\varepsilon(-t)U_j^\varepsilon(t)\right)=U_k^\varepsilon(-t)\left(\lambda_j(x)-\lambda_k(x)\right)U_j^\varepsilon(t). \tag{3.3}$$

Indeed, if $f\in L^2(\mathbf{R})$ and $f^\varepsilon(t)=U_k^\varepsilon(-t)U_j^\varepsilon(t)f$. We have

$$\begin{aligned} i\varepsilon\partial_t f^\varepsilon(t,x) &= -\left(-\frac{\varepsilon^2}{2}\partial_x^2+\lambda_k(x)\right)f^\varepsilon(t)+U_k^\varepsilon(-t)\left(-\frac{\varepsilon^2}{2}\partial_x^2+\lambda_j(x)\right)U_j^\varepsilon(t)f\\ &= U_k^\varepsilon(-t)\left(\lambda_j(x)-\lambda_k(x)\right)U_j^\varepsilon(t)f\end{aligned}$$

because $U_k^\varepsilon(-t)$ commutes with $-\frac{\varepsilon^2}{2}\partial_x^2+\lambda_k(x)$. We use Equation (3.3) to perform an integration by parts:

$$\begin{aligned} U_k^\varepsilon(-t)U_j^\varepsilon(t) &= U_k^\varepsilon(-t)\left(\lambda_j-\lambda_k\right)^{-1}U_k^\varepsilon(t)U_k^\varepsilon(-t)\left(\lambda_j-\lambda_k\right)U_j^\varepsilon(t)\\ &= i\varepsilon\, U_k^\varepsilon(-t)\left(\lambda_j-\lambda_k\right)^{-1}U_k^\varepsilon(t)\;\partial_t\left(U_k^\varepsilon(-t)U_j^\varepsilon(t)\right).\end{aligned}$$

Therefore,

$$\begin{aligned} \frac{1}{i\varepsilon}\int_0^t U_k^\varepsilon(-s)U_j^\varepsilon(s)ds &= \left[U_k^\varepsilon(-s)\left(\lambda_j-\lambda_k\right)^{-1}U_j^\varepsilon(s)\right]_0^t\\ &\quad -\int_0^t\partial_s\left(U_k^\varepsilon(-s)\left(\lambda_j-\lambda_k\right)^{-1}U_k^\varepsilon(s)\right)U_k^\varepsilon(-s)U_j^\varepsilon(s)\,ds.\end{aligned}$$

Set

$$\gamma_{j,k}=\left(\lambda_k-\lambda_j\right)^{-1}. \tag{3.4}$$

The behavior as x goes to infinity of these functions is studied in Appendix C (see Lemma C.1). It is proven there that for all $\beta\in\mathbf{N}$,

$$\left|\partial_x^\beta\gamma_{j,k}(x)\right|\lesssim\langle x\rangle^{n_0+|\beta|(1+n_0)}.$$

Since the propagators $U_k^\varepsilon(t)$ and $U_j^\varepsilon(t)$ map continuously Σ_ε^p into itself uniformly with respect to ε, we have

$$\left\|\left[U_k^\varepsilon(-s)\gamma_{j,k}U_j^\varepsilon(s)\right]_0^t\right\|_{\mathcal{L}\left(\Sigma_\varepsilon^{(p+1)n_0+p},\Sigma_\varepsilon^p\right)}\lesssim C(p),$$

where in all this paragraph, $C(p)$ denotes a generic constant depending only on the parameter $p\in\mathbf{N}$. Besides, we observe that

$$\partial_s\left(U_k^\varepsilon(-s)\gamma_{j,k}U_k^\varepsilon(s)\right)=\frac{1}{i\varepsilon}U_k^\varepsilon(-s)\left[-\frac{\varepsilon^2}{2}\partial_x^2+\lambda_k\ ,\ \gamma_{j,k}\right]U_k^\varepsilon(s).$$

In view of

$$\frac{1}{i\varepsilon}\left[-\frac{\varepsilon^2}{2}\partial_x^2+\lambda_k\ ,\ \gamma_{j,k}\right]=\frac{1}{i\varepsilon}\left[-\frac{\varepsilon^2}{2}\partial_x^2\ ,\ \gamma_{j,k}\right]=i\gamma'_{j,k}(x)\varepsilon\partial_x+i\varepsilon\gamma''_{j,k}(x),$$

and of

$$\begin{aligned}&\left\|U_k^\varepsilon(-s)\gamma'_{j,k}(x)\varepsilon\partial_xU_j^\varepsilon(s)\right\|_{\mathcal{L}\left(\Sigma_\varepsilon^{(p+3)n_0+p+2},\Sigma_\varepsilon^p\right)}\\ &+\left\|U_k^\varepsilon(-s)\gamma''_{j,k}(x)U_j^\varepsilon(s)\right\|_{\mathcal{L}\left(\Sigma_\varepsilon^{(p+2)n_0+p+2},\Sigma_\varepsilon^p\right)}\lesssim\mathrm{e}^{Cs},\end{aligned}$$

which comes from (3.2) and Lemma C.1, we get

$$\|\partial_s \left(U_k^\varepsilon(-s)\gamma_{j,k}U_k^\varepsilon(s)\right)\|_{\mathcal{L}(\Sigma_\varepsilon^{p+2},\Sigma_\varepsilon^p)} \lesssim \mathrm{e}^{Ct},$$

which concludes the proof. □

We now prove the following proposition.

Proposition 3.2. *For $p \in \mathbf{N}$, there exists $C(p)$ such that for all $j \geqslant 2$, and all $\ell \in \{1, \dots, d_j\}$,*

$$\|g_{j,\ell}^\varepsilon(t)\|_{\Sigma_\varepsilon^p} \lesssim \mathrm{e}^{C(p)t}, \quad \forall t \geqslant 0,$$

where $g_{j,\ell}^\varepsilon$ is defined in (1.16).

