Asymptotics of a non-planar rod in non-linear elasticity

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Abstract. We study the asymptotic behavior of a non-linear elastic material lying in a thin neighborhood of a non-planar line when the diameter of the section tends to zero. We first estimate the rigidity constant in such a domain then we prove the convergence of the three-dimensional model to a one-dimensional model. This convergence is established in the framework of Γ-convergence. The resulting model is the one classically used in mechanics. It corresponds to a non-extensional line subjected to flexion and torsion. The torsion is an internal parameter which can eventually by eliminated but this elimination leads to a non-local energy. Indeed the non-planar geometry of the line couples the flexion and torsion terms.

Keywords: beam, rod, non-linear elasticity, 3D–1D, Γ-convergence

1. Introduction

Thin elastic objects like beams or plates are of crucial importance in structural design for their low weight and cost. Their properties and the link with the properties of the material they are made of, was a constant subject of study for mechanicians. Approximations for the displacements at small scale which are induced by a global displacement of the structure are well known from the pioneer works of Euler, Bernoulli and Navier. The energy induced by such a global displacement can be estimated and then the behavior of the structure is known. The mathematical justifications of these approximations are more recent. A very wide literature is devoted to this subject. Here we are more particularly interested by the so-called 3D–1D reduction, the limit object is one-dimensional (the reader can refer, for instance, to [2] or [15] for a review of one-dimensional elastic models).

Non-planar thin objects like shells or curved rods are also important. Here we restrict our attention to non-planar curved rods: the limit object is a one-dimensional non-planar curve. The mechanical applications are numerous: let us simply mention that helicoidal springs are nothing else but curved elastic rods and that they are widely used in mechanisms.

The mathematical literature devoted to curved rods is much more restricted than to the straight ones. This lack seems essentially due to the difficulty for obtaining fine a priori estimates in this complex geometry. First works based on formal expansions are due to Jamal et al. [6] or Sanchez et al. [13]. Complete studies have been performed in the last years by Jurak et al. [7], by Griso [5] and by Pideri et al. [14].

These studies, like besides the major part of the studies devoted to the straight case, are performed in the framework of linear elasticity. The first reason is mathematical: a fundamental tool was missing for obtaining a priori estimates in the context of non-linear elasticity where Korn inequality is of no help. This tool has been provided very recently by Friesecke, James and Muller [4]. The second reason comes from mechanics: all theories for rods involves only the parameters (Lamé coefficients or Young modulus) which describe the linear behavior of the material the rod is made of.

This is remarkable. Rods are weak structures: submitted to reasonable forces they move far from their initial position. The classical approximation of small displacements is in general not valid, but the parameters which describe the global behavior can be efficiently computed in the linear framework. This may seem a paradox. The point is that the rigidity constant is very small in thin domains: the displacement can be large while its gradient remains close to a rotation. We are in a case of large displacement but with small strain. Starting with a non-linear three-dimensional model of elasticity, we end up with a non-linear model for a bending and torsion line but the material coefficients of this elastic line depend only of the linear approximation of the starting model.

The study of straight rods in the non-linear case has recently been performed by Mora et al. [8] by Mora [9] and by Pantz [12]. The way followed in [14] for estimating the Korn constant of a non-straight rod and the way followed by Mora et al. for estimating the rigidity constant in a straight but non-linear rod are very close: the idea is to use Korn (or Rigidity) inequality on small sections of the rod in order to obtain a step by step estimate for the displacement. The present work is based on this observation. A major part of it is devoted to the estimation of the rigidity constant on the considered domain. The transcription of the method used by Mora and Muller is not straightforward as each small part of the rod is no more similar to a fixed domain. The method used in [14] for estimating the Korn constant cannot either be transcripted. Indeed the homogeneity property of Korn inequality is a crucial tool in [14] and nothing similar can be used in the non-linear case.

We make here an intensive use of the paper by Mora and Muller [8]. We use similar assumptions and try to use close notation. For sake of simplicity, we restrict our attention to homogeneous and isotropic materials but further generalization to anisotropic materials like it is done in [8] does not present great difficulties. Our method for proving the Γ-convergence result is also very close to the one called "refined Γ-convergence and director theories" in this paper. We think that this method is the closest to the method used by mechanicians and the most natural.

The major originality of our work lies in the application of the rigidity lemma in a special geometry: it is a neighborhood of a regular non-planar curve (note that we do not try to write the weakest assumptions for the regularity of the domain and that some of assumptions we make can probably be relaxed). We do not restrict to circular sections: so we deal with a possible rotation of the section along the line. Rotation relative to what basis? There is no canonical choice for such a basis. We emphasize that choice of the Frenet basis is neither canonical neither even possible in general. Our choice has the only but fundamental advantage that the basis is always well defined. As we desire to take into account a possible rotation of the section with respect to this basis, there is no further difficulty to take also into account a possible deformation of the section. That is what we do. The resulting geometry looks like Fig. 4.

We describe precisely this geometry together with the initial and resulting energies in Section 2.

The initial energy corresponds to a general three-dimensional hyperelastic material. We assume for sake of simplicity that the material is homogeneous and isotropic but this assumption is not fundamental. The scaling assumed for the energy defines the order of magnitude of the forces which may be applied to the structure. Our choice is such that the structure resists with a finite energy to displacements the order of magnitude of which is the length of the rod. In the case of straight rods the choice of the scaling and its implications on the resulting model has been the subject of long debates but the case of straight rods is quite special: one can consider a scaling which ensures that the structure resists with a finite energy to transverse displacements of order one (this is what is done in [8], leading to a bending–torsion but inextensional model) or a scaling which ensures that the structure resists with a finite energy to longitudinal displacements of order one (this is what is done in [1], leading to an extensional string theory). Here we have no real choice: when the rod is curved this decomposition is not so simple and we think it is quite unrealistic to take into account forces (as it is done in [5]) which lead to extensional and bending displacements with the same order of magnitude.

The limit energy is similar to the one obtained in the straight case. It takes into account bending and torsion. We write it, like mechanicians do, by invoking the rotation of the section as an internal parameter. This parameter can of course be eliminated. In the simplest case of homogeneous and isotropic straight rods with circular section, this elimination is easy. The resulting energy is then a functional of the displacement only. But in many cases, this elimination leads to a non-local functional. This due to a bending–torsion coupling (which can result for instance from anisotropy [11]). In our case the coupling is, even in the isotropic case, complete and it is due to the non-planar geometry of the rod.

At the end of Section 2 we state our main theorem while Sections 3 and 4 are devoted to the proof. In Section 3 we study the application of the rigidity lemma of Frieseke et al. [4] to the considered domain. We first prove (Lemma 1) that the rigidity constants of two almost similar domains are close. Then we prove that the rigidity constant of the considered domain is of order ε^{-1} (Theorem 2). The situation is not very different from the case of straight rods. The compactness result follows easily. Section 4 is devoted to the proof of the Γ-convergence result.

2. The main result

2.1. Description of the beam

First let us define the mean line of the considered beam. Let $\mathcal{L}$ be a curve (a regular one-dimensional manifold) in the physical space $\mathbb{R}^3$ and let $\varphi \in C^3([0,\ell],\mathbb{R}^3)$ be a curvilinear parametrization of $\mathcal{L}$.

For any $x_1 \in [0,\ell]$, we denote $t(x_1) := \varphi'(x_1)$ the unit vector tangent to the curve. We complete $t(x_1)$ in order to get an orthonormal basis $(t,n,b)(x_1)$. There are many choices for such a basis. The Frenet basis is a classical choice. However, it is not defined when the curvature of $\mathcal{L}$ vanishes. Moreover this choice is arbitrary: even when considering a beam with constant section, there is no reason for this section to be constant in this particular basis. Here we consider the possibility of varying section so the choice of the basis has no importance. Let us choose it in such a way that $n' \wedge t = 0$: the basis is then well defined by a first-order differential equation as soon as $n(0)$ is given (arbitrarily in the plane orthogonal to $t(0)$). We can introduce the two functions τ, ξ in $C^1([0,\ell],\mathbb{R})$ such that

$$\begin{aligned} t'(x_1) &= \tau(x_1)n(x_1) + \xi(x_1)b(x_1),\\ n'(x_1) &= -\tau(x_1)t(x_1),\\ b'(x_1) &= -\xi(x_1)t(x_1). \end{aligned} \tag{1}$$

Note that the curvature of the line can be easily recognized as $\sqrt{\tau^2+\xi^2}$ and the torsion of the line as $\frac{\tau\xi'-\xi\tau'}{\tau^2+\xi^2}$.

Now let us describe the section of the considered beam. Let ω be the "prototype section": a piecewise C^1 domain in $\mathbb{R}^2$. It is a bounded connected open set. Let C be the cylinder of $\mathbb{R}^3$: $C := [0,\ell] \times \omega$ and

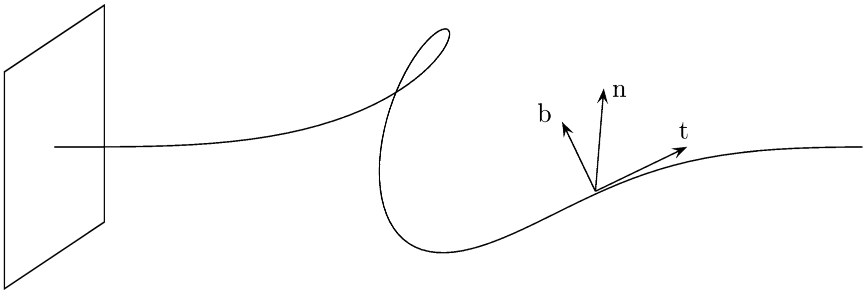

Fig. 1. The mean line $\mathcal{L}$ of the beam, a non-planar curve.

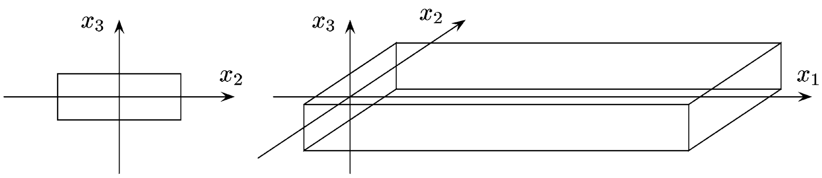

Fig. 2. The prototype section ω and the reference cylinder C.

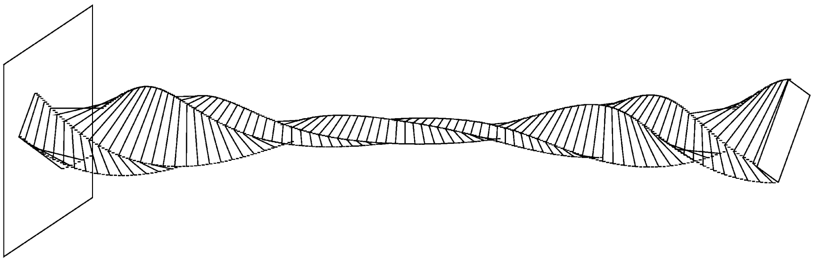

Fig. 3. The reference domain Ω.

let $y = (y_2, y_3)$ be a function in $C^2(C, \mathbb{R}^2)$ such that, for any x_1, the mapping $(x_2, x_3) \to y(x_1, x_2, x_3)$ is a positive diffeomorphism from ω onto its image ω_{x_1}. The domain ω_{x_1} describes the rescaled section at abscissa x_1. We assume that 0 is the inertial center of ω_{x_1}. For the convenience of notations we consider y as a function from C to $\mathbb{R}^3$ by setting $y_1(x_1, x_2, x_3) := x_1$. Then the image of y is a domain in $\mathbb{R}^3$ denoted Ω that we call *the reference domain*. In Fig. 2 we represent a possible prototype section (a rectangle), the corresponding reference cylinder C, while in Fig. 3 we represent the reference domain Ω. For this figure we have assumed a particular choice of y. For any x_1, the mapping $(x_2, x_3) \to y(x_1, x_2, x_3)$ is a similarity: for sake of simplicity of the drawing, we did not use the possibility of varying the shape of the section but only its scaling and position. Note that the possibility of a rotating section shows that the choice we made for the basis (t, n, b) has no importance.

