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Abstract

We study self-similar ultrametric Cantor sets arising from stationary Bratteli diagrams. We prove that such a Cantor set C is bi-Lipschitz embeddable in $\mathbb{R}^{[\dim_H(C)]+1}$, where $[\dim_H(C)]$ denotes the integer part of its Hausdorff dimension. We compute this Hausdorff dimension explicitly and show that it is the abscissa of convergence of a *zeta*-function associated with a natural sequence of refining coverings of C (given by the Bratteli diagram). As a corollary we prove that the transversal of a (primitive) substitution tiling of $\mathbb{R}^d$ is bi-Lipschitz embeddable in $\mathbb{R}^{d+1}$.

We also show that C is bi-Hölder embeddable in the real line. The image of C in $\mathbb{R}$ turns out to be the ω-spectrum (the limit points of the set of eigenvalues) of a Laplacian on C introduced by Pearson-Bellissard *via* noncommutative geometry.

1. Introduction and summary of the results

In this article, we study self-similar ultrametric Cantor sets, and their embeddings in Euclidean spaces. A prototype of such a space, and our motivating example, is the canonical transversal of a substitution tiling space. The main motivation for this work was to derive embedding results for tilings tranversals. We refer the reader to [1, 14, 9], as well as to [10] Section 2, for basic notions about substitution tilings and tiling spaces. Bratteli diagrams are important tools for the study of substitution tilings, and we use some of their techniques here.

Ultrametrics are very natural for tilings transversals as well as totally disconnected spaces in general. The condition for a metric to be a ultrametric (see Equation (1.1)) is a very strong requirement: for instance if two ultrametric balls intersect then one contains the other. Our techniques, and results, are not transferable *as is* to the study of general metric Cantor sets. The proofs of our embedding theorems are fairly elementary in comparison to what is known for metric spaces. The reader can compare for example with Assouad theorem for metric spaces with the doubling property (see [4] section 8.1).

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Ultrametric Cantor sets and Bratteli diagrams. A *ultrametric Cantor set* (C, ρ) is a compact, Hausdorff, perfect (no isolated points), and totally disconnected space C, equipped with a metric ρ which satisfies a stronger form of triangle inequality, namely

$$\forall z \in C\,, \quad \rho(x,y) \le \max\{\rho(x,z), \rho(z,y)\} \tag{1.1}$$

By Michon's theorem [12] there is an isometric equivalence between ultrametric Cantor sets and weighted Cantorian trees. Given such a Cantor set, the ultrametric allows to define a sequence of refining partitions by clopen sets (closed and open sets) which gives rise to a Cantorian weighted tree: its vertices represent the clopen sets and the weight encode their diameters. Conversely, any such weighted Cantorian tree defines a ultrametric Cantor set.

The authors showed in [10] that one can equivalently use *Bratteli diagrams* (Definition 2.1) instead of Cantorian trees: paths in the diagram (like vertices of the graph) encode the clopen sets. The formalism with Bratteli diagrams turns out to be very handy for *self-similar* Cantors which correspond to *stationary* Bratteli diagrams. Such a self-similar Cantor set can be viewed as an iterated function system (IFS) directed by the graph giving the Bratteli diagram (see Remark 2.7).

The zeta-function. Consider a self-similar ultrametric Cantor set (C, ρ). Its Bratteli diagram naturally defines a sequence of refining partitions by clopens sets $(\Pi_n)_{n\in\mathbb{N}}$. For each n, Π_n is viewed as a set of paths γ of length n in the diagram, whose corresponding clopen sets $[\gamma]$ partition C. There is a natural *zeta*-function associated with (C, ρ) (Definition 2.10), namely the complex power series:

$$\zeta(s) = \sum_{n=0}^{\infty} \sum_{\gamma\in\Pi_n} \mathrm{diam}_\rho([\gamma])^s\,.$$

When it exists, its *abscissa of convergence* is the real number $s_0 > 0$ such that $\zeta(s)$ is holomorphic in the half-plane $\Re(s) > s_0$, and singular at s_0. This function is exactly the *zeta*-function of the spectral triple that Pearson and Bellissard first defined for ultrametric Cantor sets [13, 10]. In noncommutative geometry (NCG) [6], this abscissa of convergence is interpreted as the dimension of the noncommutative space (here the C^*-algebra of continuous functions on C with the sup-norm). In Lemma 2.11, we show that $s_0 < +\infty$, and identify s_0 explicitly in terms of the contractions of the IFS. We further prove the following.

Theorem 2.12. *The Hausdorff dimension of a self-similar ultrametric Cantor set is equal to the abscissa of convergence of its zeta-function.*

As a corollary of this and previous results in [10] we compute the Hausdorff dimension of the transversal of a (primitive) substitution tiling of $\mathbb{R}^d$.

Theorem 4.1. *Consider a primitive substitution tiling of $\mathbb{R}^d$, with canonical transversal Ξ. The abscissa of convergence s_0 of the zeta-function of Ξ is equal to its Hausdorff dimension $\dim_H(\Xi)$, and moreover one has*

$$s_0 = \dim_H(\Xi) = d\,.$$

Embeddings and the Pearson-Bellissard's Laplacians. We prove several embedding results for self-similar ultrametric Cantor sets: Lipschitz embedding in $\mathbb{R}^n$, and Hölder embedding in $\mathbb{R}$. Our main result is the following.

Theorem 3.3. *Let (C, ρ) be a self-similar ultrametric Cantor set. There exists a bi-Lipschitz embedding*

$$C \hookrightarrow \mathbb{R}^{[\dim_H(C)]+1},$$

where $\dim_H(C)$ is the Hausdorff dimension of C, and $[\dim_H(C)]$ denotes its integer part.