Proof. We use Duhamel's formula and write

$$g_{j,\ell}^\varepsilon(t) = \frac{1}{i\varepsilon}\int_0^t U_j^\varepsilon(t-s)\left(\varphi^\varepsilon(s)r_{j,\ell}(s)\right)ds.$$

Besides, if $\tilde{\varphi}_{j,\ell}^\varepsilon(t,x) = \varphi^\varepsilon(t,x)r_{j,\ell}(t,x)$, then we have,

$$\left(i\varepsilon\partial_t + \frac{\varepsilon^2}{2}\partial_x^2 - \lambda_1(x)\right)\tilde{\varphi}_{j,\ell}^\varepsilon = \underbrace{i\varepsilon\partial_t r_{j,\ell}\varphi^\varepsilon + r_{j,\ell}\varepsilon^{3/2}|\varphi^\varepsilon|^2\varphi^\varepsilon + \frac{\varepsilon^2}{2}\left[\partial_x^2, r_{j,\ell}(t,x)\right]\varphi^\varepsilon}_{=:\varepsilon\widetilde{r}^\varepsilon(t,x)}.$$

Therefore, we can write

$$\tilde{\varphi}_{j,\ell}^\varepsilon(t) = U_1^\varepsilon(t)\tilde{\varphi}_{j,\ell}^\varepsilon(0) - i\int_0^t U_1^\varepsilon(t-s)\widetilde{r}^\varepsilon(s)ds,$$

whence

$$\begin{aligned} g_{j,\ell}^\varepsilon(t) &= \frac{1}{i\varepsilon}\int_0^t U_j^\varepsilon(t-s)U_1^\varepsilon(s)ds\tilde{\varphi}_{j,\ell}^\varepsilon(0) - \frac{1}{\varepsilon}\int_0^t\int_0^s U_j^\varepsilon(t-s)U_1^\varepsilon(s-\tau)\widetilde{r}^\varepsilon(\tau)d\tau ds \\ &= \frac{1}{i\varepsilon}\int_0^t U_j^\varepsilon(t-s)U_1^\varepsilon(s)ds\tilde{\varphi}_{j,\ell}^\varepsilon(0) - \int_0^t\left[\frac{1}{\varepsilon}\int_\tau^t U_j^\varepsilon(t-s)U_1^\varepsilon(s-\tau)ds\right]\widetilde{r}^\varepsilon(\tau)d\tau. \end{aligned}$$

Lemma 3.1 yields

$$\|g_{j,\ell}^\varepsilon(t)\|_{\Sigma_\varepsilon^p} \lesssim \mathrm{e}^{Ct} + \int_0^t \mathrm{e}^{C\tau}\|\widetilde{r}^\varepsilon(\tau)\|_{\Sigma_\varepsilon^q}d\tau,$$

with $q = p+2+(p+3)(1+n_0)$. Let us now study $\widetilde{r}^\varepsilon$. We write $\widetilde{r}^\varepsilon = \widetilde{r}_1^\varepsilon + \widetilde{r}_2^\varepsilon$ with

$$\widetilde{r}_1^\varepsilon(t,x) = i\partial_t r_{j,\ell}\varphi^\varepsilon + \frac{\varepsilon}{2}\left[\partial_x^2, r_{j,\ell}(t,x)\right]\varphi^\varepsilon.$$

In view of Corollary 2.2 and of (3.1), we have for all $q \in \mathbf{N}$,

$$\|\widetilde{r}_1^\varepsilon(t)\|_{\Sigma_\varepsilon^q(\mathbf{R})} \lesssim \mathrm{e}^{C(q)t}.$$

A very rough estimate yields

$$\begin{aligned} \|\widetilde{r}_2^\varepsilon(t)\|_{\Sigma_\varepsilon^q} = \|\sqrt{\varepsilon}r_{j,\ell}|\varphi^\varepsilon|^2\varphi^\varepsilon\|_{\Sigma_\varepsilon^q} &\lesssim \sqrt{\varepsilon}\left\|r_{j,\ell}\langle x\rangle^q\varphi^\varepsilon\right\|_{\Sigma_\varepsilon^q}\left\|\langle\varepsilon\partial_x\rangle^q\varphi^\varepsilon\right\|_{L^\infty}^2 \\ &\lesssim \sqrt{\varepsilon}\mathrm{e}^{Ct}\left\|\langle x\rangle^{q+(1+q)(n_0+1)}\varphi^\varepsilon\right\|_{\Sigma_\varepsilon^q}\left\|\langle\varepsilon\partial_x\rangle^q\varphi^\varepsilon\right\|_{L^\infty}^2, \end{aligned}$$

where we have used Corollary 2.2. Now with (1.13) and (3.1), we conclude

$$\|\widetilde{r}_2^\varepsilon(t)\|_{\Sigma_\varepsilon^q} \lesssim \mathrm{e}^{Ct}.$$

This completes the proof of Proposition 3.2. □

4. Consistency

We now prove Theorem 1.9. We go back to Equation (1.18), that we recall:

$$\begin{cases} i\varepsilon\partial_t\theta^\varepsilon(t,x) + \dfrac{\varepsilon^2}{2}\partial_x^2\theta^\varepsilon(t,x) = V(x)\theta^\varepsilon(t,x) + \varepsilon NL^\varepsilon(t,x) + \varepsilon L^\varepsilon(t,x), \\ \theta^\varepsilon_{\mid t=0} = r_0^\varepsilon, \end{cases}$$

where NL^ε and L^ε are defined in (1.19) and (1.20), respectively. The standard L^2-estimate yields:

$$\|\theta^\varepsilon(t)\|_{L^2} \leqslant \|r_0^\varepsilon\|_{L^2} + \int_0^t \left(\|NL^\varepsilon(s)\|_{L^2} + \|L^\varepsilon(s)\|_{L^2}\right) ds.$$

In view of (1.20), Proposition 2.1 and Proposition 3.2, we have

$$\|L^\varepsilon(t)\|_{L^2} \lesssim \sqrt{\varepsilon}\mathrm{e}^{Ct}.$$

Besides, we observe

$$\begin{aligned}\|NL^\varepsilon(t)\|_{L^2} &\lesssim \sqrt{\varepsilon}\left\|\left(|\varphi^\varepsilon(t)|^2 + |\theta^\varepsilon(t)|^2_{\mathbf{C}^N} + \varepsilon^2|g^\varepsilon(t)|^2_{\mathbf{C}^N}\right)\left(\theta^\varepsilon(t) - \varepsilon g^\varepsilon(t)\right)\right\|_{L^2} \\ &\lesssim \sqrt{\varepsilon}\left(\|\varphi^\varepsilon(t)\|^2_{L^\infty} + \|\theta^\varepsilon(t)\|^2_{L^\infty} + \varepsilon^2\|g^\varepsilon(t)\|^2_{L^\infty}\right)\left(\|\theta^\varepsilon(t)\|_{L^2} + \varepsilon\|g^\varepsilon(t)\|_{L^2}\right).\end{aligned}$$