The regularity assumptions we made for y ensures that ∇y is continuous on the compact C with a positive determinant: there exist positive constants C_1, C_2 and C_3 such that, for all $x \in C$,

$$\|\nabla y(x)\| \leqslant C_1 \tag{2}$$

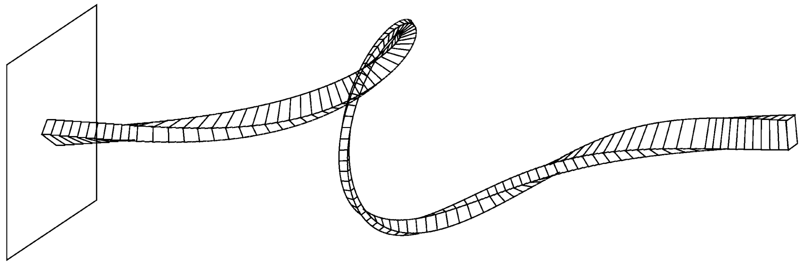

Fig. 4. The beam Ω_ε at rest.

and

$$C_2 \leqslant \left|\det\left(\nabla y(x)\right)\right| \leqslant C_3. \tag{3}$$

As we desire to describe a thin non-planar beam, we have now to rescale the sections by introducing a small parameter ε and to plug them along the line $\mathcal{L}$. Throughout this paper ε denotes a sequence tending to zero. Without loss of generality, we assume that ε is such that $\varepsilon^{-1}\ell$ is an integer. Then we denote Ψ_ε the function defined on Ω by

$$\Psi_\varepsilon(y_1, y_2, y_3) := \varphi(y_1) + \varepsilon\left(y_2 n(y_1) + y_3 b(y_1)\right) \tag{4}$$

and Φ_ε the composition $\Phi_\varepsilon := \Psi_\varepsilon \circ y$. For ε sufficiently small, Ψ_ε is a C^2-diffeomorphism from Ω onto its image denoted Ω_ε (cf. Fig. 4) (and Φ_ε is a C^2-diffeomorphism from C onto Ω_ε). In the sequel the set Ω_ε will be refereed as "the beam" and we will use in it the parameterizations Ψ_ε or Φ_ε. In order to simplify some expressions appearing in our computations we will use, when no confusion can arise, the notation y for $y(x)$ and **x** for $\Psi_\varepsilon(y) = \Phi_\varepsilon(x)$.

The geometry of the beam is then determined by the line $\mathcal{L}$, a unit vector $n(0)$ orthogonal to $t(0)$, the prototype section ω, the function y and the scaling parameter ε.

In Fig. 4 we represent the beam corresponding to the line $\mathcal{L}$ represented in Fig. 1, to the prototype section represented in Fig. 2 (a rectangle) and to the particular choice of y represented in Fig. 3. Note that the way the section "turns" around the mean line is a type of "torsion" of the beam which should not be confused with the torsion of the line $\mathcal{L}$ nor with the mechanical torsion which can result from the displacement of the beam. We emphasize the fact that, in Fig. 4, the beam is at rest.

The regularity assumptions we made for φ lead to the following estimates for $\nabla\Psi_\varepsilon$: there exist positive constants $\widetilde{C}_1$, $\widetilde{C}_2$ and $\widetilde{C}_3$ such that, for any $y \in \Omega$,

$$\left|\left\|\nabla\Psi_\varepsilon(x)\right\| - 1\right| \leqslant \widetilde{C}_1\varepsilon \tag{5}$$

and

$$(1 - \widetilde{C}_2\varepsilon)\varepsilon^2 \leqslant \left|\det\left(\nabla\Psi_\varepsilon(x)\right)\right| \leqslant (1 + \widetilde{C}_3\varepsilon)\varepsilon^2. \tag{6}$$

Indeed

$$\begin{aligned}
&\partial_1\Psi_\varepsilon(y) = \big(1 - \varepsilon\tau(y_1)y_2 - \varepsilon\xi(y_1)y_3\big)t(y_1),\\
&\partial_2\Psi_\varepsilon(y) = \varepsilon n(y_1),\\
&\partial_3\Psi_\varepsilon(y) = \varepsilon b(y_1).
\end{aligned} \tag{7}$$

This leads also to estimates of the area of any section $\omega_a^\varepsilon := \Phi_\varepsilon(\{a\} \times \omega)$ and of the volume of any part of the beam defined by $a < x_1 < b$. We have (possibly modifying the positive constants C_2 and C_3)

$$C_2|\omega|\varepsilon^2 \leqslant \big|\phi_\varepsilon\big(\{a\} \times \omega\big)\big| \leqslant C_3|\omega|\varepsilon^2, \tag{8}$$

$$C_2(b-a)|\omega|\varepsilon^2 \leqslant \big|\phi_\varepsilon\big([a,b] \times \omega\big)\big| \leqslant C_3(b-a)|\omega|\varepsilon^2. \tag{9}$$

Note that, when the context allows no confusion, we denote indifferently $|\cdot|$ the two-dimensional or three-dimensional Hausdorff measure.

2.2. *Elastic energies*

2.2.1. *3-D elastic energy*

Our goal is to study the behavior of the beam Ω_ε in the framework of non-linear elasticity. A motion of the beam is a function $\mathbf{u}$ belonging to the Sobolev space $\mathrm{H}^1(\Omega_\varepsilon, \mathbb{R}^3)$ which describes the position of the deformed beam. We assume that the beam is fixed on its basis $\{x_1 = 0\}$. So any motion $\mathbf{u}$ has to satisfy $\mathbf{u}(\mathbf{x}) = \mathbf{x}$ when $x_1 = 0$. The space of admissible motions $\mathbf{u}$ is denoted:

$$\mathrm{H}^1_b(\Omega_\varepsilon) := \big\{\mathbf{u} \in \mathrm{H}^1\big(\Omega_\varepsilon, \mathbb{R}^3\big);\ \mathbf{u}(\mathbf{x}) = \mathbf{x} \text{ when } x_1 = 0\big\}.$$

The vector field $\mathbf{u} - \mathrm{Id}$ is usually called the displacement field. It belongs to $\mathrm{H}^1_0(\Omega_\varepsilon) := \{\mathbf{v} \in \mathrm{H}^1(\Omega_\varepsilon, \mathbb{R}^3);\ \mathbf{v}(\mathbf{x}) = 0 \text{ when } x_1 = 0\}$.

The elastic energy is a functional $\mathbf{E}_\varepsilon$ on $\mathrm{H}^1_b(\Omega_\varepsilon)$ of the form

$$\mathbf{E}_\varepsilon(\mathbf{u}) := \frac{1}{\varepsilon^4}\int_{\Omega_\varepsilon} W\big(\nabla\mathbf{u}(\mathbf{x})\big)\,\mathrm{d}\mathbf{x},$$

where the energy density W is regular, objective and non-degenerated: it satisfies [8]

$$\begin{aligned}
&W \in C^0\big(\mathbb{R}^{3\times3}, \mathbb{R}\big),\\
&\forall F \in \mathbb{R}^{3\times3},\ \forall R \in SO(3), \quad W(RF) = W(F),\\
&W(\mathrm{Id}) = 0,\\
&W \text{ is of class } C^2 \text{ in a neighborhood of Id},\\
&\forall F \in \mathbb{R}^{3\times3}, \quad W(F) \geqslant C\big(d(F, SO(3))\big)^2.
\end{aligned} \tag{10}$$

This energy $\mathbf{E}_\varepsilon$ is defined on $\mathrm{H}^1_b(\Omega_\varepsilon)$. It is naturally extended on $\mathrm{L}^2(\Omega_\varepsilon, \mathbb{R}^3)$ by setting $\mathbf{E}_\varepsilon(\mathbf{u}) := +\infty$ if $\mathbf{u}$ does not belong to $\mathrm{H}^1_b(\Omega_\varepsilon)$.

The scaling ε^{-4} is needed, as we will see later, to obtain a finite energy when passing to the limit. From the mechanical point of view this scaling can be interpreted as a choice for the force unit (and therefore for the energy unit) which is adapted to the weak rigidity of the rod we consider.

We denote

$$W^{\ell\mathrm{in}}(G) := \frac{1}{2}\frac{\partial^2 W}{\partial F^2}(\mathrm{Id})(G,G). \tag{11}$$

When $\nabla\mathbf{u}$ is close to Id then $W^{\ell\mathrm{in}}(\nabla\mathbf{u} - \mathrm{Id})$ is a good approximation for $W(\nabla\mathbf{u})$: it is the linearized elastic energy associated to W. It plays an important role in this study. The point is that a thin structure, like the one we consider, allows for large displacements while the strain remains small. This explains why the limit model for the rod is completely non-linear but depends only on the linearized part $W^{\ell\mathrm{in}}$ of the elastic energy W of the material the rod is made of.

We assume for sake of simplicity that the considered material is homogeneous and isotropic:

$$\forall F \in \mathbb{R}^{3\times 3},\ \forall R \in SO(3), \quad W(FR) = W(F).$$

Then the linearized energy takes the form

$$W^{\ell\mathrm{in}}\big(\nabla(\mathbf{u} - \mathrm{Id})\big) = \mu\|\mathbf{e}\|^2 + \frac{\lambda}{2}\big(\mathrm{tr}(\mathbf{e})\big)^2,$$

where $\mathbf{e}$ is linearized strain tensor (the symmetric part of $\nabla\mathbf{u} - \mathrm{Id}$) and the Lamé coefficients, λ and μ, satisfy $\mu > 0$ and $3\lambda + 2\mu > 0$.

2.2.2. *The limit one-dimensional model*

The limit energy we obtain is the one classically used in mechanics for describing the motion of rods. Let us describe it: it is a one-dimensional model for the line $\mathcal{L}$. The motion of $\mathcal{L}$ is described by a vector field $\mathbf{u}$ on $\mathcal{L}$ but the mechanical description is made easier by the introduction of an extra matrix-valued field $\mathbf{r}$ on $\mathcal{L}$. In mechanics, $\mathbf{r}$ is interpreted as the rotation of the section of the beam. The space of admissible couples $(\mathbf{u},\mathbf{r})$ is

$$\begin{aligned} \mathrm{H}^{\mathrm{ad}} := \big\{(\mathbf{u},\mathbf{r}) \in \mathrm{H}^2\big(\mathcal{L},\mathbb{R}^3\big) \times \mathrm{H}^1\big(\mathcal{L},SO(3)\big);\ \mathbf{u}' = \mathbf{r}\cdot t \text{ along } \mathcal{L}; \\ \mathbf{u}\big(\varphi(0)\big) = \varphi(0),\ \mathbf{r}\big(\varphi(0)\big) = \mathrm{Id}\big\}. \end{aligned} \tag{12}$$

Here $\mathcal{L}$ is endowed with the one-dimensional Hausdorff measure and the derivatives are relative to the curvilinear abscissa. The boundary conditions $\mathbf{u}(\varphi(0)) = \varphi(0)$, $\mathbf{r}(\varphi(0)) = \mathrm{Id}$ are known as the "clamping conditions". For any $(\mathbf{u},\mathbf{r}) \in \mathrm{H}^{\mathrm{ad}}$, the function $\mathbf{r}^{-1}\mathbf{r}'$ belongs to $L^2(\mathcal{L},\mathbb{R}^{3\times 3})$ and takes values among skew-symmetric matrices. It is classical to associate to such a skew-symmetric matrix the vector[1] $\dot{\mathbf{r}}$ such that, for any $V \in \mathbb{R}^3$, $\dot{\mathbf{r}} \wedge V = \mathbf{r}^{-1}\mathbf{r}' \cdot V$. The elastic energy $\mathbf{F}$ is characterized by a field of definite positive symmetric matrices A and reads:

$$\mathbf{F}(\mathbf{u},\mathbf{r}) := \frac{1}{2}\int_{\mathcal{L}} (A\cdot\dot{\mathbf{r}})\cdot\dot{\mathbf{r}}\,\mathrm{d}\mathcal{H}^1. \tag{13}$$

[1]This vector is interpreted in rods theories as "the variation of the rotation vector of the sections". Such an interpretation is valid in linear elasticity. In the non-linear case considered here, the vector $\dot{\mathbf{r}}$ does not derive from any "rotation vector" (which has no sense).

