Structural optimization of thin elastic plates: the three dimensional approach

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Abstract

The natural way to find the most compliant design of an elastic plate, is to consider the three-dimensional elastic structures which minimize the work of the loading term, and pass to the limit when the thickness of the design region tends to zero. In this paper, we study the asymptotic of such compliance problem, imposing that the volume fraction remains fixed. No additional topological constraint is assumed on the admissible configurations. We determine the limit problem in different equivalent formulations, and we provide a system of necessary and sufficient optimality conditions. These results were announced in [18]. Furthermore, we investigate the vanishing volume fraction limit, which turns out to be consistent with the results in [16, 17]. Finally, some explicit computation of optimal plates are given.

1 Introduction

The simplest and most common way for designing elastic plates is to use plane layers with constant thickness and made of a single material. But if one desires to improve the resistance-weight ratio, the use of more sophisticated structures is requested. A first possibility is to allow for a varying thickness. The search of optimal designs in this context has been the object of several studies. Without any attempt of being complete, we refer to [4, 5, 8, 9, 10, 23, 24, 26, 32]. In these works it is assumed that any section of the plate is a segment and that the thickness variations are smooth enough for the classical dimension reduction analysis to be applied. Clearly more exotic and more efficient structures can be designed if one refutes such restrictive geometrical assumptions. This is what we do in this paper, where we only assume that the structure is made of a single material lying in some subset of a thin layer. The total volume of the subset being fixed, we look for an optimal design and we study its limit when the thickness of the layer tends to zero. Thus our study is at the junction of two research directions: the so-called *3D-2D asymptotic analysis* and *shape optimization*. Before describing the novelty of our approach and results, let us give a brief recall on these two topics and some related bibliography.
The *compliance* of a given amount of elastic material occupying a domain $\Omega \subset \mathbb{R}^3$, characterized by a strain potential j, and subject to a given system of loads $F \in H^{-1}(\Omega;\mathbb{R}^3)$, is the opposite of the total energy at equilibrium: the higher is this compliance, the smaller is the resistance of the

material to the load F. More precisely, it is given by

$$\mathcal{C}_{j,F}(\Omega) := \sup\left\{ \langle F, u\rangle_{\mathbb{R}^3} - \int_\Omega j(e(u))\,dx \ : \ u \in \mathcal{C}^\infty(\mathbb{R}^3;\mathbb{R}^3) \right\} . \tag{1.1}$$

Here and throughout the paper we adopt the framework of small-displacement hypothesis, so the strain tensor $e(u)$ of the displacement field $u : \Omega \to \mathbb{R}^3$ coincides with the symmetrical part of its jacobian (in the sequel, for any vector field $v : \mathbb{R}^n \to \mathbb{R}^n$, we set $2\,e(v) := \nabla v + (\nabla v)^T$). The function j characterizes the material properties of the elastic body: we assume it to be a convex, 2-homogeneous and coercive function defined on the space $\mathbb{R}^{3\times 3}_{\rm sym}$ of 3×3 symmetric tensors. In the classical case of homogeneous linear elastic materials, it takes the form $j(z) = \lambda(tr(z))^2 + (\mu/2)\|z\|^2$ where λ, μ are called the Lamé coefficients of the material and $tr(z)$ is the trace of the matrix z. The existence of a solution $\bar u \in H^1(\mathbb{R}^3;\mathbb{R}^3)$ to problem (1.1) is ensured provided the load is balanced (that is $-\langle F, u\rangle = 0$ whenever $e(u) = 0$), and it is well-known that $\mathcal{C}_{j,F}(\Omega) = -\langle F, \bar u\rangle$. A solution exists as well in variants of problem (1.1) where some Dirichlet constraint is added on the admissible functions u in a suitably large part of $\overline{\Omega}$; in these cases, the balance condition on the load is not needed. Our results extend without difficulties to such variants, so we do not treat them in this paper.

Background on 3D-2D reduction analysis. Since thin structures play an important role in mechanical engineering, the asymptotic study of the compliance when the domain Ω is a cylinder $Q_\delta := \overline{D} \times [-\delta/2, +\delta/2]$ of infinitesimal height δ, has been the subject of numerous works. Clearly, a finite value for the limit compliance can be expected only if the applied load is adapted to the thin design region Q_δ. If the spatial variable $x \in \mathbb{R}^3$ is identified with the couple $(x', x_3) \in \mathbb{R}^2 \times \mathbb{R}$, the usual scaling adopted for elastic plates undergoing small deformations is

$$F^\delta(x) = \left(\frac{F_1(x', \delta^{-1}x_3)}{\sqrt{\delta}}, \frac{F_2(x', \delta^{-1}x_3)}{\sqrt{\delta}}, \sqrt{\delta}\, F_3(x', \delta^{-1}x_3) \right) . \tag{1.2}$$

This scaling ensures that, as $\delta \to 0$, membrane and bending energies will remain finite and with the same magnitude order. Then it turns out that the limit of $\mathcal{C}_{j,F^\delta}(Q_\delta)$ as $\delta \to 0$ can be written in terms of descriptors depending only on the 2D transverse spatial variables x'. Indeed, as $\delta \to 0$, the displacement field u takes the special structure of a *Kirchoff-Love displacement*, that is it belongs to the space

$$H^1_{KL}(Q;\mathbb{R}^3) \ := \ \left\{ u \in H^1(Q;\mathbb{R}^3) \ : \ e_{i3}(u) = 0 \ , \ i = 1,2,3 \right\} . \tag{1.3}$$

As well known, any $u \in H^1_{KL}(Q;\mathbb{R}^3)$ can be written as

$$u_\alpha(x) = v_\alpha(x') - \frac{\partial v_3}{\partial x_\alpha}(x')x_3 \ , \qquad u_3(x) = v_3(x') \tag{1.4}$$

for some $v_\alpha \in H^1(D)$ ($\alpha = 1,2$) and $v_3 \in H^2(D)$. These are the above mentioned 2D descriptors, and in terms of their derivatives the nonvanishing part of the symmetric gradient of u can be expressed as

$$e_{\alpha\beta}(u) = e(v_1, v_2) - x_3 \nabla^2 v_3 \tag{1.5}$$

(here and below the greek symbols α, β are used for indices running in $\{1,2\}$). The representation (1.4) of the limit strain is the key point of the well-established theory of elastic plates for homogeneous materials (see for instance [19, 20]). It remains true in the non-homogenous case provided the stiffness tensor remains lower bounded over all the domain Ω. In presence of voids (that is Ω

is a proper subset of the thin cylinder), this result remains true only on very specific geometric restrictions [19, 26].

Background on shape optimization. A fundamental request for engineers is to have at their disposal the lightest possible structures which can resist to a given load. Then one looks for domains Ω which, for a given compliance, have the smallest possible volume, or equivalently for domains which, for a given volume, minimize the compliance. In general an additional geometric constraint is set by imposing that the admissible domains Ω are contained into some fixed compact subset Q, called the *design region.* Thus the shape optimization problem reads:

$$\inf \Big\{ \mathcal{C}_{j,F}(\Omega) \ : \ \Omega \subseteq Q \ , \ |\Omega| = m \Big\} \ . \tag{1.6}$$

In most cases such a problem is ill-posed. Indeed minimizing sequences of domains Ω_n tend to become more and more intricate structures consisting of a fine mixture of voids and elastic material: the characteristic functions of Ω_n converge to a function θ with values in $[0, 1]$, and no optimal shape exists as soon as intermediate values $\theta(x) \in (0, 1)$ are attained. Here relaxation theory comes into play: by allowing such intermediate values, the new unknown becomes the local material density θ and the new task is finding the variational problem solved by θ. A first guess in this direction (usually called *fictitious materials approach*) consists simply in replacing in (1.1) the characteristic function of Ω by the varying density θ. We adopt the notation $\widetilde{\mathcal{C}}_{j,F}(\theta)$ for the corresponding generalized compliance, which is given by

$$\widetilde{\mathcal{C}}_{j,F}(\theta) := \sup \Big\{ \langle F, u\rangle_{\mathbb{R}^3} - \int j(e(u))\, \theta(x)\, dx \ : \ u \in \mathcal{C}^\infty(Q; \mathbb{R}^3) \Big\}. \tag{1.7}$$

The problem of optimizing, under an integral constraint on θ, the convex functional $\theta \mapsto \widetilde{\mathcal{C}}_{j,F}(\theta)$ is well-posed. But unfortunately homogenization theory teach us that the compliance of minimizing sequences Ω_n for problem (1.1) converge to a more complex limit compliance, which is obtained by replacing in (1.7) $j(e(u))\, \theta(x)$ by a strictly smaller effective integrand $j^{\text{eff}}(\theta(x), e(u))$. The explicit computation of $j^{\text{eff}}(\theta, \cdot)$ is a challenging and partially open problem. Some characterization has been given in [1, 3] by modelling the void region as artificially filled with a soft material of infinitesimal stiffness.

Setting of the problem: shape optimization in thinning domains. The aim of this paper is to study the asymptotics of the optimal shape problem when the design region is the flattening cylinder $Q_\delta = \overline{D} \times [-\delta/2, \delta/2]$, with $\delta \to 0$. In this limit process, the load will be rescaled as in (1.2). Obviously, also the volume constraint for the optimal shape problem (1.6) has to be adapted to the thin design region: we assume that the volume fraction $\tau \in [0, 1]$ is fixed, thus we impose $|\Omega| = \tau\, |Q_\delta| = \tau\, \delta$. Summarizing we are interested in the asymptotics, as δ tends to zero, of the following optimization problem

$$\mathcal{I}^\delta_{j,F}(\tau) := \inf \Big\{ \mathcal{C}_{j,F^\delta}(\Omega) \ : \ \Omega \subseteq Q_\delta \ , \ |\Omega| = \tau\delta \Big\} \ . \tag{1.8}$$

It is convenient to remark that the volume constraint on the admissible sets in (1.8) can be dropped by enclosing in the cost a volume penalization through a Lagrange multiplier. For a fixed $k \in \mathbb{R}$, we set

$$\phi^\delta_{j,F}(k) := \inf \Big\{ \mathcal{C}_{j,F^\delta}(\Omega) + \frac{k}{\delta}|\Omega| \ : \ \Omega \subseteq Q_\delta \Big\} \ . \tag{1.9}$$

Indeed, as it will be shown later, the asymptotics of problem (1.8) can be easily deduced once the asymptotics of (1.9) is known.

Previous contributions on the problem. As mentioned at the beginning of the paper, the most significant contributions recently appeared on optimization of thin plates concern the case of plates with varying thickness: in this framework, problem (1.8) is studied under the additional constraint that admissible domains Ω are of the form $\{|x_3| < \delta h(x')\}$. As $\delta \to 0$, this leads to a variational problem where the unknown profile function h appears with a cubic dependence and whose relaxation has been efficiently treated through the use of Young measures [29]. The papers [17, 18] are first tries to get rid of topological constraints. In particular, [17] deals with the asymptotics of problem (1.6) when the total volume of the material is firstly sent to zero, and then the 3D-2D limit is performed. The outcoming model involves material distributions represented by possibly concentrated measures, and the optimal configurations are shown to be related with a simple linear Hessian-constrained problem, of the type considered in [16, 27]. As we shall see, this model can be recovered within the approach of the present paper, in the special situation when the filling ratio τ in (1.8) becomes infinitesimal (or equivalently when the parameter k in (1.9) goes to $+\infty$).

Synopsis of the results and comments. We prove that, for every fixed $k \in \mathbb{R}$, the limit as $\delta \to 0$ of $\phi^{\delta}_{j,F}(k)$ is given by

$$\phi(k) := \inf \left\{ \mathcal{C}^{lim}(\theta) + k \int_Q \theta \, dx \ : \ \theta \in L^\infty(Q; [0,1]) \right\} ,$$

where the limit compliance $\mathcal{C}^{\rm lim}(\theta)$ is the *convex* functional

$$\mathcal{C}^{lim}(\theta) = \sup \left\{ \langle F, u \rangle_{\mathbb{R}^3} - \int_Q \overline{j}(e_{\alpha\beta}(u)) \, \theta \, dx \ : \ u \in H^1_{KL}(Q; \mathbb{R}^3) \right\} .$$

Here $Q = \overline{D} \times [-1/2, 1/2]$, $\overline{j}$ is the 2D-energy density in the plane stress case, and $H^1_{KL}(Q; \mathbb{R}^3)$ is the space of Kirkhoff-Love displacements defined in (1.3). Representing the competitors u in terms of $2D$-descriptors v as in (1.4), the computation of $\phi(k)$ reduces to solving a very simple inf-sup problem:

$$\inf_{\theta \in L^\infty(Q;[0,1])} \sup \left\{ \langle \overline{F}, v \rangle_{\mathbb{R}^2} - \int_D \int_{-1/2}^{1/2} \left[\overline{j}(e(v_1, v_2) - x_3 \nabla^2 v_3) - k \right] \theta \, dx \ : \ v_\alpha \in H^1(D) \ , \ v_3 \in H^2(D) \right\} ,$$

where $\overline{F}$ is a suitable $2D$-average load. By this way, the initial nonconvex problem becomes in the limit a classical saddle problem for a convex-concave Lagrangian. It is then straightforward that a saddle point $(\overline{\theta}, \overline{v})$ exists, and one is allowed to exchange the supremum in v and the infimum in θ. It follows that the optimal density $\overline{\theta}(x', \cdot)$ is uniquely determined, for all $x' \in D$ such that $\nabla^2 \overline{v}_3(x') \neq 0$, as the characteristic function of the set

$$\left\{ x_3 \in [-1/2, 1/2] \ : \ \overline{j}(e(\overline{v}_1, \overline{v}_2) - x_3 \nabla^2 \overline{v}_3) > k \right\} .$$

The rather unexpected consequence is that no homogenization region appears in the limit optimal shape: in bending regime (that is when $\nabla^2 \overline{v}_3(x') \neq 0$ on D), the optimal material distribution is simply made by two layers concentrated along the top and bottom faces of the design region. The thickness of these two layers is not constant and depends on the applied load through the optimal displacement configuration $\overline{v}$. The efficiency of concentrating the material on the top and bottom faces in order to design plates with the best resistance to bending is well known in applied mechanics: sandwich structures are commonly used, as well as "I" or "T" structures in case of bending rods (see for instance [30]). The more suprising feature is that, in the mathematical limit as $\delta \to 0$, these top and bottom layers are not connected. A more detailed discussion of this point is postponed to the last section of the paper.

Outline of the paper. The paper is organized as follows.

In Section 2 we state and prove the main convergence results to the limit compliance problem.

In Section 3 we present a saddle approach, in which the limit problem is reformulated in terms of Kirchoff-Love fields, or associated $2D$ descriptors.

In Section 4 we present a duality approach, in which the limit problem is reformulated in terms of $3D$ or $2D$ stress tensors.

In Section 5 we define optimal triples for the limit problem: they consist of a material distribution, a Kirchoff-Love field, and a stress tensor, which are optimal for the different formulations of the limit problem given respectively in Section 2, Section 3, and Section 4. We are able to derive the optimality conditions for such triples and to deduce qualitative properties of optimal material distributions.

In Section 6 we focus attention on the asymptotic behaviour of the limit problem when the volume fraction becomes infinitesimal.

In Section 7 we show how straightforward the results allow to find explicitly optimal mass distributions in concrete model situations, and we conclude with some perspectives.

2 Convergence to the limit compliance problem

For convenience, let uilas sum up the basic notation and assumptions, which have been partly already introduced along Section 1 and will be kept throughout the paper without any further mention. Let $Q := \overline{D} \times I$, where D is an open bounded connected subset of $\mathbb{R}^2$, and $I := [-1/2, 1/2]$. We assume with no loss of generality that $|Q| = 1$, and we denote the spatial variable in $\mathbb{R}^3$ by $(x', x_3) \in \mathbb{R}^2 \times \mathbb{R}$. We set $Q_\delta := \overline{D} \times \delta I = \overline{D} \times [-\delta/2, \delta/2]$, where $\delta > 0$ is an infinitesimal parameter. We assume that the stored energy density j is a convex, 2-homogeneous, and coercive integrand, defined on the space $\mathbb{R}^{3\times3}_{\rm sym}$ of 3×3 symmetric tensors. We denote by j^* the Fenchel conjugate of j:

$$j^*(\xi) := \sup \left\{ z \cdot \xi - j(z) \ : \ z \in \mathbb{R}^{3\times3}_{\rm sym} \right\} ,$$

where $\cdot$ denotes the Euclidean scalar product. For any $z \in \mathbb{R}^{2\times2}_{\rm sym}$, we denote by $E_0 z \in \mathbb{R}^{3\times3}_{\rm sym}$ the matrix defined by $(E_0 z)_{ij} = z_{ij}$ if $i, j < 3$ and $(E_0 z)_{i3} = 0$ for $i = 1, 2, 3$.
Finally, we assume that the load $F = (F_1, F_2, F_3)$ acting on the material belongs to the space $H^{-1}(Q; \mathbb{R}^3)$ and is "balanced", namely

$$\langle F, u \rangle_{\mathbb{R}^3} = 0 \quad \text{whenever} \quad 2e(u) := \nabla u + (\nabla u)^T = 0 \ .$$

An additional condition on F will be introduced later on (see (2.27)), in order to deal with arbitrarily small volume fractions $\tau \in (0, 1]$.
Our goal is to determine, for fixed $k \in \mathbb{R}$ and $\tau \in [0, 1]$, the asymptotic behaviour as $\delta \to 0$ of the problems introduced in (1.8) and (1.9). We proceed as follows.
After rescaling the problems under study on the fixed domain Q (subsection 2.1), we study the asymptotic behaviour of the corresponding problems for fictitious materials (subsection 2.2). In

subsection 2.3 we give bounds for the initial problems in terms of the corresponding fictitious ones. In subsection 2.4 we state the main results of convergence to the limit compliance problem, and we prove them relying on the contents of previous subsections.

2.1 Scaling

In order to deal with the asymptotics as $\delta \to 0$ of problems (1.8)-(1.9), it is convenient to restate them on the fixed domain Q. Therefore, for $u \in \mathcal{C}^\infty(Q;\mathbb{R}^3)$, we introduce the rescaled strain tensor $e^\delta(u)$ defined by

$$e^\delta(u) := \begin{bmatrix} e_{\alpha\beta}(u) & \delta^{-1} e_{\alpha 3}(u) \\ \\ \delta^{-1} e_{\alpha 3}(u) & \delta^{-2} e_{33}(u) \end{bmatrix} , \tag{2.1}$$

and, for any $\omega \subseteq Q$, we set

$$\mathcal{C}^\delta_{j,F}(\omega) := \sup \left\{ \langle F, u\rangle_{\mathbb{R}^3} - \int_\omega j(e^\delta(u))\, dx \ : \ u \in \mathcal{C}^\infty(Q;\mathbb{R}^3) \right\} . \tag{2.2}$$

Then our rescaled problems read as follows.