In view of (1.13), we have $\|\varphi^\varepsilon(t)\|_{L^\infty} \lesssim \varepsilon^{-1/4}\mathrm{e}^{Ct}$. On the other hand, Proposition 3.2 implies, in view of the Gagliardo-Nirenberg inequality

$$\|f\|_{L^\infty} \lesssim \varepsilon^{-1/2}\|f\|^{1/2}_{L^2}\|\varepsilon\partial_x f\|^{1/2}_{L^2}, \tag{4.1}$$

the estimate

$$\varepsilon^2\|g^\varepsilon(t)\|^2_{L^\infty} \lesssim \varepsilon\mathrm{e}^{Ct}.$$

Therefore, it is natural to perform a bootstrap argument assuming, say

$$\|\theta^\varepsilon(t)\|_{L^\infty} \leqslant \varepsilon^{-1/4}\mathrm{e}^{Ct}. \tag{4.2}$$

Note that we fixed the value of the constant in factor of the right hand side equal to one. We did so because θ^ε, as an error term, is expected to be smaller than φ^ε (the approximate solution) in the limit $\varepsilon \to 0$. As long as (4.2) holds, the L^2-estimate implies, in view of (1.5)

$$\|\theta^\varepsilon(t)\|_{L^2} \lesssim \varepsilon^\kappa + \int_0^t \left(\sqrt{\varepsilon}\mathrm{e}^{Cs} + \mathrm{e}^{Cs}\|\theta^\varepsilon(s)\|_{L^2}\right) ds.$$

By Gronwall Lemma, we obtain

$$\|\theta^\varepsilon(t)\|_{L^2} \leqslant C\left(\varepsilon^\kappa + \sqrt{\varepsilon}\right)\mathrm{e}^{\mathrm{e}^{Ct}}. \tag{4.3}$$

It remains to check how long the bootstrap assumption (4.2) holds. For this, we use Gagliardo-Nirenberg inequality (4.1), and we look for a control of the norm of $\theta^\varepsilon(t)$ in Σ^1_ε. Differentiating the system (1.18) with respect to x, we find

$$i\varepsilon\partial_t(\varepsilon\partial_x\theta^\varepsilon) + \frac{\varepsilon^2}{2}\partial_x^2(\varepsilon\partial_x\theta^\varepsilon) = V(x)\varepsilon\partial_x\theta^\varepsilon + \varepsilon V'(x)\theta^\varepsilon + \varepsilon^2\partial_x NL^\varepsilon + \varepsilon^2\partial_x L^\varepsilon,$$

We observe that since V is at most quadratic, $|V'(x)\theta^\varepsilon|_{\mathbf{C}^N} \lesssim \langle x\rangle\,|\theta^\varepsilon|_{\mathbf{C}^N}$. Therefore, in order to obtain a closed system of estimates, we consider the equation satisfied by $x\theta^\varepsilon$: multiply (1.18) by x,

$$i\varepsilon\partial_t(x\theta^\varepsilon) + \frac{\varepsilon^2}{2}\partial_x^2(x\theta^\varepsilon) = V(x)(x\theta^\varepsilon) + \varepsilon^2\partial_x\theta^\varepsilon + \varepsilon xNL^\varepsilon + \varepsilon xL^\varepsilon.$$

By Proposition 3.2, we have

$$\|xL^\varepsilon(t)\|_{L^2} + \|\varepsilon\partial_x L^\varepsilon(t)\|_{L^2} \lesssim \sqrt{\varepsilon}\mathrm{e}^{Ct}.$$

Besides,

$$\begin{aligned}
|xNL^\varepsilon(t,x)|_{\mathbf{C}^N} &\lesssim \left(|\phi^\varepsilon(t,x)|^2 + |\theta^\varepsilon(t,x)|^2_{\mathbf{C}^N} + \varepsilon^2|g^\varepsilon(t,x)|^2_{\mathbf{C}^N}\right) \times \\
&\quad \times \left(|x\theta^\varepsilon(t,x)|_{\mathbf{C}^N} + \varepsilon|xg^\varepsilon(t,x)|_{\mathbf{C}^N}\right), \\
|\varepsilon\partial_x NL^\varepsilon(t,x)|_{\mathbf{C}^N} &\lesssim \left(|\phi^\varepsilon(t,x)|^2 + |\theta^\varepsilon(t,x)|^2_{\mathbf{C}^N} + \varepsilon^2|g^\varepsilon(t,x)|^2_{\mathbf{C}^N}\right)|\varepsilon\partial_x\theta^\varepsilon(t,x)|_{\mathbf{C}^N} \\
&\quad + \varepsilon\left(|\phi^\varepsilon(t,x)|^2 + |\theta^\varepsilon(t,x)|^2_{\mathbf{C}^N} + \varepsilon^2|g^\varepsilon(t,x)|^2_{\mathbf{C}^N}\right)|\varepsilon\partial_x g^\varepsilon(t,x)|_{\mathbf{C}^N} \\
&\quad + |\varepsilon\partial_x\phi^\varepsilon(t,x)| \times |\phi^\varepsilon(t,x)| \times |\theta^\varepsilon(t,x)|_{\mathbf{C}^N} \\
&\quad + \varepsilon|\phi^\varepsilon(t,x)|^2 \times |\partial_x\chi^1(t,x)|_{\mathbf{C}^N} \times |\theta^\varepsilon(t,x)|_{\mathbf{C}^N}.
\end{aligned}$$

Arguing as before and using again (1.13), we obtain that under (1.2) we have

$$\|\varepsilon\partial_x\theta^\varepsilon(t)\|_{L^2} + \|x\theta^\varepsilon(t)\|_{L^2} \lesssim \left(\varepsilon^\kappa + \sqrt{\varepsilon}\right)\mathrm{e}^{\mathrm{e}^{Ct}}.$$

Gagliardo–Nirenberg inequality then implies

$$\|\theta^\varepsilon(t)\|_{L^\infty} \lesssim \varepsilon^{-1/2}\left(\varepsilon^\kappa + \sqrt{\varepsilon}\right)\mathrm{e}^{\mathrm{e}^{Ct}}.$$