The energy is defined on H^{ad} but we extend it on $L^2(\mathcal{L}, \mathbb{R}^3 \times SO(3))$ by setting $\mathbf{F}(\mathbf{u}, \mathbf{r}) := +\infty$ when $(\mathbf{u}, \mathbf{r})$ does not belong to the admissible space.

Note that the dependence upon $\mathbf{u}$ is hidden in the constraint $\mathbf{u}' = \mathbf{r} \cdot t$. Note also that this constraint implies $\|\mathbf{u}'\| = 1$: the line is non-extensional.

In terms of the motion of the line only, the energy reads

$$\widetilde{\mathbf{F}}(\mathbf{u}) := \min_{\mathbf{r} \in L^2(\mathcal{L}, SO(3))} \mathbf{F}(\mathbf{u}, \mathbf{r}). \tag{14}$$

Computing this infimum is in general very intricate and leads to a non-local functional. Here the coupling between $\mathbf{u}$ and $\mathbf{r}$ is total. This coupling refereed in mechanics as the "flexion–torsion coupling" is due both to the non-planar geometry of the line $\mathcal{L}$ and to the fact that the section is varying.

The matrix A defining $\mathbf{F}$ in (13) is related to the geometry of the section and to the material properties by

$$A(\varphi(x_1)) := \mu J t \otimes t + Y\big(I_2 n \otimes n + I_3 b \otimes b + I_{23}(b \otimes n + n \otimes b)\big), \tag{15}$$

where Y is the Young modulus of the material $Y := \mu(3\lambda + 2\mu)(\lambda + \mu)^{-1}$, I_2, I_3, I_{23} are the inertial moments of the section

$$I_2(x_1) := \int_{\omega_{x_1}} (y_3)^2 \, \mathrm{d}y_2 \, \mathrm{d}y_3, \qquad I_3(x_1) := \int_{\omega_{x_1}} (y_2)^2 \, \mathrm{d}y_2 \, \mathrm{d}y_3, \tag{16}$$

$$I_{23}(x_1) := -\int_{\omega_{x_1}} y_2 y_3 \, \mathrm{d}y_2 \, \mathrm{d}y_3, \tag{17}$$

and J is the solution of the classical problem for torsional rigidity:

$$J(x_1) := \min\left\{ \int_{\omega_{x_1}} \left((\partial_3\psi + y_2)^2 + (\partial_2\psi - y_3)^2\right) \mathrm{d}y_2 \, \mathrm{d}y_3;\ \psi \in H^1(\omega_{x_1}, \mathbb{R}) \right\}. \tag{18}$$

The parameters J, I_2, I_3 and I_{23} depend only on the geometry of the rescaled section ω_{x_1}. Finally, on H^{ad}, $\mathbf{F}$ reads

$$\mathbf{F}(\mathbf{u}, \mathbf{r}) = \frac{1}{2} \int_{\mathcal{L}} \left(\mu J (\dot{\mathbf{r}} \cdot t)^2 + Y\big(I_2(\dot{\mathbf{r}} \cdot n)^2 + I_3(\dot{\mathbf{r}} \cdot b)^2 + 2I_{23}(\dot{\mathbf{r}} \cdot n)(\dot{\mathbf{r}} \cdot b)\big)\right) \mathrm{d}\mathcal{H}^1. \tag{19}$$

2.3. *The main result*

Let $|\omega^\varepsilon_{x_1}|$ denote the (two-dimensional) Hausdorff measure of the section $\omega^\varepsilon_{x_1}$. For any $f \in L^2(\Omega_\varepsilon, \mathbb{R}^p)$, we denote $\bar{f} \in L^2(\mathcal{L}, \mathbb{R}^p)$ its mean value on each section:

$$\bar{f}(\varphi(x_1)) := \frac{1}{|\omega^\varepsilon_{x_1}|} \int_{\omega^\varepsilon_{x_1}} f \, \mathrm{d}\mathcal{H}^2.$$

Our main theorem is the following

Theorem 1.

(i) *If* $(\mathbf{u}_\varepsilon)$ *is a sequence in* $L^2(\Omega_\varepsilon, \mathbb{R}^3)$ *with a bounded energy (i.e.,* $\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) < M$*), then there exists a subsequence still denoted* $(\mathbf{u}_\varepsilon)$ *such that* $(\bar{\mathbf{u}}_\varepsilon, \overline{\nabla \mathbf{u}_\varepsilon})$ *converges weakly in* $\mathrm{L}^2(\mathcal{L}, \mathbb{R}^3 \times \mathbb{R}^{3\times3})$.

(ii) *For any sequence* $(\mathbf{u}_\varepsilon)$ *in* $\mathrm{L}^2(\Omega_\varepsilon, \mathbb{R}^3)$ *such that* $(\bar{\mathbf{u}}_\varepsilon, \overline{\nabla \mathbf{u}_\varepsilon})$ *converges weakly to* $(\mathbf{u}, \mathbf{r})$ *in* $\mathrm{L}^2(\mathcal{L}, \mathbb{R}^3 \times \mathbb{R}^{3\times3})$*, we have*

$$\liminf \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) \geqslant \mathbf{F}(\mathbf{u}, \mathbf{r}). \tag{20}$$

(iii) *For any* $(\mathbf{u}, \mathbf{r})$ *in* $\mathrm{L}^2(\mathcal{L}, \mathbb{R}^3 \times \mathbb{R}^{3\times3})$*, there exists a sequence in* $\mathrm{L}^2(\Omega_\varepsilon, \mathbb{R}^3)$*,* $(\mathbf{u}_\varepsilon)$ *such that* $(\bar{\mathbf{u}}_\varepsilon, \overline{\nabla \mathbf{u}_\varepsilon})$ *converges to* $(\mathbf{u}, \mathbf{r})$ *in* $\mathrm{L}^2(\mathcal{L}, \mathbb{R}^3 \times \mathbb{R}^{3\times3})$ *and*

$$\limsup \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) \leqslant \mathbf{F}(\mathbf{u}, \mathbf{r}). \tag{21}$$

One may prefer to reformulate the limit energy in terms of the motion of the line only. A trivial consequence of Theorem 1 reads

Corollary 1.

(i) *For any sequence* $(\mathbf{u}_\varepsilon)$ *in* $\mathrm{L}^2(\Omega_\varepsilon, \mathbb{R}^3)$ *such that* $\bar{\mathbf{u}}_\varepsilon$ *converges to* $\mathbf{u}$ *in* $\mathrm{L}^2(\mathcal{L}, \mathbb{R}^3)$*, we have*

$$\liminf \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) \geqslant \widetilde{\mathbf{F}}(\mathbf{u}). \tag{22}$$

(ii) *For any* $\mathbf{u}$ *in* $\mathrm{L}^2(\mathcal{L}, \mathbb{R}^3)$*, there exists a sequence* $(\mathbf{u}_\varepsilon)$ *in* $\mathrm{L}^2(\Omega_\varepsilon, \mathbb{R}^3)$ *such that* $\bar{\mathbf{u}}_\varepsilon$ *converges to* $\mathbf{u}$ *in* $\mathrm{L}^2(\mathcal{L}, \mathbb{R}^3)$ *and*

$$\limsup \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) \leqslant \widetilde{\mathbf{F}}(\mathbf{u}). \tag{23}$$

Remark 1. We have decided to formulate this theorem in terms of the actual displacement fields, those which arise from the physical problem and are defined on Ω_ε and $\mathcal{L}$. One may prefer to refer to a fixed functional space. This is what is usually done in the study of straight beams [] and this is actually what we will do in the proof. Formulating the theorem in a fixed functional space has an important advantage: it can then be written in terms of Γ-convergence. A first disadvantage is that the choice of the fixed functional space is somehow arbitrary and one could then wonder whether the theorem is still valid for a different choice. A second disadvantage is the very intricate expression of the energy in the fixed space.

Remark 2. There is however a canonical way to reformulate the previous theorem in terms of Γ-convergence. Indeed let us associate to any function $\mathbf{u} \in L^2(\Omega_\varepsilon, \mathbb{R}^3)$ the vector valued measure $|\omega^\varepsilon_{x_1}|^{-1}\mathbf{u}(\mathbf{x})1_{\Omega_\varepsilon}(\mathbf{x})\,\mathrm{d}\mathbf{x}$, where 1_{Ω_ε} denotes the characteristic function of Ω_ε. In the same way let us associate to any $\mathbf{u} \in L^2(\mathcal{L}, \mathbb{R}^3)$ the vector valued measure $\mathbf{u}\,\mathrm{d}\mathcal{H}^1_{|\mathcal{L}}$, where $\mathcal{H}^1$ denotes the one-dimensional Hausdorff measure. Let us endow the space of such vector valued measures with the weak* topology. A slightly different version of the previous theorem states the relative compactness of sequences with bounded energy and the Γ-convergence of $\mathbf{E}_\varepsilon$ to $\widetilde{\mathbf{F}}$. Indeed it is easy to check that the convergence of $\mathbf{u}_\varepsilon$ to $\mathbf{u}$ in the sense of these measures implies, when the energy is bounded, the convergence of $\bar{\mathbf{u}}_\varepsilon$ to $\mathbf{u}$ in $L^2(\mathcal{L}, \mathbb{R}^3)$.

Remark 3. Let f be a continuous field of forces. A property of Γ-convergence (for details about the definition and the properties of Γ-convergence the reader can refer to [3]) shows that Theorem 1 remains valid when adding[2] in $\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon)$ and $\widetilde{\mathbf{F}}(\mathbf{u})$ respectively $-\varepsilon^{-2}\int_{\Omega_\varepsilon} f\cdot\mathbf{u}_\varepsilon$ and $-\int_{\mathcal{L}}|\omega_{x_1}|f\cdot\mathbf{u}$. A second property of Γ-convergence shows that a sequence of equilibrium displacements for the beam (i.e., of minimizers of $\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon)-\varepsilon^{-2}\int_{\Omega_\varepsilon} f\cdot\mathbf{u}_\varepsilon$) converges to an equilibrium solution for the line $\mathcal{L}$ (i.e., a minimizer of $\widetilde{\mathbf{F}}(\mathbf{u})-\int_{\mathcal{L}}|\omega_{x_1}|f\cdot\mathbf{u}$).

3. Rigidity lemma and compactness

3.1. Uniform estimation of the rigidity constant

For any connected bounded Lipschitz domain $\mathcal{D}\subset\mathbb{R}^3$ we consider the Lebesgue space $L^2(\mathcal{D},\mathbb{R}^{3\times3})$ of matrix valued functions and we denote $d_{\mathcal{D}}$ the standard distance induced by the L^2-norm. We denote $\mathcal{S}_{\mathcal{D}}$ the subset of those functions which take values among rotations $\mathcal{S}_{\mathcal{D}}:=\mathrm{L}^2(\mathcal{D},SO(3))$ and we denote $s_{\mathcal{D}}\subset\mathcal{S}_{\mathcal{D}}$ the subset of such functions which are constant.