Lemma 2.1 *For every fixed $\delta > 0$, $k \in \mathbb{R}$ and $\tau \in [0,1]$, there holds*

$$\phi^\delta_{j,F}(k) = \inf \left\{ \mathcal{C}^\delta_{j,F}(\omega) + k|\omega| \ : \ \omega \subseteq Q \right\} , \tag{2.3}$$

$$\mathcal{I}^\delta_{j,F}(\tau) = \inf \left\{ \mathcal{C}^\delta_{j,F}(\omega) \ : \ \omega \subseteq Q \ , \ |\omega| = \tau \right\} . \tag{2.4}$$

PROOF. First rewrite any admissible domain $\Omega \subseteq Q_\delta$ in formulae (1.8)-(1.9) as

$$\Omega = \left\{ \left(x', \delta x_3\right) \ : \ (x', x_3) \in \omega \right\}$$

(so that $\omega \subseteq Q$). Then insert in the definition of $\mathcal{C}_{j,F^\delta}(\Omega)$ in (1.8)-(1.9) the following change of variables: for any competitor $\tilde{u} \in \mathcal{C}^\infty(Q_\delta;\mathbb{R}^3)$ set

$$\tilde{u}(x) = \Big(u_1\big(x', \delta^{-1}x_3\big),\ u_2\big(x', \delta^{-1}x_3\big),\ \delta^{-1}u_3\big(x', \delta^{-1}x_3\big) \Big) .$$

The lemma follows immediately, thanks to definition (2.1).

□

2.2 Asymptotic behaviour of fictitious problems

A first step towards our main results is the study of the "fictitious counterparts" of problems (2.3)-(2.4). For fictitious materials, represented by varying densities θ in $L^\infty(Q;[0,1])$, the generalized rescaled compliance takes the form

$$\widetilde{\mathcal{C}}^\delta_{j,F}(\theta) := \sup \left\{ \langle F, u\rangle_{\mathbb{R}^3} - \int_Q j(e^\delta(u))\, \theta\, dx \ : \ u \in \mathcal{C}^\infty(Q;\mathbb{R}^3) \right\} . \tag{2.5}$$

Accordingly, we set

$$\widetilde{\phi}^{\delta}_{j,F}(k) = \inf\left\{\widetilde{\mathcal{C}}^{\delta}_{j,F}(\theta) + k\int_Q \theta\,dx \ : \ \theta \in L^\infty(Q;[0,1])\right\}\,, \tag{2.6}$$

$$\widetilde{\mathcal{I}}^{\delta}_{j,F}(\tau) = \inf\left\{\widetilde{\mathcal{C}}^{\delta}_{j,F}(\theta) \ : \ \theta \in L^\infty(Q;[0,1])\,,\ \int_Q \theta\,dx = \tau\right\}\,. \tag{2.7}$$

In Proposition 2.2 below, we determine the Γ-limit $\mathcal{C}^{lim}(\theta)$ of the sequence $\widetilde{\mathcal{C}}^{\delta}_{j,F}(\theta)$ in the weak star topology of $L^\infty(Q;[0,1])$, and we deduce the asymptotic behaviour of the infima in (2.6)-(2.7).
In order to precise the expression of the limit compliance $\mathcal{C}^{lim}(\theta)$, we need to introduce the $2D$-strain energy density in the plane stress case, that is the integrand $\overline{j} : \mathbb{R}^{2\times 2}_{\rm sym} \to \mathbb{R}$ defined by

$$\overline{j}(z) := \inf\left\{j\big(z + \sum_{i=1}^{3}\xi_i(e_i\otimes e_3 + e_3\otimes e_i)\big) \ : \ \xi_i \in \mathbb{R}\right\}\,. \tag{2.8}$$

Moreover, we need to introduce the $2D$-average load $\overline{F} = (\overline{F}_1, \overline{F}_2, \overline{F}_3)$ defined by

$$\overline{F}_\alpha := [[F_\alpha]] \qquad \text{and} \qquad \overline{F}_3 := [[F_3 + x_3\sum_{\alpha=1}^{2}\frac{\partial F_\alpha}{\partial x_\alpha}]]\,, \tag{2.9}$$

where for a given real measure ν on Q, $[[\nu]]$ is the real measure on $\overline{D}$ defined by the identity $\langle[[\nu]],\varphi\rangle_{\mathbb{R}^2} := \langle\nu,\varphi\rangle_{\mathbb{R}^3}$ holding for all $\varphi \in \mathcal{C}^\infty(\mathbb{R}^2;\mathbb{R})$. Note that definition (2.9) implies that the identity $\langle F,u\rangle_{\mathbb{R}^3} = \langle\overline{F},v\rangle_{\mathbb{R}^2}$ holds whenever u and v are related to each other by (1.4).

Proposition 2.2 *As $\delta \to 0$, the following results hold:*

(i) The sequence $\widetilde{\mathcal{C}}^{\delta}_{j,F}(\theta)$ defined by (2.5) *Γ-converges, in the weak star topology of $L^\infty(Q;[0,1])$, to the limit compliance defined by*

$$\mathcal{C}^{lim}(\theta) := \sup\left\{\langle F,u\rangle_{\mathbb{R}^3} - \int_Q \overline{j}(e_{\alpha\beta}(u))\,\theta\,dx \ : \ u \in H^1_{KL}(Q;\mathbb{R}^3)\right\} \tag{2.10}$$

$$= \sup\left\{\langle\overline{F},v\rangle_{\mathbb{R}^2} - \int_Q \overline{j}(e(v_1,v_2) - x_3\nabla^2 v_3)\,\theta\,dx \ : \ v_\alpha \in H^1(D),\ v_3 \in H^2(D)\right\}. \tag{2.11}$$

(ii) For each $k \in \mathbb{R}$, the sequence $\widetilde{\phi}^{\delta}_{j,F}(k)$ defined by (2.6) *converges to the limit $\phi(k)$ defined by*

$$\phi(k) := \inf\left\{\ \mathcal{C}^{lim}(\theta) + k\int_Q \theta\,dx \ : \ \theta \in L^\infty(Q;[0,1])\right\}\,, \tag{2.12}$$

(iii) For each $\tau \in [0,1]$, the sequence $\widetilde{\mathcal{I}}^{\delta}_{j,F}(\tau)$ defined by (2.7) *converges to the limit $\mathcal{I}(\tau)$ defined by*

$$\mathcal{I}(\tau) := \sup_{k\in\mathbb{R}}\left\{\Phi(k) - k\tau\right\}\,. \tag{2.13}$$

Remark 2.3 The properties of $\phi(k)$ as a function of the real variable k will be studied in detail in Section 6. In particular, we shall prove that $k \mapsto \phi(k)$ is a finite concave function on $\mathbb{R}$ and that, for every $\tau \in [0,1]$, the supremum in over k in (2.13) is attained at a certain $k(\tau)$, *cf.* Theorem 6.1 and Corollary 6.4.

Let us start with the preliminary lemmas needed for the proof of Proposition 2.2. The first one establishes two asymptotic properties, hereafter named (AP1) and (AP2), for the sequence of stored energies $\int_Q j(e^\delta(u))\theta\,dx$ in terms of pairs (θ, u). Properties (AP1) and (AP2) will give the Γ-convergence statement (i) of Proposition 2.2. Moreover, they will be used once again in Section 3 in order to prove a convergence result for saddle points.

Lemma 2.4 *Let $\theta \in L^\infty(Q;[0,1])$, and $u \in H^1(Q;\mathbb{R}^3)$. Then*

(AP1) $\forall\, \theta^\delta \overset{*}{\rightharpoonup} \theta$, $\exists\, u^\delta \rightharpoonup u$ *such that*

$$\limsup_\delta \int_Q j(e^\delta(u^\delta))\theta^\delta\,dx \le \begin{cases} \int_Q \overline{j}(e_{\alpha\beta}(u))\theta\,dx & \text{if } u \in H^1_{KL}(Q;\mathbb{R}^3) \\ +\infty & \text{otherwise ;} \end{cases} \tag{2.14}$$

(AP2) *whenever* $\inf_Q \theta > 0$, $\forall\, u^\delta \rightharpoonup u$ *there holds*

$$\liminf_\delta \int_Q j(e^\delta(u^\delta))\theta\,dx \ge \begin{cases} \int_Q \overline{j}(e_{\alpha\beta}(u))\theta\,dx & \text{if } u \in H^1_{KL}(Q;\mathbb{R}^3) \\ +\infty & \text{otherwise .} \end{cases}$$

PROOF. (AP1). Let $\theta^\delta \overset{*}{\rightharpoonup} \theta$, and let $u \in H^1_{KL}(Q;\mathbb{R}^3)$. In order to construct a sequence $u^\delta \rightharpoonup u$ such that (2.14) holds, we take smooth functions $\xi_i = \xi_i(x)$ such that

$$\overline{j}\big(e_{\alpha\beta}(u)\big) = j\big(e_{\alpha\beta}u + \sum_{i=1}^3 \xi_i(e_i\otimes e_3 + + e_3\otimes e_i)\big) \ ,$$

and we denote by $\Phi = (\Phi_1, \Phi_2, \Phi_3)$ the field of their primitives with respect to the x_3 variable:

$$\Phi_i(x', x_3) := \int_0^{x_3} \xi_i(x', s)\,ds \ , \qquad i = 1,2,3 \ .$$

We then define the sequence $\{u^\delta\}$ componentwise by

$$u_1^\delta = u_1 + \delta\Phi_1 \ , \qquad u_2^\delta = u_2 + \delta\Phi_2 \ , \qquad u_3^\delta = u_3 + \delta^2\Phi_3 \ ,$$

so that

$$e^\delta(u^\delta) = e_{\alpha\beta}(u) + \delta\, e_{\alpha\beta}(\Phi) + \sum_{i=1}^2 \Big(\xi_i + \delta\frac{\partial\Phi_3}{\partial x_i}\Big)(e_i\otimes e_3 + e_3\otimes e_i) + \xi_3(e_3\otimes e_3) \ ,$$

By dominated convergence, we have $j(e^\delta(u^\delta)) \to \overline{j}(e_{\alpha\beta}(u))$ strongly in $L^1(Q)$. Therefore, recalling that by assumption $\theta^\delta \overset{*}{\rightharpoonup} \theta$, the integrand in the left hand side of (2.14) is the product between a strongly and a weakly convergent sequence. We deduce that (2.14) is satisfied (with equality sign).

(AP2). Assume $\inf_Q \theta > 0$, and let $u^\delta \rightharpoonup u$. For every $\xi \in L^2(Q;\mathbb{R}^{2\times 2}_{\rm sym})$, using the Fenchel inequality and the assumption $u^\delta \rightharpoonup u$, we obtain

$$\begin{aligned} \liminf_\delta \int_Q j(e^\delta(u^\delta))\theta\,dx &\ge \liminf_\delta \Big\{ \int_Q e^\delta(u^\delta)\cdot E_0\xi\,\theta\,dx - \int_Q j^*(E_0\xi)\,\theta\,dx \Big\} \\ &= \int_Q e_{\alpha\beta}(u)\cdot\xi\,\theta\,dx - \int_Q j^*(E_0\xi)\,\theta\,dx \ . \end{aligned}$$

Starting from the definition (2.8) of $\overline{j}$, one can easily check the algebraic identity

$$(\overline{j})^*(\xi) = j^*(E_0\xi) \qquad \forall \xi \in \mathbb{R}^{2\times 2}_{\rm sym} \ . \tag{2.15}$$

Using (2.15) and the arbitrariness of ξ in the previous inequality, we deduce that

$$\liminf_\delta \int_Q j(e^\delta(u^\delta))\,\theta\,dx \geq \sup_\xi \Big\{ \int_Q e_{\alpha\beta}(u)\cdot\xi\,\theta\,dx - \int_Q \overline{j}^*(\xi)\,\theta\,dx \Big\} \ .$$

By passing the supremum over ξ under the sign of integral (see *e.g.* [15, Lemma A.2]), and taking into account that $\overline{j} = (\overline{j})^{**}$, we get the required inequality

$$\liminf_\delta \int_Q j(e^\delta(u^\delta))\,\theta\,dx \geq \int_Q \overline{j}(e_{\alpha\beta}(u))\,\theta\,dx \ .$$

Let us now show that, if $u \notin H^1_{\rm KL}(Q;\mathbb{R}^3)$, the left hand side of the above inequality is actually $+\infty$. Assume that $e_{i3}(u) \neq 0$ for some i: for instance, let $e_{13}(u) \neq 0$ (the other cases are completely analogous). Using the coercivity of j and the definition $e^\delta(u^\delta)$ according to (2.1), we have

$$\liminf_\delta \int_Q j(e^\delta(u^\delta))\,\theta\,dx \geq c\liminf_\delta \int_Q |e^\delta(u^\delta)|^2\theta\,dx \geq c\liminf_\delta \delta^{-2}\int_Q |e_{13}(u^\delta)|^2\theta\,dx \ . \tag{2.16}$$

Thanks to the weak lower semicontinuity on $H^1(Q;\mathbb{R}^3)$ of the map $u \mapsto \int_Q |e_{i3}(u)|^2\theta\,dx$, and to the assumptions $e_{i3}(u) \neq 0$, $\inf_Q \theta > 0$, we have

$$\liminf_\delta \int_Q |e_{13}(u^\delta)|^2\theta\,dx \geq \int_Q |e_{13}(u)|^2\theta\,dx > 0 \ . \tag{2.17}$$

Combining (2.16) and (2.17), we obtain $\liminf_\delta \int_Q j(e^\delta(u^\delta))\,\theta\,dx = +\infty$. $\square$

Next lemma allows to relate, in the fictitious framework, the volume constrained and the volume penalized problems.

Lemma 2.5 *For every $\delta > 0$,*
– *the map $k \mapsto \widetilde{\phi}^\delta_{j,F}(k)$ is concave on $\mathbb{R}$;*
– *the map $\tau \mapsto \widetilde{\mathcal{I}}^\delta_{j,F}(\tau)$ is convex on $[0,1]$;*
– *if we extend $\widetilde{\mathcal{I}}^\delta_{j,F}$ by setting to $\widetilde{\mathcal{I}}^\delta_{j,F}(\tau) = +\infty$ for $\tau \notin [0,1]$, there holds*

$$\widetilde{\phi}^\delta_{j,F}(k) = -(\widetilde{\mathcal{I}}^\delta_{j,F})^*(-k) \qquad \text{on } \mathbb{R} \ . \tag{2.18}$$

Proof. The concavity of the map $k \mapsto \widetilde{\phi}^\delta_{j,F}(k)$ on $\mathbb{R}$ follows directly from its definition (2.6), as it is the infimum of linear functions of k. Similarly, the convexity of the map $\tau \mapsto \widetilde{\mathcal{I}}^\delta_{j,F}(\tau)$ on $[0,1]$ is obtained from its definition (2.7), by using the convexity of the map $\theta \mapsto \widetilde{\mathcal{C}}^\delta_{j,F}(\theta)$ on $L^\infty(Q;[0,1])$. Finally, to prove (2.18), we apply the definition of Fenchel conjugate and we obtain

$$\begin{aligned}
(\widetilde{\mathcal{I}}^\delta_{j,F})^*(-k) &= \sup\Big\{ -k\tau - \widetilde{\mathcal{I}}^\delta_{j,F}(\tau) \ : \ \tau \in \mathbb{R} \Big\} = \sup\Big\{ -k\tau - \widetilde{\mathcal{I}}^\delta_{j,F}(\tau) \ : \ \tau \in [0,1] \Big\} \\
&= \sup\Big\{ -k\tau - \widetilde{\mathcal{C}}^\delta_{j,F}(\theta) \ : \ \tau \in [0,1] \ , \ \theta \in L^\infty(Q;[0,1]) \ , \ \int_Q \theta\,dx = \tau \Big\} \\
&= \sup\Big\{ -k\int_Q \theta\,dx - \widetilde{\mathcal{C}}^\delta_{j,F}(\theta) \ : \ \theta \in L^\infty(Q;[0,1]) \Big\} \\
&= -\inf\Big\{ k\int_Q \theta\,dx + \widetilde{\mathcal{C}}^\delta_{j,F}(\theta) \ : \ \theta \in L^\infty(Q;[0,1]) \Big\} = -\widetilde{\phi}^\delta_{j,F}(k) \ .
\end{aligned}$$

□

We are now in a position to give the

Proof of Proposition 2.2.
(i) By definition, to show that $\widetilde{\mathcal{C}}^{\delta}_{j,F}(\theta)$ Γ-converges to $\mathcal{C}^{lim}(\theta)$ in the weak star topology, we have to prove that for each fixed $\theta \in L^\infty(Q;[0,1])$ the following two inequalities hold:

$$\inf \Big\{ \liminf \widetilde{\mathcal{C}}^{\delta}_{j,F}(\theta^\delta) \ : \ \theta^\delta \overset{*}{\rightharpoonup} \theta \Big\} \geq \mathcal{C}^{lim}(\theta) \tag{2.19}$$

$$\inf \Big\{ \limsup \widetilde{\mathcal{C}}^{\delta}_{j,F}(\theta^\delta) \ : \ \theta^\delta \overset{*}{\rightharpoonup} \theta \Big\} \leq \mathcal{C}^{lim}(\theta) \ . \tag{2.20}$$

Consider an arbitrary sequence $\theta^\delta \overset{*}{\rightharpoonup} \theta$. Let $\overline{u} \in H^1_{KL}(Q;\mathbb{R}^3)$ be an optimal Kirchoff-Love displacement for the variational definition (2.10) of $\mathcal{C}^{lim}(\theta)$. By (AP1) in Lemma 2.4, there exists a sequence $u^\delta \rightharpoonup \overline{u}$ such that (2.14) holds. Then we have

$$\liminf_\delta \widetilde{\mathcal{C}}^{\delta}_{j,F}(\theta^\delta) \geq \liminf_\delta \Big\{ \langle F, u^\delta \rangle - \int_Q j(e^\delta(u^\delta))\theta^\delta \, dx \Big\} \geq \langle F, \overline{u} \rangle_{\mathbb{R}^3} - \int_Q \overline{j}(e_{\alpha\beta}(\overline{u}))\, \theta \, dx = \mathcal{C}^{lim}(\theta) \ .$$

This proves (2.19). In order to prove (2.20), we have to find a recovery sequence $\theta^\delta \overset{*}{\rightharpoonup} \theta$ such that $\limsup_\delta \widetilde{\mathcal{C}}^{\delta}_{j,F}(\theta^\delta) \leq \mathcal{C}^{lim}(\theta)$. Let us first show that this is possible under the additional assumption $\inf_Q \theta > 0$. In this case, we claim that we are done simply by taking $\theta^\delta \equiv \theta$. Indeed, let $\overline{u}^\delta \in H^1_{KL}(Q;\mathbb{R}^3)$ be such that

$$\limsup_\delta \widetilde{\mathcal{C}}^{\delta}_{j,F}(\theta) = \limsup_\delta \Big\{ \langle F, \overline{u}^\delta \rangle - \int_Q j(e^\delta(\overline{u}^\delta))\theta \, dx \Big\} \ . \tag{2.21}$$

We may assume with no loss of generality that $\limsup_\delta \widetilde{\mathcal{C}}^{\delta}_{j,F}(\theta^\delta) > -\infty$. Then, taking into account that j is coercive and that $\inf_Q \theta > 0$, (2.21) implies that the sequence $e^\delta(\overline{u}^\delta)$ is uniformly bounded in L^2-norm. By applying the Korn inequality (after possibly substracting a rigid displacement, which is not restrictive thanks to the assumption that F is balanced), we obtain that the sequence $\overline{u}^\delta$ is precompact in $H^1(Q;\mathbb{R}^3)$. Then we may pass to a weakly convergent subsequence. By applying to such subsequence property (AP2) in Lemma 2.4, we infer that $\limsup_\delta \widetilde{\mathcal{C}}^{\delta}_{j,F}(\theta^\delta) \leq \mathcal{C}^{lim}(\theta)$. It remains to get rid of the additional assumption $\inf_Q \theta > 0$. This can be done via a standard density argument. Indeed, for any θ we may find a sequence θ^h with $\inf_Q \theta^h > 0$ such that $\theta^h \overset{*}{\rightharpoonup} \theta$. Then, since the left hand side of (2.20) (usually called $\Gamma - \limsup \widetilde{\mathcal{C}}^{\delta}_{j,F}(\theta)$), is weakly star lower semicontinuous, and $\mathcal{C}^{lim}(\theta)$ is weakly star continuous, we obtain

$$(\Gamma - \limsup_\delta \widetilde{\mathcal{C}}^{\delta}_{j,F})(\theta) \leq \liminf_h (\Gamma - \limsup_\delta \widetilde{\mathcal{C}}^{\delta}_{j,F})(\theta^h) \leq \lim_h \mathcal{C}^{lim}(\theta^h) = \mathcal{C}^{lim}(\theta) \ .$$

(ii) Since the term $\theta \mapsto \int_Q \theta$ is weakly star continuous, and since the space $L^\infty(Q;[0,1])$ is weakly star compact, the convergence of $\widetilde{\phi}^{\delta}_{j,F}(k)$ to $\phi(k)$ follows immediately from statement (i) thanks to well-known properties of Γ-convergence.