We infer that (4.2) holds (at least) as long as

$$\left(\varepsilon^{\kappa-1/2} + 1\right)\mathrm{e}^{\mathrm{e}^{Ct}} \ll \varepsilon^{-1/4}\mathrm{e}^{Ct},$$

which is ensured provided that $t \leqslant C\log\log\left(\frac{1}{\varepsilon}\right)$, for some suitable constant C, since $\kappa > 1/4$. This concludes the bootstrap argument: we infer

$$\sup_{|t|\leqslant C\log\log\left(\frac{1}{\varepsilon}\right)} \left(\|\theta^\varepsilon(t)\|_{L^2} + \|x\theta^\varepsilon(t)\|_{L^2} + \|\varepsilon\partial_x\theta^\varepsilon(t)\|_{L^2}\right) \underset{\varepsilon\to 0}{\longrightarrow} 0.$$

Theorem 1.9 then follows from the above asymptotics, together with the relation $\theta^\varepsilon = w^\varepsilon + \varepsilon g^\varepsilon$, and Proposition 3.2.

Remark 4.1. In the case where $V^\varepsilon = D + \varepsilon W$, as in Remark 1.10, the proof can be adapted, in order to reproduce the argument given in [4]. The main point to notice is that (local in time) Strichartz estimates are available for the propagator associated to $-\frac{\varepsilon^2}{2}\partial_x^2 + D(x)$, thanks to [6]. Then in the presence of the power ε in front of W, the potential εW can be considered as a source term in the error estimates: the factor ε is crucial to avoid a singular power of ε due to the presence of ε in front of the time derivative in (1.18). The proof in [4, Section 6] for the cubic, one-dimensional Schrödinger equation can be reproduced: another bootstrap argument can be invoked, which does not involve Gagliardo–Nirenberg inequalities, since a useful *a priori* estimate for the envelope u is available.

5. Superposition

As explained in the introduction, the only difficulty in the proof of Theorem 1.11 is to treat a nonlinear interaction term. Indeed, we set

$$w^\varepsilon = \psi^\varepsilon - \varphi_1^\varepsilon\chi^1 - \varphi_2^\varepsilon\chi^2 + \varepsilon g^\varepsilon$$

where g^ε is the sum of two correction terms, similar to the one introduced in §1.3. More precisely, set $p(1) = 1$, and $p(2) = 1$ if $\tilde\lambda_1 = \tilde\lambda_2$, $p(2) = 2$ otherwise. Define

$g^\varepsilon = g_1^\varepsilon + g_2^\varepsilon$, with

$$g_1^\varepsilon = \sum_{1\leqslant j\leqslant P,\, j\neq p(1)} \sum_{1\leqslant \ell\leqslant d_j} g^\varepsilon_{j,1,\ell}(t,x)\chi_j^\ell(t,x),$$
$$g_2^\varepsilon = \sum_{1\leqslant j\leqslant P,\, j\neq p(2)} \sum_{1\leqslant \ell\leqslant d_j} g^\varepsilon_{j,2,\ell}(t,x)\chi_j^\ell(t,x),$$

where for $k=\{1,2\}$, $j\neq p(k)$ and $1\leqslant \ell\leqslant d_j$, the function $g^\varepsilon_{j,k,\ell}(t,x)$ solves the scalar Schrödinger equation

$$i\varepsilon\partial_t g^\varepsilon_{j,k,\ell} + \frac{\varepsilon^2}{2}\partial_x^2 g^\varepsilon_{j,k,\ell} - \lambda_j(x) g^\varepsilon_{j,k,\ell} = \varphi^\varepsilon r_{j,k,\ell} \quad ; \quad g^\varepsilon_{j,k,\ell|t=0}=0, \tag{5.1}$$

where

$$r_{j,k,\ell}(t,x) = -i\left(\partial_t\chi^k(t,x) + \xi_{p(k)}(t)\partial_x\chi^k(t,x)\ ,\ \chi_j^\ell(t,x)\right)_{\mathbf{C}^N}. \tag{5.2}$$

The function $w^\varepsilon(t)$ then solves

$$i\varepsilon\partial_t w^\varepsilon + \frac{\varepsilon^2}{2}\partial_x^2 w^\varepsilon = V(x)w^\varepsilon + \varepsilon NL^\varepsilon + \varepsilon L^\varepsilon \quad ; \quad w^\varepsilon_{|t=0}=0,$$

with

$$L^\varepsilon = \mathcal{O}(\sqrt{\varepsilon}\mathrm{e}^{Ct}) + \sum_{k=1,2}\sum_{\substack{1\leqslant j\leqslant P\\ j\neq p(k)}}\sum_{1\leqslant\ell\leqslant d_j}\left[\frac{\varepsilon^2}{2}\partial_x^2,\chi_j^\ell\right] g^\varepsilon_{j,k,\ell} = \mathcal{O}(\sqrt{\varepsilon}\mathrm{e}^{Ct}).$$

Here, the $\mathcal{O}(\sqrt{\varepsilon}\mathrm{e}^{Ct})$ holds in Σ^1_ε, from Proposition 3.2. Besides,

$$NL^\varepsilon = \sqrt{\varepsilon}\Big(\left|w^\varepsilon + \varphi_1^\varepsilon\chi^1 + \varphi_2^\varepsilon\chi^2 + \varepsilon g^\varepsilon\right|^2\left(w^\varepsilon + \varphi_1^\varepsilon\chi^1 + \varphi_2^\varepsilon\chi^2 + \varepsilon g_\varepsilon\right) - |\varphi_1^\varepsilon|^2\varphi_1^\varepsilon\chi^1 - |\varphi_2^\varepsilon|^2\varphi_2^\varepsilon\chi^2\Big)$$