Part of the technicalities in the proof of this (Theorem 3.1) are inspired by a result of E. Christensen and C. Ivan in [5] (Theorem 4.10 in Section 4.3). As a corollary of this and Theorem 4.1, we prove an embedding of tilings transversals.

Theorem 3.4. *The transversal of a substitution tiling of $\mathbb{R}^d$ is bi-Lipschitz embeddable in $\mathbb{R}^{d+1}$.*

We also prove a Hölder embedding on the real line.

Theorem 5.1. *A self-similar ultrametric Cantor set is bi-Hölder embeddable in the real line.*

This is interesting in the case of the transversal Ξ of a substitution tiling space of $\mathbb{R}^d$. The Pearson-Bellissard spectral triple allows to define a one-parameter family $(\Delta_s)_{s\in\mathbb{R}}$ of Laplace-Beltrami-like operators on Ξ (as "squares" of the Dirac operator). Those operators were introduced in [13] and studied in details in [10]. Under some techniqueal but fairly general assumptions (Remark 4.3) we prove that their spectra "contains" the transversal as follows.

Corollary 5.4. *For all s greater than $2(d+1)$, the ω-spectrum of Δ_s (the limit points of its pure point spectrum) is bi-Hölder homeomorphic to Ξ.*

2. Self-similar Cantor sets and stationary Bratteli diagrams

Bratteli diagrams were first introduced in [3] to classify *AF* C^*-algebras. They provide also a handy formalism for substitutions [8, 7, 10], and tilings in general [2].

Definition 2.1. A *Bratteli diagram* is an infinite oriented graph $\mathcal{B} = (E, V)$ where the sets of edges E and vertices V are given by the disjoint unions

$$E = \coprod_{n\in\mathbb{N}\cup\{0\}} E_n\,, \qquad V = \coprod_{n\in\mathbb{N}\cup\{0\}} V_n\,,$$

where E_n and V_n are finite sets. $V_0 = \{\circ\}$ is called the root of the diagram, and E_0 contains exactly one edge e_v for each $v \in V_0$, which links $\circ$ to v. The range and source maps, which define adjacency, ar maps $r : E_n \to V_{n+1}$, $s : E_n \to V_n$. The set E_n is the set of edges of *level* n.

A *path* is a sequence $\gamma = (e_0, e_1, e_2, \dots)$ of composable edges: $e_i \in E_i$, $r(e_i) = s(e_{i+1})$. One lets Π_n denote the sets of paths down to level n, and for $\gamma \in \Pi_n$ one calls $|\gamma| = n$ its *length*[2]. Also, one sets $\Pi = \bigcup_n \Pi_n$ for the set of finite paths, and Π_∞ for the set of infinite paths (always starting from the root). For $\gamma \in \Pi_m$, $m \in \mathbb{N} \cup \{+\infty\}$, and $1 \le n < m$, one sets $\gamma_{[0,n]}$ for the restriction of γ to Π_n. One extends the range map to Π as follows: if $\gamma = (e_0, e_1, \dots e_n) \in \Pi_n$ then $r(\gamma) = r(e_n)$.

The diagram is *stationary* if the sets E_n and V_n are respectively pairwise isomorphic for all $n \ge 1$. One then writes $E_n \cong \mathcal{E}$, $V_n \cong \mathcal{V}$, $n \ge 1$. In this case, the diagram is entirely determine by its *adjacency matrix*

$$A_{vv'} = \#\{e \in \mathcal{E} \,:\, s(e) = v\,, r(e) = v'\}\,,\ v, v' \in \mathcal{V}\,. \tag{2.2}$$

With the product topology of the E_n, $n \in \mathbb{N}$, the set Π_∞ is compact and totally disconnected. A base of clopen sets (closed and open sets) for this topology is given by the *cylinders*

$$[\gamma] = \{x \in \Pi_\infty \,:\, x_{[0,n]} = \gamma\}\,, \quad \gamma \in \Pi_n,\, n \in \mathbb{N}\,. \tag{2.3}$$

If Π_∞ has no isolated point, then it is a Cantor set. This is the case whenever the following condition is satisfied.

Hypothesis 2.2. *For any $n \in \mathbb{N}$ and any $v \in V_n$, there exists $m > n$ such that for any $v' \in V_m$ there is a path from v to v'.*

Definition 2.3. A *weight function* on a Bratteli diagram $\mathcal{B} = (E, V)$, is a function $w : \Pi \to (0, +\infty)$ which satisfies

(i) $w(\gamma) \le w(\gamma_{[0,m]})$ for any $\gamma \in \Pi_n$ and any $m < n$,

(ii) $\sup\{w(\gamma) \,:\, \gamma \in \Pi_n\} \to 0$ as $n \to +\infty$.

A *weighted Bratteli diagram*, is a Bratteli diagram equipped with a weight function.

A weight function w on a Bratteli diagram $\mathcal{B} = (E, V)$ defines a ultrametric ρ_w on Π_∞ as follows:

$$\mathrm{diam}_w([\gamma]) = w(\gamma)\,, \qquad \rho_w(x, y) = \begin{cases} w(x \wedge y) & \text{if} \quad x \ne y \\ 0 & \text{if} \quad x = y \end{cases} \tag{2.4}$$

where $x \wedge y$ is the longest common prefix of x and y.

In this manner any weighted Bratteli diagram satisfying Hypothesis 2.2. defines a ultrametric Cantor set. The converse is a result of Michon [12] (rephrased here in terms of Bratteli diagrams).

[2]Formally, $(e_0, e_1, \dots, e_n) \in \Pi_n$ is made of $n+1$ edges. However, by convention, we say its length is n

Theorem 2.4. *For any ultrametric Cantor set C, there exists a weighted Bratteli diagram satisfying Hypothesis 2.2, whose set of infinite path is isometric to C.*

One has the obvious following Lemma.