(iii) By Lemma 2.5, we have

$$\widetilde{\mathcal{I}}^{\delta}_{j,F}(\tau) = (\widetilde{\mathcal{I}}^{\delta}_{j,F})^{**}(\tau) = \sup_{k\in\mathbb{R}} \big[- (\widetilde{\mathcal{I}}^{\delta}_{j,F})^*(-k) - k\tau \big] = \sup_{k\in\mathbb{R}} \big[\widetilde{\phi}^{\delta}_{j,F}(k) - k\tau \big] \ .$$

Then we are reduced to compute the pointwise limit of the Fenchel conjugates of the sequence of functions $k \mapsto -\widetilde{\phi}^{\delta}_{j,F}(-k)$. By Lemma 2.5 we know that these functions are convex, and by statement (ii) already proved we know that they converge pointwise to $-\phi(-k)$. Since such limit remains finite (*cf.* Remark 2.3), we deduce that $-\widetilde{\phi}^{\delta}_{j,F}(-k)$ Γ-converge on $\mathbb{R}$ to $-\phi(-k)$ (see [22, Example 5.13]). Now recall that the Fenchel conjugate is continuous with respect to the Mosco convergence, hence to the Γ-convergence on $\mathbb{R}$ (see [6, Theorem 3.18]). Thereforewe have

$$\lim_{\delta \to 0} \widetilde{\mathcal{I}}^{\delta}_{j,F}(\tau) = \sup_{k \in \mathbb{R}} \big[\phi(k) - k\tau\big] = \mathcal{I}(\tau) \ .$$

□

2.3 Bounding the relaxed compliance with ficticious problems

The next crucial step consists in bounding the relaxed compliance, both from above and from below, in terms of fictitious problems. To this aim, we introduce the weak star lower semicontinuous envelope $\overline{\mathcal{C}}_{j,F}$ of the compliance, defined on $L^\infty(Q;[0,1])$ by:

$$\overline{\mathcal{C}}_{j,F}(\theta) := \inf \left\{ \liminf_h \mathcal{C}_{j,F}(\omega_h) \ : \ 1\!\!1_{\omega_h} \stackrel{*}{\rightharpoonup} \theta \right\} \qquad \forall\, \theta \in L^\infty(Q;[0,1]) \ . \tag{2.22}$$

It is immediate that $\overline{\mathcal{C}}_{j,F}(\theta)$ is bounded from below by the fictitious compliance defined in (1.7):

$$\widetilde{\mathcal{C}}_{j,F}(\theta) \le \overline{\mathcal{C}}_{j,F}(\theta) \qquad \text{on } L^\infty(Q;[0,1]) \ . \tag{2.23}$$

In contrast, it is a delicate matter to estimate $\overline{\mathcal{C}}_{j,F}(\theta)$ *from above* in terms of fictitious problems. This is the reason why we are led to deal with a modified stored energy density. Let us introduce the integrand $j_0 : \mathbb{R}^{3\times 3}_{\rm sym} \to \mathbb{R}$ defined by

$$j_0(z) := \sup \big\{ z \cdot \xi - j^*(\xi) \ : z \in \mathbb{R}^{3\times 3}_{\rm sym} \ , \ \det(\xi) = 0 \big\} \ . \tag{2.24}$$

It follows immediately from this definition that the inequality $j_0 \le j$ holds. Moreover, denoting by S_0 the set of degenerate tensors in $\mathbb{R}^{3\times 3}_{\rm sym}$ and by δ_{S_0} its indicator function, we see from (2.24) that j_0 is nothing else but the polar of $j^* + \delta_{S_0}$. As j_0 is convex continuous, the polar j_0^* agrees with the convexification of $j^* + \delta_{S_0}$, so that we have

$$j_0^*(\xi) = \inf \Big\{ \sum_i \alpha_i j^*(\xi_i) \ : \ \xi_i \in S_0 \ , \ \alpha_i \in [0,1], \ \sum_i \alpha_i = 1 \ , \ \sum_i \alpha_i \xi_i = \xi \Big\} \ . \tag{2.25}$$

In particular, as already noticed in [11, Lemma 3.1], it holds

$$j^*(\xi) = j_0^*(\xi) \qquad \forall\, \xi \in S_0 \ . \tag{2.26}$$

As a straightforward consequence, we obtain that j_0 has the crucial property stated in the next lemma.

Lemma 2.6 *There holds:*

$$\overline{j}(z) = \overline{j_0}(z) \qquad \forall\, z \in \mathbb{R}^{2\times 2}_{\rm sym} \ .$$

PROOF. Applying the algebraic identity (2.15) to the convex integrands j and j_0 yields, for every $z \in \mathbb{R}^{2\times 2}_{\rm sym}$:

$$\overline{j_0}(z) = \sup\left\{z\cdot\xi - (j_0)^*(E_0\xi) \ :\ \xi \in \mathbb{R}^{2\times 2}_{\rm sym}\right\} \quad , \quad \overline{j}(z) = \sup\left\{z\cdot\xi - j^*(E_0\xi) \ :\ \xi \in \mathbb{R}^{2\times 2}_{\rm sym}\right\} .$$

Then the lemma is proved since by (2.26) we have $(j_0)^*(E_0\xi) = j^*(E_0\xi)$ for all $\xi \in \mathbb{R}^{2\times 2}_{\rm sym}$. □

Remark 2.7 For the explicit computation of j_0, we refer to [11] and also to [1], where the Fenchel conjugate j_0^* of j_0 is considered. For instance, when $j(z) = |z|^2$, denoting by $\lambda_i(z)$ the eigenvalues of z, with $|\lambda_1(z)| \geq |\lambda_2(z)| \geq |\lambda_3(z)|$, one has $j_0(z) = \lambda_1(z)^2 + \lambda_2(z)^2$.
The effective strain potential j_0 appears to play a major role when studying the asymptotics of the optimal elastic compliance problem in the vanishing volume limit. In a 2D setting, it is shown in [3] that this limit reduces to the so called Michell's problem [12, 28]. The 3D counterpart of this compliance model was used in [2] and, in the context of 3D-2D reduction analysis, in [17] (see in particular Proposition 3.2), in order to describe the optimal design of a thin plate in the small volume case. A rough intuitive picture of the way this effective stress potential j_0^* comes out is the following: when the total volume of material vanishes, concentrations on subsets of lower dimension are necessary (possibly at a small scale) and on these sets stress tensors become degenerate. However, macroscopic full rank stress tensors ξ can be reached by using fine mixtures of degenerate tensors $\xi_i \in S_0$. The minimal stress energy needed to build these mixtures turns out to be given precisely by (2.25).

We are now going to exploit the integrand j_0 in order to obtain the desired upper bound for the relaxed compliance (as a counterpart of the lower bound (2.23)). At this stage, having in mind to handle possibly small volume fractions, we introduce the following additional assumption on the topological support of the load F: we ask that the Lebesgue measure of its r-neighbourhood is infinitesimal as $r \to 0^+$, namely

$$\lim_{r\to 0^+} \left|\{x \in Q \ :\ \mathrm{dist}(x, \mathrm{spt}(F)) < r\}\right| = 0 . \tag{2.27}$$

For instance, (2.27) holds whenever $\mathrm{spt}(F)$ is compactly contained into a 2-rectifiable subset of $\mathbb{R}^3$.

Proposition 2.8 *Let j_0 given by* (2.24). *Then, under assumption* (2.27)*, the following upper bound holds*

$$\overline{\mathcal{C}}_{j,F}(\theta) \ \leq\ \widetilde{\mathcal{C}}_{j_0,F}(\theta) \qquad \text{on } L^\infty(Q;[0,1]) . \tag{2.28}$$

PROOF. *Step 1.* We first show that (2.28) holds under the assumption that $\theta(x) \equiv \theta$, where is θ is a constant in $(0,1]$. To that aim we need to find a sequence $\{\theta_\varepsilon\}$ in $L^\infty(\Omega, \{0,1\})$ such that $\theta_\varepsilon \overset{*}{\rightharpoonup} \theta$ and $\limsup_\varepsilon \mathcal{C}_{j,F}(\theta_\varepsilon) \leq \widetilde{\mathcal{C}}_{j_0,F}(\theta)$. Our construction starts with a finite family $\{\Omega_i, 1 \leq i \leq I\}$ of connected disjoint regular open subsets of Q such that $\cup\Omega_i$ is of full measure in Q. Inside each subset Ω_i, we approach the constant θ by the characteristic function $\chi_i(y)$ of a periodic subset of $\mathbb{R}^3$. We ask that, for each $i \in I$, the function $\chi = \chi_i$ satisfies $\int_Y \chi(y) = \theta$ and is admissible in the following sense: for a suitable constant $C > 0$ and for all $k > 0$

$$\int_{kY} \chi(y)\, |Du|^2\, dy \ \leq\ C\, k^3 \int_{kY} \chi(y)\, \left(|e(u)|^2 + |u|^2\right)\, dy \qquad \forall u \in H^1_{\rm per}(kY) . \tag{2.29}$$

The latter condition is a scaled uniform Korn's inequality allowing to apply classical homogenization theory. It implies in particular that the set $\{\chi = 1\}$ is connected in a suitably strong way. However, the connectedness of the set $\{\chi_i = 1\}$ inside each Ω_i is not enough to our purposes. In fact, the approximation of θ under construction must be made of sets which are globally connected (included at the boundaries of the sets Ω_i), and also compatible with the load F. To ensure these properties, we consider a small r-neighbourhood Σ_r of all the boundaries of the sets Ω_i, and a r-neighborhood K_r of $\mathrm{spt}(F)$. Then we define

$$\chi^r(x,y) := \chi_i(y) \quad \text{if } x \in \Omega_i \setminus (\Sigma_r \cup K_r) \ , \qquad \chi^r(x,y) := 1 \quad \text{if } x \in K_r \cup \Sigma_r \ ,$$

and we consider the double-indexed sequence

$$\theta_\varepsilon^r(x) := \chi^r\big(x, \frac{x}{\varepsilon}\big) \ .$$

By construction, as $\varepsilon \to 0$, we have $\theta_\varepsilon^r \overset{*}{\rightharpoonup} \theta^r$, with

$$\theta^r(x) = \theta \quad \text{if } x \in Q \setminus (\Sigma_r \cup K_r) \ , \qquad \theta^r(x) = 1 \quad \text{if } x \in K_r \cup \Sigma_r \ .$$

In turn, as $r \to 0$, since by the assumption (2.27) and the regularity of the Ω_i's the measure of $\Sigma_r \cup K_r$ is infinitesimal, we have $\theta^r \overset{*}{\rightharpoonup} \theta$.
Since $\theta_\varepsilon^r = 1$ on $K_r \cup \Sigma_r$, and since all the χ_i's satisfy (2.29), the solutions u_ε which achieve the maximum value $\mathcal{C}_{j,F}(\theta_\varepsilon^r)$ in (2.22) have uniformly controlled norm in $H^1(K_r \cup \Sigma_r; \mathbb{R}^3)$ and can be extended in the void subregion $\{\theta_\varepsilon^r = 0\}$ so that they are precompact in $L^2(Q;\mathbb{R}^3)$. We can therefore pass to the limit in the bracket $\langle F, u_\varepsilon\rangle_{\mathbb{R}^3}$ and apply well-known homogenization results for the elastic energies on the perforated domain $Q \setminus (K_r \cup \Sigma_r)$ (see for instance [14, 21]). We obtain

$$\lim_{\varepsilon\to 0} \mathcal{C}_{j,F}(\theta_\varepsilon^r) \ = \ \sup_u \Big\{ \langle F, u\rangle_{\mathbb{R}^3} - \int_Q j_r^{hom}(x, e(u))\, dx \Big\} \tag{2.30}$$

where $j_r^{hom}(x,z) := \sum_{i\in I} 1\!\!1_{\Omega_i\setminus(K_r\cup\Sigma_r)}(x) j_{\chi_i}^{hom}(z) + 1\!\!1_{K_r\cup\Sigma_r}(x) j(z)$, being, for any χ, $j_\chi^{hom}(z) = \inf\limits_{\varphi\, Y-per} \Big\{ \int_Y j(z + \nabla\varphi(y))\chi(y)\, dy \Big\}$. It is convenient to majorize the right hand side of (2.30) in terms of the conjugate potential given by $(j_r^{hom})^*(x,\xi) := \sum_{i\in I} 1\!\!1_{\Omega_i\setminus(K_r\cup\Sigma_r)}(x)(j_{\chi_i}^{hom})^*(\xi) + 1\!\!1_{K_r\cup\Sigma_r}(x) j^*(\xi)$, being, for any χ,

$$(j_\chi^{hom})^*(\xi) \ = \ \inf \Big\{ \int_Y j^*(\zeta)\, dy \ : \ \zeta \ Y\text{-per}, \ \int_Y \zeta = \xi \ , \ \mathrm{div}\, \zeta = 0 \ , \ \zeta = 0 \text{ on } \{\chi = 0\} \Big\} \ .$$

Fix an arbitrary stress field $\sigma \in L^2(Q;\mathbb{R}^{3\times 3}_{\mathrm{sym}})$ such that $-\mathrm{div}(\sigma) = F$. Then, since $\theta_\varepsilon^r \overset{*}{\rightharpoonup} \theta^r$, by (2.30) and by standard duality, we get

$$\overline{\mathcal{C}}_{j,F}(\theta^r) \ \le \ \sum_{i=1}^{I} \int_{\Omega_i\setminus(\Sigma_r\cup K_r)} (j_{\chi_i}^{hom})^*(\sigma)\, dx \ + \ \int_{\Sigma_r\cup K_r} j^*(\sigma)\, dx \ .$$

Eventually we send r to zero in the last inequality. As $\theta^r \overset{*}{\rightharpoonup} \theta$, by the lower semicontinuity of relaxed functional $\overline{\mathcal{C}}_{j,F}$ and by using Beppo-Levi's Theorem in the right hand side, we obtain

$$\overline{\mathcal{C}}_{j,F}(\theta) \quad \le \quad \sum_{i=1}^{I} \int_{\Omega_i} (j_{\chi_i}^{hom})^*(\sigma)\, dx \ . \tag{2.31}$$

So far, we have obtained that the above inequality holds for all the admissible choices of the family $\{\Omega_i, \chi_i\}$. We notice that, by approximation, it can be extended to all measurable partitions $\{\Omega_i\}$ (not necessarily smooth), and all $\chi_i \in \mathcal{A}_{\theta_i}$ satisfying (2.29), being

$$\mathcal{A}_\theta := \big\{\chi \in \{0,1\}\ ,\ \chi\ Y\text{-periodic}\ ,\ \int_Y \chi(y)\,dy = \theta\big\}\ .$$

Now, we are going to optimize (2.31) with respect to the choice of the family $\{\Omega_i, \chi_i\}$. Following [1], we introduce the following stress potential, corresponding to the stiffest composite material that one can reach with a volume fraction θ when the average stress is given by ξ:

$$g(\xi,\theta)\ :=\ \inf\Big\{(j_\chi^{hom})^*(\xi)\ ,\ \chi \in \mathcal{A}_\theta\Big\}\ . \tag{2.32}$$

Assume for a moment that, for such potential g, the following claim holds true:

$$g(\xi,\theta)\ \le\ \inf\Big\{(j_\chi^{hom})^*(\xi)\ ,\ \chi \in \mathcal{A}_\theta\ ,\ \chi \text{ satisfies (2.29)}\Big\}\ \le\ \theta^{-1}\, j_0^*(\xi)\ . \tag{2.33}$$

Let $\mathcal{F}$ be the subfamily of $L^1(Q)$ consisting of all functions of the kind $f = \sum_i 1\!\!1_{\Omega_i}\,(j_{\chi_i}^{hom})^*(\sigma)$ (where Ω_i are the elements of a partition of Q and χ_i runs over $\mathcal{A}_{\theta_i}$). Then, from (2.31) and (2.33), we deduce that

$$\overline{\mathcal{C}}_{j,F}(\theta)\ \le\ \inf_{f\in\mathcal{F}} \int_Q f(x)\,dx\ =\ \int_Q (\operatorname*{ess\,inf}_{\mathcal{F}} f)\,dx\ \le\ \int_Q \theta^{-1} j_0^*(\sigma)\,dx\ . \tag{2.34}$$

Here in the second equality we passed the infimum under the integral thanks to a classical argument (see for instance[25]) which relies on the following stability property of $\mathcal{F}$: for any Borel set $B \subset Q$ and any couple (f_1, f_2) of elements of $\mathcal{F}$, we have $1\!\!1_B\, f_1 + 1\!\!1_{Q\setminus B}\, f_2 \in \mathcal{F}$. We conclude by taking the infimum in (2.34) with respect to all $\sigma \in L^2(Q; \mathbb{R}^{3\times3}_{\rm sym})$ such that $-\operatorname{div}(\sigma) = F$, which is nothing else but looking for the optimal stress tensor associated with the fictitious compliance problem (see (4.2)) after choosing j_0 to be the strain potential. This achieves the first step of the proof of Proposition 2.8.