In a remarkable paper [4] G. Friesecke, D. James and S. Muller proved a *rigidity lemma* which states the existence of a positive constant K such that

$$\forall v\in\mathrm{H}^1(\mathcal{D},\mathbb{R}^3),\quad d_{\mathcal{D}}(\nabla v,s_{\mathcal{D}})\leqslant K d_{\mathcal{D}}(\nabla v,\mathcal{S}_{\mathcal{D}}).\tag{24}$$

We denote by $K_{\mathcal{D}}$ the rigidity constant: i.e., the smallest constant K which satisfies (24).

In order to estimate the asymptotic behavior of the rigidity constant K_{Ω_ε} when ε tends to zero, we first compare the rigidity constant of two almost identical domains.

Lemma 1. *Let $\mathcal{D}$ be a domain in $\mathbb{R}^N$ and $(\mathcal{D}_\delta)$ be a sequence of such domains. Assume that, for any $\delta>0$, there exists a C^1-diffeomorphism Ψ_δ from $\mathcal{D}$ onto $\mathcal{D}_\delta$ satisfying, at every point $x\in\mathcal{D}$, $\|\nabla\Psi_\delta(x)-\mathrm{Id}\|\leqslant\delta$. Then, for any $K\geqslant K_{\mathcal{D}}$, we have*

$$K_{\mathcal{D}_\delta}\leqslant\frac{K(1+\delta)}{1-6\delta-2K\delta}\tag{25}$$

which clearly implies

$$\limsup_{\delta\to0}K_{\mathcal{D}_\delta}\leqslant K_{\mathcal{D}}.\tag{26}$$

Proof. Let $v_\delta\in\mathrm{H}^1(\mathcal{D}_\delta,\mathbb{R}^3)$.

• Let $r_\delta\in s_{\mathcal{D}_\delta}$ such that $d_{\mathcal{D}_\delta}(\nabla v_\delta,s_{\mathcal{D}_\delta})=\|\nabla v_\delta-r_\delta\|_{\mathrm{L}^2(\mathcal{D}_\delta)}$ and let us define

$$w_\delta:=r_\delta^{-1}\circ v_\delta\quad\text{and}\quad\bar{w}_\delta:=w_\delta\circ\Psi_\delta+\mathrm{Id}-\Psi_\delta\tag{27}$$

which belong respectively to $\mathrm{H}^1(\mathcal{D}_\delta,\mathbb{R}^3)$ and $\mathrm{H}^1(\mathcal{D},\mathbb{R}^3)$. Then we have

$$d_{\mathcal{D}_\delta}(\nabla v_\delta,s_{\mathcal{D}_\delta})=d_{\mathcal{D}_\delta}(\nabla w_\delta,s_{\mathcal{D}_\delta})=\|\nabla w_\delta-\mathrm{Id}\|_{\mathrm{L}^2(\mathcal{D}_\delta)},\tag{28}$$

$$d_{\mathcal{D}_\delta}(\nabla v_\delta,\mathcal{S}_{\mathcal{D}_\delta})=d_{\mathcal{D}_\delta}(\nabla w_\delta,\mathcal{S}_{\mathcal{D}_\delta}).\tag{29}$$

[2]Different choices of forces could be considered as in [10] to the price of a different formulation of the theorem.

Moreover, defining $\overline{\nabla w_\delta}$ by $\overline{\nabla w_\delta}(x) = \nabla w_\delta(\Psi_\delta(x))$, we have

$$\begin{aligned}(\nabla \bar{w}_\delta - \mathrm{Id}) &= (\overline{\nabla w_\delta} - \mathrm{Id})(\nabla \Psi_\delta),\\ &= (\overline{\nabla w_\delta} - \mathrm{Id}) + (\overline{\nabla w_\delta} - \mathrm{Id})(\nabla \Psi_\delta - \mathrm{Id}).\end{aligned} \tag{30}$$

Using the assumption $\|\nabla \Psi_\delta - \mathrm{Id}\| \leqslant \delta$ which implies also that $|\det(\nabla \Psi_\delta) - 1| \leqslant \sqrt{3}\delta$ we get by integration, change of variables and for δ sufficiently small, the following estimates:

$$\|\nabla \bar{w}_\delta - \mathrm{Id}\|_{\mathrm{L}^2(\mathcal{D})} \leqslant (1 + 3\delta)\|\nabla w_\delta - \mathrm{Id}\|_{\mathrm{L}^2(\mathcal{D}_\delta)}, \tag{31}$$

$$\|\nabla \bar{w}_\delta - \mathrm{Id}\|_{\mathrm{L}^2(\mathcal{D})} \geqslant (1 - 3\delta)\|\nabla w_\delta - \mathrm{Id}\|_{\mathrm{L}^2(\mathcal{D}_\delta)}. \tag{32}$$

• Now, let $\rho_\delta \in s_{\mathcal{D}}$ such that $d_{\mathcal{D}}(\nabla \bar{w}_\delta, s_{\mathcal{D}}) = \|\nabla \bar{w}_\delta - \rho_\delta\|_{\mathrm{L}^2(\mathcal{D})}$. We have

$$\begin{aligned}d_{\mathcal{D}}(\nabla \bar{w}_\delta, s_{\mathcal{D}}) &= \|\overline{\nabla w_\delta}\nabla \Psi_\delta + \mathrm{Id} - \nabla \Psi_\delta - \rho_\delta\|_{\mathrm{L}^2(\mathcal{D})}\\ &= \big\|(\overline{\nabla w_\delta} - \rho_\delta) + (\mathrm{Id} - \nabla \Psi_\delta)(\mathrm{Id} - \rho_\delta) + (\overline{\nabla w_\delta} - \rho_\delta)\big\|_{\mathrm{L}^2(\mathcal{D})}.\end{aligned}$$

By a change of variables, we get the inequality

$$d_{\mathcal{D}}(\nabla \bar{w}_\delta, s_{\mathcal{D}}) \geqslant (1 - 3\delta)\|\nabla w_\delta - \rho_\delta\|_{\mathrm{L}^2(\mathcal{D}_\delta)} - \delta\|\mathrm{Id} - \rho_\delta\|_{\mathrm{L}^2(\mathcal{D})}. \tag{33}$$

The definition of ρ_δ implies

$$\|\nabla \bar{w}_\delta - \rho_\delta\|_{\mathrm{L}^2(\mathcal{D})} \leqslant \|\nabla \bar{w}_\delta - \mathrm{Id}\|_{\mathrm{L}^2(\mathcal{D})}.$$

Using triangular inequality, we get

$$\|\mathrm{Id} - \rho_\delta\|_{\mathrm{L}^2(\mathcal{D})} \leqslant 2\|\nabla \bar{w}_\delta - \mathrm{Id}\|_{\mathrm{L}^2(\mathcal{D})}.$$

From (28) and (31) we deduce, for δ sufficiently small,

$$\|\mathrm{Id} - \rho_\delta\|_{\mathrm{L}^2(\mathcal{D})} \leqslant 3\|\nabla w_\delta - \mathrm{Id}\|_{\mathrm{L}^2(\mathcal{D}_\delta)} \leqslant 3 d_{\mathcal{D}_\delta}(\nabla w_\delta, s_{\mathcal{D}_\delta}).$$

Then inequality (33) leads to

$$d_{\mathcal{D}}(\nabla \bar{w}_\delta, s_{\mathcal{D}}) \geqslant (1 - 6\delta) d_{\mathcal{D}_\delta}(\nabla w_\delta, s_{\mathcal{D}_\delta}). \tag{34}$$

• Now, let $R_\delta \in \mathcal{S}_{\mathcal{D}_\delta}$ such that $d_{\mathcal{D}_\delta}(\nabla w_\delta, \mathcal{S}_{\mathcal{D}_\delta}) = \|\nabla w_\delta - R_\delta\|_{\mathrm{L}^2(\mathcal{D}_\delta)}$. We have, for δ sufficiently small,

$$\begin{aligned}d_{\mathcal{D}}(\nabla \bar{w}_\delta, \mathcal{S}_{\mathcal{D}}) &\leqslant \|\nabla \bar{w}_\delta - R_\delta \circ \Psi_\delta\|_{\mathrm{L}^2(\mathcal{D})}\\ &\leqslant \|\overline{\nabla w_\delta}\nabla \Psi_\delta + \mathrm{Id} - \nabla \Psi_\delta - R_\delta \circ \Psi_\delta\|_{\mathrm{L}^2(\mathcal{D})}\\ &\leqslant \big\|\overline{\nabla w_\delta} - R_\delta \circ \Psi_\delta + (\mathrm{Id} - \overline{\nabla w_\delta})(\mathrm{Id} - \nabla \Psi_\delta)\big\|_{\mathrm{L}^2(\mathcal{D})}\\ &\leqslant (1 + \delta)\|\nabla w_\delta - R_\delta\|_{\mathrm{L}^2(\mathcal{D}_\delta)} + \delta(1 + \delta)\|\nabla w_\delta - \mathrm{Id}\|_{\mathrm{L}^2(\mathcal{D}_\delta)},\end{aligned}$$

and, using (28),

$$d_{\mathcal{D}}(\nabla \bar{w}_\delta, \mathcal{S}_{\mathcal{D}}) \leqslant (1+\delta) d_{\mathcal{D}_\delta}(\nabla w_\delta, \mathcal{S}_{\mathcal{D}_\delta}) + 2\delta d_{\mathcal{D}_\delta}(\nabla w_\delta, s_{\mathcal{D}_\delta}). \tag{35}$$

Applying the rigidity lemma (24) to $\mathcal{D}$ we get

$$d_{\mathcal{D}}(\nabla \bar{w}_\delta, s_{\mathcal{D}}) \leqslant K_{\mathcal{D}}(1+\delta) d_{\mathcal{D}_\delta}(\nabla w_\delta, \mathcal{S}_{\mathcal{D}_\delta}) + 2K_{\mathcal{D}}\delta d_{\mathcal{D}_\delta}(\nabla w_\delta, s_{\mathcal{D}_\delta})$$

then, using (34),

$$(1-6\delta) d_{\mathcal{D}_\delta}(\nabla w_\delta, s_{\mathcal{D}_\delta}) \leqslant K_{\mathcal{D}}(1+\delta) d_{\mathcal{D}_\delta}(\nabla w_\delta, \mathcal{S}_{\mathcal{D}_\delta}) + 2K_{\mathcal{D}}\delta d_{\mathcal{D}_\delta}(\nabla w_\delta, s_{\mathcal{D}_\delta}),$$

$$d_{\mathcal{D}_\delta}(\nabla w_\delta, s_{\mathcal{D}_\delta}) \leqslant \frac{K_{\mathcal{D}}(1+\delta)}{1-6\delta-2K_{\mathcal{D}}\delta} d_{\mathcal{D}_\delta}(\nabla w_\delta, \mathcal{S}_{\mathcal{D}_\delta}), \tag{36}$$

$$K_{\mathcal{D}_\delta} \leqslant \frac{K_{\mathcal{D}}(1+\delta)}{1-6\delta-2K_{\mathcal{D}}\delta}. \tag{37}$$

If K is such that $K \geqslant K_\delta$ then

$$K_{\mathcal{D}_\delta} \leqslant \frac{K(1+\delta)}{1-6\delta-2K\delta}. \tag{38}$$

Proof is concluded as the right-hand side of this inequality tends to K as δ tends to zero. □

As the rigidity inequality (24) is invariant when rescaling the domain and obviously invariant when rotating it, we easily get the following corollary:

Corollary 2. *Lemma* 1 *remains valid if we only assume that each domain $\mathcal{D}_\delta$ is almost similar to $\mathcal{D}$. More precisely if there exist a sequence a_δ satisfying $\lim_{\delta\to 0} \delta a_\delta^{-1} = 0$, a rotation r_δ and a diffeomorphism Ψ_δ from $\mathcal{D}$ onto $\mathcal{D}_\delta$ satisfying, at every point $x \in \mathcal{D}_\delta$, $\|\nabla\Psi_\delta(x) - a_\delta r_\delta\| \leqslant \delta$, then inequality* (25) *still holds with δ replaced by δa_δ^{-1}.*

Indeed, denoting $\widetilde{\mathcal{D}_\delta} := {a_\delta}^{-1} r_\delta^{-1}(\mathcal{D}_\delta)$, the diffeomorphism $\widetilde{\Psi_\delta} := {a_\delta}^{-1} r_\delta^{-1} \circ \Psi_\delta$ maps $\mathcal{D}$ on $\widetilde{\mathcal{D}_\delta}$ and satisfies $\|\nabla\widetilde{\Psi_\delta} - \mathrm{Id}\| \leqslant \delta {a_\delta}^{-1}$.