Adding and subtracting the term $\sqrt{\varepsilon}|\varphi_1^\varepsilon\chi^1+\varphi_2^\varepsilon\chi^2|^2(\varphi_1^\varepsilon\chi^1+\varphi_2^\varepsilon\chi^2)$, we have

$$|NL^\varepsilon| \leqslant N_S^\varepsilon + N_I^\varepsilon,$$

where we have the pointwise estimates

$$N_I^\varepsilon \lesssim \sqrt{\varepsilon}\left(|\varphi_1^\varepsilon|^2|\varphi_2^\varepsilon| + |\varphi_2^\varepsilon|^2|\varphi_1^\varepsilon|\right),$$
$$N_S^\varepsilon \lesssim \sqrt{\varepsilon}\left(|\varphi_1^\varepsilon|^2 + |\varphi_2^\varepsilon|^2 + |w^\varepsilon|^2 + \varepsilon^2|g^\varepsilon|^2\right)\left(|w^\varepsilon| + \varepsilon|g^\varepsilon|\right).$$

The semilinear term N_S^ε can be treated exactly in the same manner as in Section 4. It remains to analyze $\displaystyle\int_0^t \|NL_I^\varepsilon(s)\|_{\Sigma^1_\varepsilon}\,ds$. We observe

$$\sqrt{\varepsilon}\int_0^t \left\||\varphi_1^\varepsilon(s)|^2\varphi_2^\varepsilon(s)\right\|_{L^2}ds = \int_0^t\left\|\left|u_1\left(s, y-\frac{x_1(s)-x_2(s)}{\sqrt{\varepsilon}}\right)\right|^2 u_2(s,y)\right\|_{L^2}ds,$$

and we note that the contribution of $|\varphi_1^\varepsilon|^2\varphi_2^\varepsilon$ and that of $|\varphi_2^\varepsilon|^2\varphi_1^\varepsilon$ play the same role. Also, we leave out the other terms which are needed in view of a Σ^1_ε estimate, since they create no trouble. Arguing as in [4, Lemma 6.1], we obtain:

Lemma 5.1. *Let $T\in\mathbf{R}$, $0<\gamma<1/2$ and*

$$I^\varepsilon(T) = \{t\in[0,T],\ \ |x_1(t)-x_2(t)|\leqslant\varepsilon^\gamma\}.$$

Then, for all $k \in \mathbf{N}$, there exists a constant C_k such that

$$\int_0^T \|NL_I^\varepsilon(t)\|_{\Sigma_\varepsilon^1} dt \lesssim (M_{k+2}(T))^3 \left(T\varepsilon^{k(1/2-\gamma)} + |I^\varepsilon(T)|\right) \mathrm{e}^{C_k T},$$

with

$$M_k(T) = \sup \left\{ \| \langle x \rangle^\alpha \partial_x^\beta u_j \|_{L^\infty([0,T],L^2(\mathbf{R}))};\ j \in \{1,2\}, \quad \alpha+\beta \leqslant k \right\}.$$

In view of this lemma and of Equation (1.12), we obtain

$$\int_0^T \|NL_I^\varepsilon(t)\|_{\Sigma_\varepsilon^1} dt \lesssim \mathrm{e}^{CT} \left(T\varepsilon^{k(1/2-\gamma)} + |I^\varepsilon(T)|\right),$$

and the next lemma yields the conclusion.

Lemma 5.2. *Set*

$$\Gamma = \inf_{x\in\mathbf{R}} \left| \tilde{\lambda}_1(x) - \tilde{\lambda}_2(x) - (E_1 - E_2) \right|,$$

and suppose $\Gamma > 0$. Then for $0 < \gamma < 1/2$, there exists $C_0, C_1 > 0$ such that

$$|I^\varepsilon(t)| \lesssim \varepsilon^\gamma \Gamma^{-2} \mathrm{e}^{C_0 t}, \quad 0 \leqslant t \leqslant C_1 \log\left(\frac{1}{\varepsilon}\right).$$

Proof. Consider $J^\varepsilon(t)$ an interval of maximal length included in $I^\varepsilon(t)$, and $N^\varepsilon(t)$ the number of such intervals. The result comes from the estimate

$$|I^\varepsilon(t)| \leqslant N^\varepsilon(t) \times \max |J^\varepsilon(t)|,$$

with

$$|J^\varepsilon(t)| \lesssim \varepsilon^\gamma e^{Ct}\Gamma^{-1} \text{ and } N^\varepsilon(t) \lesssim te^{Ct}\Gamma^{-1}, \tag{5.3}$$

provided that $\varepsilon^\gamma e^{Ct} \ll 1$. Let us prove the first property: consider $\tau, \sigma \in J^\varepsilon(t)$. There exists $t^* \in [\tau, \sigma]$ such that

$$|(x_1(\tau) - x_2(\tau)) - (x_1(\sigma) - x_2(\sigma))| = |\tau - \sigma| \, |\xi_1(t^*) - \xi_2(t^*)|,$$

whence

$$|\tau - \sigma| \leqslant |\xi_1(t^*) - \xi_2(t^*)|^{-1} \times 2\varepsilon^\gamma.$$

On the other hand,

$$|\xi_1(t^*) - \xi_2(t^*)| \geqslant ||\xi_1(t^*)| - |\xi_2(t^*)|| \geqslant \frac{\left||\xi_1(t^*)|^2 - |\xi_2(t^*)|^2\right|}{|\xi_1(t^*)| + |\xi_2(t^*)|}.$$

We use

$$|\xi_1(t^*)| + |\xi_2(t^*)| \lesssim e^{Ct},$$
$$|\xi_1(t^*)|^2 - |\xi_2(t^*)|^2 = 2\left(E_1 - E_2 - \tilde{\lambda}_1(x_1(t^*)) + \tilde{\lambda}_2(x_2(t^*))\right),$$

and infer

$$\begin{aligned}\left|E_1 - E_2 - \tilde{\lambda}_1(x_1(t^*)) + \tilde{\lambda}_2(x_2(t^*))\right| &\geqslant \left|E_1 - E_2 - \tilde{\lambda}_1(x_1(t^*)) + \tilde{\lambda}_2(x_1(t^*))\right| \\ &\quad - \left|\tilde{\lambda}_2(x_1(t^*)) + \tilde{\lambda}_2(x_2(t^*))\right| \\ &\geqslant \Gamma - C\varepsilon^\gamma \mathrm{e}^{Ct},\end{aligned}$$

where we have used the fact that $\tilde\lambda_2$ is at most quadratic. Therefore, if $\varepsilon^\gamma \mathrm{e}^{Ct}$ is sufficiently small,

$$\left|E_1 - E_2 - \tilde\lambda_1(x_1(t^*)) + \tilde\lambda_2(x_2(t^*))\right| \geqslant \frac{\Gamma}{2}.$$

We infer

$$|\tau-\sigma| \lesssim \varepsilon^\gamma \mathrm{e}^{Ct}\Gamma^{-1},$$

provided $\varepsilon^\gamma e^{Ct} \ll 1$.

