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Frédéric Nataf, Hua Xiang, Victorita Dolean

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Frédéric Nataf[a], Hua Xiang[b], Victorita Dolean[c]

[a] *UPMC Université Paris-06, UMR 7598, Laboratoire J.L. Lions, F-75006 Paris, France*

[b] *School of Mathematics and Statistics, Wuhan University, Wuhan 430072, P. R. China.*

[c] *Laboratoire J.-A. Dieudonné, UMR 6621, Université de Nice-Sophia Antipolis, 06108 Nice Cedex 02, France*

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Abstract

Coarse grid correction is a key ingredient in order to have scalable domain decomposition methods. In this work we construct the coarse grid space using the low frequency modes of the subdomain DtN (Dirichlet-Neumann) maps, and apply the obtained two-level preconditioner to the linear system arising from an overlapping domain decomposition. Our method is suitable for the parallel implementation and its efficiency is demonstrated by numerical examples on problems with high heterogeneities.

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Résumé

Les grilles grossières sont un ingrédient important pour obtenir des méthodes de décomposition de domaine qui passent à l'échelle. Dans ce travail on propose la construction d'un espace grossier en utilisant les modes basses fréquence des opérateurs DtN (Dirichlet-Neumann) et on applique le préconditionneur à deux niveaux ainsi obtenu au système linéaire issu d'une décomposition de domaine avec recouvrement. Notre méthode est adaptée à une implémentation parallèle et son efficacité est montrée à l'aide des exemples numériques sur des problèmes avec des grandes hétérogénéités.

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Email addresses: `nataf@ann.jussieu.fr` (Frédéric Nataf), `xiang@ann.jussieu.fr` (Hua Xiang), `dolean@unice.fr` (Victorita Dolean).

On s'intéresse à la résolution d'un système linéaire issu de la discrétisation d'un problème aux limites elliptique (1) où κ est la diffusion qui peut être discontinue. Afin d'obtenir des méthodes de décomposition de domaine qui passent à l'échelle (robuste vis vis du nombre des sous-domaines), on a besoin des méthodes à deux niveaux, voir [10] et les références correspondantes. Ces méthodes sont étroitement liées aux méthodes multigrille et correction par déflation, voir [9]. Elles sont définies par deux ingrédients : une matrice de rang maximal $Z \in \mathbb{R}^{p\times m}$ avec $m \ll p$ et une formulation algébrique de la correction et elles impliquent la résolution d'un problème de taille réduite $m \times m$ appellé problème grossier.

Le preconditionneur à deux niveaux Balancing Neumann-Neumann (BNN), a été intialement introduit dans [6] et peut s'écrire sous une forme algébrique (selon [4]) comme dans (2). Ici M^{-1} est le preconditionneur grille fine de type Schwarz, $E = Z^T AZ$, $\Xi = ZE^{-1}Y^T$ est la matrice de correction grille grossière et $P_D = I - A\Xi = I - AZ(Y^T AZ)^{-1}Y^T$, $Q_D = I - \Xi A = I - Z(Y^T AZ)^{-1}Y^T A$ sont des opérateurs de projection.

Notre but est la construction d'un espace grossier Z afin d'obtenir une méthode de décomposition de domaine robuste pour un problème de type (1) avec des coefficients très hétérogènes. Quand les sauts des coefficients sont à l'interieur des domaines (et pas à l'interface) ou à travers l'interface qui sépare les différentes régions, l'utilisation d'un nombre fixe de vecteurs pas sous-domaine dans Z donne de très bons résultats, voir [3] ou [2]. Par contre, quand les discontinuités sont le long des interfaces entre les sous-domaines, les résultats ne sont plus satisfaisants. Notre méthode est basée sur les modes basse fréquence associés à l'opérateur Dirichlet-Neumann dans chaque sous-domaine. Elle est très efficace pour des problèmes avec grandes discontinuités dans les coefficients.