Proof of Claim (2.33). First we establish a bound for $g(\xi,\theta)$ when ξ belongs to the class S_0 of degenerated tensors. In this case ξ has rank strictly less than 3 and there exists some nonzero vector e in the kernel of ξ. Therefore there exists some $\chi_e \in \mathcal{A}_\theta$ associated with a strip whose boundary is orthogonal to e so that $\zeta := \frac{\xi}{\theta}\,\chi_e$ is admissible in the variational problem which defines $\big(j_{\chi_e}^{hom}\big)^*(\xi)$. By (2.32), this implies that

$$g(\xi,\theta)\ \le\ \big(j_{\chi_e}^{hom}\big)^*(\xi)\ \le\ \theta^{-1} j^*(\xi) \qquad \forall \xi \in S_0\ . \tag{2.35}$$

By passing to the Fenchel conjugates with respect to ξ in (2.35) and recalling the definition (2.24), we get for all $z \in \mathbb{R}^{3\times3}_{\rm sym}$:

$$g^*(z,\theta)\ \ge\ \sup_{\xi\in S_0}\big\{z\cdot\xi - \theta^{-1} j^*(\xi)\big\}\ =\ \theta^{-1}\, j_0(\theta\, z)\ .$$

To conclude we exploit Lemma 4.2.5 in [1] which ensures that the map $\xi \mapsto g(\xi,\theta)$ is *convex*. Therefore, by passing again to the Fenchel conjugate,

$$g(\xi,\theta)\ =\ g^{**}(\xi,\theta)\ \le\ \theta^{-1} j_0^*(\xi) \qquad \forall \xi \in \mathbb{R}^{3\times3}_{\rm sym}\ . \tag{2.36}$$

Set now $\widetilde{g}(\xi,\theta) := \inf\left\{(j_\chi^{hom})^*(\xi)\ ,\ \chi\in\mathcal{A}_\theta\ ,\ \chi \text{ satifies (2.29)}\right\}$. Clearly $\widetilde{g}\geq g$. Conversely, given ξ and $\theta'\in(0,\theta)$, for every $\eta>0$ we can choose $\chi'\in\mathcal{A}_{\theta'}$ so that $\left(j_{\chi'}^{hom}\right)^*(\xi)\leq\ g(\xi,\theta')+\eta$. Further we may increase the set $\{\chi'=1\}$ to a set of total volume θ so that its characteristic function χ belongs to $\mathcal{A}_\theta$ and satisfies (2.29). It is straightforward to check that $\left(j_\chi^{hom}\right)^*(\xi)\leq\left(j_{\chi'}^{hom}\right)^*(\xi)$. Thus, recalling (2.32) and exploiting (2.36), we deduce that

$$\widetilde{g}(\xi,\theta)\ \leq\ \left(j_\chi^{hom}\right)^*(\xi)+\eta\ \leq\ (\theta')^{-1}\,(j_0^*(\xi)+\eta)\ .$$

The second inequality in (2.33) follows by sending η to zero and then θ' to θ.

Step 2. We show now that (2.28) holds for θ piecewise constant and positive. Fix a finite family $\{Q_i, i\in I\}$ of connected disjoint regular open subsets of Q such that $\cup Q_i$ is of full measure in Q. Given $\theta_i\in(0,1]$, we then consider $\theta\in L^\infty(Q,[0,1])$ defined by $\theta(x)=\theta_i$ on each Q_i. As in step 1, we consider an optimal stress field for the dual form (4.2) of the fictitious compliance, namely a field $\sigma\in L^2(Q;\mathbb{R}^{3\times3}_{\rm sym})$ such that $-\operatorname{div}(\sigma)=F$ and $\int_Q\theta^{-1}j_0^*(\sigma)\,dx=\ \widetilde{\mathcal{C}}_{j_0,F}(\theta)$. We are looking for an approximating sequence $\theta_\varepsilon\overset{*}{\rightharpoonup}\theta$ with $\theta_\varepsilon\in\{0,1\}$ and such that

$$\limsup_\varepsilon \mathcal{C}_{j,F}(\theta_\varepsilon)\leq\int_Q\theta^{-1}j_0^*(\sigma)\,dx \tag{2.37}$$

To that aim, we apply step 1 substituting Q with Q_i and F with F_i given by

$$\langle F_i,u\rangle\ :=\ \int_{Q_i}\sigma\cdot e(u)\,dx\qquad\forall u\in H^1(Q;\mathbb{R}^3).$$

By the regularity assumption on the Q_i's, all the F_i's are balanced loads in $H^{-1}(Q_i;\mathbb{R}^3)$ and share the property (2.27). Therefore, there exist sequences $\theta_{i,\varepsilon}\in L^\infty(Q_i,\{0,1\})$ and $\sigma_{i,\varepsilon}\in L^2(Q_i;\mathbb{R}^{3\times3}_{\rm sym})$ such that, for every i, $\theta_{i,\varepsilon}\overset{*}{\rightharpoonup}\theta_i$, $\sigma_{i,\varepsilon}=0$ a.e. on $\{\theta_{i,\varepsilon}=0\}$, $-\operatorname{div}\sigma_{i,\varepsilon}=F_i$ in $\mathbb{R}^3$ and

$$\limsup_\varepsilon\int_{Q_i}j^*(\sigma_{i,\varepsilon})\,dx\ \leq\ (\theta_i)^{-1}\int_{Q_i}j_0^*(\sigma)\,dx\ . \tag{2.38}$$

We then define $\sigma_\varepsilon\in L^2(Q;\mathbb{R}^{3\times3}_{\rm sym})$ and $\theta_\varepsilon\in L^\infty(Q;\{0,1\})$ by setting

$$\theta_\varepsilon=\theta_{i,\varepsilon}\ ,\qquad\sigma_\varepsilon=\sigma_{i,\varepsilon}\qquad\text{on }Q_i\ .$$

As $F=\sum_i F_i$, the stess field σ_ε is supported on $\{\theta_\varepsilon=1\}$ and satisfies $-\operatorname{div}\sigma_\varepsilon=F$. Thus, by (2.38), we obtain inequality (2.37):

$$\limsup_\varepsilon\mathcal{C}_{j,F}(\theta_\varepsilon)\ \leq\ \sum_i\limsup_\varepsilon\int_{Q_i}j^*(\sigma_{i,\varepsilon})\,dx\ \leq\ \sum_i(\theta_i)^{-1}\int_{Q_i}j_0^*(\sigma)\,dx\ =\ \int_Q\theta^{-1}j_0^*(\sigma)\,dx\ .$$

Step 3. In step 2, we have shown that the subset $E:=\left\{\theta\in L^\infty(Q,[0,1])\ :\ \overline{\mathcal{C}}_{j,F}(\theta)\leq\widetilde{\mathcal{C}}_{j_0,F}(\theta)\right\}$ contains all piecewise constants functions θ related to smooth finite partitions of our domain Q. To show the equality $E=L^\infty(Q,[0,1])$, we use the following implication holding for every sequence $\{\theta_n\}\subset L^\infty(Q,[0,1])$:

$$\theta_n\in E\ ,\quad\theta_n\geq\theta\text{ a.e. },\quad\theta_n\overset{*}{\rightharpoonup}\theta\quad\Longrightarrow\quad\theta\in E\ . \tag{2.39}$$

This can be easily checked by observing that $\mathcal{C}_{j_0,F}(\theta_n)\leq\mathcal{C}_{j_0,F}(\theta)$ whenever $\theta_n\geq\theta$ and by using the lower semicontinuity of $\overline{\mathcal{C}}_{j,F}$. Thus, from (2.39), we find successively that E contains all continuous functions, then all upper semicontinuous functions and finally, by the regularity of Lebesgue measure, all measurable functions $\theta:Q\mapsto[0,1]$. $\square$

2.4 Main results

By combining the lower and upper bounds (2.23) and (2.28) for the relaxed compliance, with the results on fictitious problems established in subsection 2.2, and with the key Lemma 2.6, we are able to establish the asymptotic behaviour of problems (2.3)-(2.4).

Theorem 2.9 (**convergence of volume penalized problems**)
Assume that the load F is balanced, belongs to $H^{-1}(Q;\mathbb{R}^3)$ and satisfies condition (2.27). Then for every fixed $k \in \mathbb{R}^+$, the following results hold:

(i) As $\delta \to 0$, the sequence $\phi^{\delta}_{j,F}(k)$ defined by (2.3) converges to the limit $\phi(k)$ defined by (2.12).

(ii) If $\omega^{\delta} \subseteq Q$ is a sequence of domains such that $\phi^{\delta}_{j,F}(k) = \mathcal{C}^{\delta}_{j,F}(\omega^{\delta}) + k|\omega^{\delta}| + o(1)$, up to subsequences there holds

$$\lim_{\delta\to 0} 1\!\!1_{\omega^{\delta}} = \overline{\theta} \quad \textit{weakly star in } L^{\infty}(Q;[0,1]) \ ,$$

where $\overline{\theta}$ solves problem (2.12).

PROOF. (i) From (2.23) and (2.28) we obtain the following double bounding for $\phi^{\delta}_{j,F}(k)$:

$$\widetilde{\phi}^{\delta}_{j,F}(k) \le \phi^{\delta}_{j,F}(k) \le \widetilde{\phi}^{\delta}_{j_0,F}(k) \ .$$

The thesis follows since, by Proposition 2.2 (ii) and Lemma 2.6, both sequences $\widetilde{\phi}^{\delta}_{j,F}(k)$ and $\widetilde{\phi}^{\delta}_{j_0,F}(k)$ converge to $\phi(k)$ as $\delta \to 0$.

(ii) Since the sequences $\widetilde{\phi}^{\delta}_{j,F}(k)$ and $\phi^{\delta}_{j,F}(k)$ have the same limit as $\delta \to 0$, the assumption $\phi^{\delta}_{j,F}(k) = \mathcal{C}^{\delta}_{j,F}(\omega^{\delta}) + k|\omega^{\delta}| + o(1)$ implies $\widetilde{\phi}^{\delta}_{j,F}(k) = \widetilde{\mathcal{C}}^{\delta}_{j,F}(1\!\!1_{\omega^{\delta}}) + k\int_Q 1\!\!1_{\omega^{\delta}}\,dx + o(1)$. As already noticed, by Proposition 2.2 (i), the sequence $\widetilde{\mathcal{C}}^{\delta}_{j,F}(\theta) + k\int_Q \theta\,dx$ Γ-converges to $\mathcal{C}^{lim}(\theta) + k\int_Q \theta\,dx$ in the weak star topology $L^{\infty}(Q;[0,1])$. Therefore, we may identify any cluster point of $1\!\!1_{\omega^{\delta}}$ with a solution $\overline{\theta}$ to problem (2.12).

□

Corollary 2.10 (**convergence of volume constrained problems**)
For every fixed $\tau \in [0,1]$ and for F satisfying the same assumptions as in Theorem 2.9, we have:

(i) As $\delta \to 0$, the sequence $\mathcal{I}^{\delta}_{j,F}(\tau)$ defined by (2.4) converges to the limit $\mathcal{I}(\tau)$ defined by (2.13).

(ii) If $\omega^{\delta} \subseteq Q$ is a sequence of domains with $|\omega^{\delta}| = \tau$ such that $\mathcal{I}^{\delta}_{j,F}(\tau) = \mathcal{C}^{\delta}_{j,F}(\omega^{\delta}) + o(1)$, then there exists $k \in \mathbb{R}$ such that $\phi^{\delta}_{j,F}(k) = \mathcal{C}^{\delta}_{j,F}(\omega^{\delta}) + k|\omega^{\delta}| + o(1)$ (so that Theorem 2.9 (ii) can be applied).

PROOF. (i) From (2.23) and (2.28) we obtain the following double bounding for $\mathcal{I}^{\delta}_{j,F}(\tau)$:

$$\widetilde{\mathcal{I}}^{\delta}_{j,F}(\tau) \le \mathcal{I}^{\delta}_{j,F}(\tau) \le \widetilde{\mathcal{I}}^{\delta}_{j_0,F}(\tau) \ .$$

The thesis follows since, by Proposition 2.2 (iii) and Lemma 2.6, both sequences $\widetilde{\mathcal{I}}^{\delta}_{j,F}(\tau)$ and $\widetilde{\mathcal{I}}^{\delta}_{j_0,F}(\tau)$ converge to $\mathcal{I}(\tau)$ as $\delta \to 0$.

(ii) Let now $\omega^\delta \subseteq Q$ be a sequence such that $|\omega^\delta| = \tau$ and $\mathcal{I}^\delta_{j,F}(\tau) = \mathcal{C}^\delta_{j,F}(\omega^\delta) + o(1)$. We have to show that there exists $k \in \mathbb{R}$ such that $\phi^\delta_{j,F}(k) = \mathcal{C}^\delta_j(\omega^\delta) + k\tau + o(1)$, or equivalently $\phi^\delta_{j,F}(k) - k\tau = \mathcal{I}^\delta_{j,F}(\tau) + o(1)$. The latter condition is satisfied because there exists $k \in \mathbb{R}$ such that $\phi(k) - k\tau = \mathcal{I}(\tau)$ (*cf.* first statement of Corollary 6.4). $\square$

3 A saddle point approach.

Let us firstly recall the notion of saddle point. Let X, Y be topological spaces, and let $\mathcal{L} : X \times Y \to \mathbb{R} \cup \{\pm\infty\}$. By definition $(\overline{x}, \overline{y})$ is a *saddle point* for $\mathcal{L}$ if

$$\mathcal{L}(\overline{x}, y) \le \mathcal{L}(\overline{x}, \overline{y}) \le \mathcal{L}(x, \overline{y}) \qquad \forall (x, y) \in X \times Y \ .$$

It is not difficult to check that this occurs if and only if

$$\overline{x} \in \operatorname{argmin}_X \Big(\sup_Y \mathcal{L}(x,y) \Big), \quad \overline{y} \in \operatorname{argmax}_Y \Big(\inf_X \mathcal{L}(x,y) \Big), \quad \inf_X \sup_Y \mathcal{L}(x,y) = \sup_Y \inf_X \mathcal{L}(x,y) \ . \tag{3.1}$$

We now take $X = L^\infty(Q; [0,1])$, endowed with the weak star topology, and $Y = H^1(Q; \mathbb{R}^3)$ endowed with the weak topology. On the product space $X \times Y$ we consider, for a fixed $k \in \mathbb{R}$, the Lagrangians

$$\mathcal{L}^\delta(\theta, u) := \langle F, u \rangle_{\mathbb{R}^3} - \int_Q \big[j(e^\delta(u)) - k \big] \, \theta \, dx$$

$$\mathcal{L}(\theta, u) := \begin{cases} \langle F, u \rangle_{\mathbb{R}^3} - \displaystyle\int_Q \big[\overline{j}\big(e_{\alpha\beta}(u)\big) - k \big] \theta \, dx & \text{if } u \in H^1_{KL}(Q; \mathbb{R}^3) \\ -\infty & \text{otherwise .} \end{cases}$$

Notice that the sequence of fictitious problems $\widetilde{\phi}^\delta_{j,F}(k)$ in (2.6) and the limit problem $\phi(k)$ in (2.12) satisfy respectively

$$\widetilde{\phi}^\delta_{j,F}(k) = \inf_X \sup_Y \mathcal{L}^\delta(\theta, u) \qquad \text{and} \qquad \phi(k) = \inf_X \sup_Y \mathcal{L}(\theta, u) \ .$$

Let us define $f_k : \mathbb{R}^{2\times2}_{\text{sym}} \times \mathbb{R}^{2\times2}_{\text{sym}} \to \mathbb{R}$ by

$$f_k(z, \xi) := \int_I \big[\overline{j}(z - x_3 \xi) - k \big]_+ \, dx_3 \ , \tag{3.2}$$

where $[\,\cdot\,]_+$ indicates the positive part.

With this notation, we can state the main result of this section.

Theorem 3.1

(i) For every $\delta > 0$, there exist saddle points $(\theta^\delta, u^\delta)$ for $\mathcal{L}^\delta$; we have that θ^δ is optimal for $\widetilde{\phi}^\delta_{j,F}(k)$ and u^δ is optimal for $\widetilde{\mathcal{C}}^\delta_{j,F}(\theta^\delta)$ in (2.5).

(ii) Up to subsequences, $(\theta^\delta, u^\delta)$ converge to a saddle point $(\overline{\theta}, \overline{u})$ for $\mathcal{L}$; we have that $\overline{\theta}$ is optimal for $\phi(k)$, and $\overline{u}$ is optimal for both $\mathcal{C}^{lim}(\overline{\theta})$ in (2.10) *and the following reformulation* (3.3) *holding for $\phi(k)$:*

$$\phi(k) = \sup\Big\{\langle F, u\rangle_{\mathbb{R}^3} - \int_Q \big[\overline{j}(e_{\alpha\beta}(u)) - k\big]_+ \, dx \ : \ u \in H^1_{KL}(Q;\mathbb{R}^3)\Big\} \tag{3.3}$$

$$= \sup\Big\{\langle \overline{F}, v\rangle_{\mathbb{R}^2} - \int_D f_k\big(e(v_1, v_2), \nabla^2 v_3\big)\, dx' \ : \ v_\alpha \in H^1(D)\ ,\ v_3 \in H^2(D)\Big\}\ . \tag{3.4}$$

Proof of Theorem 3.1. We set

$$\begin{aligned} G^\delta(\theta) &:= \sup\nolimits_Y \mathcal{L}^\delta(\theta, u) = \widetilde{\mathcal{C}}^\delta_{j,F}(\theta) + k\int_Q \theta\, dx \\ J^\delta(u) &:= \inf\nolimits_X \mathcal{L}^\delta(\theta, u) = \langle F, u\rangle_{\mathbb{R}^3} - \int_Q [j(e^\delta(u)) - k]_+\, dx\ . \end{aligned}$$

and

$$\begin{aligned} G(\theta) &:= \sup\nolimits_Y \mathcal{L}(\theta, u) = \mathcal{C}^{lim}(\theta) + k\int_Q \theta\, dx \\ J(u) &:= \inf\nolimits_X \mathcal{L}(\theta, u) = \begin{cases} \langle F, u\rangle_{\mathbb{R}^3} - \displaystyle\int_Q [\overline{j}(e_{\alpha\beta}(u)) - k]_+\, dx & \text{on } H^1_{KL}(Q;\mathbb{R}^3) \\ -\infty & \text{otherwise.} \end{cases} \end{aligned}$$

We now prove separately statements (i) and (ii).

(i) It is enough to show that

$$\exists\, \theta^\delta \in \text{argmin}_X G^\delta\ , \qquad \exists\, u^\delta \in \text{argmax}_Y J^\delta\ , \qquad \inf\nolimits_X G^\delta = \sup\nolimits_Y J^\delta\ . \tag{3.5}$$

Indeed by the characterization (3.1) this ensures that $(\theta^\delta, u^\delta)$ are saddle points for $\mathcal{L}^\delta$. Moreover θ^δ is optimal for $\widetilde{\phi}^\delta_{j,F}(k)$ by construction, and u^δ is optimal for $\mathcal{C}^\delta_{j,F}(\theta^\delta)$ because $G^\delta(\theta^\delta) = \mathcal{L}^\delta(\theta^\delta, u^\delta)$, so that

$$\widetilde{\mathcal{C}}^\delta_{j,F}(\theta^\delta) = \langle F, u^\delta\rangle_{\mathbb{R}^3} - \int_Q j(e^\delta(u^\delta))\, \theta^\delta\, dx\ .$$

Let us check the three conditions in (3.5).
The existence of $\theta^\delta \in \text{argmin}_X G^\delta$ is a consequence of the compactness of X, combined with the weak star lower semicontinuity of the map $\theta \mapsto G^\delta(\theta)$ (which is in fact the supremum over u of the weakly star continuous maps $\theta \mapsto \mathcal{L}^\delta(\theta, u)$).
Let us show that a minimizer in Y exists for the functional

$$-J^\delta(u) = \int_Q \big[j(e^\delta(u)) - k]_+\, dx - \langle F, u\rangle_{\mathbb{R}^3}\ .$$

By the convexity of the integrand $[j(z) - k]_+$, $-J^\delta$ is weakly lower semicontinuous on Y. Moreover, the quadratic growth from below imposed on j implies that $[j(z) - k]_+$ is coercive, so that any minimizing sequence $\{u_n\}$ for $-J^\delta$ satisfies $\sup_n \|e^\delta(u_n)\|_{L^2(Q)} < +\infty$. Then, by the Korn inequality, $\{u_n\}$ is weakly precompact (up to subtracting a rigid displacement, which is not restrictive thanks to the assumption that F is balanced). Hence $-J^\delta$ attains its minimum on Y.