Lemma 2.5. *Let w, w' be two weights on a Bratteli diagram, and assume that there exist constants $0 < c_1 < c_2$ such that $c_1 w'(\gamma) \leq w(\gamma) \leq c_2 w'(\gamma)$, for all $\gamma \in \Pi$. Then the ultrametrics associated with w and w' are bi-Lipschitz equivalent, namely one has: $c_1 \rho_{w'} \leq \rho_w \leq c_2 \rho_{w'}$.*

Michon's theorem and Lemma 2.5 motivate the following definition.

Definition 2.6. A self-similar ultrametric Cantor set is the set Π_∞ of infinite paths in a stationary Bratteli diagram with ultrametric ρ given, for some $\alpha \in (0,1)$ and some family of positive numbers $(a_\gamma; \gamma \in \Pi)$, by:

$$\rho(x,y) = a_\gamma \alpha^{|x \wedge y|}\,, \qquad \text{and the condition} \qquad 0 < \inf_{\gamma \in \Pi} a_\gamma \leq \sup_{\gamma \in \Pi} a_\gamma < +\infty\,.$$

A self-similar ultrametric Cantor set is called *regular* if $a_\gamma = 1$ for all $\gamma \in \Pi$.

Remark 2.7. (i) In view of Lemma 2.5, any self-similar ultrametric Cantor set is bi-Lipschitz equivalent to a regular one. If one writes $a = \inf_\gamma a_\gamma$ and $a' = \sup_\gamma a_\gamma$, then the two constants c_1, c_2, in Lemma 2.5 can be taken to be a/a' and a'/a respectively.

(ii) A regular self-similar Cantor set as defined above can be seen as the invariant set of a graph directed IFS. The maps are in one-to-one correspondence with the edges in $\mathcal{E}$, and have all contraction factor α. The directing graph is the graph of adjacencies of $\mathcal{E}$, and its paths correspond to Π.

Example 2.8. Given a substitution on a finite alphabet $\mathcal{A}$ (that is a map which to each letter of $\mathcal{A}$ associates a finite word over $\mathcal{A}$), define the *Abelianization matrix*

$$A_{ij} = \text{ number of occurences of } j \text{ is the substitution of } i. \tag{2.5}$$

Then one can associate stationary Bratteli diagram with $\mathcal{V} \cong \mathcal{A}$ and adjacency matrix equal to A. Under some conditions like primitivity (see below) and border forcing, see [11], it turns out that there is a canonical map between Π_∞ and the subshift of $\mathcal{A}^{\mathbb{Z}}$ associated with the substitution.

A square matrix with non-negative entries is *primitive* if some power A^n has positive entries. When A is primitive, then Π_∞ satisfies Hypothesis 2.2 and is thus a Cantor set, and the the subshift of $\mathcal{A}^{\mathbb{Z}}$ associated with the substitution is minimal. Also, A has a Perron-Frobenius eigenvalue Λ: it is real, simple, and its modulus is maximal amongst the other eigenvalues. Here Λ is also greater than one, and is the dilation factor of the substitution. If one normalizes the Perron-Frobenius eigenvector, such that its coordinates add up to 1, the latter are the frequencies of the letters: $\nu_i, i \in \mathcal{V}$. A natural choice of ultrametric is then given by the following weights:

$$w(\gamma) = \nu_{r(\gamma)} \Lambda^{-|\gamma|}\,, \qquad i.e. \qquad a_\gamma = \nu_{r(\gamma)}\,, \quad \alpha = \Lambda^{-1}\,. \tag{2.6}$$

In the case of substitution tilings of $\mathbb{R}^d$, in order to recover the metric generally used on the canonical transversal Ξ of the tiling space, one can take the following

$$a_\gamma = (\nu_{r(\gamma)})^{1/d}\,, \quad \alpha = \Lambda^{-1/d}\,. \tag{2.7}$$

With such ultrametric, Π_∞ is bi-Lipschitz homeomorphic to Ξ (see [10], Section 2).

Given a Bratteli diagram $\mathcal{B} = (E, V)$, and an integer $k \geq 1$, one defines its k-th *telescoping* $\mathcal{B}^{(k)}$ as follows:

(i) $V_0^{(k)} = V_0$ and $E_0^{(k)} = E_0$;

(ii) $V_n^{(k)} \cong V_{1+(n-1)k}$;

(iii) $E_n^{(k)} \cong$ set of paths from $V_{1+(n-1)k}$ down to V_{1+nk}.

If the ultrametric of ρ on Π_∞ has parameter α, then one sees that the inherited ultrametric $\rho^{(k)}$ on $\Pi_\infty^{(k)}$ has parameter α^k. Also if A is the adjacency matrix of $\mathcal{B}$, then A^k is that of $\mathcal{B}^k$.

Lemma 2.9. *Let $\mathcal{B} = (E, V)$ be a stationary Bratteli diagram, and (Π_∞, ρ) its associated self-similar Cantor set, with regular ultrametric ρ of parameter α. Then (Π_∞, ρ) and $(\Pi_\infty^{(k)}, \rho^{(k)})$ are bi-Lipschitz equivalent for all $k \geq 1$, namely one has:*

$$\alpha^k \rho^{(k)} < \rho \leq \rho^{(k)}\,.$$

Proof. Let $x, y \in \Pi_\infty$ with $|x \wedge y| = n$ so that $\rho(x, y) = \alpha^n$. Write $n = qk + r$ with $0 \leq r < k$. We have $\rho^{(k)}(x, y) = (\alpha^k)^q$. Hence we get $\rho(x, y) = \alpha^{qk+r} \leq \alpha^{qk} = \rho^{(k)}(x, y)$ and $\alpha^k \rho^{(k)}(x, y) = \alpha^{k+(n-r)} < \alpha^n = \rho(x, y)$. This completes the proof. $\square$

Definition 2.10. The *zeta*-function of a self-similar utrametric Cantor set (Π_∞, ρ) is the (formal) complex series

$$\zeta(s) = \sum_{n=0}^{\infty} \sum_{\gamma \in \Pi_n} \mathrm{diam}_\rho([\gamma])^s\,.$$

When it exists, its *abscissa of convergence* is the real $s_0 > 0$ such that $\zeta(s)$ is holomorphic in the half-plane $\Re(s) > s_0$, and singular at s_0.