1. Introduction

We consider the solution of the linear system $Ax = b \in \mathbb{R}^p$ arising from the discretization of an elliptic boundary value problem of the type

$$\eta u - \operatorname{div}(\kappa \nabla u) = f \tag{1}$$

where κ is the diffusion tensor which can be discontinuous. In order to obtain a scalable domain decomposition algorithm (weakly dependent on the number of subdomains), one usually needs a two-level method, see [10] and references therein. These methods are closely related to multigrid and deflation corrections (see [9] and references therein). They are defined by two ingredients: a full rank matrix $Z \in \mathbb{R}^{p\times m}$ with $m \ll p$ and an algebraic formulation of the correction and they imply solving a reduced size problem of order $m \times m$ called a coarse grid problem.

As a two-level preconditionner we mention the balancing Neumann-Neumann which was proposed by Mandel [6], whose algebraic form for non-symmetric systems can be found in [4]

$$P_{BNN} = Q_D M^{-1} P_D + ZE^{-1}Y^T. \tag{2}$$

Here M^{-1} designs the additive Schwarz preconditionner, $E = Z^T AZ$, $\Xi = ZE^{-1}Y^T$ is coarse-grid correction matrix and $P_D = I - A\Xi = I - AZ(Y^T AZ)^{-1}Y^T$ and $Q_D = I - \Xi A = I - Z(Y^T AZ)^{-1}Y^T A$ are the projection operators.

We focus now on the choice of the coarse space Z in order to obtain a robust domain decomposition method for problems of type (1) with highly heterogeneous coefficients. When the jumps in the coefficients are inside the subdomains (and not on the interface) or across the interface separating the subregions, the use of a fixed number of vectors per subdomain in Z gives good results, see [3] or [2]. When the discontinuities are along the interfaces between the subdomains, results are not so good. Our method is based on the low-frequency modes associated with the Dirichlet-to-Neumann (DtN) map on each subdomain. After obtaining the eigenvectors associated with the small eigenvalues of DtN, we use the harmonic extension to the whole subdomain. It is quite efficient even for the problem with large discontinuities in the coefficients. Moreover, it is suitable for the parallel implementation.

The paper is organized as follows. In Section 2, we introduce the two-level preconditioners using the restricted additive Schwarz (RAS) with the coarse grid correction. In Section 3 numerical tests on the model problem demonstrate the efficiency of our method.

2. Coarse Grid Correction for Algebraic Domain Decomposition Methods

Without loss of generality, we consider here a decomposition of a domain Ω into two overlapping subdomains Ω_1 and Ω_2 and denote by A_j the local discretization matrices and by R_j, the rectangular restriction matrices. A classical preconditioner to problem $Au = b$ issued from the discretization of the BVP (1) is the restricted additive Schwarz (RAS) method, see [1].

$$u^{n+1} = u^n + (\tilde{R}_1^T A_1^{-1} R_1 + \tilde{R}_2^T A_2^{-1} R_2)(f - Au^n), \tag{3}$$

where $A_i := R_i A R_i^T$, $i = 1, 2$ and $\tilde{R}_j$ are obtained by setting some ones in R_j to zeros, such that the operators $\tilde{R}_j$ correspond to a non-overlapping decomposition. Note that the RAS preconditioner is nonsymmetric. From the iterative scheme (3), we can define the preconditioner

$$M_{RAS}^{-1} := \tilde{R}_1^T A_1^{-1} R_1 + \tilde{R}_2^T A_2^{-1} R_2.$$

Using the M_{RAS} preconditionner, we can remove the very large eigenvalues of the coefficient matrix, which correspond to high frequency modes. But the small eigenvalues still exist and hamper the convergence. These small eigenvalues correspond to low frequency modes and represent some global information. We need a suitable coarse grid space to efficiently deal with them. Ideally, we can choose the deflation subspace Z which consists of the eigenvectors associated with the small eigenvalues. But the lower part of the spectrum of a matrix is costly to obtain. Thus, there is a need to choose a priori the coarse space. For instance in [8], Nicolaides defines the deflation subspace Z as follows

$$(z_j)_i = \begin{cases} 1, \text{ if } i \in \Omega_j, \\ 0, \text{ if } i \notin \Omega_j \,. \end{cases} \tag{4}$$

This coarse space performs well in the constant coefficient case. But when there are jumps in the coefficients, it cannot prevent stagnation in the convergence. We propose now a construction of the coarse space that will be suitable for parallel implementation and efficient for accelerating the convergence for the problem with highly heterogeneous coefficients and arbitrary domain decompositions. We still choose Z such that it has the form

$$(z_j)_i = \begin{cases} (w_j)_i, \text{ if } i \in \Omega_j, \\ 0, \text{ if } i \notin \Omega_j \,. \end{cases} \tag{5}$$

The vectors w_i are associated to the eigenvectors corresponding to the smallest eigenvalues of the DtN map in each subdomain Ω_i.