Finally, since $\mathcal{L}^\delta(\theta, u)$ is convex in θ on the compact space X and concave in u on Y, the equality $\inf_X G^\delta = \sup_Y J^\delta$ holds by a standard commutation argument, see for instance [31, Proposition A.8].

(ii) We first claim that the sequences θ^δ and u^δ are precompact in X and Y respectively. The sequence θ^δ is clearly precompact since X is compact. Concerning u^δ, since it maximizes J^δ, we have

$$\int_Q [j(e^\delta(u^\delta)) - k]_+ \, dx - \langle F, u^\delta \rangle_{\mathbb{R}^3} = -J^\delta(u^\delta) \le -J^\delta(0) = [-k]_+ .$$

Since as noticed above the integrand $[j(z) - k]_+$ is coercive, the sequence $\{e^\delta(u^\delta)\}$ is bounded in $L^2(Q; \mathbb{R}^{3\times 3}_{\rm sym})$. Then, similarly as above, the Korn inequality ensures that $\{u^\delta\}$ is precompact in Y up to subtracting a rigid displacement.
Thus, up to subsequences, we know that the saddle points $(\theta^\delta, u^\delta)$ converge to some limit $(\overline{\theta}, \overline{u})$ in $X \times Y$. By applying [7, Theorem 2.4], we can conclude that $(\overline{\theta}, \overline{u})$ is a saddle point for $\mathcal{L}$ provided the following two inequalities hold for any pair $(\theta, u) \in X \times Y$:

$$\inf_{\theta^\delta \overset{*}{\rightharpoonup} \theta} \ \sup_{u^\delta \rightharpoonup u} \Big\{ \liminf_\delta \mathcal{L}^\delta(\theta^\delta, u^\delta) \Big\} \ge \mathcal{L}(\theta, u) \ , \tag{3.6}$$

$$\mathrm{lsc}_X \Big(\sup_{u^\delta \rightharpoonup u} \ \inf_{\theta^\delta \overset{*}{\rightharpoonup} \theta} \Big\{ \limsup_\delta \mathcal{L}^\delta(\theta^\delta, u^\delta) \Big\} \Big) \le \mathcal{L}(\theta, u) \tag{3.7}$$

where lsc_X denotes the lower semicontinuous envelope in X.
Inequalities (3.6) and (3.7) may be reformulated in a much simpler way. Indeed, thanks to the continuity of the maps $u \mapsto \langle F, u \rangle_{\mathbb{R}^3}$ on Y and $\theta \mapsto \int_Q \theta \, dx$ on X, inequality (3.6) holds provided, $\forall\, \theta^\delta \overset{*}{\rightharpoonup} \theta$, $\exists\, u^\delta \rightharpoonup u$ such that

$$\limsup_\delta \int_Q j(e^\delta(u^\delta)) \theta^\delta \, dx \le \int_Q \overline{j}(e_{\alpha\beta}(u)) \theta \, dx \ .$$

This is exactly the asymptotic property (AP1) proved in Lemma 2.4.
Concerning (3.7), let ut first show that it holds under the assumption $\inf_Q \theta > 0$. Similarly as above, inequality (3.7) holds provided, $\forall\, u^\delta \rightharpoonup u$, $\exists\, \theta^\delta \overset{*}{\rightharpoonup} \theta$ such that

$$\liminf_\delta \int_Q j(e^\delta(u^\delta)) \theta^\delta \, dx \ge \begin{cases} \displaystyle\int_Q \overline{j}(e_{\alpha\beta}(u)) \theta \, dx & \text{if } u \in H^1_{KL}(Q; \mathbb{R}^3) \\ +\infty & \text{otherwise} \ . \end{cases}$$

This is implied (with $\theta^\delta \equiv \theta$), by the asymptotic property (AP2) proved in Lemma 2.4.
It remains to get rid of the additional assumption $\inf_Q \theta > 0$. This can be done via a standard density argument. Indeed, for any θ we may find a sequence θ^h with $\inf_Q \theta^h > 0$ such that $\theta^h \overset{*}{\rightharpoonup} \theta$. Then, since the left hand side of (3.7) (call it for brevity $\mathcal{L}'(\theta, u)$) is weakly star lower semicontinuous in θ, and $\mathcal{L}(\theta, u)$ is weakly * continuous in θ, we obtain

$$\mathcal{L}'(\theta, u) \le \liminf_h \mathcal{L}'(\theta^h, u) \le \lim_h \mathcal{L}(\theta^h, u) = \mathcal{L}(\theta, u) \ .$$

Finally, since we have proved that $(\overline{\theta}, \overline{u})$ is a saddle point for $\mathcal{L}$, by using the characterization (3.1) we obtain immediately that $\overline{\theta}$ is optimal for $\phi(k)$, that $\phi(k)$ may be rewritten as in (3.3) (or in its $2D$-reformlation (3.4)), and that $\overline{u}$ is optimal for both $\mathcal{C}^{lim}(\overline{\theta})$ and problem (3.3).

4 A stress approach

In this section we focus attention on an alternative approach for studying the limiting behaviour of problems (2.3)-(2.4), based on duality arguments. Let us first recall a useful Convex Analysis lemma.

Lemma 4.1 *Let X, Y be Banach spaces. Let $A : X \to Y$ be a linear operator with dense domain $D(A)$. Let $\Phi : X \to \mathbb{R} \cup \{+\infty\}$ be convex, and $\Psi : Y \to \mathbb{R} \cup \{+\infty\}$ be convex lower semicontinuous. Assume there exists $u_0 \in D(A)$ such that $\Phi(u_0) < +\infty$ and Ψ is continuous at $A(u_0)$. Let Y^* denote the dual space of Y, A^* the adjoint operator of A, and Φ^*, Ψ^* the Fenchel conjugates of Φ, Ψ. Then*

$$-\inf_{u \in X} \Big\{ \Psi(Au) + \Phi(u) \Big\} = \inf_{\sigma \in Y^*} \Big\{ \Psi^*(\sigma) + \Phi^*(-A^*\sigma) \Big\} ,$$

where the infimum on the right hand side is achieved.

PROOF. See [13, Proposition 14]. □

By applying repeatedly Lemma 4.1, we deduce that:

– the compliance $\mathcal{C}_{j,F}(\Omega)$ defined by (1.1) and its fictitious counterpart $\widetilde{\mathcal{C}}_{j,F}(\theta)$ defined by (1.7) can be rewritten in dual form as

$$\mathcal{C}_{j,F}(\Omega) = \inf \Big\{ \int_Q j^*(\sigma)\, dx \ : \ \sigma \in L^2(Q; \mathbb{R}^{3\times3}_{\rm sym}) \ , \ \sigma = 0 \text{ on } Q \setminus \Omega \ , \ -\operatorname{div}(\sigma) = F \Big\} \tag{4.1}$$

$$\widetilde{\mathcal{C}}_{j,F}(\theta) = \inf \Big\{ \int_Q \theta^{-1} j^*(\sigma)\, dx \ : \ \sigma \in L^2(Q; \mathbb{R}^{3\times3}_{\rm sym}) \ , \ -\operatorname{div}(\sigma) = F \Big\} \ ; \tag{4.2}$$

– the rescaled compliance $\mathcal{C}^\delta_{j,F}(\omega)$ defined by (2.2) and its fictitious counterpart $\widetilde{\mathcal{C}}^\delta_{j,F}(\theta)$ defined by (2.5) can be rewritten in dual form as

$$\mathcal{C}^\delta_{j,F}(\omega) = \inf \Big\{ \int_Q j^*(\Pi^\delta(\sigma))\, dx \ : \ \sigma \in L^2(Q; \mathbb{R}^{3\times3}_{\rm sym}) \ , \ \sigma = 0 \text{ on } Q \setminus \omega \ , \ -\operatorname{div}(\sigma) = F \Big\} \tag{4.3}$$

$$\widetilde{\mathcal{C}}^\delta_{j,F}(\theta) = \inf \Big\{ \int_Q \theta^{-1} j^*(\Pi^\delta(\sigma))\, dx \ : \ \sigma \in L^2(Q; \mathbb{R}^{3\times3}_{\rm sym}) \ , \ -\operatorname{div}(\sigma) = F \Big\} \ , \tag{4.4}$$

where the operator $\Pi^\delta(\sigma)$ is defined by

$$\Pi^\delta \sigma := \begin{bmatrix} \sigma_{\alpha\beta} & \delta\sigma_{\alpha3} \\ \delta\sigma_{\alpha3} & \delta^2\sigma_{33} \end{bmatrix} \ ;$$

– the limit compliance $\mathcal{C}^{lim}(\theta)$ defined in (2.10) can be rewritten in dual form as

$$\mathcal{C}^{lim}(\theta) = \inf \Big\{ \int_Q \theta^{-1} \overline{j}^*(\sigma) dx \ : \ \sigma \in L^2(Q; \mathbb{R}^{2\times2}_{\rm sym}), \ -\operatorname{div}[[\sigma]] = (\overline{F}_1, \overline{F}_2), \ -\operatorname{div}^2[[x_3\sigma]] = \overline{F}_3 \Big\}; \tag{4.5}$$

– the formulations (3.3) and (3.4) of the limit problem $\phi(k)$ defined in (2.12) can be rewritten in dual form as

$$\phi(k) = \inf\Big\{\int_Q [\overline{j}-k]^*_+(\sigma)\,dx \;:\; \sigma\in L^2(Q;\mathbb{R}^{2\times2}_{\rm sym}), -\operatorname{div}[[\sigma]] = (\overline{F}_1,\overline{F}_2), -\operatorname{div}^2[[x_3\sigma]] = \overline{F}_3\Big\} \quad (4.6)$$

$$= \inf\Big\{\int_D f_k^*(\lambda,\eta)\,dx' \;:\; \lambda,\eta\in L^2(D;\mathbb{R}^{2\times2}_{\rm sym}), -\operatorname{div}\lambda = (\overline{F}_1,\overline{F}_2), \operatorname{div}^2\eta = \overline{F}_3\Big\}, \quad (4.7)$$

where the integrand f_k is defined by (3.2).

We are now in a position to give the main result of this section.

Theorem 4.2 (**convergence of stress tensors**)
Let $k\in\mathbb{R}$ be fixed. Let $\omega^\delta\subseteq Q$ be a sequence of domains such that $\phi^\delta_{j,F}(k) = \mathcal{C}^\delta_{j,F}(\omega^\delta)+k|\omega^\delta|+o(1)$, and $\mathbb{1}_{\omega^\delta}$ converges weakly star to $\overline{\theta}$ in $L^\infty(Q;[0,1])$. For every δ, let $\sigma^\delta\in L^2(Q;\mathbb{R}^{3\times3}_{\rm sym})$ be optimal for the dual form (4.3) of $\mathcal{C}^\delta_{j,F}(\omega^\delta)$. Then, up to subsequences, there holds

$$\lim_{\delta\to0}\Pi^\delta\sigma^\delta = E_0\overline{\sigma} \quad \textit{weakly in } L^2(Q;\mathbb{R}^{3\times3}_{\rm sym}) \ ,$$

where $\overline{\sigma}$ is optimal for the dual form (4.5) of $\mathcal{C}^{lim}(\overline{\theta})$, and also for the 3D dual form (4.6) of $\phi(k)$.

In order to prove Theorem 4.2, we begin to establish some preliminary lemmas.

Lemma 4.3 *Let $\{\sigma^\delta\}\subset L^2(Q;\mathbb{R}^{3\times3}_{\rm sym})$ be such that*

$$-\operatorname{div}(\sigma^\delta) = F \qquad \textit{and} \qquad \Pi^\delta\sigma^\delta \rightharpoonup \sigma \quad \textit{in } L^2(Q;\mathbb{R}^{3\times3}_{\rm sym}) \ .$$

Then

(i) $\sigma = E_0\overline{\sigma}$ for some $\overline{\sigma}\in L^2(Q;\mathbb{R}^{2\times2}_{\rm sym})$;

(ii) $-\operatorname{div}[[\overline{\sigma}]] = (\overline{F}_1,\overline{F}_2)$ and $-\operatorname{div}^2[[x_3\overline{\sigma}]] = \overline{F}_3$ [or equivalently $\operatorname{div}(E_0\overline{\sigma})+F\in\big(H^1_{KL}(Q;\mathbb{R}^3)\big)^\perp$].

Proof. (i) For a given smooth vector field $\xi: Q\to\mathbb{R}^3$, define $\Phi: Q\to\mathbb{R}^3$ by

$$\Phi_i(x',x_3) := \int_0^{x_3}\xi_i(x',s)\,ds \qquad i=1,2,3 \ .$$

Let $\{u^\delta\}$ be the sequence defined componentwise by:

$$u_1^\delta = \delta\Phi_1 \ , \qquad u_2^\delta = \delta\Phi_2 \ , \qquad u_3^\delta = \delta^2\Phi_3 \ ;$$

we have:

$$e^\delta(u^\delta) = \delta e_{\alpha\beta}(\Phi) + \sum_{i=1}^2\Big(\xi_i + \delta\frac{\partial\Phi_3}{\partial x_i}\Big)(e_i\otimes e_3 + e_3\otimes e_i) + \xi_3(e_3\otimes e_3) \ .$$

Thus

$$u^\delta\to0 \ \text{ in } H^1(Q;\mathbb{R}^3) \qquad \text{and} \qquad e^\delta(u^\delta)\to\sum_{i=1}^3\xi_i(e_i\otimes e_3+e_3\otimes e_i) \ \text{ in } L^2(Q;\mathbb{R}^{3\times3}_{\rm sym}) \ .$$

Hence

$$\begin{aligned}
0 &= \lim_\delta \langle F, u^\delta\rangle_{\mathbb{R}^3} = -\lim_\delta \langle \operatorname{div}(\sigma^\delta), u^\delta\rangle_{\mathbb{R}^3} = \lim_\delta \int_Q \sigma^\delta \cdot e(u^\delta)\, dx \\
&= \lim_\delta \int_Q \Pi^\delta \sigma^\delta \cdot e^\delta(u^\delta)\, dx = \int_Q \sigma \cdot \sum_{i=1}^3 \xi_i (e_i \otimes e_3 + e_3 \otimes e_i)\, dx = \sum_{i=1}^3 \int_Q \sigma_{3i}\xi_i \, dx \ .
\end{aligned}$$

By the arbitrariness of the functions ξ_i, we deduce that $\sigma_{3i} = 0$, and therefore $\sigma = E_0\overline{\sigma}$ for some $\overline{\sigma} \in L^2(Q; \mathbb{R}^{2\times 2}_{\rm sym})$.

(ii) Take $(v_1, v_2, v_3) \in H^1(D) \times H^1(D) \times H^2(D)$ and $u \in H^1_{KL}(Q;\mathbb{R}^3)$ related to each other through formula (1.4). Recalling (2.9) and (1.5), we get:

$$\langle \overline{F}, v\rangle_{\mathbb{R}^2} = \langle F, u\rangle_{\mathbb{R}^3} = -\langle \operatorname{div}(\sigma^\delta), u\rangle_{\mathbb{R}^3} = \int_Q \sigma^\delta \cdot e(u)\, dx = \int_Q \sigma^\delta_{\alpha\beta} \cdot \big(e(v_1,v_2) - x_3\nabla^2 v_3\big)\, dx \ .$$

Passing to the limit as $\delta \to 0$ in the right hand side, we infer

$$\begin{aligned}
\langle \overline{F}, v\rangle_{\mathbb{R}^2} &= \int_Q \overline{\sigma} \cdot \big(e(v_1,v_2) - x_3 \nabla^2 v_3\big)\, dx = \int_D \big([[\overline{\sigma}]] \cdot e(v_1,v_2) - [[x_3\overline{\sigma}]] \cdot \nabla^2 v_3\big)\, dx' \\
&= -\langle \operatorname{div}[[\overline{\sigma}]], (v_1,v_2)\rangle_{\mathbb{R}^2} - \langle \operatorname{div}^2[[x_3\overline{\sigma}]], v_3\rangle_{\mathbb{R}^2} \ ,
\end{aligned}$$

so that $-\operatorname{div}[[\overline{\sigma}]] = (\overline{F}_1, \overline{F}_2)$ and $-\operatorname{div}^2[[x_3\overline{\sigma}]] = \overline{F}_3$. These conditions may be equivalently formulated as $\operatorname{div}(E_0\overline{\sigma}) + F \in \big(H^1_{KL}(Q;\mathbb{R}^3)\big)^\perp$ by taking into account that $\langle F, u\rangle_{\mathbb{R}^3} = \langle \overline{F}, v\rangle_{\mathbb{R}^2}$ and

$$-\langle \operatorname{div}[[\overline{\sigma}]], (v_1,v_2)\rangle_{\mathbb{R}^2} - \langle \operatorname{div}^2[[x_3\overline{\sigma}]], v_3\rangle_{\mathbb{R}^2} = \int_Q (E_0\overline{\sigma}) \cdot e(u)\, dx = -\langle \operatorname{div}(E_0\overline{\sigma}), u\rangle_{\mathbb{R}^3} \ .$$

□

It will be useful to consider the (nonconvex) function $g_k : \mathbb{R}^{3\times 3}_{\rm sym} \to \mathbb{R}$ defined by

$$g_k(\xi) := \begin{cases} j^*(\xi) + k & \text{if } \xi \neq 0 \\ 0 & \text{otherwise.} \end{cases} \tag{4.8}$$

Next lemma allows to compute the convex envelope of g_k.

Lemma 4.4 *The Fenchel conjugate and biconjugate of the function $g_k : \mathbb{R}^{3\times 3}_{\rm sym} \to \mathbb{R}$ defined in* (4.8) *are given respectively by*

$$g_k^*(z) = [j(z) - k]_+ \tag{4.9}$$

$$g_k^{**}(\xi) = \inf_{\theta \in [0,1]} \big(\theta^{-1} j^*(\xi) + k\theta\big) \ . \tag{4.10}$$

Proof. There holds:

$$\begin{aligned}
g_k^*(z) &= \sup_{\xi \in \mathbb{R}^{3\times 3}_{\rm sym}} \big\{ z\cdot\xi - g_k(\xi)\big\} = \sup\Big\{ \sup_{\xi \neq 0} \big\{ z \cdot \xi - j^*(\xi) - k\big\} \ , \ 0\Big\} \\
&= \sup\big\{ j^{**}(z) - k, 0\big\} = [j(z) - k]_+ \ ,
\end{aligned}$$

where in the last equality we have used the identity $j = j^{**}$, holding since $z \mapsto j(z)$ is convex, proper, and lower semicontinuous on $\mathbb{R}^{3\times 3}_{\rm sym}$.
For the computation of f_k^{**}, we observe that the optimal θ in the r.h.s. of (4.10) is given by

$$\min \left\{ 1, \sqrt{\frac{j^*(\xi)}{k}} \right\} .$$

The corresponding value of the infimum turns out to be

$$\begin{cases} j^*(\xi) + k & \text{if } j^*(\xi) \geq k \\ 2\sqrt{k}\sqrt{j^*(\xi)} & \text{if } j^*(\xi) \leq k \ , \end{cases}$$

which agrees with the expression of the convex envelope of g_k.