Now we can state a rigidity theorem for Ω_ε. Let us introduce a new notation: we denote $\overline{\mathcal{S}}_{\Omega_\varepsilon}$ the subset of those functions $\mathbf{r}$ in $\mathcal{S}_{\Omega_\varepsilon}$ which depend only on the x_1 coordinate. This set can clearly be identify to $\mathrm{L}^2(\mathcal{L}, SO(3))$. In the same way, the set of such functions with bounded variations $\overline{\mathcal{S}}_{\Omega_\varepsilon} \cap \mathrm{BV}(\Omega_\varepsilon)$ can be identified with $\mathrm{BV}(\mathcal{L}, SO(3))$. We have

Theorem 2. *There exists a constant K such that, for ε sufficiently small and for any $\mathbf{u}$ in $\mathrm{H}^1(\Omega_\varepsilon)$,*

$$d_{\Omega_\varepsilon}(\nabla\mathbf{u}, \overline{\mathcal{S}}_{\Omega_\varepsilon}) \leqslant K d_{\Omega_\varepsilon}(\nabla\mathbf{u}, \mathcal{S}_{\Omega_\varepsilon}), \tag{39}$$

$$d_{\Omega_\varepsilon}(\nabla\mathbf{u}, s_{\Omega_\varepsilon}) \leqslant \frac{K}{\varepsilon} d_{\Omega_\varepsilon}(\nabla\mathbf{u}, \mathcal{S}_{\Omega_\varepsilon}). \tag{40}$$

Moreover, inequality (39) *can be precised in the following way: there exists* $\mathbf{r}_\varepsilon$ *in* $\overline{\mathcal{S}}_{\Omega_\varepsilon} \cap \mathrm{BV}(\Omega_\varepsilon, SO(3))$ *such that*

$$\|\nabla \mathbf{u} - \mathbf{r}_\varepsilon\|_{\mathrm{L}^2(\Omega_\varepsilon)} \leqslant K d_{\Omega_\varepsilon}(\nabla \mathbf{u}, \mathcal{S}_{\Omega_\varepsilon}), \tag{41}$$

$$\|\mathbf{r}_\varepsilon\|_{\mathrm{BV}(\Omega_\varepsilon)} \leqslant K d_{\Omega_\varepsilon}(\nabla \mathbf{u}, \mathcal{S}_{\Omega_\varepsilon}). \tag{42}$$

Proof. Recalling that ω_z denotes the (rescaled) section at abscissa z, let us define $\Omega^z := [0,1] \times \omega_z$. Let f_z be the diffeomorphism from ω onto ω_z defined by $f_z(x_2, x_3) := y(z, x_2, x_3)$. Then $g_\delta := f_{z+\delta} \circ (f_z)^{-1}$ is a diffeomorphism from ω_z onto $\omega_{z+\delta}$ which induces a natural diffeomorphism from Ω^z onto $\Omega^{z+\delta}$. Its satisfies

$$\nabla g_\delta - \mathrm{Id} = (\nabla f_{z+\delta} - \nabla f_z)(\nabla f_z)^{-1}$$

and so $\|\nabla g_\delta - \mathrm{Id}\|$ tends to zero as δ tends to zero. Lemma 1 ensures that $\limsup_{\delta \to 0} K_{\Omega^{z+\delta}} \geqslant K_{\Omega^z}$: the function $z \to K_{\Omega^z}$ is upper-semi-continuous. Therefore the rigidity constant K_{Ω^z} is upper-bounded by some positive real $\mathcal{K}$ for z in the compact $[-\ell, \ell]$.

Now, for any $z \in [-\ell, \ell - \varepsilon]$, let us consider the part Ω_ε^z of Ω_ε defined by

$$\Omega_\varepsilon^z := \Phi_\varepsilon\big([z, z+\varepsilon] \times \omega\big). \tag{43}$$

It is the image of Ω^z by the mapping F_ε^z:

$$F_\varepsilon^z(u, v, w) := \Phi_\varepsilon\big(z + \varepsilon u, f_z^{-1}(v, w)\big),$$

which can be rewritten

$$F_\varepsilon^z(u, v, w) = \varphi(z + \varepsilon u) + \varepsilon (g_{\varepsilon u})_2(v, w) n(z + \varepsilon u) + \varepsilon (g_{\varepsilon u})_3(v, w) b(z + \varepsilon u).$$

Let us check that the gradient of F_ε^z is close to a similarity. We denote r^z the rotation which transforms the canonical basis (e^1, e^2, e^3) in $(t(z), n(z), b(z))$. We have, for some constant C, by using the results (7),

$$\big\|\partial_1 F_\varepsilon^z - \varepsilon r^z \cdot e^1\big\| = \big\|\varepsilon \partial_1 \Phi_\varepsilon\big(z + \varepsilon u, f_z^{-1}(v, w)\big) - \varepsilon t(z)\big\| \leqslant C\varepsilon^2.$$

It is easier to estimate the two other terms together: we have

$$\begin{aligned}
&\big\|\partial_2 F_\varepsilon^z - \varepsilon r^z \cdot e^2\big\| + \big\|\partial_3 F_\varepsilon^z - \varepsilon r^z \cdot e^3\big\| \\
&\quad \leqslant \varepsilon\big(\|\nabla g_{\varepsilon u} - \mathrm{Id}\| + \big\|n(z + \varepsilon u) - n(z)\big\| + \big\|b(z + \varepsilon u) - b(z)\big\|\big) \\
&\quad \leqslant C\varepsilon^2.
\end{aligned}$$

Thus $\|\nabla F_\varepsilon^z - \varepsilon r^z\| \leqslant 2C\varepsilon^2$ and we can apply Corollary 2. As, for any $z \in [-\ell, \ell]$, $K_{\Omega^z} < \mathcal{K}$, then, for ε sufficiently small,

$$\forall z \in [-\ell, \ell - \varepsilon], \quad K_{\Omega_\varepsilon^z} \leqslant \frac{\mathcal{K}(1 + 2C\varepsilon^2)}{1 - (6 + 2\mathcal{K})2C\varepsilon^2} \leqslant 2\mathcal{K}. \tag{44}$$

A similar reasoning shows, for $\widetilde{\Omega}_\varepsilon^z := \Phi_\varepsilon([z, z+2\varepsilon] \times \omega)$, the existence of a positive constant $\widetilde{\mathcal{K}}$ such that

$$\forall z \in [-\ell, \ell - 2\varepsilon], \quad K_{\widetilde{\Omega}_\varepsilon^z} < 2\widetilde{\mathcal{K}}. \tag{45}$$

Let us now denote $n_\varepsilon := \frac{\ell}{\varepsilon}$ (which was assumed to be an integer). For any integer $i \in \{0, \ldots, n_\varepsilon - 1\}$, let us consider the set $\Omega_\varepsilon^{i\varepsilon}$ and introduce the rotation r_ε^i such that $\|\nabla \mathbf{u} - r_\varepsilon^i\|_{\mathrm{L}^2(\Omega_\varepsilon^{i\varepsilon})} = d_{\Omega_\varepsilon^{i\varepsilon}}(\nabla \mathbf{u}, s_{\Omega_\varepsilon^{i\varepsilon}})$. Note that for $i < 0$, we have $r_\varepsilon^i = \mathrm{Id}$. The function

$$\mathbf{r}_\varepsilon := \sum_{i=0}^{n_\varepsilon - 1} r_\varepsilon^i 1_{\Omega_\varepsilon^{i\varepsilon}}$$

is a piecewise constant function which belongs to $\overline{\mathcal{S}}_{\Omega_\varepsilon}$. Rigidity lemma applied to $\Omega_\varepsilon^{i\varepsilon}$ reads

$$\int_{\Omega_\varepsilon^{i\varepsilon}} \|\nabla \mathbf{u} - r_\varepsilon^i\|^2 \,\mathrm{d}\mathbf{x} \leqslant 4\mathcal{K}^2 \big(d_{\Omega_\varepsilon^{i\varepsilon}}(\nabla \mathbf{u}, \mathcal{S}_{\Omega_\varepsilon^{i\varepsilon}})\big)^2. \tag{46}$$

By summing over i, we get

$$\|\nabla \mathbf{u} - \mathbf{r}_\varepsilon\|_{\mathrm{L}^2(\Omega_\varepsilon)} \leqslant 2\mathcal{K} d_{\Omega_\varepsilon}(\nabla \mathbf{u}, \mathcal{S}_{\Omega_\varepsilon}). \tag{47}$$

Inequality (39) is proved.

Now let us apply Rigidity lemma to $\widetilde{\Omega}_\varepsilon^{i\varepsilon} = \Omega_\varepsilon^{i\varepsilon} \cup \Omega_\varepsilon^{(i+1)\varepsilon}$ states the existence of a rotation $\tilde{r}_\varepsilon^i$ such that

$$\int_{\widetilde{\Omega}_\varepsilon^{i\varepsilon}} \|\nabla \mathbf{u} - \tilde{r}_\varepsilon^i\|^2 \,\mathrm{d}\mathbf{x} \leqslant 4\widetilde{\mathcal{K}}^2 \big(d_{\widetilde{\Omega}_\varepsilon^{i\varepsilon}}(\nabla \mathbf{u}, \mathcal{S}_{\widetilde{\Omega}_\varepsilon^{i\varepsilon}})\big)^2. \tag{48}$$

Restricting the integral at the left-hand side of this inequality to $\Omega_\varepsilon^{i\varepsilon}$ and using (46) we get

$$\int_{\Omega_\varepsilon^{i\varepsilon}} \|r_\varepsilon^i - \tilde{r}_\varepsilon^i\|^2 \,\mathrm{d}\mathbf{x} \leqslant 8\mathcal{K}^2 \big(d_{\Omega_\varepsilon^{i\varepsilon}}(\nabla \mathbf{u}, \mathcal{S}_{\Omega_\varepsilon^{i\varepsilon}})\big)^2 + 8\widetilde{\mathcal{K}}^2 \big(d_{\widetilde{\Omega}_\varepsilon^{i\varepsilon}}(\nabla \mathbf{u}, \mathcal{S}_{\widetilde{\Omega}_\varepsilon^{i\varepsilon}})\big)^2. \tag{49}$$

The measure of $\Omega_\varepsilon^{i\varepsilon}$ was estimated in (9). Thus

$$\|r_\varepsilon^i - \tilde{r}_\varepsilon^i\|^2 \leqslant \frac{8}{C_2|\omega|\varepsilon^3} \big(\mathcal{K}^2 \big(d_{\Omega_\varepsilon^{i\varepsilon}}(\nabla \mathbf{u}, \mathcal{S}_{\Omega_\varepsilon^{i\varepsilon}})\big)^2 + \widetilde{\mathcal{K}}^2 \big(d_{\widetilde{\Omega}_\varepsilon^{i\varepsilon}}(\nabla \mathbf{u}, \mathcal{S}_{\widetilde{\Omega}_\varepsilon^{i\varepsilon}})\big)^2\big). \tag{50}$$