Let us now consider $N^\varepsilon(t)$. We use that as t is large, $N^\varepsilon(t)$ is comparable to the number of distinct intervals of maximal size where $|x_1(t)-x_2(t)| \geqslant \varepsilon^\gamma$. More precisely, $N^\varepsilon(t)$ is smaller than t divided by the minimal size of these intervals. Therefore, we consider one interval $]\tau,\sigma[$ of this type and we look for lower bound of $\sigma-\tau$. We have

$$|x_1(\tau)-x_2(\tau)| = |x_1(\sigma)-x_2(\sigma)| = \varepsilon^\gamma, \text{ and } \forall t\in[\tau,\sigma], \quad |x_1(t)-x_2(t)| \geqslant \varepsilon^\gamma.$$

Besides, inside $]\tau,\sigma[$, $x_1(t)-x_2(t)$ has a constant sign that we can suppose to be $+$ (one argues similarly if it is $-$). Under this assumption, we have

$$\xi_1(\tau)-\xi_2(\tau) > 0 \text{ and } \xi_1(\sigma)-\xi_2(\sigma) < 0.$$

Using the exponential control of $\lambda_j'(x_j(t))$ for $j\in\{1,2\}$, we obtain

$$(\xi_1(\tau)-\xi_2(\tau)) - (\xi_1(\sigma)-\xi_2(\sigma)) \lesssim \mathrm{e}^{Ct}(\sigma-\tau). \tag{5.4}$$

We write

$$\begin{aligned}\xi_1(\tau)-\xi_2(\tau) = |\xi_1(\tau)-\xi_2(\tau)| &\geqslant \frac{\left||\xi_1(\tau)|^2-|\xi_2(\tau)|^2\right|}{|\xi_1(\tau)|+|\xi_2(\tau)|}\\ &\gtrsim e^{-Ct}\left||\xi_1(\tau)|^2-|\xi_2(\tau)|^2\right|\end{aligned} \tag{5.5}$$

and

$$-\xi_1(\sigma)+\xi_2(\sigma) = |\xi_1(\tau)-\xi_2(\tau)| \gtrsim \mathrm{e}^{-Ct}\left||\xi_1(\sigma)|^2-|\xi_2(\sigma)|^2\right|. \tag{5.6}$$

As before, we prove

$$\left||\xi_1(\tau)|^2-|\xi_2(\tau)|^2\right| + \left||\xi_1(\sigma)|^2-|\xi_2(\sigma)|^2\right| \gtrsim \Gamma,$$

provided that $\varepsilon^\gamma \mathrm{e}^{Ct} \ll 1$. Therefore, plugging the latter equation, (5.5) and (5.6) into (5.4), we obtain

$$\sigma-\tau \gtrsim e^{-2Ct}\Gamma \text{ thus } N^\varepsilon(t) \lesssim te^{Ct}\Gamma^{-1} \lesssim \mathrm{e}^{Ct}\Gamma^{-1}$$

which completes the proof of Theorem 1.11. □

Remark 5.3. The proof shows that if the approximation of Theorem 1.9 is proven to be valid on some time interval $[0, C\log(1/\varepsilon)]$, then Theorem 1.11 will also be valid on a time interval of the same form.

Appendix A. Global existence of the exact solution

The proof of Lemma 1.3 follows classical arguments; see [15] (or [5]) for more details. We suppose $\varepsilon = 1$ without loss of generality. We use the decomposition $V(x) = D(x) + W(x)$ of Assumption 1.2 and we denote by $U(t)$ the unitary propagator of $-\frac{1}{2}\partial_x^2 + D(x)$. Let X_T be the set

$$X_T = \left\{\psi \in \mathcal{C}(I_T, \Sigma_1^1),\ \psi, x\psi, \nabla\psi \in L^8(I_T, L^4(\mathbf{R}, \mathbf{C}^N))\right\},\quad I_T =]s-T, s+T[$$

for $s \in \mathbf{R}$ and $T \in \mathbf{R}$ to be fixed later. The proof consists in a fixed point argument for the function

$$\Phi_s : \psi \mapsto \Phi_s(\psi)$$

where for $s \in \mathbf{R}$, the function $\Phi_s(\psi)$ is defined by

$$\Phi_s(\psi)(t) = U(t-s)\psi(s) - i\Lambda \int_s^t U(t-\tau)\left(|\psi|^2_{\mathbf{C}^N}\psi\right)(\tau)d\tau - i\int_s^\tau U(t-\tau)\left(W\psi(\tau)\right)d\tau.$$

By [6], local in time Strichartz estimates are available for U. Strichartz estimates and Hölder inequality imply that there exists a constant $C > 0$ such that

$$\begin{aligned}\|\Phi_s(\psi)\|_{L^8(I_T,L^4)\cap L^\infty(I_T,L^2)} \leqslant C\|\psi(s)\|_{L^2} + C\|\psi\|^2_{L^{8/3}(I_T,L^4)}\|\psi\|_{L^8(I_T,L^4)}\\ + C\|W\psi\|_{L^1(I_T,L^2)}.\end{aligned}$$

Using the boundedness of the coefficients of W and Hölder inequality in time, we obtain

$$\|\Phi_s(\psi)\|_{L^8(I_T,L^4)\cap L^\infty(I_T,L^2)} \leqslant C\|\psi(s)\|_{L^2} + C\sqrt{T}\|\psi\|^3_{L^8(I_T,L^4)} + CT\|\psi\|_{L^\infty(I_T,L^2)}.$$

We can then infer that Φ_s is a contraction on a ball of X_T for some T which depends only on $\|\psi(s)\|_{L^2}$. Then, the conservation of $\|\psi(t)\|_{L^2}$ yields the lemma.