□

Lemma 4.5 *The Fenchel conjugate f_k^* of the function $f_k : \mathbb{R}^{2\times 2}_{\rm sym} \times \mathbb{R}^{2\times 2}_{\rm sym} \to \mathbb{R}$ defined in (3.2) is given by*

$$f_k^*(\lambda, \eta) = \inf \left\{ \int_I [\overline{j} - k]_+^*(\sigma)\, dx_3 \ : \sigma \in L^2(I; \mathbb{R}^{2\times 2}_{\rm sym}) \ , \ [[\sigma]] = \lambda \ , \ -[[x_3 \sigma]] = \eta \right\} . \tag{4.11}$$

PROOF. Let us denote by $\psi_k(\lambda, \eta)$ the right hand side of (4.11), and let us show that $\psi_k^*(z, \xi) = f_k(z, \xi)$. We have indeed:

$$\begin{aligned} \psi_k^*(z,\xi) &= \sup_{\lambda,\eta} \Big\{ z\cdot\lambda + \xi\cdot\eta - \inf \Big\{ \int_I [\overline{j} - k]_+^*(\sigma)\, dx_3 \ : \sigma \in L^2(I; \mathbb{R}^{2\times 2}_{\rm sym}) \ , \ [[\sigma]] = \lambda \ , \ -[[x_3\sigma]] = \eta \Big\} \Big\} \\ &= \sup \Big\{ \int_I \big((z - x_3\xi)\cdot\sigma - [\overline{j} - k]_+^*(\sigma) \big)\, dx_3 \ : \sigma \in L^2(I; \mathbb{R}^{2\times 2}_{\rm sym}) \Big\} \\ &= \int_I [\overline{j} - k]_+^{**}(z - x_3\xi)\, dx_3 = f_k(z,\xi) \ . \end{aligned}$$

Passing to the Fenchel conjugate we obtain $f_k^*(\lambda, \eta) = \psi_k^{**}(\lambda, \eta)$. This gives the thesis by the identity $\psi_k^{**}(\lambda, \eta) = \psi_k(\lambda, \eta)$, holding since ψ_k is convex, proper, and lower semicontinuous on $\mathbb{R}^{2\times 2}_{\rm sym} \times \mathbb{R}^{2\times 2}_{\rm sym}$.
□

PROOF OF THEOREM 4.2. We claim that $\phi^\delta_{j,F}(k)$ and $\widetilde{\phi}^\delta_{j,F}(k)$ admit asymptotically the following integral representations:

$$\phi^\delta_{j,F}(k) = \int_Q g_k(\Pi^\delta \sigma^\delta)\, dx + o(1) \ . \tag{4.12}$$

$$\widetilde{\phi}^\delta_{j,F}(k) = \int_Q g_k^{**}(\Pi^\delta \sigma^\delta)\, dx + o(1) \ , \tag{4.13}$$

where g_k is the nonconvex integrand defined in (4.8) and g_k^{**} is its convex envelope which can be computed thanks to Lemma 4.4.
In order to prove (4.12), let us consider the sets $E^\delta := \{x \in \omega^\delta \ : \ \sigma^\delta \neq 0\} = \{x \in \omega^\delta \ : \ \Pi^\delta \sigma^\delta \neq 0\}$.
We first notice that

$$\mathcal{C}^\delta_{j,F}(E^\delta) = \mathcal{C}^\delta_{j,F}(\omega^\delta) \ . \tag{4.14}$$

Indeed, since $E^\delta \subseteq \omega^\delta$, there holds $\mathcal{C}^\delta_{j,F}(E^\delta) \geq \mathcal{C}^\delta_{j,F}(\omega^\delta)$. On the other hand, σ^δ is admissible in the dual form of $\mathcal{C}^\delta_{j,F}(E^\delta)$ because it vanishes outside E^δ, and by assumption it is optimal for the dual form (4.3) of $\mathcal{C}^\delta_{j,F}(\omega^\delta)$. Therefore we have $\mathcal{C}^\delta_{j,F}(E^\delta) \leq \int_Q j^*(\Pi^\delta \sigma^\delta)\, dx = \mathcal{C}^\delta_{j,F}(\omega^\delta)$.
Using (4.14) and the assumption made on ω^δ, we deduce that

$$\phi^\delta_{j,F}(k) \leq \mathcal{C}^\delta_{j,F}(E^\delta) + k|E^\delta| = \mathcal{C}^\delta_{j,F}(\omega^\delta) + k|E^\delta| \leq \mathcal{C}^\delta_{j,F}(\omega^\delta) + k|\omega^\delta| = \phi^\delta_{j,F}(k) + o(1)\ .$$

and hence

$$|E^\delta| = |\omega^\delta| + o(1)\ . \tag{4.15}$$

Then the assumptions made on ω^δ and σ^δ, combined with (4.15), give:

$$\begin{aligned}\phi^\delta_{j,F}(k) = \mathcal{C}^\delta_{j,F}(\omega^\delta) + k|\omega^\delta| + o(1) &= \int_Q \big\{ j^*(\Pi^\delta \sigma^\delta) + k 1\!\!1_{\omega^\delta} \big\}\, dx + o(1) \\ &= \int_Q \big\{ j^*(\Pi^\delta \sigma^\delta) + k 1\!\!1_{E^\delta} \big\}\, dx + o(1) \\ &= \int_Q g_k(\Pi^\delta \sigma^\delta)\, dx + o(1)\ .\end{aligned}$$

In order to prove (4.13), let us insert the dual form of $\widetilde{\mathcal{C}}^\delta_{j,F}(\theta)$ in the definition of $\widetilde{\phi}^\delta_{j,F}(k)$, and then let us we exchange the infima in θ and σ. We get

$$\widetilde{\phi}^\delta_{j,F}(k) = \inf \Big\{ \inf_{\theta \in L^\infty(Q;[0,1])} \int_Q \big(\theta^{-1} j^*(\Pi^\delta \sigma) + k\theta\big)\, dx \ :\ \sigma \in L^2(Q;\mathbb{R}^{3\times 3}_{\rm sym})\ ,\ -\operatorname{div}\sigma = F \Big\}\ .$$

Taking into account (4.10), we deduce that

$$\widetilde{\phi}^\delta_{j,F}(k) = \inf\{ \int_Q g_k^{**}(\Pi^\delta \sigma)\, dx \ :\ \sigma \in L^2(Q;\mathbb{R}^{3\times 3}_{\rm sym})\ ,\ -\operatorname{div}\sigma = F \Big\}\ . \tag{4.16}$$

By using (4.12), (4.16), and the convergence of both $\phi^\delta_{j,F}(k)$ and $\widetilde{\phi}^\delta_{j,F}(k)$ to $\phi(k)$, we obtain

$$\phi(k) = \lim_{\delta\to 0} \int_Q g_k(\Pi^\delta \sigma^\delta)\, dx \geq \liminf_{\delta\to 0} \int_Q g_k^{**}(\Pi^\delta \sigma^\delta)\, dx \geq \liminf_{\delta\to 0} \widetilde{\phi}^\delta_{j,F}(k) = \phi(k)\ ,$$

which implies (4.13).
Now we observe that, since the integrals in the right hand side of (4.13) remain bounded and since g_k^{**} is coercive, the sequence $\Pi^\delta \sigma^\delta$ turns out to be bounded in $L^2(Q;\mathbb{R}^{3\times 3}_{\rm sym})$. Up to subsequences, it converges weakly to some limit which, by Lemma 4.3, is of the form $E_0\overline{\sigma}$ for some $\overline{\sigma} \in L^2(Q;\mathbb{R}^{2\times 2}_{\rm sym})$

such that $-\operatorname{div}[[\overline{\sigma}]] = (\overline{F}_1, \overline{F}_2)$ and $-\operatorname{div}^2[[x_3\overline{\sigma}]] = \overline{F}_3$. Then:

$$\begin{aligned}
\phi(k) &=_{(1)} \lim_\delta \widetilde{\phi}^\delta_{j,F}(k) =_{(2)} \lim_\delta \int_Q g_k^{**}(\Pi^\delta \sigma^\delta)\, dx \geq_{(3)} \int_Q g_k^{**}(E_0 \overline{\sigma})\, dx \\
&=_{(4)} \inf_{\theta \in L^\infty(Q;[0,1])} \int_Q \big(\theta^{-1} \overline{j}^{\,*}(\overline{\sigma}) + k\theta\big)\, dx =_{(5)} \int_Q [j-k]^*_+(E_0\overline{\sigma})\, dx =_{(6)} \int_Q [\overline{j}-k]^*_+(\overline{\sigma})\, dx \\
&\geq_{(7)} \inf\Big\{ \int_Q [\overline{j}-k]^*_+(\sigma)\, dx \ : \ \sigma \in L^2(Q;\mathbb{R}^{2\times 2}_{\rm sym}) \ , \ -\operatorname{div}[[\sigma]] = (\overline{F}_1, \overline{F}_2) \ , \ -\operatorname{div}^2[[x_3\sigma]] = \overline{F}_3 \Big\} \\
&\geq_{(8)} \inf\Big\{ \int_D \Big(\inf_{\substack{\sigma \in L^2(Q;\mathbb{R}^{2\times 2}_{\rm sym}) \\ [[\sigma]] = \lambda \ , \ -[[x_3\sigma]] = \eta}} \int_I [\overline{j}-k]^*_+(\sigma)\, dx_3 \Big)\, dx' \ : \ -\operatorname{div}\lambda = (\overline{F}_1, \overline{F}_2) \ , \ \operatorname{div}^2 \eta = \overline{F}_3 \Big\} \\
&=_{(9)} \inf\Big\{ \int_D f_k^*(\lambda, \eta)\, dx' \ : \ \lambda, \eta \in L^2(D;\mathbb{R}^{2\times 2}_{\rm sym}) \ , \ -\operatorname{div}\lambda = (\overline{F}_1, \overline{F}_2) \ , \ \operatorname{div}^2 \eta = \overline{F}_3 \Big\} =_{(10)} \phi(k)
\end{aligned}$$

Here:

– equality (1) follows from Proposition 2.2 (ii);

– equality (2) follows from (4.13);

– inequality (3) follows from the convexity of the map g_k^{**} which ensures the weak lower semicontinuity on $L^2(Q;\mathbb{R}^{3\times 3}_{\rm sym})$ of the integral functional $\xi \mapsto \int_Q g_k^{**}(\xi)\, dx$;

– equalities (4) and (5) follow from (4.9) and (4.10) in Lemma 4.4;

– equality (6) follows by an easy algebraic calculation (*cf.* (2.15));

– inequality (7) follows from Lemma 4.3;

– inequality (8) follows by passing an infimum under the sign of integral;

– equality (9) follows from Lemma 4.5;

– equality (10) follows from the $2D$ dual formulation (4.7) of $\phi(k)$.

Since all the inequalities in the above chain must turn into equalities, we infer that $\overline{\sigma}$ is optimal for both the dual form (4.5) of $\mathcal{C}^{lim}(\overline{\theta})$ and the $3D$ dual form (4.6) of $\phi(k)$. □

5 Characterization of optimal limit configurations

We now turn attention to the optimality conditions for the limit problem. We give first the definition of optimal triples, and then their characterization as solutions of a suitable optimality system (see Theorem 5.2). As a remarkable consequence, we deduce that the limit compliance problem *always* admits a classical solution. Moreover, for *any* admissible load, such solution is *always* given by the characteristic function of two layers leaning on the top and bottom faces of the design region (see Corollary 5.4).

Definition 5.1 Let $(\overline{\theta}, \overline{u}, \overline{\sigma}) \in L^\infty(Q;[0,1]) \times H^1_{\rm KL}(Q;\mathbb{R}^3) \times L^2(Q;\mathbb{R}^{2\times 2}_{\rm sym})$. We say that $(\overline{\theta}, \overline{u}, \overline{\sigma})$ is an *optimal triple for* $\phi(k)$ if:

$(\cdot)$ $\overline{\theta}$ solves problem (2.12);

$(\cdot)$ $\overline{u}$ solves problem (3.3) and is optimal for $\mathcal{C}^{lim}(\overline{\theta})$ in its primal form (2.10);

$(\cdot)$ $\overline{\sigma}$ solves problem (4.6) and is optimal for $\mathcal{C}^{lim}(\overline{\theta})$ in its dual form (4.5).

Theorem 5.2 **(optimality conditions)**
Let $(\overline{\theta},\overline{u},\overline{\sigma}) \in L^\infty(Q;[0,1]) \times H^1_{\rm KL}(Q;\mathbb{R}^3) \times L^2(Q;\mathbb{R}^{2\times 2}_{\rm sym})$. *We have that* $(\overline{\theta},\overline{u},\overline{\sigma})$ *is an optimal triple for* $\phi(k)$ *if and only if it satisfies the conditions:*

$$-\operatorname{div}[[\overline{\sigma}]] = (\overline{F}_1,\overline{F}_2)\ , \quad -\operatorname{div}^2[[x_3\overline{\sigma}]] = \overline{F}_3 \tag{5.1}$$

$$\overline{\sigma} = \overline{\theta}\ \overline{j}\,'\big(e_{\alpha\beta}(\overline{u})\big) \tag{5.2}$$

$$\overline{\sigma} \in \partial\Big([\overline{j}(e_{\alpha\beta}(\overline{u})) - k]_+\Big)\ . \tag{5.3}$$

$$\overline{\theta}\,\big(\overline{j}(e_{\alpha\beta}(\overline{u})) - k\big) = [\overline{j}(e_{\alpha\beta}(\overline{u})) - k]_+ \tag{5.4}$$

Remark 5.3 It can be checked that, in terms of $2D$ variables, the inclusion (5.3) is equivalent to the following pair of conditions:

$$(\overline{\lambda},\overline{\eta}) \in \partial f_k\big(e(\overline{v}_1,\overline{v}_2),\nabla^2\overline{v}_3\big) \quad \text{ and } \quad f_k^*(\overline{\lambda},\overline{\eta}) = \int_I \big[\overline{j}-k\big]^*_+(\overline{\sigma}(x',x_3))\,dx_3\ ,$$

where $\overline{v}$ is related to $\overline{u}$ through the equality $e_{\alpha\beta}(\overline{u}) = e(\overline{v}_1,\overline{v}_2) - x_3\nabla^2\overline{v}_3$, whereas $\overline{\lambda}$ and $\overline{\eta}$ are given respectively by $[[\overline{\sigma}]]$, and $-[[x_3\overline{\sigma}]]$.

Proof of Theorem 5.2. Firstly note that, for every u and σ admissible respectively in the primal and dual forms (2.10) and (4.5) of $\mathcal{C}^{lim}(\overline{\theta})$, there holds:

$$\langle F,u\rangle_{\mathbb{R}^3} = -\langle \operatorname{div}(E_0\sigma),u\rangle_{\mathbb{R}^3} = \int_Q E_0\sigma\cdot e(u)\,dx = \int_Q \sigma\cdot e_{\alpha\beta}(u)\,dx\ . \tag{5.5}$$

Assume now that $(\overline{\theta},\overline{u},\overline{\sigma})$ is an optimal triple for $\phi(k)$ according to Definition 5.1. Then clearly (5.1) holds since $\overline{\sigma}$ must be admissible for problem (4.6). Moreover, since $\overline{\sigma}$ is optimal for the dual form (4.5) of $\mathcal{C}^{lim}(\overline{\theta})$, necessarily it must vanish on the set $\{\overline{\theta}=0\}$. Then, using the equivalence between the primal and the dual forms (2.10) and (4.5) of $\mathcal{C}^{lim}(\overline{\theta})$, we obtain:

$$\begin{aligned} 0 &= \int_Q \Big\{\overline{\sigma}\cdot e_{\alpha\beta}(\overline{u}) - \overline{\theta}\,\overline{j}(e_{\alpha\beta}(\overline{u})) - \overline{\theta}^{\,-1}\,\overline{j}^{\,*}(\overline{\sigma})\Big\}\,dx \\ &= \int_{Q\cap\{\overline{\theta}>0\}} \Big\{\overline{\theta}^{\,-1}\overline{\sigma}\cdot e_{\alpha\beta}(\overline{u}) - \overline{j}(e_{\alpha\beta}(\overline{u})) - \overline{j}^{\,*}(\overline{\theta}^{\,-1}\overline{\sigma})\Big\}\,\overline{\theta}\,dx\ , \end{aligned}$$

which yields (5.2) thanks to the Fenchel inequality.
Similarly, again using (5.5), the equivalence between (3.3) and (4.6) implies:

$$\int_Q \Big\{\overline{\sigma}\cdot e_{\alpha\beta}(\overline{u}) - [\overline{j}-k]_+(e_{\alpha\beta}(\overline{u})) - [\overline{j}-k]^*_+(\overline{\sigma})\Big\}\,dx = 0\ ,$$

which yields (5.3) thanks to the Fenchel inequality.
Finally, the equivalence between (2.12) and (3.3) implies:

$$\int_Q \Big\{\big(\overline{j}(e_{\alpha\beta}(\overline{u})) - k\big)\,\overline{\theta} - \big[\overline{j}-k\big]_+(e_{\alpha\beta}(\overline{u}))\Big\}\,dx = 0\ ,$$

which yields (5.4) since the integrand is nonpositive.

Viceversa, assume that $(\overline{\theta}, \overline{u}, \overline{\sigma})$ satisfy the optimality conditions (5.1)-(5.2)-(5.3)-(5.4).
By (5.1), $\overline{\sigma}$ is admissible for $\mathcal{C}^{lim}(\overline{\theta})$ in its dual form (4.5). Hence,

$$\begin{aligned}
&\langle F, \overline{u}\rangle_{\mathbb{R}^3} - \int_Q \overline{j}(e_{\alpha\beta}(\overline{u}))\,\overline{\theta}\,dx \\
&\leq \sup\Big\{\langle F, u\rangle_{\mathbb{R}^3} - \int_Q \overline{j}(e_{\alpha\beta}(u))\,\overline{\theta}\,dx \ :\ u \in H^1_{KL}(Q;\mathbb{R}^3)\Big\} \\
&= \inf\Big\{\int_Q \overline{\theta}^{\,-1}\overline{j}^{\,*}(\sigma)\,dx \ :\ \sigma \in L^2(Q;\mathbb{R}^{2\times 2}_{\rm sym})\ ,\quad -\mathrm{div}[[\sigma]] = (\overline{F}_1, \overline{F}_2)\ ,\ -\mathrm{div}^2[[x_3\sigma]] = \overline{F}_3\Big\} \\
&\leq \int_Q \overline{\theta}^{\,-1}\overline{j}^{\,*}(\overline{\sigma})\,dx\ .
\end{aligned}$$

Using (5.5) one sees that, thanks to (5.2), the first and the last term in the above chain of inequalities agree. Hence $\overline{u}$ and $\overline{\sigma}$ are optimal respectively for the primal and the dual forms (2.10) and (4.5) of $\mathcal{C}^{lim}(\overline{\theta})$.
Similarly, by (5.1), $\overline{\sigma}$ is admissible also for problem (4.6). Hence,

$$\begin{aligned}
&\langle F, \overline{u}\rangle_{\mathbb{R}^3} - \int_Q \big[\overline{j} - k\big]_+(e_{\alpha\beta}(\overline{u}))\,dx \\
&\leq \sup\Big\{\langle F, u\rangle_{\mathbb{R}^3} - \int_Q \big[\overline{j}(e_{\alpha\beta}(u)) - k\big]_+\,dx \ :\ u \in H^1_{KL}(Q;\mathbb{R}^3)\Big\} \\
&= \inf\Big\{\int_Q \big[\overline{j} - k\big]_+^*(\sigma)\,dx \ :\ \sigma \in L^2(Q;\mathbb{R}^{2\times 2}_{\rm sym})\ ,\ -\mathrm{div}[[\sigma]] = (\overline{F}_1, \overline{F}_2)\ ,\ -\mathrm{div}^2[[x_3\sigma]] = \overline{F}_3\Big\} \\
&\leq \int_Q \big[\overline{j} - k\big]_+(\overline{\sigma})\ .
\end{aligned}$$

Using (5.5) one sees that, thanks to (5.3), the first and the last term in the above chain of inequalities agree. Hence $\overline{u}$ and $\overline{\sigma}$ are optimal respectively for problems (3.3) and (4.6).
It remains to check that $\overline{\theta}$ is optimal for problem (2.12). Indeed we have

$$\begin{aligned}
\mathcal{C}^{lim}(\overline{\theta}) + k\int_Q \overline{\theta}\,dx &= \sup\Big\{\langle F, u\rangle_{\mathbb{R}^3} - \int_Q \big(\overline{j}(e_{\alpha\beta}(u) - k)\big)\,\overline{\theta}\,dx \ :\ u \in H^1_{KL}(Q;\mathbb{R}^3)\Big\} \\
&= \langle F, \overline{u}\rangle_{\mathbb{R}^3} - \int_Q \big(\overline{j}(e_{\alpha\beta}(\overline{u})) - k\big)\,\overline{\theta}\,dx \\
&= \langle F, \overline{u}\rangle_{\mathbb{R}^3} - \int_Q \big[\overline{j} - k\big]_+(e_{\alpha\beta}(\overline{u})) = \phi(k)\ ,
\end{aligned}$$

where in the first equality we have used the primal form (2.10) of $\mathcal{C}^{lim}(\overline{\theta})$, in the second equality the already proved optimality of $\overline{u}$ for such formulation, in the third equality the optimality condition (5.4), and finally in the fourth equality the already proved optimality of $\overline{u}$ for problem (3.3). □

Thanks to Theorem 5.2, it is an easy task to deduce qualitative properties of optimal material distributions $\overline{\theta}$, and to realize that classical shape solutions always exist, namely that $\overline{\theta}$ can be identified with characteristic functions.