We get a similar estimate for $\|r_\varepsilon^{i+1} - \tilde{r}_\varepsilon^i\|^2$ by restricting the integral at the left-hand side of (48) to $\Omega_\varepsilon^{(i+1)\varepsilon}$. Using the fact that

$$\big(d_{\widetilde{\Omega}_\varepsilon^{i\varepsilon}}(\nabla \mathbf{u}, \mathcal{S}_{\widetilde{\Omega}_\varepsilon^{i\varepsilon}})\big)^2 = \big(d_{\Omega_\varepsilon^{i\varepsilon}}(\nabla \mathbf{u}, \mathcal{S}_{\Omega_\varepsilon^{i\varepsilon}})\big)^2 + \big(d_{\Omega_\varepsilon^{(i+1)\varepsilon}}(\nabla \mathbf{u}, \mathcal{S}_{\Omega_\varepsilon^{(i+1)\varepsilon}})\big)^2$$

we obtain

$$\|r_\varepsilon^{i+1} - r_\varepsilon^i\|^2 \leqslant \frac{16}{C_2|\omega|\varepsilon^3} \big(\mathcal{K}^2 + 2\widetilde{\mathcal{K}}^2\big) \big(d_{\widetilde{\Omega}_\varepsilon^{i\varepsilon}}(\nabla \mathbf{u}, \mathcal{S}_{\widetilde{\Omega}_\varepsilon^{i\varepsilon}})\big)^2. \tag{51}$$

We get by summation, for any index i,

$$\left(\sum_{j=0}^{i}\|r_\varepsilon^j - r_\varepsilon^{j-1}\|\right)^2 \leqslant i\sum_{j=0}^{i}\|r_\varepsilon^j - r_\varepsilon^{j-1}\|^2$$

$$\leqslant n_\varepsilon \frac{16(\mathcal{K}^2 + 2\widetilde{\mathcal{K}}^2)}{C_2|\omega|\varepsilon^3}\sum_{j=0}^{i}\big(d_{\widetilde{\Omega}_\varepsilon^{j\varepsilon}}(\nabla\mathbf{u}, \mathcal{S}_{\widetilde{\Omega}_\varepsilon^{j\varepsilon}})\big)^2$$

$$\leqslant n_\varepsilon \frac{32(\mathcal{K}^2 + 2\widetilde{\mathcal{K}}^2)}{C_2|\omega|\varepsilon^3}\big(d_{\Omega_\varepsilon}(\nabla\mathbf{u}, \mathcal{S}_{\Omega_\varepsilon})\big)^2$$

$$\leqslant \ell\frac{32(\mathcal{K}^2 + 2\widetilde{\mathcal{K}}^2)}{C_2|\omega|\varepsilon^4}\big(d_{\Omega_\varepsilon}(\nabla\mathbf{u}, \mathcal{S}_{\Omega_\varepsilon})\big)^2. \tag{52}$$

And so,

$$\|r_\varepsilon^i - r_\varepsilon^0\|^2 \leqslant \ell\frac{32(\mathcal{K}^2 + 2\widetilde{\mathcal{K}}^2)}{C_2|\omega|\varepsilon^4}\big(d_{\Omega_\varepsilon}(\nabla\mathbf{u}, \mathcal{S}_{\Omega_\varepsilon})\big)^2. \tag{53}$$

Multiplying by the volume $|\Omega_\varepsilon^{i\varepsilon}|$ and summing from $i = 0$ to $n_\varepsilon - 1$ leads to

$$\|\mathbf{r}_\varepsilon - r_\varepsilon^0\|_{L^2(\Omega_\varepsilon)} \leqslant \frac{\ell}{\varepsilon}\sqrt{32C_3C_2^{-1}(\mathcal{K}^2 + 2\widetilde{\mathcal{K}}^2)}\,d_{\Omega_\varepsilon}(\nabla\mathbf{u}, \mathcal{S}_{\Omega_\varepsilon}). \tag{54}$$

Proof of the second inequality is concluded using (47) and triangular inequality.
From (52), we can also deduce an estimate for the variations of $\mathbf{r}_\varepsilon$. We have

$$\|\mathbf{r}_\varepsilon\|_{\mathrm{BV}(\Omega_\varepsilon)} = \sum_{j=0}^{n_\varepsilon}|\omega_{j\varepsilon}^\varepsilon|\|r_\varepsilon^j - r_\varepsilon^{j-1}\|$$

$$\leqslant \sqrt{32\ell|\omega|C_3^2C_2^{-1}(\mathcal{K}^2 + 2\widetilde{\mathcal{K}}^2)}\,d_{\Omega_\varepsilon}(\nabla\mathbf{u}, \mathcal{S}_{\Omega_\varepsilon}). \tag{55}$$

Hence $\mathbf{r}_\varepsilon$ satisfies the two last inequalities of the theorem. □

Corollary 3. *There exists a constant K such that, for ε sufficiently small and for any $\mathbf{u}$ in $H_b^1(\Omega_\varepsilon)$,*

$$\|\mathbf{u} - \mathrm{Id}\|_{\mathrm{H}^1(\Omega_\varepsilon)} \leqslant \frac{K}{\varepsilon}d_{\Omega_\varepsilon}(\nabla\mathbf{u}, \mathcal{S}_{\Omega_\varepsilon}). \tag{56}$$

Proof. In order to take easily into account the boundary condition, let us extend (without changing the notations) the domain Ω_ε by considering suitable extensions of φ on $[-\ell, \ell]$, of y and Φ_ε on $[-\ell, \ell] \times \omega$. We also extend any $\mathbf{u} \in \mathrm{H}_b^1(\Omega_\varepsilon)$ by setting $\mathbf{u} = \mathrm{Id}$ on the new part $\widetilde{\Omega}_\varepsilon := \Phi_\varepsilon([-\ell, 0] \times \omega]$. Theorem 2 states the existence of $\mathbf{r} \in s_{\widetilde{\Omega}_\varepsilon}$ satisfying

$$\int_{\widetilde{\Omega}_\varepsilon\cup\Omega_\varepsilon}\|\nabla\mathbf{u} - \mathbf{r}\|^2\,\mathrm{d}\mathbf{x} \leqslant \frac{K^2}{\varepsilon^2}\big(d_{\widetilde{\Omega}_\varepsilon\cup\Omega_\varepsilon}(\nabla\mathbf{u}, \mathcal{S}_{\widetilde{\Omega}_\varepsilon\cup\Omega_\varepsilon})\big)^2.$$

Using the fact that $\nabla\mathbf{u} = \mathrm{Id}$ on $\widetilde{\Omega}_\varepsilon$, we have

$$d_{\widetilde{\Omega}_\varepsilon\cup\Omega_\varepsilon}(\nabla\mathbf{u}, \mathcal{S}_{\widetilde{\Omega}_\varepsilon\cup\Omega_\varepsilon}) = d_{\Omega_\varepsilon}(\nabla\mathbf{u}, \mathcal{S}_{\Omega_\varepsilon}).$$

Restricting the integral on the left-hand side on the previous inequality to $\widetilde{\Omega}_\varepsilon$ leads to

$$|\widetilde{\Omega}_\varepsilon| \|\mathrm{Id} - \mathbf{r}\|^2 \leqslant \frac{K^2}{\varepsilon^2}\big(d_{\Omega_\varepsilon}(\nabla\mathbf{u}, \mathcal{S}_{\Omega_\varepsilon})\big)^2.$$

Thus

$$\|\mathrm{Id} - \mathbf{r}\|^2_{\mathrm{L}^2(\Omega_\varepsilon)} \leqslant \frac{2C_3}{C_2}\frac{K^2}{\varepsilon^2}\big(d_{\Omega_\varepsilon}(\nabla\mathbf{u}, \mathcal{S}_{\Omega_\varepsilon})\big)^2$$

and triangular inequality gives,

$$\|\nabla\mathbf{u} - \mathrm{Id}\|_{\mathrm{L}^2(\Omega_\varepsilon)} \leqslant \left(\sqrt{\frac{2C_3}{C_2}} + 1\right)\frac{K}{\varepsilon} d_{\Omega_\varepsilon}(\nabla\mathbf{u}, \mathcal{S}_{\Omega_\varepsilon}).$$

The result will be obtained when checked that the Poincaré constant in $\mathrm{H}^1_0(\Omega_\varepsilon)$ is bounded by a constant independent of ε. Indeed, let us use the change of variables Φ_ε. Using estimations (5), (6) and the fact that ℓ is an obvious upper-bound for the Poincaré constant on $\mathrm{H}^1_0(C)$,

$$\begin{aligned}
\|\mathbf{u} - \mathrm{Id}\|^2_{L^2(\Omega_\varepsilon)} &\leqslant C_2\varepsilon^2 \int_C \big\|\mathbf{u}\big(\Phi_\varepsilon(x)\big) - \mathrm{Id}\big\|^2 \varepsilon^2\,\mathrm{d}x \\
&\leqslant C_2\ell^2\varepsilon^2 \int_C \big\|\nabla(\mathbf{u}\circ\Phi_\varepsilon)(x) - \mathrm{Id}\big\|^2\,\mathrm{d}x \\
&\leqslant 2C_2\ell^2\varepsilon^2 \int_C \big\|\nabla\mathbf{u}\big(\Phi_\varepsilon(x)\big) - \mathrm{Id}\big\|^2\,\mathrm{d}x \\
&\leqslant \frac{2C_2}{C_3}\ell^2 \int_{\Omega_\varepsilon} \big\|\nabla\mathbf{u}(\mathbf{x}) - \mathrm{Id}\big\|^2\,\mathrm{d}\mathbf{x}. \qquad \square
\end{aligned}$$

3.2. Compactness

We use the notations defined in Theorem 1. Let us begin by a general remark:

Remark 4. If $\mathbf{v}_\varepsilon$ is a sequence in $\mathrm{L}^2(\Omega_\varepsilon, \mathbb{R}^p)$ such that, for some constant M, $\|\mathbf{v}_\varepsilon\|_{\mathrm{L}^2(\Omega_\varepsilon)} < \varepsilon M$, then up to a subsequence, the sequences $\mathbf{v}_\varepsilon \circ \Psi_\varepsilon$ and $\bar{\mathbf{v}}_\varepsilon$ converges weakly respectively in $\mathrm{L}^2(\Omega)$ and $\mathrm{L}^2(\mathcal{L})$.

Indeed, owing to estimation (5) the sequence $\mathbf{v}_\varepsilon \circ \Psi_\varepsilon$ is bounded in $\mathrm{L}^2(\Omega)$ and, up to a subsequence converges weakly to some v_0 in $\mathrm{L}^2(\Omega)$. Owing also to estimation (8), the sequence of functions $y \to \mathbf{v}_\varepsilon \circ \Psi_\varepsilon(y)\det(\nabla\Psi_\varepsilon(y))|\omega^\varepsilon_{y_1}|$ is still bounded in $\mathrm{L}^2(\Omega)$. A subsequence converges weakly to some w_0. It is then easy to check that $\bar{\mathbf{v}}_\varepsilon$ converges weakly to the function $x_1 \to \int_{\omega_{y_1}} w_0(y)\,\mathrm{d}y_2\,\mathrm{d}y_3$.