Lemma 2.11. *Let* (Π_∞, ρ) *be a self-similar ultrametric Cantor set. Let* α *be the parameter of* ρ*, and* Λ *the Perron-Frobenius eigenvalue of the adjacency matrix as in equation* (2.2)*. The abscissa of convergence of the zeta-function exists and is given by*

$$s_0 = \frac{\log(\Lambda)}{-\log(\alpha)} \,. \tag{2.8}$$

Proof. The *zeta*-function of (Π_∞, ρ) reads

$$\zeta(s) = \sum_{n=0}^{\infty} \sum_{\gamma \in \Pi_n} a_\gamma^s \, \alpha^{ns} \,,$$

where the a_γ are the parameters of ρ as in Definition 2.6. Set $c_1 = \inf_\gamma a_\gamma$ and $c_2 = \sup_\gamma a_\gamma$. For all $s > 0$ we have the inequalities

$$c_1^s \sum_{n=0}^{\infty} \#(\Pi_n) \, \alpha^{ns} \;\le\; \zeta(s) \;\le\; c_2^s \sum_{n=0}^{\infty} \#(\Pi_n) \, \alpha^{ns} \,.$$

Now $\#(\Pi_n)$ grows like Λ^n with n, that is, there are (uniform) constants $0 < c_1' < c_2'$ such that

$$c_1' \Lambda^n \le \#(\Pi_n) \le c_2' \Lambda^n \,.$$

Therefore one gets the inequalities

$$c_1^s c_1' \sum_{n=0}^{\infty} (\Lambda \alpha^s)^n \le \zeta(s) \le c_2^s c_2' \sum_{n=0}^{\infty} (\Lambda \alpha^s)^n \,,$$

and one sees that the power series is divergent for $\Re(s) = -\log(\Lambda)/\log(\alpha)$, and absolutely convergent for $\Re(s) > -\log(\Lambda)/\log(\alpha)$. $\square$

Theorem 2.12. *The Hausdorff dimension of a self-similar ultrametric Cantor set is equal to the abscissa of convergence of its zeta-function.*

Proof. Since the Hausdorff dimension is invariant under bi-Lipschitz homeomorphisms, one can assume that the self-similar Cantor set $C = (\Pi_\infty, \rho)$ is regular. Denote by α the parameter of ρ. We recall the definition of the Hausdorff dimension of C. For $\delta > 0, d \ge 0$ set

$$H_\delta^d(C) = \inf \Big\{ \sum_{i \in I} \mathrm{diam}(u_i)^d \;:\; (u_i)_{i \in I} \text{ is a covering by balls of diameters less than } \delta \Big\} \,, \tag{2.9}$$

and define the d-dimensional Hausdorff content of C to be

$$H^d(C) = \inf_\delta H_\delta^d(C) \,. \tag{2.10}$$

The Hausdorff dimension is given by the critical value of the Hausdorff content, namely:

$$\dim_H(C) = \inf\{d \,:\, H^d(C) = 0\} = \sup\{d \,:\, H^d(C) > 0\} . \tag{2.11}$$

Consider the case $d > s_0$ first. For any $n \in \mathbb{N}$ there is a clopen partition of C by the cylinders $[\gamma], \gamma \in \Pi_n$, so we have

$$H^d(C) \le H^d_{\alpha^n}(C) \le \sum_{\gamma\in\Pi_n} (\alpha^n)^d \le c\Lambda^n \alpha^{nd} = c\alpha^{n(d-s_0)} ,$$

where we used equation (2.8) for the last equality. This inequality holds for all n, and as n tends to infinity the term $\alpha^{n(d-s_0)}$ tends to zero. Hence $H^d(C) = 0$ for $d > s_0$, and this proves

$$\dim_H(C) \le s_0 . \tag{2.12}$$

Consider now the case $d < s_0$. Notice to begin with, that one can restrict to coverings that are finite clopen partitions of C, *i.e.* partitions by cylinders $[\gamma], \gamma \in \Pi$. This is because, first: as C is compact, one can consider finite coverings only, and second: since ρ is a ultrametric, if two balls intersect then one is contained in the other.

Further, we will consider the k-th telescope $C^{(k)} = (\Pi_\infty^{(k)}, \rho^{(k)})$ of C as in Lemma 2.9, for an integer $k \ge 1$ to be determined later. For ay $k \ge 1$, $C^{(k)}$ is bi-Lipschitz equivalent to C, so they have the same Hausdorff dimension:

$$\forall k \ge 1 , \quad \dim_H(C) = \dim_H(C^{(k)}) . \tag{2.13}$$

Consider a path $\gamma \in \Pi_n^{(k)}$, and let $\mathrm{ext}_1^{(k)}(\gamma)$ be the set of paths in $\Pi_{n+1}^{(k)}$ that extend γ: $\eta \in \mathrm{ext}_1^{(k)}(\gamma) \iff \eta_{[0,n]} = \gamma$. The cardinality of $\mathrm{ext}_1^{(k)}(\gamma)$ equals the number of edges in $\mathcal{B}^{(k)}$ that extend γ one generation further: it is given then by the sum of elements in a line of the k-th power of the adjacency matrix of $\mathcal{B}$, and behaves like Λ^k as k tends to infinity. More precisely, there exists a constant $c_{r(\gamma)} > 0$ (that only depends on the range vertex of γ) such that one has

$$\#\mathrm{ext}_1^{(k)}(\gamma) = c_{r(\gamma)}\Lambda^k + o(\Lambda^k) . \tag{2.14}$$