To be more precise, let us associate with a given $u \in H^1_{KL}(Q;\mathbb{R}^3)$ the following subsets of Q:

$$\Sigma_k^+(u) := \{\overline{j}(e_{\alpha\beta}(u)) > k\} \ , \qquad \Sigma_k^-(u) := \{\overline{j}(e_{\alpha\beta}(u)) < k\} \ , \qquad \Sigma_k^0(u) := \{\overline{j}(e_{\alpha\beta}(u)) = k\} \ .$$

Recalling that $e_{\alpha\beta}(u) = e(v_1, v_2) - x_3\nabla^2 v_3$ for some $v_\alpha \in H^1(D)$ and $v_3 \in H^2(D)$, we deduce that, for any $x' \in D$, the fiber $\{x_3 \in I \, : \, (x', x_3) \in \Sigma_k^+(u)\}$ turns out to be the complement of an interval. Moreover, up to a set of null Lebesgue measure in $\mathbb{R}^3$, $\Sigma_k^0(u)$ may be identified with the cylinder

$$\big\{(x', x_3) \ : \ \nabla^2 v_3(x') = 0 \ , \ \overline{j}(e(v_1, v_2)(x')) = k\big\} \ ;$$

in particular, $\Sigma_k^0(u)$ is Lebesgue negligible in $\mathbb{R}^3$ if it happens that the set $\{x' \in D \, : \, \nabla^2 v_3(x') = 0\}$ is Lebesgue negligible in $\mathbb{R}^2$.

Corollary 5.4 *Let $(\overline{\theta}, \overline{u}, \overline{\sigma})$ be any optimal triple for $\phi(k)$. Then*

$$\overline{\theta} = 0 \quad on \ \Sigma_k^-(\overline{u}) \quad and \quad \overline{\theta} = 1 \quad on \ \Sigma_k^+(\overline{u}) \ .$$

Then two cases may occur, each one leading to the existence of a two-layers classical solution:

- *if $\Sigma_k^0(\overline{u})$ has null measure, $\overline{\theta}$ itself is a classical solution: it is indeed uniquely determined as the characteristic function of the set $\overline{\omega} = \Sigma_k^+(\overline{u})$, whose fibers are the complement of a (possibly empty) interval;*
- *if $\Sigma_k^0(\overline{u})$ has positive measure, the initial optimal triple $(\overline{\theta}, \overline{u}, \overline{\sigma})$ can be modified into another optimal triple $(\theta^*, \overline{u}, \sigma^*)$ such that θ^* is a classical solution, and more specifically it is the characteristic function of a set $\overline{\omega}$, whose fibers are the complement of a (possibly empty) interval.*

PROOF. Equation (5.4) implies that $\overline{\theta} = 0$ on $\Sigma_k^-(\overline{u})$, and that $\overline{\theta} = 1$ on $\Sigma_k^+(\overline{u})$. In case $\Sigma_k^0(\overline{u})$ has null measure, it follows straighforward that $\overline{\theta}$ is a classical solution : it agrees with the characteristic function of the set $\Sigma_k^+(\overline{u})$, whose fibers are the complement of an interval. In case $\Sigma_k^0(\overline{u})$ has positive measure, the optimal triple $(\overline{\theta}, \overline{u}, \overline{\sigma})$ can be modified into another optimal triple $(\theta^*, \overline{u}, \sigma^*)$ as follows. We leave $(\overline{\theta}, \overline{u}, \overline{\sigma})$ unchanged on the complement of $\Sigma_k^0(\overline{u})$. For $x = (x', x_3) \in \Sigma_k^0(\overline{u})$ we define:

$$\theta^*(x', x_3) := 1 - 1\!\!1_{I^*(x')}(x_3) \qquad \text{and} \qquad \sigma^* := \theta^* \overline{j}\,'(e_{\alpha\beta}(\overline{u})) \ ,$$

where $I^*(x') = (-\delta^*(x'), \delta^*(x'))$ is an interval chosen so that $[[\theta^*]] = [[\overline{\theta}]]$.
By construction, θ^* is the characteristic function of a set whose fibers are the complement of an interval. We have only to check that the triple $(\theta^*, \overline{u}, \sigma^*)$ satisfies the optimality system on $\Sigma_k^0(\overline{u})$.
Equation (5.1) holds because $[[\sigma^*]] = [[\overline{\sigma}]]$ and $[[x_3\sigma^*]] = [[x_3\overline{\sigma}]]$ (this follows from the definition of σ^*, taking into account that $[[\theta^*]] = [[\overline{\theta}]]$, and that $\overline{j}\,'(e_{\alpha\beta}(\overline{u}))$ is independent of x_3 on $\Sigma_k^0(\overline{u})$).
Equation (5.2) holds by the definition of σ^*.
Equation (5.3) holds because we know from (5.2) that σ^* equals either 0 or $\overline{j}\,'(e_{\alpha\beta}(\overline{u}))$, so that in both cases it belongs to $\partial\Big([\overline{j}(e_{\alpha\beta}(\overline{u})) - k]_+\Big)$.
Finally equation (5.4) holds trivially on $\Sigma_k^0(\overline{u})$.

□

6 Varying Lagrange multiplier and vanishing volume limit

The volume penalized version of the limit problem obtained in Section 2 is described, in terms of the Lagrange parameter k, by the function $\phi(k)$ in (2.12). By Corollary 5.4, $\phi(k)$ admits a classical solution $\overline{\theta}$, which is the characteristic function of some set. From now on, we will denote such a set by ω_k, since we want to emphasize its dependence on k. Indeed, we want to let the parameter k vary, and to examine the asymptotic behaviour as k tends to $+\infty$ of both the function $\phi(k)$ and the sets ω_k. This is an important issue, as it corresponds to study the asymptotic behaviour, as the volume fraction τ tends to 0^+, of the function $\mathcal{I}(\tau)$ defined in (2.13). We are going to show that this limit process allows to link our results with the models previously studied in [16, 17, 27].
Let us begin by extending the limit compliance in (2.11) to the class $\mathcal{M}^+(Q)$ of positive measures on Q, by setting

$$\mathcal{C}^{\lim}(\mu) := \sup_{v\in C^\infty(D;\mathbb{R}^3)} \Big\{ \langle \overline{F}, v\rangle_{\mathbb{R}^2} - \int_Q \overline{j}\big(e(v_1,v_2) - x_3\nabla^2 v_3\big)\, d\mu \Big\} \qquad \forall \mu \in \mathcal{M}^+(Q)\ . \tag{6.1}$$

The function $\phi(k)$ can be then recast as:

$$\phi(k) = \inf \Big\{ \mathcal{C}^{\lim}(\mu) + k \int d\mu \ :\ \mu = \theta\, dx\ ,\ \theta \in L^\infty(Q;[0,1]) \Big\}\ .$$

For $k > 0$, by multiplying μ by the scalar factor $\sqrt{2k}$, we are led to the equality

$$\frac{\phi(k)}{\sqrt{2k}} = \inf \Big\{ \mathcal{C}^{\lim}(\mu) + \frac{1}{2} \int d\mu \ :\ \mu = \theta\, dx\ ,\ \theta \in L^\infty(Q;[0,\sqrt{2k}]) \Big\}\ . \tag{6.2}$$

Therefore the positive quantity $\frac{\phi(k)}{\sqrt{2k}}$ remains finite and is monotone nonincreasing in k. It is natural to expect that its limit as $k \to \infty$ is positive and given by

$$\overline{m} := \inf \Big\{ \mathcal{C}^{\lim}(\mu) + \frac{1}{2} \int d\mu \ :\ \mu \in \mathcal{M}^+(Q) \Big\} \tag{6.3}$$

This fact is proved in the next theorem where in addition the asymptotics as $k \to +\infty$ of the optimal sets ω_k is established.

Theorem 6.1 **(asymptotics as $k \to +\infty$)**

(i) The function $\phi(k)$ defined by (2.12) *is concave continuous. For $k \le 0$ it is affine, whereas for $k > 0$ it has the following behaviour (being $\overline{m}$ defined by* (6.3)*):*

$$\frac{\phi(k)}{\sqrt{2k}} \quad \textit{is nonincreasing}\ , \qquad \lim_{k\to+\infty} \frac{\phi(k)}{\sqrt{2k}} \ =\ \overline{m}\ . \tag{6.4}$$

(ii) Let ω_k be an optimal set for $\phi(k)$. Then there holds $|\omega_k| \le \frac{C}{\sqrt{k}}$ for a suitable constant C and, up to a subsequence, as $k \to +\infty$ we have:

$$\sqrt{2k}\, \mathbb{1}_{\omega_k} \overset{*}{\rightharpoonup} \overline{\mu}\ , \qquad \textit{with } \overline{\mu} \textit{ optimal for problem } (6.3)\ . \tag{6.5}$$

In view of this result, it is important to understand how to deal in practice with problem (6.3).
In Proposition 6.2 below, we show that $\overline{m}$ can also be obtained through the linear constrained problem studied in [16, 17, 27], or equivalently through its dual formulation. This allows to determine optimal measures $\overline{\mu}$ for (6.3). Moreover, Proposition 6.2 will be useful for the proof of Theorem 6.1.
Let $\overline{\rho} : \mathbb{R}^{2\times 2}_{\rm sym} \to \mathbb{R}$ be defined by the identity $\overline{j}(z) = \frac{1}{2}\overline{\rho}^2(z)$, and let $\overline{\rho}^o$ be its polar function, namely $\overline{\rho}^o(\xi) = \sup\{z \cdot \xi \ :\ \overline{\rho}(z) \le 1\}$.

Proposition 6.2 *With the above notation,*

(i) problem (6.3) *has solutions and* $\overline{m}$ *is positive. Furthermore*

$$\overline{m} = \sup\Big\{\langle F,u\rangle_{\mathbb{R}^3} \ :\ u\in H^1_{KL}(Q;\mathbb{R}^3)\ ,\ \overline{\rho}(e_{\alpha\beta}(u))\le 1 \ \textit{a.e. on } Q\Big\} \tag{6.6}$$

$$= \sup\Big\{\langle \overline{F},v\rangle_{\mathbb{R}^2} \ :\ v_\alpha\in H^1(D), v_3\in H^2(D)\ ,\ \overline{\rho}\big(e(v_1,v_2)\pm \tfrac{1}{2}\nabla^2 v_3\big)\le 1 \ \textit{a.e. on } D\Big\} \tag{6.7}$$

and any optimal measure $\overline{\mu}$ *in* (6.3) *satisfies* $\mathcal{C}^{\lim}(\overline{\mu}) = \frac{1}{2}\int d\overline{\mu} = \frac{\overline{m}}{2}$*;*

(ii) it holds

$$\overline{m} = \min\Big\{\int \overline{\rho}^o\big(\frac{\lambda}{2}+\eta\big) + \int \overline{\rho}^o\big(\frac{\lambda}{2}-\eta\big) \ :\ \lambda,\eta\in\mathcal{M}(D,\mathbb{R}^{2\times 2}_{\rm sym})\,, \\ -\operatorname{div}\lambda = (\overline{F}_1,\overline{F}_2)\,,\ \operatorname{div}^2(\eta)=\overline{F}_3\Big\}\ ; \tag{6.8}$$

(iii) if $(\overline{\lambda},\overline{\eta})$ *is a solution of* (6.8)*, then a solution of* (6.3) *is given by*

$$\overline{\mu} := \overline{\rho}^o\Big(\frac{\overline{\lambda}}{2}+\overline{\eta}\Big)\otimes\delta_{\{x_3=1/2\}} + \overline{\rho}^o\Big(\frac{\overline{\lambda}}{2}-\overline{\eta}\Big)\otimes\delta_{\{x_3=-1/2\}}\ . \tag{6.9}$$

Remark 6.3 Theorem 6.1 and Proposition 6.2 give a rather complete picture of the asymptotic behaviour of the sets ω_k. Firstly, by Theorem 6.1, we know that the rescaled measures $\sqrt{2k}|\omega_k|$ converge to $\overline{m}$ as $k\to+\infty$. Moreover, according to Corollary 5.4, we can look at the asymptotic behaviour of the "profile functions" $h_k^{\pm}(x'): D\to[0,1]$ such that

$$\omega_k = \Big\{(x',x_3)\in Q \ :\ x_3\in\Big[-\frac{1}{2},-\frac{1}{2}+h_k^-(x')\Big]\cup\Big[\frac{1}{2}-h_k^+(x'),\frac{1}{2}\Big]\Big\} \tag{6.10}$$

(notice that, for those $x'\in D$ such that $\{x'\}\times(-\frac{1}{2},\frac{1}{2})\subset\omega_k$, by convention we may set $h_k^{\pm}(x')=\frac{1}{2}$). By Theorem 6.1, we have $\sqrt{2k}(h_k^+ + h_k^-)\overset{*}{\rightharpoonup}[[\overline{\mu}]]$ where $\overline{\mu}$ solves (6.3). Furthermore, if this solution is unique (which is the most usual case), then it is given by (6.9) and the asymptotics of h_k^+, h_k^- can be explicited separately. Indeed, in this case one can easily deduce by localizing the convergence (6.5) that

$$\sqrt{2k}\,h_k^+ \overset{*}{\rightharpoonup} \overline{\rho}^o\Big(\frac{\overline{\lambda}}{2}+\overline{\eta}\Big) \qquad\text{and}\qquad \sqrt{2k}\,h_k^- \overset{*}{\rightharpoonup} \overline{\rho}^o\Big(\frac{\overline{\lambda}}{2}-\overline{\eta}\Big)$$

Proof of Proposition 6.2. The functional $\mathcal{C}^{\lim}(\mu)+\frac{1}{2}\int d\mu$ is convex and lower semicontinuous with respect to the weak star convergence of measures. The existence of solutions for (6.3) follows from the direct method of calculus variations since minimizing sequences are tight on the compact subset Q. The positivity of $\mathcal{C}^{\lim}(\mu)$ implies that $\overline{m}>0$ (note that $\mathcal{C}^{\lim}(0)=+\infty$).
Let $m\in\mathbb{R}^+$ be arbitrary. Recalling the definition (6.1) of $\mathcal{C}^{\lim}(\mu)$, we apply the same inf-sup commutation argument as in Section 3, and by using the 2-homogeneity of $\overline{j}$, we obtain:

$$\inf\Big\{\mathcal{C}^{\lim}(\mu)\ :\ \int d\mu\le m\Big\} = \sup_{v\in C^\infty(D;\mathbb{R}^3)} \inf\Big\{\langle\overline{F},v\rangle_{\mathbb{R}^2} - \int_Q \overline{j}\big(e(v_1,v_2)-x_3\nabla^2 v_3\big)\,d\mu\ ,\ \int d\mu\le m\Big\}$$
$$= \sup_{v\in C^\infty(D;\mathbb{R}^3)}\Big\{\langle\overline{F},v\rangle_{\mathbb{R}^2} - m\sup_{x\in Q}\big|\overline{j}\big(e(v_1,v_2)-x_3\nabla^2 v_3\big)\big|\Big\} = \frac{m_0^2}{2m}\ ,$$

where m_0 denotes the right hand side of (6.7). Then, by the definition (6.3) of $\overline{m}$, since the function $m \mapsto \left(\frac{m_0^2}{2m} + \frac{m}{2}\right)$ attains its minimum on $\mathbb{R}^+$ at $m = m_0$, we deduce that $\overline{m} = m_0$. Thus any optimal μ satisfies $\mathcal{C}^{\lim}(\mu) = \frac{\overline{m}}{2} = \frac{1}{2}\int d\mu$ which proves statement (i).
Statement (ii) follows directly from [16, Lemma 3.9].
Finally, let $\overline{\mu}$ be given by (6.9), with $(\overline{\lambda}, \overline{\eta})$ solution of (6.8). If $\overline{v} = (\overline{v}_1, \overline{v}_2, \overline{v}_3)$ is a solution of (6.7), by [16, Proposition 3.10], it holds

$$\overline{\rho}^o\Big(\frac{\overline{\lambda}}{2} \pm \overline{\eta}\Big) = \langle \frac{\overline{\lambda}}{2} \pm \overline{\eta}\,,\, e(\overline{v}_1, \overline{v}_2) \pm \frac{1}{2}\nabla^2 \overline{v}_3 \rangle_{\mathbb{R}^2} \ .$$

By the definition of $\overline{\rho}^o$, this implies $\overline{\rho}\Big(e(\overline{v}_1, \overline{v}_2) \pm \frac{1}{2}\nabla^2 \overline{v}_3\Big) \geq 1$ a.e. on D. On the other hand, since by construction $\overline{v}$ is admissible in problem (6.7), it holds $\overline{\rho}\big(e(\overline{v}_1, \overline{v}_2) \pm \frac{1}{2}\nabla^2 \overline{v}_3\big) \leq 1$ a.e. on D. We conclude that $\overline{\rho}\big(e(\overline{v}_1, \overline{v}_2) \pm \frac{1}{2}\nabla^2 \overline{v}_3\big) = 1$ a.e. on D, and hence that $\overline{j}\big(e(\overline{v}_1, \overline{v}_2) \pm \frac{1}{2}\nabla^2 \overline{v}_3\big) = 1/2$ a.e. on D. Combined with the equality $\langle F, \overline{v}\rangle = \overline{m}$, this gives

$$\mathcal{C}^{\lim}(\overline{\mu}) = \overline{m} - \frac{1}{2}\int d\overline{\mu} \ ,$$

and statement (iii) follows. □

Proof of Theorem 6.1.