More precisely, let us consider first at the continuous level the Dirichlet to Neumann map DtN_{Ω_i}. Let $u_{\Gamma_i} : \Gamma_i \mapsto \mathbb{R}$, where Γ_i is the interface boundary,

$$\text{DtN}_{\Omega_i}(u_{\Gamma_i}) = \left. \frac{\partial v}{\partial n_i} \right|_{\Gamma_i},$$

where v satisfies

$$\begin{cases} \mathcal{L}(v) := (\eta - \operatorname{div}(k\nabla))v = 0, & \text{in } \Omega_i, \\ v = u_{\Gamma_i}, & \text{on } \Gamma_i\,. \end{cases} \tag{6}$$

If the subdomain is not a floating one (i.e. $\partial\Omega_i \cap \partial\Omega \neq \emptyset$), it is necessary to add on this part of the boundary, the boundary condition coming from the original problem. To construct the coarse grid subspace, we use the low frequency modes associated with the DtN operator:

$$\mathrm{DtN}_{\Omega_i}(w_i) = \lambda\, w_i$$

with λ small and then we take their harmonic extension to the whole subdomain.

A rationale for this choice is that a fast decay of the error of the RAS method corresponds to large eigenvalues of the DtN map whereas a slow decay corresponds to small eigenvalues of this map. Thus the small eigenvalues of the DtN map are responsible for the slow convergence of the algorithm and it is natural to incorporate them in the coarse grid space.

If the spectrum of the DtN map has several isolated small eigenvalues we take all of them in the coarse space. We denote by N_{smeig} the number of eigenvectors per subdomain we incorporate in the coarse space. We call this procedure the Z_{D2N} method. We also use Z_{D2N} to denote the coarse grid space constructed by this method. Its construction is fully parallel. Similarly we call Z_{Nico} the method of Nicolaides or the corresponding coarse grid space. Let us remark that when $\eta = 0$ and the subdomain is a floating one, the lowest eigenvalue of the DtN map is zero and the corresponding eigenvector is a constant vector. Thus, if N_{smeig}=1, Z_{Nico} and Z_{D2N} coincide. As we shall see in the next section, when a subdomain has several jumps of the coefficient, taking Z_{Nico} is not efficient and it is necessary to take Z_{D2N} with more than one vector per subdomain.

3. Numerical Results

We solve now the model problem (1) on the domain $\Omega = [0,1]^2$ discretized by a P_1 finite element method. The diffusion κ is a function of x and y. The corresponding discretizations and data structures were obtained by using the software FreeFem++ [5] in connection with Metis partitioner [7]. In the following we will compare the RAS preconditioner with and without Nicolaides coarse space to the new preconditioner based on the harmonic extension of the eigenvectors of the local DtN operators. We consider here two situations in the case of highly heterogeneous viscosity, skyscraper and alternating case, (see Figure 2), defined as follows

- skyscraper viscosity: for x and y such that for $[10x] \equiv 0 (\mathrm{mod}\ 2)$ or $[10y] \equiv 0 (\mathrm{mod}\ 2)$, $\kappa = 10^3 \cdot ([10y]+1)$, and $\kappa = 1$ elsewhere.
- alternating viscosity: for y such that for $[10y] \equiv 0 (\mathrm{mod}\ 2)$, $\kappa = 10^3 \cdot ([10y]+1)$, and $\kappa = 1$ elsewhere.