Since $\mathrm{diam}([\gamma]) = \alpha^{kn}$ we have

$$\begin{aligned} \sum_{\eta\in\mathrm{ext}_1^{(k)}(\gamma)} \mathrm{diam}([\eta])^d &= \#\mathrm{ext}_1^{(k)}(\gamma)\; (\alpha^{k(n+1)})^d \\ &= (c_{r(\gamma)}\Lambda^k + o(\Lambda^k))\alpha^{kd}\, \mathrm{diam}([\gamma])^d , \end{aligned}$$

where the last equality holds by equation (2.14). Now, for $d < s_0$, equation (2.8) implies that $\Lambda\alpha^d > 1$. So there exists k_1 such that for all $k \ge k_1$ the term $(c_{r(\gamma)}\Lambda^k + o(\Lambda^k))\alpha^{kd}$ in the previous equation to be strictly greater than 1. And since the constant $c_{r(\gamma)}$ only depends on the range vertex of γ, and

$\#\mathcal{V}^{(k)} = \#\mathcal{V}$ does not depend on k, there exists $k_2 \geq k_1$, such that for all $k \geq k_2$ this inequality holds for all paths $\gamma \in \Pi^{(k)}$. Hence for $k \geq k_2$ we have

$$\forall \gamma \in \Pi^{(k)}\,, \quad \mathrm{diam}([\gamma])^d < \sum_{\eta \in E^{(k)}(\gamma)} \mathrm{diam}([\eta])^d\,. \tag{2.15}$$

We have proven that for any path γ, the diameter of its cylinder to the power d is strictly smaller than the sum of those of its extensions. This means that for any $k \geq k_2$ the trivial covering of $C^{(k)}$ by itself (which corresponds to a path of length zero) minimizes the sum in equation (2.9). Therefore, using equation (2.13), we deduce that for $k \geq k_2$

$$H^d(C) = H^d(C^{(k)}) \geq \mathrm{diam}(C^{(k)})^d = 1 \quad \text{ for } d < s_0.$$

And with equation (2.11) we conclude that

$$\dim_H(C) \leq s_0\,. \tag{2.16}$$

Together with equation (2.12) this completes the proof. □

3. Bi-Lipschitz embedding in $\mathbb{R}^n$

The following theorem is inspired by a result of E. Christensen and C. Ivan in [5] (Theorem 4.10 in Section 4.3)

Theorem 3.1. *A self-similar ultrametric Cantor set is bi-Lipschitz embeddable in $\mathbb{R}^n$, for some n.*

Proof. Consider a self-similar ultrametric Cantor set (Π_∞, ρ). We can assume that ρ is regular, with parameter $\alpha \in (0,1)$. Make a choice of a one-to-one mapping $\beta : \mathcal{E} \to (0, +\infty)$, and define

$$\delta_{\min} = \min_{e,f \in \mathcal{E}} |\beta(e) - \beta(f)|\,, \quad \delta_{\max} = \max_{e,f \in \mathcal{E}} |\beta(e) - \beta(f)|\,. \tag{3.17}$$

For an integer $n \in \mathbb{N}$ we define a map $F : \Pi_\infty \to \mathbb{R}^n$, whose i-th coordinate, $1 \leq i \leq n$, is given by

$$F^i(x) = \sum_{j=0}^{\infty} \beta(x_{i+nj})\alpha^{nj}\,, \qquad x = (x_j)_{j\geq 0} \in \Pi_\infty\,. \tag{3.18}$$

Fix $x, y \in \Pi_\infty$, with $|x \wedge y| = m$, so that we have $\rho(x,y) = \alpha^m$. We write $m = nj_0 + i_0$, for some $j_0 \in \mathbb{N}$ and $0 \leq i_0 < n$. We have the inequalities $m < nj_0 \leq m - n$, and get the bounds

$$\alpha^m < \alpha^{nj_0} \leq \alpha^{-n}\alpha^m\,. \tag{3.19}$$

We consider F^i and look at the case $i \leq i_0$ first. All the terms in $F^i(x) - F^i(y)$ vanish for $j \leq j_0$, so we get the upper bound

$$|F^i(x) - F^i(y)| \leq \delta_{\max} \sum_{j \geq j_0+1} \alpha^{nj} = \frac{\delta_{\max}}{1-\alpha^n}\alpha^{n(j_0+1)} \leq \frac{\delta_{\max}}{1-\alpha^n}\alpha^m\,,$$

where we used equation (3.19) for the last inequality. We get the upper Lipschitz bound

$$|F^i(x) - F^i(y)| \leq \frac{\delta_{\max}}{1-\alpha^n}\alpha^m = c_1\rho(x,y)\,, \quad i \leq i_0 \tag{3.20}$$

For the lower bound we single out the first non-vanishing term in the series and write

$$\begin{aligned}|F^i(x) - F^i(y)| &\geq \delta_{\min}\alpha^{n(j_0+1)} - \Big|\sum_{j\geq j_0+2}(\beta(x_{i+nj}) - \beta(y_{i+nj}))\alpha^{nj}\Big| \\ &\geq \delta_{\min}\alpha^{n(j_0+1)} - \delta_{\max}\frac{\alpha^{n(j_0+2)}}{1-\alpha^n} = \frac{\delta_{\min} - \alpha^n(\delta_{\min}+\delta_{\max})}{1-\alpha^n}\alpha^{n(j_0+1)}\,.\end{aligned} \tag{3.21}$$