(i) The function given in (2.12) is an infimum of affine functions of k. It is therefore concave. Moreover it is nonnegative for $k \geq 0$ whereas, for $k \leq 0$, the infimum in formulation (2.12) is attained for $\theta \equiv 1$, so that $\phi(k) - k\tau = k(1-\tau) + \mathcal{C}_{j,F}(Q)$. Thus ϕ is continuous as it is concave and finite over the whole set $\mathbb{R}$. On the other hand, owing to (6.2) and (6.3), we have trivially

$$\lim_{k\to+\infty} \frac{\phi(k)}{\sqrt{2k}} \ = \ \inf_{k>0} \frac{\phi(k)}{\sqrt{2k}} \ \geq \ \overline{m} \ .$$

In order to show the converse inequality, we exploit the formulation (3.3) for $\phi(k)$ in which we insert the change of variable $w = u/\sqrt{2k}$. We obtain, for every $k \in \mathbb{R}^+$,

$$\frac{\phi(k)}{\sqrt{2k}} = \sup_{w\in H^1_{KL}(Q;\mathbb{R}^3)} \Big\{ \langle F, w\rangle_{\mathbb{R}^3} - \sqrt{2k}\int_Q \big[\overline{j}(e_{\alpha\beta}(w)) - \frac{1}{2}\big]_+ \,dx \Big\} \ , \tag{6.11}$$

Thanks to (6.11), there exists a sequence $\{w_k\}$ in $H^1_{KL}(Q;\mathbb{R}^3)$ such that

$$\lim_{k\to+\infty} \frac{\phi(k)}{\sqrt{2k}} = \lim_{k\to+\infty} \Big\{ \langle F, w_k\rangle_{\mathbb{R}^3} - \sqrt{2k}\int_Q \big[\overline{j}(e_{\alpha\beta}(w_k)) - \frac{1}{2}\big]_+ \,dx \Big\} \ . \tag{6.12}$$

Up to a rigid displacement (which is not restrictive thanks to the assumption that F is balanced), the sequence $\{w_k\}$ is bounded in the H^1-norm. Indeed, by using the Korn inequality, the coercivity of $[\overline{j}(z) - k]_+$, the inequality $\phi(k) \geq 0$, and the assumption $F \in H^{-1}(Q;\mathbb{R}^3)$, we may find positive constants C_i such that

$$\|w_k\|^2_{H^1} \leq C_1 \|e(w_k)\|^2_{L^2} \leq C_2\sqrt{2k}\int_Q \big[\overline{j}(e_{\alpha\beta}(w_k)) - \frac{1}{2}\big]_+ \leq C_2 \langle F, w_k\rangle_{\mathbb{R}^3} \leq C_3 \|w_k\|_{H^1} \ .$$

Then up to a subsequence w_k converge weakly in $H^1(Q;\mathbb{R}^3)$ to some function $\overline{w}$, which satisfies

$$\int_Q \left[\overline{j}(e_{\alpha\beta}(\overline{w})) - \frac{1}{2}\right]_+ dx \ \leq \ \liminf_{k\to+\infty} \int_Q \left[\overline{j}(e_{\alpha\beta}(w_k)) - \frac{1}{2}\right]_+ dx \ = \ 0 \ .$$

In particular, we have $\overline{j}(e_{\alpha\beta}(\overline{w})) \leq \frac{1}{2}$ a.e. in Q. Recalling that $F \in H^{-1}(Q;\mathbb{R}^3)$, by using (6.12) and the characterization (6.6), it follows that

$$\lim_{k\to+\infty} \frac{\phi(k)}{\sqrt{2k}} \ \leq \ \lim_{k\to+\infty} \langle F, w_k\rangle_{\mathbb{R}^3} \ = \ \langle F, \overline{w}\rangle_{\mathbb{R}^3} \ \leq \ \overline{m} \ ,$$

which completes the proof of (6.4).

(ii) Let now ω_k be an optimal set for $\phi(k)$. Then for each k the measure $\mu_k := \sqrt{2k}\,1\!\!1_{\omega_k}$ is optimal in (6.2), namely

$$\frac{\phi(k)}{\sqrt{2k}} = \mathcal{C}(\mu_k) + \frac{1}{2}\int d\mu_k \ .$$

Since $\mathcal{C}(\mu_k) \geq 0$ and since by the monotonicity property $\phi(k) \leq \phi(1)\sqrt{k}$, the above equation implies that the integral $\int d\mu_k$ remains uniformly bounded. Then up to a subsequence there exists $\overline{\mu}$ such that $\mu_k \overset{*}{\rightharpoonup} \overline{\mu}$. By using statement (6.4) already proved, the weak star semicontinuity of the map $\mu \mapsto \mathcal{C}(\mu)$, and the definition (6.3) of $\overline{m}$, we obtain

$$\overline{m} = \lim_{k\to+\infty} \frac{\phi(k)}{\sqrt{2k}} = \lim_{k\to+\infty} \left\{ \mathcal{C}(\mu_k) + \frac{1}{2}\int d\mu_k \right\} \geq \mathcal{C}(\overline{\mu}) + \frac{1}{2}\int d\overline{\mu} \geq \overline{m} \ . \tag{6.13}$$

We deduce that all the inequalities in (6.13) must turn into equalities, so that $\overline{\mu}$ is optimal for problem (6.3). □

We turn now to our initial goal which consists in determining the vanishing volume fraction limit that is the limit behavior as $\tau \to 0$ of $\mathcal{I}(\tau)$ defined by (2.13). We notice that, by (6.4), the function $k \mapsto \phi(k) - k\tau$ behaves like $\sqrt{2k}\overline{m} - k\tau$ for $k \to +\infty$ and therefore the Lagrange multiplier $k(\tau)$ which for a given τ maximizes (2.13) should behave asymptotically like $k(\tau) = \overline{m}^2/(2\tau^2)$, so that $k(\tau) \to +\infty$ as $\tau \to 0$. In fact it is easy to deduce from Theorem 6.1 the following

Corollary 6.4 **(asymptotics as $\tau \to 0^+$)**
For every $\tau \in (0,1)$, the maximal value $\mathcal{I}(\tau)$ in formulation (2.13) is achieved. Furthermore we have

$$\lim_{\tau\to 0} \ \tau\,\mathcal{I}(\tau) \ = \ \frac{\overline{m}^2}{2} \ . \tag{6.14}$$

PROOF. First we notice that

$$\mathcal{I}(\tau) \ = \ \sup\left\{\phi(k) - k\tau \ : \ k \geq 0\right\} \ .$$

Indeed, as seen in the proof of Theorem 6.1, for $k \leq 0$, we have $\phi(k) - k\tau = k(1-\tau) + \mathcal{C}_{j,F}(Q)$ which is is maximal for $k = 0$. On the other hand, due to (6.4), the sublinear growth at infinity for positive k implies that $\phi(k) - k\tau \to -\infty$ as $k \to +\infty$. Thus, by the continuity of the function $\phi(k)$, the maximum in (2.13) is attained. Futhermore, by (6.4), we have for every $\tau \in [0,1]$ the following lower bound

$$\mathcal{I}(\tau) \ = \ \sup_{k\geq 0} \ \{\phi(k) - k\tau\} \ \geq \ \sup_{k\geq 0} \ \overline{m}\sqrt{2k} - k\,\tau \ = \ \frac{\overline{m}^2}{2\tau} \ . \tag{6.15}$$

Then the proof of (6.14) is achieved if we are able to show that

$$\limsup_{\tau\to 0} \ \tau\,\mathcal{I}(\tau) \ \leq \ \frac{\overline{m}^2}{2} \ . \tag{6.16}$$

Consider sequences $\tau_n \to 0$ and $k_n \in \mathbb{R}^+$ such that

$$\limsup_{\tau\to 0} \tau\,\mathcal{I}(\tau) \ \leq \ \limsup_{\tau\to 0} \big[\sup_{k\in\mathbb{R}^+} \{\tau\phi(k) - k\tau^2\}\big] \ = \ \lim_n \big(\tau_n\phi(k_n) - k_n\tau_n^2\big). \tag{6.17}$$

Possibly passing to a subsequence, we may assume that the sequence $\{k_n\}$ admits a limit $\overline{k} \in \mathbb{R}^+ \cup \{+\infty\}$. If $\overline{k} \in \mathbb{R}^+$, then $\phi(k_n)$ remains bounded and, by (6.17), we find that $\lim_{\tau\to 0} \tau\,\mathcal{I}(\tau) = 0$ which is in contradiction with (6.15). Therefore $k_n \to +\infty$ and, exploiting (6.4) once more, we deduce (6.16) by passing to the limit in the following upper bound

$$\tau_n\phi(k_n) - k_n\tau_n^2 \quad \leq \quad \frac{1}{2}\left[\frac{\phi(k_n)}{\sqrt{2k_n}}\right]^2 \ .$$

□

7 An explicit example of optimal plate and some perspectives

The $2D$ stress formulation (4.7) of $\phi(k)$ and the optimality conditions given by Theorem 5.2 make easy to design optimal plates. Even if some numerical computation may be needed, the difficulties are in no way comparable with those encountered when designing optimal structures in the general case (see [1]). Indeed, by solving the stress problem (4.7), we can get the equilibrium stress fields $(\bar\lambda, \bar\eta)$. As the optimal displacement field $\bar u$ is a Kirchoff-Love displacement, $e(\bar u)$ is, owing to (1.5), a linear function of x_3. The coefficients of this linear function can easily be deduced from the knowledge of $(\overline{\lambda}, \overline{\eta})$. Then an optimal set ω_k for $\phi(k)$ can be explicitly determined by using the optimality conditions.

Let us now describe a case where all these computations can be performed in an analytical way. We first restrict to the case when no function depends on the x_2 variable. Indeed we take $Q = [-2, 2] \times \mathbb{R} \times [-1/2, 1/2]$, which is infinite in the x_2-direction, and a balanced load $F = (F_1, 0, F_3)$, with F_1, F_3 independent of the x_2 variable. Therefore in the sequel we drop any reference to the variable x_2.

In such framework, the stress problem (4.7) becomes trivial since there is a unique competitor $(\overline{\lambda}, \overline{\eta})$: indeed the differential equations $\lambda' = \overline{F}_1$, $\eta'' = \overline{F}_3$ (where the derivation is relative to the x_1 variable) uniquely determine stresses supported in the compact interval $[-2, 2]$. If we further assume that $F_1 = 0$ then, by (2.9), $\overline{F}_1 = 0$ and so the solution of the first of these differential equation is $\overline{\lambda} = 0$.

Now, in order to make further computations as simple as possible, we assume that the considered material is an isotropic linear elastic one with a vanishing Poisson coefficient : the energy density j is given, for any $z \in \mathbb{R}^{3\times 3}_{\rm sym}$, by $j(z) = \frac{1}{2}|z|^2$. Then for any $z \in \mathbb{R}^{2\times 2}_{\rm sym}$, equation (2.8) reduces to $\bar j(z) = \frac{1}{2}|z|^2$ and the optimality condition (5.2) simply reads $\overline{\sigma} = 1\!\!1_{\omega_k}\, e(\overline{u})$.

Owing to equation (1.5) we know that $e_{\alpha\beta}(\overline{u})$ takes the form: $e_{\alpha\beta}(\overline{u}) = A + Bx_3$ where A and B are rank one matrices depending only on the x_1 variable. Owing to Corollary 5.4 we also know that

ω_k may be written as in (6.10) for suitable profile functions $h_k^\pm$. Thus the condition $[[\overline{\sigma}]] = \overline{\lambda} = 0$ reads

$$A(h_k^+ + h_k^-) - \frac{B}{2}((h_k^- - h_k^+)(1 - (h_k^+ + h_k^-))) = 0 \ . \tag{7.1}$$

Moreover Corollary 5.4 states that $[-\frac{1}{2} + h_k^-, \frac{1}{2} - h_k^+]$ coincides with the set $\{|e(\overline{u})|^2 \leq 2k\}$, so that $(-\frac{1}{2} + h_k^-)$ and $(\frac{1}{2} - h_k^+)$ are the two solutions of the following quadratic equation

$$|B|^2\, x^2 + 2(B \cdot A)\, x + (|A|^2 - 2k) = 0 \ . \tag{7.2}$$

We deduce that $B(h_k^+ - h_k^-) = 2A$. Substituting this equality into (7.1) we get $A = 0$. Hence the two solutions of (7.2) are $(-\frac{1}{2} + h_k^-) = -\frac{\sqrt{2k}}{|B|}$, $(\frac{1}{2} - h_k^+) = \frac{\sqrt{2k}}{|B|}$. Finally the condition $[[-x_3\overline{\sigma}]] = \overline{\eta}$ gives

$$\overline{\eta} = \frac{B}{3}\Big((\frac{1}{2} - h_k^+)^3 - (-\frac{1}{2} + h_k^-)^3 - \frac{1}{4}\Big) = \frac{B}{12}\Big(8(\frac{1}{2} - h_k^+)^3 - 1\Big) \ .$$

Therefore, the total thickness $2h_k^+ = 2h_k^- = h_k^+ + h_k^-$ can be obtained in terms of $\overline{\eta}$ as the unique solution of the third degree polynomial equation

$$\frac{6|\overline{\eta}|}{\sqrt{2k}}(x-1) + 3x - 3x^2 + x^3 = 0 \ .$$

As a consequence, when $k \to +\infty$, $h_k^\pm$ are infinitesimal, and more precisely we have the asymptotical equivalence:

$$h_k^+ = h_k^- \sim \frac{|\overline{\eta}|}{\sqrt{2k}} \tag{7.3}$$

Such equivalence is consistent with the results obtained in Section 6, *cf.* in particular Remark 6.3 by taking into account that, in the present framework, $(0, \overline{\eta})$ is also the unique competitor for problem (6.8).
Now, let us consider for instance the case when F_3 is given by

$$F_3 = -4\,\mathcal{H}^1 \llcorner \Big([-1,1] \times \{1/2\}\Big) + 10\,\mathcal{H}^1 \llcorner \Big(([-2,-1] \cup [1,2]) \times \{-1/2\}\Big)$$
$$-6\,\mathcal{H}^1 \llcorner \Big((\{-2\} \cup \{+2\}) \times [-1/2, 1/2]\Big) \ .$$

The domain and the loads under consideration are represented in Figure 1. Using (2.9), we have

$$\overline{F}_3 = -4\,\mathcal{H}^1 \llcorner [-1,1] + 10\,\mathcal{H}^1 \llcorner \big([-2,-1] \cup [1,2]\big) - 6\,\delta_{-2} - 6\delta_2 \ ,$$

where δ_a denotes the Dirac mass concentrated at point a. The unique solution of the differential equation $\eta'' = \overline{F}_3$ reads

$$\overline{\eta} = \begin{cases} -2x_1^2 + 1 & \text{if } x_1 \in [-1,1], \\ 5x_1^2 - 14x_1 + 8 & \text{if } x_1 \in [-2,-1] \cup [1,2] \\ 0 & \text{otherwise.} \end{cases}$$

If we assume that the total volume of material is much smaller than the volume of the design region, namely that the parameter k is large enough, we can use the approximation (7.3) to deduce that the profile $h_k^\pm$ is proportional to $|\overline{\eta}|$. Thus we may draw an optimal design as done in Figure 1.

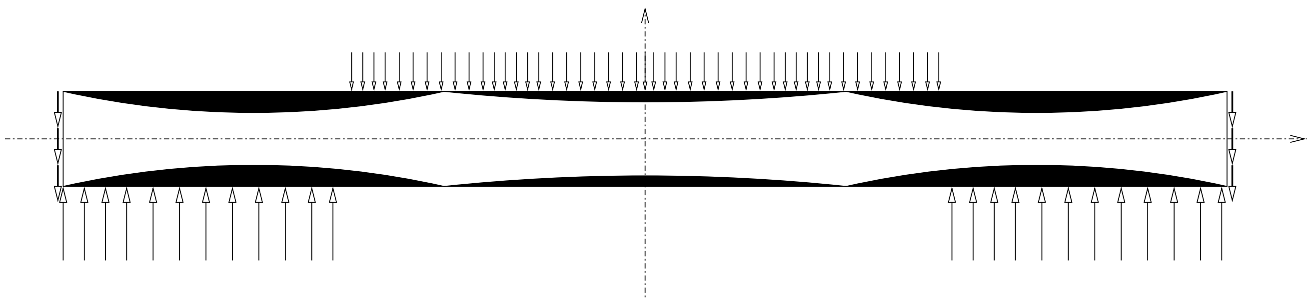

Figure 1: Optimal shape for a plate submitted to bending forces

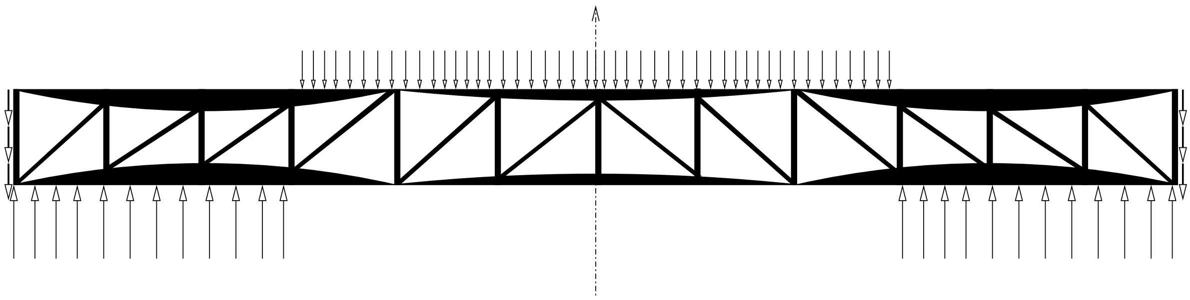

Figure 2: Completed optimal shape for a plate submitted to bending forces

Such optimal design thwarts our intuition. A first striking fact is that the support of the lateral forces is not included neither in the optimal design nor in its boundary. How can such forces be supported by the material? A second striking fact is that the optimal design is divided in two disconnected parts: how can these two parts support the top and bottom forces without collapsing? To understand our results, one has to remember that the optimal design under consideration is just the limit as δ tends to zero of the (rescaled) optimal shape. When δ is small but finite the structure is close to the presented design but it is somehow different: a small amount of material can be diverted in order to solve the aforementioned problems. As this amount is very small (negligible in the limit), it is not necessary to use it in a very clever way: we believe this is likely the reason why so many different structures are commonly used to connect the upper and lower layers. In Figure 2 we have added such a light structure, which should be sufficient for this connection, and we have added as well a boundary layer, which should be enough to support the lateral forces.

To conclude, we wish to point out some related open questions which are out of the scope of this article and in our opinion deserve interest for future investigation.
A first natural problem is the optimization of the intermediate light structure that we have added arbitrarily, according to our feeling, in Figure 2 above. In fact, this corresponds to a higher order problem in the asymptotic expansion with respect to the parameter δ: a second order Γ-limit should be studied in order to find the optimal layout of the material connecting the top and bottom layers.
It is also of interest to understand how our work could be adapted if one desires to include the additional constraint of connected sections, *i.e.* to add in our starting problem (1.6) the constraint that Ω takes the form $\Omega = \{h^-(x') < x_3 < h^+(x')\}$. This variant of the problem has some similarities with the one considered in [8, 9, 10]. However, the comparison with our approach discloses deep differences. Indeed, in [8, 9, 10], the 3D-2D limit is taken *before* any optimization is performed;

therefore, plane stress state is somehow an implicit assumption, while in subsequent optimization rapid oscillations of the thickness appear. In contrast, in our approach, we carry out a real 3D shape optimization process (in a thinning domain), in which thickness oscillations would likely prevent a plane stress state. We foresee that this will lead to a more complex situation, whose study should require a challenging work.

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