Appendix B. Some formulas involving the projectors

In this section, we list and prove some formulas which will be used in the course of the computations in the next appendix. We consider here the more general case $x \in \mathbf{R}^d$, with $d \geqslant 1$. Fix once and for all in this paragraph $j \in \{1, \dots, P\}$ and $\ell \in \{1, \dots, d\}$. First, recall that we have seen in §2 that since $\Pi_j^2 = \Pi_j$,

$$\Pi_j\left(\partial_\ell \Pi_j\right)\Pi_j = 0. \tag{B.1}$$

Differentiating the relation $\Pi_j^2 = \Pi_j$, we find: $\forall j \in \{1, \dots, P\},\ \forall \ell \in \{1, \dots, d\}$,

$$\partial_\ell \Pi_j = (\partial_\ell \Pi_j)\Pi_j + \Pi_j(\partial_\ell \Pi_j). \tag{B.2}$$

We now show: $\forall j \in \{1, \dots, P\},\ \forall \ell \in \{1, \dots, d\}$,

$$\begin{aligned}\partial_\ell \Pi_j &= \sum_{k\neq j}\left(\Pi_k(\partial_\ell \Pi_j)\Pi_j + \Pi_j(\partial_\ell \Pi_j)\Pi_k\right)\\ &= \sum_{1\leqslant k\leqslant P}\left(\Pi_k(\partial_\ell \Pi_j)\Pi_j + \Pi_j(\partial_\ell \Pi_j)\Pi_k\right),\end{aligned} \tag{B.3}$$

where the last equality stems from (B.1). To prove (B.3), simply write

$$\partial_\ell \Pi_j = \sum_{k,m}\Pi_k(\partial_\ell \Pi_j)\Pi_m$$

where we have used $\sum_k \Pi_k = \mathrm{Id}$. Then, observe that $\Pi_k\Pi_j = \delta_{jk}\Pi_j$ yields

$$\Pi_k(\partial_\ell \Pi_j) + (\partial_\ell \Pi_k)\Pi_j \ \text{ whence }\ \Pi_k(\partial_\ell \Pi_j) = -(\partial_\ell \Pi_k)\Pi_j.$$

The fact that $\Pi_j\Pi_m = 0$ for all $m \neq j$ gives (B.3).

The last formulas we wish to establish involve the spectral gap. Since we have a basis of eigenfunctions, we have

$$V\Pi_j = \Pi_j V = \lambda_j \Pi_j.$$

Differentiating with respect to x_ℓ, we infer

$$(\partial_\ell \Pi_j)V + \Pi_j \partial_\ell V = \lambda_j \partial_\ell \Pi_j + (\partial_\ell \lambda_j)\Pi_j.$$

For $k \in \{1, \dots, P\}$, multiply this relation by Π_k on the right, and use the property $V\Pi_k = \lambda_k \Pi_k$:

$$\lambda_k(\partial_\ell \Pi_j)\Pi_k + \Pi_j\left(\partial_\ell V - \partial_\ell \lambda_j\right)\Pi_k = \lambda_j(\partial_\ell \Pi_j)\Pi_k,$$

hence

$$(\lambda_j - \lambda_k)(\partial_\ell \Pi_j)\Pi_k = \Pi_j\left(\partial_\ell V - \partial_\ell \lambda_j\right)\Pi_k. \tag{B.4}$$

Similarly, we have

$$(\lambda_j - \lambda_k)\Pi_k(\partial_\ell \Pi_j) = \Pi_k\left(\partial_\ell V - \partial_\ell \lambda_j\right)\Pi_j. \tag{B.5}$$

Appendix C. About the growth of the eigenvectors at infinity

This section is devoted to the proof of estimates at infinity for the eigenprojectors associated with a potential V satisfying Assumption 1.2. We will use a lemma on the derivatives of the inverse of the gap between two different eigenvalues. For $j, k \in \{1, \dots, P\}$, $j \neq k$, we recall that we have set (see (3.4))

$$\forall x \in \mathbf{R}, \;\; \gamma_{j,k}(x) = (\lambda_j(x) - \lambda_k(x))^{-1}.$$

Since the results are not specific to the space dimension one, we prove them for potentials depending on $x \in \mathbf{R}^d$, $d \geqslant 1$.

Lemma C.1. *Assume* (1.2) *is satisfied with $n_0 \in \mathbf{N}$ and that the functions V and λ_j ($j \in \{1, \cdots, P\}$) are at most quadratic. Then, for $\beta \in \mathbf{N}^d$ and for $j, k \in \{1, \dots, P\}$ with $j \neq k$,*

$$\left|\partial_x^\beta \gamma_{j,k}(x)\right| \;\; \lesssim \;\; \langle x \rangle^{n_0 + |\beta|(1+n_0)}. \tag{C.1}$$

Proof. For $\beta = \mathbf{1}_\ell$, we immediately obtain

$$|\partial_\ell \gamma_{j,k}(x)| = \left|\frac{\partial_\ell\left(\lambda_j(x) - \lambda_k(x)\right)}{\left(\lambda_j(x) - \lambda_k(x)\right)^2}\right| \lesssim \langle x \rangle^{1+2n_0},$$

from (1.2), and the fact that λ_j and λ_k are at most quadratic.

Set $\Lambda_{j,k} = \lambda_j - \lambda_k$: it is at most quadratic. Besides, for $\beta \in \mathbf{N}^d$, we have

$$\partial_x^\beta\left(\gamma_{j,k}\right) = \sum_{\substack{\alpha_1 + \cdots + \alpha_p = \beta \\ |\alpha_\ell| \geqslant 1, \;\, p \leqslant |\beta|}} a_{\alpha_1, \dots, \alpha_p} \Lambda_{j,k}^{-1-p} \partial_x^{\alpha_1}\Lambda_{j,k} \cdots \partial_x^{\alpha_p}\Lambda_{j,k}$$

for some real-numbers $a_{\alpha_1, \dots, \alpha_p}$. The result then follows by observing that

$$\left|\Lambda_{j,k}^{-1-p} \partial_x^{\alpha_1}\Lambda_{j,k} \cdots \partial_x^{\alpha_p}\Lambda_{j,k}\right| \lesssim \langle x \rangle^{n_0(1+p)} \langle x \rangle^{p},$$

from (1.2), and the property $|\partial_x^\alpha \Lambda_{j,k}| \lesssim \langle x \rangle^{(2-|\alpha|)_+}$, which follows from the fact that $\Lambda_{j,k}$ is at most quadratic, in the sense of Definition 1.1. $\square$

We now consider the eigenprojectors Π_j associated with the eigenvalues λ_j of the matrix V. Because of the gap condition, these functions are smooth in $\mathbf{R}^d$. We prove the following

Lemma C.2. *Let Π_j be an eigenprojector of V for $j \in \{1, \dots, P\}$, we have for $\beta \in \mathbf{N}^d$*

$$|\partial_x^\beta \Pi|_{\mathbf{C}^{N,N}} \quad \lesssim \quad \langle x \rangle^{|\beta|(1+n_0)}, \tag{C.2}$$

where the norm $|\cdot|_{\mathbf{C}^{N,N}}$ denotes the matricial norm.