Since $\alpha \in (0,1)$ we can find n large enough for the numerator of the last fraction to be non-negative. Using equation (3.19) we get the lower Lipschitz bound

$$|F^i(x) - F^i(y)| \geq \frac{\delta_{\min} - \alpha^n(\delta_{\min}+\delta_{\max})}{1-\alpha^n}\alpha^n\,\alpha^m = c_2\rho(x,y)\,, \quad i \leq i_0 \tag{3.22}$$

The case $i > i_0$ is similar and yields the same bounds, but with constants multiplied by α^{-n}. That is the upper bound is the same as in equation (3.20) but with constant $c_1\alpha^{-n}$, and the lower bound is the same as in equation (3.22) but with constant $c_2\alpha^{-n}$. We let $c_- = c_2$ and $c_+ = c_1\alpha^{-n}$ and get the bounds

$$c_-\rho(x,y) \leq |F^i(x) - F^i(y)| \leq c_+\rho(x,y)\,, \quad i = 1,\cdots n\,. \tag{3.23}$$

From this we deduce immediately the Lipschitz bounds for F . For the Euclidean norm, one gets for instance

$$c_-\sqrt{n}\,\rho(x,y) \leq \|F(x) - F(y)\| \leq c_+\sqrt{n}\,\rho(x,y)\,.$$

□

We now give lower bounds on n.

Lemma 3.2. *Let $p = \#\mathcal{E}$, and set $n = [-\log(p)/\log(\alpha)] + 1$. Then (Π_∞, ρ) can be bi-Lipschitz embedded in $\mathbb{R}^n$.*

Proof. The condition on n in equation (3.21) in the proof of Theorem 3.1 reads

$$n > \log\big(\frac{\delta_{\min}}{\delta_{\max}+\delta_{\min}}\big)/\log(\alpha)\,.$$

Now clearly, the minimum possible value of the quotient $\delta_{\min}/(\delta_{\min}+\delta_{\max})$ is reached if one takes the constants $\beta(e)$ to be uniformly distributed, in which case it equals $1/p$. □

Theorem 3.3. *Let $C = (\Pi_\infty, \rho)$ be a self-similar ultrametric Cantor set. There exists a bi-Lipschitz embedding*

$$C \hookrightarrow \mathbb{R}^{[\dim_H(C)]+1},$$

where $\dim_H(C)$ is the Hausdorff dimension of C, and $[\dim_H(C)]$ denotes its integer part.

Proof. By Theorem 2.12 and Lemma 2.11 we have $d_H = \dim_H(C) = -\log(\Lambda)/\log(\alpha)$. According to Lemma 3.2, the condition on n for the lower Lipschitz bound in equation (3.21) in the proof of Theorem 3.1 reads

$$n > -\log(1 + \delta_{\max}/\delta_{\min})/\log(\alpha) = d_H \log(p)/\log(\Lambda)\,. \tag{3.24}$$

where p denotes the cardinality of $\mathcal{E}$. For an integer $k \geq 1$, consider the k-th telescope $C^{(k)} = (\Pi_\infty^{(k)}, \rho^{(k)})$ of C as in Lemma 2.9. The parameter of $\rho^{(k)}$ is α^k. The cardinality $p^{(k)}$ of $\mathcal{E}^{(k)}$ grows like Λ^k with k:

$$p^{(k)} = c\Lambda^k + o(\Lambda^k)\,, \tag{3.25}$$

where the constant c is uniformly bounded (does not depend on k). Hence the lower Lipschitz bound for $\mathcal{B}^{(k)}$ in equation (3.24) reads

$$n > d_H \log(c\Lambda^k + o(\Lambda^k))/\log(\Lambda^k) = d_H + c'/k + o(1/k)\,, \quad c' = \log(c)/\log(\Lambda)\,.$$

By Lemma 2.9, C is Lipschitz equivalent to $C^{(k)}$ for all k. Hence for any $0 < \varepsilon < 1$, by choosing k large enough, this shows that C is bi-Lipschitz embeddable in $\mathbb{R}^n$, for $n > d_H + \varepsilon$. This completes the proof. □

As a corollary of this and Theorem 4.1 one gets the following.

Theorem 3.4. *The transversal of a substitution tiling of $\mathbb{R}^d$ is bi-Lipschitz embeddable in $\mathbb{R}^{d+1}$.*

4. The Laplacians and their spectra

Pearson and Bellissard built in [13] a spectral triple (the data of Riemaniann noncommutative geometry [6]) for ultrametric Cantor sets, and derived a family of Laplace-Beltrami like operators. The authors revisited their construction in [10] for the transversals of substitution tilings which are particular self-similar ultrametric Cantor sets. We remind the reader here of the one parameter family of Laplace operators $(\Delta_s)_{s\in\mathbb{R}}$ that one obtains in this case, and refer the reader to [10] for the details.

Consider a substitution tiling of $\mathbb{R}^d$, with primitive Abelianization matrix A (see Example 2.8, equation (2.5)). Denote by Ξ the canonical transversal to its tiling space: it is a ultrametric Cantor set. Write Λ for its Perron-Frobenius eigenvalue, and $\nu_i, i \in \mathcal{V}$, for the normalized coordinates of its associated eigenvector. Let $\mathcal{B} = (V, E)$ be the Bratteli diagram of the substitution, with ultrametric given as in Example 2.8, equation (2.7). One has a bi-Lipschitz homeomorphism between Π_∞ and Ξ [10].

The spectral triple is the data of a Dirac operator D (self-adjoint, unbounded, with compact resolvent) on the Hilbert space $\ell^2(\Pi_\infty) \otimes \mathbb{C}^2$ together with a faithful $*$-representation of the C^*-algebra $C(\Pi_\infty)$ by bounded operators. The *ζ-function* of the spectral triple is the trace of $|D|^{-s}$:

$$\zeta(s) = \frac{1}{2}\mathrm{Tr}\left(|D|^{-s}\right) = \sum_{\gamma\in\Pi} \mathrm{diam}[\gamma]^s \,, \tag{4.26}$$

and its abscissa of convergence will be denoted by $s_0 \in \overline{\mathbb{R}}$. The authors proved in [10] that s_0 is equal to both d and the exponent of complexity of the tiling. This result is refined here. As a corollary of Theorem 2.12 one gets the following.