Now, let $\mathbf{u}_\varepsilon$ be a sequence with bounded energy ($\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) \leqslant M$). Owing to assumption (10), we have

$$C\varepsilon^{-4}\big(d_{\Omega_\varepsilon}(\nabla\mathbf{u}_\varepsilon, \mathcal{S}_{\Omega_\varepsilon})\big)^2 \leqslant \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) \leqslant M. \tag{57}$$

Then Corollary 3 implies

$$\|\nabla\mathbf{u}_\varepsilon - \mathrm{Id}\|^2_{\mathrm{H}^1(\Omega_\varepsilon)} \leqslant \varepsilon^2 \frac{M}{K^2 C}.$$

Previous remark states that, up to a subsequence, $u_\varepsilon := \mathbf{u}_\varepsilon \circ \Psi_\varepsilon$ and $\nabla\mathbf{u}_\varepsilon \circ \Psi_\varepsilon$ converge weakly in $\mathrm{L}^2(\Omega)$ and then $\bar{\mathbf{u}}_\varepsilon$ and $\overline{\nabla}\mathbf{u}_\varepsilon$ converge weakly to some $\mathbf{u}$ and $\mathbf{r}$ in $\mathrm{L}^2(\mathcal{L})$. Point (i) of Theorem 1 is proved.

4. Proof of the Γ-convergence result

4.1. Lowerbound

First, in order to take easily into account the boundary condition, we extend (without changing the notations) the domain Ω_ε by considering suitable extensions of φ on $[-\ell, \ell]$, of y and Φ_ε on $[-\ell, \ell] \times \omega$. We also extend any $\mathbf{u} \in \mathrm{H}^1_b(\Omega_\varepsilon)$ by setting $\mathbf{u} = \mathrm{Id}$ on the new part $\widetilde{\Omega}_\varepsilon := \Phi_\varepsilon([-\ell, 0] \times \omega])$.

It is clear that it is enough to consider in the proof of point (ii) of Theorem 1 only sequences $(\mathbf{u}_\varepsilon)$ with bounded energy ($\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) \leqslant M$). Moreover we can restrict our attention to a subsequence (still denoted $(\mathbf{u}_\varepsilon)$) such that $\liminf \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) = \lim \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon)$. The statements will then be proved, when proved for some subsequence.

In Section 3.2 we did not write all the implications of the rigidity theorem. Under assumption (57), Theorem 2 states also the existence of $\mathbf{r}_\varepsilon$ in $\overline{\mathcal{S}}_{\Omega_\varepsilon}$ such that

$$\|\nabla\mathbf{u} - \mathbf{r}_\varepsilon\|_{\mathrm{L}^2(\Omega_\varepsilon)} \leqslant K\sqrt{MC^{-1}}\varepsilon^2, \tag{58}$$

$$\|\mathbf{r}_\varepsilon\|_{\mathrm{BV}(\Omega_\varepsilon)} \leqslant K\sqrt{MC^{-1}}\varepsilon^2. \tag{59}$$

Inequality (59) implies that $\mathbf{r}_\varepsilon$ is bounded in $\mathrm{BV}(\mathcal{L})$ and then converges strongly in $\mathrm{L}^2(\mathcal{L})$ to some $\mathbf{r}$ which takes values in $SO(3)$ (in an equivalent way $r_\varepsilon := \mathbf{r}_\varepsilon \circ \Psi_\varepsilon$ converges strongly in $\mathrm{L}^2(\Omega)$ to $r := \mathbf{r} \circ y$). Inequality (58) implies that $\nabla\mathbf{u}_\varepsilon \circ \Psi_\varepsilon - r_\varepsilon$ converges strongly to zero in $\mathrm{L}^2(\Omega)$. Therefore $\nabla\mathbf{u}_\varepsilon \circ \Psi_\varepsilon$ converges strongly in $\mathrm{L}^2(\Omega)$ to the function $r := \mathbf{r} \circ y$ which depends only on the first variable.

We have $\nabla u_\varepsilon = (\nabla\mathbf{u}_\varepsilon \circ \Psi_\varepsilon)\nabla\Psi_\varepsilon$, and from expression (7) we know that $\nabla\Psi_\varepsilon$ converges uniformly on Ω to $t \otimes e^1$. Passing to the limit, we get, in the sense of distributions on Ω,

$$\nabla u = (r \cdot t) \otimes e^1.$$

Hence u depends only on the first variable, $\partial_1 u = r \cdot t$, and finally, all along the line $\mathcal{L}$,

$$\mathbf{u}' = \mathbf{r} \cdot t. \tag{60}$$

From inequality (58) we deduce also that $\mathbf{G}_\varepsilon := \varepsilon^{-1}\mathbf{r}_\varepsilon^{-1}(\nabla\mathbf{u}_\varepsilon - \mathbf{r}_\varepsilon)$ satisfies

$$\|\mathbf{G}_\varepsilon\| \leqslant K\sqrt{MC^{-1}}\varepsilon$$

and so, owing to Remark 4, that $G_\varepsilon := \mathbf{G}_\varepsilon \circ \Psi_\varepsilon$ converges weakly to some G in $\mathrm{L}^2(\Omega)$. We have $\nabla \mathbf{u}_\varepsilon = \mathbf{r}_\varepsilon(\mathrm{Id} + \varepsilon \mathbf{G}_\varepsilon)$ and so

$$\nabla u_\varepsilon = r_\varepsilon\,(\mathrm{Id} + \varepsilon G_\varepsilon)\nabla \Psi_\varepsilon.$$

Applying this equality to a basis vector e^i and deriving with respect to the variable y_j, we get for any $(i,j) \in \{1,2,3\}^2$,

$$\partial_j \partial_i u_\varepsilon = r_\varepsilon(\mathrm{Id} + \varepsilon G_\varepsilon) \cdot \partial_j \partial_i \Psi_\varepsilon + \partial_j\big(r_\varepsilon(\mathrm{Id} + \varepsilon G_\varepsilon)\big) \cdot \partial_i \Psi_\varepsilon. \tag{61}$$

This equality holds in the sense of distributions on Ω and the quantity has to be symmetric with respect to i and j. For $i = 1$ and $j \in \{2,3\}$, we get

$$\partial_j\big(r_\varepsilon(\mathrm{Id} + \varepsilon G_\varepsilon)\big) \cdot \partial_1 \Psi_\varepsilon = \partial_1\big(r_\varepsilon(\mathrm{Id} + \varepsilon G_\varepsilon)\big) \cdot \partial_j \Psi_\varepsilon.$$

Hence, taking $j = 2$,

$$r_\varepsilon \partial_2(G_\varepsilon) \cdot \partial_1 \Psi_\varepsilon = \big(r'_\varepsilon + \varepsilon \partial_1(r_\varepsilon G_\varepsilon)\big) \cdot n,$$

which gives, passing to the limit, $r \cdot \partial_2(G) \cdot t = r' \cdot n$. Recalling the definition of $\dot r$ ($\dot r$ is the vector such that $\forall V,\ \dot r \wedge V = r^{-1} r' \cdot V$),

$$\partial_2(G \cdot t) = \dot r \wedge n. \tag{62}$$

In the same way, taking $j = 3$ we get

$$\partial_3(G \cdot t) = \dot r \wedge b. \tag{63}$$

From these equalities, which hold in the sense of distributions, we deduce the following structure for $G \cdot t$:

$$(G \cdot t)(y) = h(y_1) + \dot r(y_1) \wedge \big(y_2 n(y_1) + y_3 b(y_1)\big). \tag{64}$$

Now let us define on Ω, $\tilde w_\varepsilon(y) := \varepsilon^{-1} u_\varepsilon(y) - r_\varepsilon(y_1) \cdot (y_2 n(y_1) + y_3 b(y_1))$, on $[0,\ell]$ $\bar w_\varepsilon(y_1) := |\omega_{y_1}|^{-1} \int_{\omega_{y_1}} \tilde w_\varepsilon(y)\,\mathrm{d}y_2\,\mathrm{d}y_3$, and, on Ω, $w_\varepsilon(y) = \tilde w_\varepsilon(y) - \bar w_\varepsilon(y_1)$. It is easy to check that

$$\partial_2 w_\varepsilon = r_\varepsilon G_\varepsilon \cdot n, \qquad \partial_3 w_\varepsilon = r_\varepsilon G_\varepsilon \cdot b. \tag{65}$$

Hence the gradient of w_ε with respect to the variables y_2 and y_3 is bounded in $\mathrm{L}^2(\Omega)$ and Poincaré Wirtinger inequality applied to each section ω_{y_1} ensures that w_ε is a bounded sequence in $\mathrm{L}^2(\Omega)$. So is $r_\varepsilon^{-1} w_\varepsilon$, a subsequence of which converges weakly to some w in $\mathrm{L}^2(\Omega)$. Passing to the limit in (65) leads to

$$G \cdot n = \partial_2 w, \qquad G \cdot b = \partial_3 w. \tag{66}$$

Note that, for almost all $y_1 \in [0,\ell]$, the function $(y_2,y_3) \to w(y_1,y_2,y_3)$ belongs to $\mathrm{H}^1(\omega_{y_1})$.

Let $A_\varepsilon \subset \Omega$ the set of points x where $\|G_\varepsilon(x)\| > \sqrt{\varepsilon}$. Its measure tends to zero with ε and $G_\varepsilon(1-1_{A_\varepsilon})$ still converges to G. Let $\eta > 0$. From the definition of $W^{\ell\text{in}}$, we know that, for ε sufficiently small,

$$\|G_\varepsilon\| < \sqrt{\varepsilon} \Rightarrow W(\mathrm{Id} + \varepsilon G_\varepsilon) \geqslant W^{\ell\text{in}}(\varepsilon G_\varepsilon)(1-\eta).$$

Recalling that $W(\mathrm{Id}) = 0$, and using the fact that the functional of linear elasticity is quadratic, positive and so lower semi-continuous, we have

$$\begin{aligned}
\liminf \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) &= \liminf\left(\frac{1}{\varepsilon^4}\int_{\Omega_\varepsilon} W(\mathrm{Id} + \varepsilon\mathbf{G}_\varepsilon)\,\mathrm{d}\mathbf{x}\right) \\
&\geqslant \liminf\left(\frac{1}{\varepsilon^2}(1-\widetilde{C}_3\varepsilon)\int_\Omega W(\mathrm{Id} + \varepsilon G_\varepsilon)\,\mathrm{d}y\right) \\
&\geqslant \liminf\left(\frac{1}{\varepsilon^2}\int_\Omega W\bigl(\mathrm{Id} + \varepsilon(1-1_{A_\varepsilon})G_\varepsilon\bigr)\,\mathrm{d}y\right) \\
&\geqslant \liminf\left((1-\eta)\int_\Omega W^{\ell\text{in}}\bigl((1-1_{A_\varepsilon})G_\varepsilon\bigr)\,\mathrm{d}y\right) \\
&\geqslant (1-\eta)\int_\Omega W^{\ell\text{in}}(G)\,\mathrm{d}y.
\end{aligned}$$

Then, passing to the limit $\eta \to 0$,

$$\liminf \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) \geqslant \int_\Omega W^{\ell\text{in}}(G)\,\mathrm{d}y. \tag{67}$$

A simple minimization with respect to the components $n\cdot G\cdot n$, $n\cdot G\cdot b$ and $b\cdot G\cdot n$ of G shows that

$$W^{\ell\text{in}}(G) \geqslant \frac{Y}{2}(t\cdot G\cdot t)^2 + \frac{\mu}{2}(t\cdot G\cdot n + n\cdot G\cdot t)^2 + \frac{\mu}{2}(t\cdot G\cdot b + n\cdot G\cdot b)^2, \tag{68}$$

where Y is the combination of μ and λ defined in Section 2.2.2. Let us now integrate this expression over a section ω_{y_1}. For the first term, using (64), and the assumption that $(y_1, 0, 0)$ is the inertial center of the section, we get

$$\int_{\omega_{y_1}} (t\cdot G\cdot t)^2\,\mathrm{d}y_2\,\mathrm{d}y_3 \geqslant I_2(y_1)(\dot r\cdot n)^2 + I_3(y_1)(\dot r\cdot b)^2 + I_{23}(y_1)(\dot r\cdot b)(\dot r\cdot n), \tag{69}$$

where I_2, I_3 and I_{23} are the inertial moments or product defined in Section 2.2.2. For the last two terms, we use (64) and (66) together. We have

$$\begin{aligned}
&\int_{\omega_{y_1}} \bigl((t\cdot G\cdot n + n\cdot G\cdot t)^2 + (t\cdot G\cdot b + n\cdot G\cdot b)^2\bigr)\,\mathrm{d}y_2\,\mathrm{d}y_3 \\
&\quad= (t\cdot\dot r)^2\int_{\omega_{y_1}} \bigl(\bigl(-y_3 + \partial_2(w\cdot t)\bigr)^2 + \bigl(y_2 + \partial_3(w\cdot t)\bigr)^2\bigr)\,\mathrm{d}y_2\,\mathrm{d}y_3 \\
&\quad\geqslant (t\cdot\dot r)^2 J(y_1),
\end{aligned} \tag{70}$$

where $J(y_1)$ is the geometrical parameter defined by the ad hoc minimization problem in 2.2.2. Collecting (67)–(70), we finally obtain

$$\liminf \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) \geqslant \mathbf{F}(\mathbf{u},\mathbf{r}).$$