Proof. The case $|\beta| = 0$ is immediate since Π_j is a projector. In view of (B.3), relations (B.5) and (B.4) imply (C.2) for $|\beta| = 1$.

We now argue by induction. We suppose that (C.2) holds for any $\gamma \in \mathbf{N}^d$ with $|\gamma| = K$ for some $K \in \mathbf{N}$ and we consider β with $|\beta| = K + 1$ and $\beta_\ell \neq 0$. Differentiation of order $\beta - \mathbf{1}_\ell$ of (B.2) and multiplication on both sides by Π_j yields

$$\Pi_j(\partial_x^\beta \Pi_j)\Pi_j = \Pi_j \left(\sum_{0<|\alpha|<|\beta|} a_\alpha \partial_x^\alpha \Pi_j \partial_x^{\beta-\alpha} \Pi_j \right) \Pi_j,$$

where all along this proof, a_α will denote real numbers whose exact value is unimportant. We obtain

$$\left| \Pi_j(\partial_x^\beta \Pi_j)\Pi_j \right|_{\mathbf{C}^{N,N}} \lesssim \langle x \rangle^{|\beta|(1+n_0)} . \tag{C.3}$$

Then, for all $k \neq j$, we estimate $(\partial_x^\beta \Pi_j)\Pi_k$. To do so, we differentiate (B.4) and get

$$\begin{aligned}(\partial_x^\beta \Pi_j)\Pi_k = &\sum_{0<|\alpha|<|\beta|} a_\alpha \partial_x^\alpha \Pi_j \partial_x^{\beta-\alpha} \Pi_k \\ &+ \sum_{\alpha_1+\dots+\alpha_4=\beta-\mathbf{1}_\ell} b_{\alpha_1,\dots,\alpha_4} \partial_x^{\alpha_1} \left((\lambda_j - \lambda_k)^{-1} \right) \partial_x^{\alpha_2} \Pi_j \partial_x^{\alpha_3} \partial_\ell \left(V - \lambda_j \right) \partial_x^{\alpha_4} \Pi_k.\end{aligned}$$

In the first sum, the induction assumption yields

$$\left\| \partial_x^\alpha \Pi_j \partial_x^{\beta-\alpha} \Pi_k \right\|_{\mathbf{C}^{N,N}} \lesssim \langle x \rangle^{|\alpha|(1+n_0)+|\beta-\alpha|(1+n_0)} = \langle x \rangle^{|\beta|(1+n_0)} \tag{C.4}$$

Besides, for each term in the second sum, we write

$$\begin{aligned}&\left\| \partial_x^{\alpha_1} \left((\lambda_j - \lambda_k)^{-1} \right) \partial_x^{\alpha_2} \Pi_j \partial_x^{\alpha_3} \partial_{x_i} \left(V - \lambda_j \right) \partial_x^{\alpha_4} \Pi_k \right\|_{\mathbf{C}^{N,N}} \\ &\qquad \lesssim \langle x \rangle^{n_0+|\alpha_1|(1+n_0)} \langle x \rangle^{(1-|\alpha_3|)_+} \langle x \rangle^{(1+n_0)(|\alpha_2|+|\alpha_4|)}\end{aligned}$$

where $r_+ = \max(r, 0)$ and where we have used the fact that V and λ_j are at most quadratic, together with the induction assumption and Lemma C.1. We have the two alternatives:

- If $\alpha_3 = 0$, then

$$\begin{aligned}n_0 + |\alpha_1|(1 + n_0) + (1 - |\alpha_3|)_+ + (1 + n_0)(|\alpha_2| + |\alpha_4|) \\ = n_0 + |\alpha_1|(1 + n_0) + 1 + (1 + n_0)(|\alpha_2| + |\alpha_4|) \\ = (1 + n_0)(1 + |\alpha_1| + |\alpha_2| + |\alpha_3|) = (1 + n_0)|\beta|,\end{aligned}$$

  since $\alpha_1 + \alpha_2 + \alpha_4 = \beta - \mathbf{1}_\ell$.

- If $\alpha_3 \neq 0$, then

$$\begin{aligned} n_0 + |\alpha_1|(1+n_0) + (1-|\alpha_3|)_+ + (1+n_0)(|\alpha_2|+|\alpha_4|) \\ &= n_0 + (1+n_0)(|\alpha_1|+|\alpha_2|+|\alpha_4|) \\ &\leqslant (1+n_0)(1+|\alpha_1|+|\alpha_2|+|\alpha_4|) \\ &\leqslant (1+n_0)|\beta|. \end{aligned}$$

We deduce

$$\left\|(\partial_x^\beta \Pi_j)\Pi_k\right\|_{\mathbf{C}^{N,N}} \lesssim \langle x\rangle^{|\beta|(1+n_0)}, \quad \forall k \neq j. \tag{C.5}$$

Similarly,

$$\left\|\Pi_k(\partial_x^\beta \Pi_j)\right\|_{\mathbf{C}^{N,N}} \lesssim \langle x\rangle^{|\beta|(1+n_0)}, \quad \forall k \neq j. \tag{C.6}$$

In view of (C.3), we infer

$$\left\|\Pi_j(\partial_x^\beta \Pi_j)\Pi_k\right\|_{\mathbf{C}^{N,N}} + \left\|\Pi_k(\partial_x^\beta \Pi_j)\Pi_j\right\|_{\mathbf{C}^{N,N}} \lesssim \langle x\rangle^{|\beta|(1+n_0)}, \quad \forall j,k. \tag{C.7}$$

Applying the operator $\partial_x^{\beta-\mathbf{1}_\ell}$ to (B.3), the induction assumption and equations (C.5), (C.6) and (C.7) yield (C.2), which concludes the induction. □

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