Theorem 4.1. *Consider a primitive substitution tiling of $\mathbb{R}^d$, with canonical transversal Ξ. The abscissa of convergence s_0 of the zeta-function of Ξ is equal to its Hausdorff dimension $\dim_H(\Xi)$, and moreover one has*

$$s_0 = \dim_H(\Xi) = d\,. \tag{4.27}$$

The spectral triple allows to define a measure μ on Π_∞ (Dixmier trace). Using the Dirac operator one can define a one parameter family $(\mathcal{D}_s)_{s\in\mathbb{R}}$ of closable Dirichlet form on $L^2(\Pi_\infty, d\mu)$. For each s, $\mathcal{D}_s$ is associated with the generator of a Markov semi-group Δ_s. One has that Δ_s is a self-adjoint, definite, and non negative operator on $L^2(\Pi_\infty, d\mu)$. For $s > s_0 + 2$, Δ_s is bounded. And for $s \le s_0 + 2$, Δ_s is unbounded and has pure point spectrum.

We recall now the spectrum of Δ_s. The eigenvalues do not depend on s and are parametrized by Π. For $\gamma \in \Pi$ the corresponding eigenspace is

$$E_\gamma = \Big\langle \frac{1}{\mu[\gamma\cdot e]}\chi_{\gamma\cdot e} - \frac{1}{\mu[\gamma\cdot e']}\chi_{\gamma\cdot e'} \; : \; e, e' \in \mathrm{ext}_1(\gamma) \Big\rangle \tag{4.28}$$

where χ_η is the characteristic function of the cylinder $[\eta]$ of a path $\eta \in \Pi$, and $\mathrm{ext}_1(\eta)$ is the set of edges that extend η one generation further. The dimension of the eigenspace is $\dim E_\gamma = n_\gamma - 1$, where $n_\gamma = \#\mathrm{ext}_1(\gamma)$. Now define

$$\Lambda_s = \Lambda^{(2+s_0-s)/s_0} = \alpha^{s-s_0-2} \tag{4.29}$$

where Λ is the Perron-Frobenius eigenvalue of A, and α is thus the parameter of the ultrametric on Π_∞. So as in Lemma 2.11 one has $s_0 = -\log(\Lambda)/\log(\alpha)$. The self-similar structure allows to compute the eigenvalues explicitly, using a Cuntz-Krieger algebra associated with the diagram $\mathcal{B}$. For a finite path $\gamma = (d_0, e_1, \cdots e_n)$ in $\mathcal{B}$, one has its associated eigenvalue $\lambda_\gamma(t)$ of Δ_t, which one can write

$$\lambda_\gamma(t) = \Lambda_t^n \lambda_{s(e_n)} + \sum_{j=1}^{n} \Lambda_t^{j-1}\beta(e_j, t)\,, \tag{4.30}$$

where $\lambda_{s(e_n)}$ is the eigenvalue associated with the path of length one from the root to $s(e_n)$ and does not depend on t, and where the constants $\beta(e,t)$ depend only of the measures of the cylinders of $s(e)$ and $r(e)$, and are uniformly bounded. In view of equation (4.29), for $s > s_0 + 2$ one has $\Lambda_s < 1$, thus as n goes to infinity the first term in (4.30) tends to zero, while the sum in the second term converges. Given an infinite path $x = (e_j)_{j\in\mathbb{N}}$ one can thus define $\lambda_x(s)$ as the sum

$$\lambda_x(s) = \sum_{j=1}^{\infty} \beta(e_j, s)\Lambda_s^{j-1}\,, \qquad s > s_0 + 2\,. \tag{4.31}$$

One defines now, for $s > s_0 + 2$, the ω-spectrum of Δ_s as

$$\mathrm{Sp}_\omega(\Delta_s) = \bigcup_{n\in\mathbb{N}} \overline{\{\lambda_k : k > n\}}\,,$$

where the eigenvalues are ordered so that if λ_k is an eigenvalue of $\Delta_s|_{\Pi_n}$ then so are all λ_l for $l \le k$. In the case where all eigenvalues are finite multiplicity, this coincides with the usual definition of the ω-spectrum as the intersection of the spectra $\overline{\mathrm{Sp}(\Delta_s) \setminus \mathrm{Sp}\big(\Delta_s|_{\Pi_n}\big)}$ over $n \in \mathbb{N}$. In our case here, this is the boundary of the pure point spectrum of Δ_s.

Under some conditions on A and the $\beta(e,s)$, and for s large enough, it will now be shown that $\mathrm{Sp}_\omega(\Delta_s)$ is bi-*Hölder* homeomorphic to Π_∞ (or Ξ), see Corollary 5.2. Using the explicit form of the constants $\beta(e,s)$ given in [10], one can derive the elementary following lemma.

Lemma 4.2. *Assume that the diagram satisfies the following: all edges are simple, and the coordinates of the Perron-Frobenius eigenvector of its adjacency matrix are pairwise distinct. Then there exists $s_1 \in \mathbb{R}$ such that for all $s > s_1$, if $e \ne f$ then either $\beta(e,s) \ne \beta(f,s)$, or $\beta(e',s) \ne \beta(f',s)$ for any $e' \in \mathrm{ext}_1(e), f' \in \mathrm{ext}_1(f)$.*

Remark 4.3. The above condition in Lemma 4.2 cannot be fulfilled *as is* if the Bratteli diagram is "too symmetrical". Examples of such diagrams include the dyadic odometer which encodes the usual triadic Cantor set, and the diagram of the Thue-Morse substitution. In such cases, the symmetry implies high degeneracies in the spectrum of Δ_s (in the dyadic case, the eigenvalues λ_γ are all equal for $\gamma \in \Pi_n$). Substitutions giving rise to such "over symmetrical" diagrams will not be considered for the purpose of Hölder embeddings of Ξ using Δ_s in Corollaries 5.2 and 5.4, and Lemma 5.3. One can deal with such diagrams however by modifying the ultrametric with constants $a_\gamma, \gamma \in \Pi$, as in Definition 2.6. Although no longer regular, the modified ultrametric is Lipschitz equivalent by Remark 2.7, and this destroys the degeneracies of the eigenvalues.