Eq. (60) already proved that the constraint $\mathbf{u}' = \mathbf{r}\cdot t$ is satisfied. The sequence of inequalities (67)–(70) shows that all components of $\dot{\mathbf{r}}$ are bounded in $\mathrm{L}^2(\mathcal{L},\mathbb{R}^3)$. Thus $\mathbf{r}^{-1}\mathbf{r}'$ belongs to $\mathrm{L}^2(\mathcal{L},\mathbb{R}^{3\times 3})$ and so does $\mathbf{r}'$. Then $\mathbf{r}$ belongs to $\mathrm{H}^1(\mathcal{L},SO(3))$ and Eq. (60) shows that $\mathbf{u}$ belongs to $\mathrm{H}^2(\mathcal{L},\mathbb{R}^3)$. The boundary conditions are naturally imposed by the fact that $\mathbf{u}(\varphi(y_1)) = \varphi(y_1)$ and $\mathbf{r}(\varphi(y_1)) = \mathrm{Id}$ on the extended part ($y_1 \in [-\ell, 0]$). The regularity of u and r imposes $\mathbf{u}(\varphi(0)) = \varphi(0)$ and $\mathbf{r}(\varphi(0)) = \mathrm{Id}$. Hence $(\mathbf{u},\mathbf{r})$ belongs to the admissible space H^{ad}. Proof of the lower-bound inequality (point (ii) of Theorem 1) is concluded.

4.2. Upperbound

As usual in Γ-convergence proofs, we restrict our attention when proving point (iii) of Theorem 1 to a function $\mathbf{u}$ such that $\widetilde{\mathbf{F}}(\mathbf{u})$ is finite. Let $\mathbf{r}$ such that $\widetilde{\mathbf{F}}(\mathbf{u}) = \mathbf{F}(\mathbf{u},\mathbf{r})$. Using a density argument we also restrict our attention to regular functions $\mathbf{u}$ and $\mathbf{r}$. With the same argument we assume that $\mathbf{r} = \mathrm{Id}$ in a neighborhood of $\varphi(0)$.

Using the curvilinear parametrization of $\mathcal{L}$, we consider $u := \mathbf{u}\circ\varphi$ and $r := \mathbf{r}\circ\varphi$ which are defined on $[0,\ell]$. We also consider the vector valued function $\dot{r}$ defined by $\forall V,\ \dot{r}(y_1)\wedge V = r(y_1)^{-1}r'(y_1)\cdot V$. We introduce the function ψ_{y_1}, solution of the minimization problem (18) and we set $\psi(y_1,y_2,y_3) := \psi_{y_1}(y_2,y_3)$. Then we define $\mathbf{u}_\varepsilon$ on Ω_ε by defining $u_\varepsilon = \mathbf{u}_\varepsilon\circ\Psi_\varepsilon$ on Ω:

$$\begin{aligned} u_\varepsilon(y) := {} & u(y_1) + \varepsilon r\cdot(y_2 n + y_3 b) \\ & + \varepsilon^2(\dot{r}\cdot t)\psi(y) r\cdot t \\ & + \varepsilon^2\frac{-\lambda}{2(\lambda+\mu)}\left[(\dot{r}\cdot b)\frac{y_3^2-y_2^2}{2} + (\dot{r}\cdot n)y_2y_3\right] r\cdot n \\ & + \varepsilon^2\frac{-\lambda}{2(\lambda+\mu)}\left[(\dot{r}\cdot n)\frac{y_3^2-y_2^2}{2} - (\dot{r}\cdot b)y_2y_3\right] r\cdot b. \end{aligned} \tag{71}$$

Let us estimate $\mathbf{G}_\varepsilon := \varepsilon^{-1}\mathbf{r}^{-1}(\nabla\mathbf{u}_\varepsilon - \mathbf{r})$ or $G_\varepsilon := \mathbf{G}_\varepsilon\circ\Psi_\varepsilon$. We have $(\varepsilon G_\varepsilon + \mathrm{Id})\nabla\Psi_\varepsilon = r^{-1}\nabla u_\varepsilon$. We can explicit the partial derivatives of u_ε and use the expression (7) for the partial derivatives of Ψ_ε. Defining G by

$$\begin{aligned} G := {} & \left[-(\dot{r}\cdot b)y_2 + (\dot{r}\cdot n)y_3\right]\left(t\otimes t + \frac{-\lambda}{2(\lambda+\mu)}(n\otimes n + b\otimes b)\right) \\ & + \left[(\dot{r}\cdot n)y_2 + (\dot{r}\cdot b)y_3\right]\frac{-\lambda}{2(\lambda+\mu)}(n\otimes b - b\otimes n) \\ & + (\dot{r}\cdot t)(-y_3 n\otimes t + y_2 b\otimes t + \partial_2\psi t\otimes n + \partial_3\psi t\otimes b) \end{aligned}$$

we have

$$(\varepsilon G_\varepsilon + \mathrm{Id})\cdot(\varepsilon n) = (\varepsilon G_\varepsilon + \mathrm{Id})\nabla\Psi_\varepsilon\cdot e^2$$

$$= r^{-1}\nabla u_\varepsilon \cdot e^2$$
$$= \varepsilon n + \varepsilon^2 G \cdot n.$$

Hence $G_\varepsilon \cdot n = G \cdot n$. In the same way we get $G_\varepsilon \cdot b = G \cdot b$. Computations are a bit more tricky in the e^1 direction: We have, using (7)

$$\begin{aligned}(\varepsilon G_\varepsilon + \mathrm{Id})\nabla \Psi_\varepsilon \cdot e^1 &= (1 - \varepsilon\tau y_2 - \varepsilon\xi y_3)(\varepsilon G_\varepsilon + \mathrm{Id}) \cdot t \\ &= (1 - \varepsilon\tau y_2 - \varepsilon\xi y_3)t + \varepsilon G_\varepsilon \cdot t + \mathrm{O}(\varepsilon^2),\end{aligned}$$

and on the other hand

$$\begin{aligned}(\varepsilon G_\varepsilon + \mathrm{Id})\nabla \Psi_\varepsilon \cdot e^1 &= r^{-1}\nabla u_\varepsilon \cdot e^1 \\ &= r^{-1}\bigl(u' + \varepsilon r' \cdot (y_2 n + y_3 b) + \varepsilon r \cdot (-y_2\tau t - y_3 \xi t)\bigr) + \mathrm{O}(\varepsilon^2).\end{aligned}$$

Recalling that $\mathbf{F}(\mathbf{u}, \mathbf{r}) < +\infty$ implies $r^{-1} \cdot u' = t$, the comparison of the two last equalities leads to $G_\varepsilon \cdot t = G \cdot t + \mathrm{O}(\varepsilon)$. Then G_ε converges uniformly to G. This implies also that $(\nabla \mathbf{u}_\varepsilon) \circ \Psi_\varepsilon - r$ converges uniformly to zero. Let $\eta > 0$. For ε sufficiently small,

$$\begin{aligned}W(\nabla \mathbf{u}_\varepsilon) = W\bigl(\mathbf{r}^{-1}\nabla \mathbf{u}_\varepsilon\bigr) &\leqslant (1 + \eta) W^{\ell\mathrm{in}}\bigl(\mathbf{r}^{-1}\nabla \mathbf{u}_\varepsilon - \mathrm{Id}\bigr) \\ &\leqslant \varepsilon^2 (1 + \eta) W^{\ell\mathrm{in}}(\mathbf{G}_\varepsilon).\end{aligned}$$

Hence

$$\begin{aligned}\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) &\leqslant \frac{1}{\varepsilon^2}(1 + \eta)\int_{\Omega_\varepsilon} W^{\ell\mathrm{in}}\bigl(\mathbf{G}_\varepsilon(\mathbf{x})\bigr)\,\mathrm{d}\mathbf{x} \\ &\leqslant (1 + \eta)(1 + \widetilde{C}_3\varepsilon)\int_\Omega W^{\ell\mathrm{in}}\bigl(G_\varepsilon(y)\bigr)\,\mathrm{d}y,\end{aligned}$$

$$\limsup \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) \leqslant (1 + \eta)\int_\Omega W^{\ell\mathrm{in}}\bigl(G(y)\bigr)\,\mathrm{d}y$$

and passing to the limit $\eta \to 0$,

$$\limsup \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) \leqslant \int_\Omega W^{\ell\mathrm{in}}\bigl(G(y)\bigr)\,\mathrm{d}y. \tag{72}$$

Let us check now that the quantity $\int_\Omega W^{\ell\mathrm{in}}(G(y))\,\mathrm{d}y$ coincides with the expression (19) for $\mathbf{F}(\mathbf{u}, \mathbf{r})$. Indeed, concerning the diagonal terms, we have

$$\begin{aligned}&\frac{\lambda}{2}(t\cdot G\cdot t + n\cdot G\cdot n + b\cdot G\cdot b)^2 + \mu\bigl((t\cdot G\cdot t)^2 + (n\cdot G\cdot n)^2 + (b\cdot G\cdot b)^2\bigr) \\ &\quad = \frac{\mu(3\lambda + 2\mu}{2(\lambda + \mu)}\bigl[-(\dot{r} \cdot b)y_2 + (\dot{r} \cdot n)y_3\bigr]^2.\end{aligned}$$

The integration of these terms over a section ω_{y_1} gives the quantity

$$\frac{Y}{2}\left[I_3(\dot{r}\cdot b)^2 + I_2(\dot{r}\cdot n)^2 + 2I_{23}(\dot{r}\cdot b)(\dot{r}\cdot n)\right].$$

Concerning the non-diagonal terms, we have $n\cdot G\cdot b + b\cdot G\cdot n = 0$ and

$$(n\cdot G\cdot t + t\cdot G\cdot n)^2 = (\dot{r}\cdot t)^2(-y_3 + \partial_2\psi)^2,$$
$$(b\cdot G\cdot t + t\cdot G\cdot b)^2 = (\dot{r}\cdot t)^2(y_2 + \partial_3\psi)^2.$$

Taking into account the definition of ψ, the integration of these two last terms over a section ω_{y_1} gives the quantity $J(\dot{r}\cdot t)^2$. Finally the expression (19) for $\mathbf{F}(\mathbf{u},\mathbf{r})$ is recovered.

It is clear that $\mathbf{u}_\varepsilon$ belongs to $\mathrm{H}^1_b(\Omega_\varepsilon)$ and that $\bar{\mathbf{u}}_\varepsilon$ converges to $\mathbf{u}$. We already noticed that $(\nabla\mathbf{u}_\varepsilon)\circ\Psi_\varepsilon$ converges uniformly to r. Thus $\overline{\nabla\mathbf{u}_\varepsilon}$ converges to $\mathbf{r}$. This concludes the proof of point (iii) of Theorem 1. □

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