5. Bi-Hölder embedding in $\mathbb{R}$

Theorem 5.1. *A self-similar ultrametric Cantor set is bi-Hölder embeddable in the real line.*

Proof. Consider a self-similar ultrametric Cantor set (Π_∞, ρ). We can assume that ρ is regular, with parameter $\alpha \in (0,1)$. As in the proof of Theorem 3.1, make a choice of a one-to-one mapping $\beta : \mathcal{E} \to (0, +\infty)$, and set as in equation (3.17)

$$\delta_{\min} = \min_{e,f\in\mathcal{E}} |\beta(e) - \beta(f)| \,, \quad \delta_{\max} = \max_{e,f\in\mathcal{E}} |\beta(e) - \beta(f)| \,.$$

For $s > 0$, define the mapping $\varphi_s : \Pi_\infty \to \mathbb{R}$ by

$$\varphi_s(x) = \sum_{j=0}^{\infty} \beta(x_j)\alpha^{sj} \,, \qquad x = (x_j)_{j\geq 0} \in \Pi_\infty \,.$$

The series converges since $\alpha^s \in (0,1)$. Fix $x, y \in \Pi_\infty$, with $|x \wedge y| = m$, so that $\rho(x,y) = \alpha^m$. We have

$$|\varphi_s(x) - \varphi_s(y)| \leq \delta_{\max} \sum_{j\geq m} \alpha^{sj} = \frac{\delta_{\max}}{1-\alpha^s}\alpha^{sm} \,,$$

and with (4.29) we get the upper Hölder bound

$$|\varphi_s(x) - \varphi_s(y)| \leq c_+ \rho(x,y)^s \,. \tag{5.32}$$

For the other bound we single out the first non-vanishing term in the series and write

$$\begin{aligned} |\varphi_s(x) - \varphi_s(y)| &\geq \delta_{\min}\alpha^{sm} - \Big| \sum_{j\geq m+1} (\beta(x_j) - \beta(y_j))\alpha^{sj)} \Big| \\ &\geq \delta_{\min}\alpha^{sm} - \delta_{\max}\frac{\alpha^{s(m+1)}}{1-\alpha^s} = \frac{\delta_{\min} - \alpha^s(\delta_{\min} + \delta_{\max})}{1-\alpha^s}\alpha^{sm} \end{aligned}$$

and since $\alpha \in (0,1)$ we can find s large enough for the numerator of the last fraction to be non-negative. That is, for

$$s > \log\big(\frac{\delta_{\min}}{\delta_{\max} + \delta_{\min}}\big) / \log(\alpha) \,, \tag{5.33}$$

we get a lower Hölder bound

$$|\varphi_s(x) - \varphi_s(y)| \geq c_- \rho(x,y)^s \,. \tag{5.34}$$

$\square$

Corollary 5.2. *Let Ξ be the transversal of a substitution tiling of $\mathbb{R}^d$. If the Bratteli diagram of the substitution satisfies the condition of Lemma 4.2, then for s large enough Ξ is bi-Hölder homeomorphic to the ω-spectrum of Δ_s.*

Proof. In this setting, the constant α in the proof of Theorem 5.1 has to be replaced by $\Lambda^{(d+2-s)/d}$ as in equation (4.29) (here $s_0 = d$ by Theorem 4.1). Fix $x, y \in \Pi_\infty$, with say $|x \wedge y| = m$. By Lemma 4.2 for s large enough, we have either $\beta(x_m, s) \neq \beta(y_m, s)$ or $\beta(x_{m+1}, s) \neq \beta(y_{m+1}, s)$. One uses the constants $\beta(e, s)$ to define the map φ_s in the proof of Theorem 5.1, and one gets that $|\varphi_s(x) - \varphi_s(y)|$ is bounded as in equations (5.32) and (5.34) up to factors α^s. The condition on s in equation (5.33) becomes here

$$s > \max\{s_1, d + 2 + d \log\big(1 + \delta_{\max}/\delta_{\min}\big) / \log(\Lambda)\} . \tag{5.35}$$

□

In some cases, namely if s_1 is not too large, one can get a better lower bound on s as follows.

Lemma 5.3. *With the settings of the Corollary 5.2, let $p = \#\mathcal{E}$, then Ξ is bi-Hölder homeomorphic to the ω-spectrum of Δ_s for all s greater than $d + 2 + d \log(p)/\log(\Lambda)$.*

Proof. Here p is the number of edges in the diagram between two generations away from the root. The maximum possible value of $1 + \delta_{\max}/\delta_{\min}$ in equation (5.35) is reached when the $\beta(e)$ are uniformly distributed, in which case this equals p. □

Corollary 5.4. *If $s_1 \leq 2(d + 1)$, then Ξ is bi-Hölder homeomorphic to the ω-spectrum of Δ_s for all s greater than $2(d + 1)$.*

Proof. Using Lemma 5.3 and telescoping k times as in the proof of Theorem 3.3, one gets that Ξ is bi-Hölder embeddable for all s greater than $d + 2 + d \log(c\Lambda^k)/\log(\Lambda^k) = 2(d + 1) + c'/k$, and one can choose k arbitrarily large. □

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