We first test the methods on overlapping decompositions into rectangular $N \times N$ domains with $N = 2, 4$ and 8 and then on decompositions into irregular domains obtained via Metis. These overlapping decompositions are build by adding layers to non-overlapping ones. The general non-overlapping decompositions can be generated for example, by using the Metis partitioner (see Figure 1 for such a decomposition into 64 subdomains). The number of discretization points for the uniform decompositions is 14×14 in every subdomain and an overlap of $\delta = 4$ mesh cells. The number of unknowns increase as N^2. We will compare these two level preconditioners on two different configurations of highly heterogeneous viscosity, as defined above and shown in Figure 2. Here the number of eigenvectors considered is constant by subdomain and equal to N_{smeig}=4 and it shows a clear advantage of the DtN preconditioner over Nicolaides one: see Table 1 for the skyscraper configuration and Table 2 for the alternating configuration.

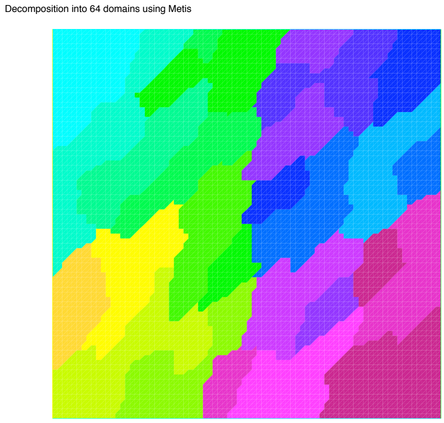

Figure 1. Decomposition into 64 subdomains using Metis.

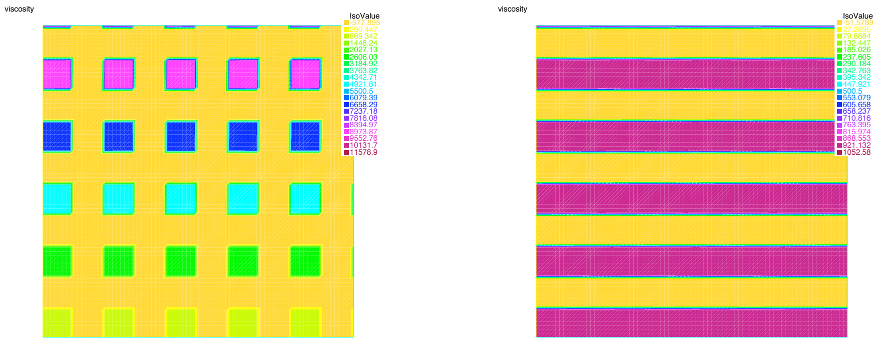

Figure 2. Heterogeneous viscosity: skyscraper and alternating cases.

	uniform $N \times N$ decomposition			$N \times N$ decomp. using Metis		
N	RAS	P_{BNN}: RAS+Z_{Nico}	P_{BNN}: RAS+Z_{D2N}	RAS	P_{BNN}: RAS+Z_{Nico}	P_{BNN}: RAS+Z_{D2N}
2	112	97	24	229	203	29
4	>400	299	18	>400	>400	32
8	>400	294	18	>400	>400	23

Table 1
Iteration counts vs. domain decomposition in the skyscraper case with N_{smeig}=4.

	uniform $N \times N$ decomposition			$N \times N$ decomp. using Metis		
N	RAS	P_{BNN}: RAS+Z_{Nico}	P_{BNN}: RAS+Z_{D2N}	RAS	P_{BNN}: RAS+Z_{Nico}	P_{BNN}: RAS+Z_{D2N}
2	30	29	11	46	41	19
4	80	58	26	116	84	25
8	180	61	31	220	110	31

Table 2
Iteration counts vs. domain decomposition in the skyscraper case with N_{smeig}=4.

A better strategy would consist in adapting N_{smeig} in function of the decomposition into subdomains. In Table 3, as the number of subdomains increases we increase N_{smeig} (the number between brackets in the table). We see that the results are further improved and are very similar to the homogeneous case.

	uniform $N \times N$ decomposition		$N \times N$ decomp. using Metis	
N	P_{BNN}: RAS+Z_{Nico}	P_{BNN}: RAS+Z_{D2N} (N_{smeig})	P_{BNN}: RAS+Z_{Nico}	P_{BNN}: RAS+Z_{D2N} (N_{smeig})
2	97	12 (6)	203	12 (6)
4	299	11 (8)	>400	12 (8)
8	294	13 (10)	>400	14 (10)

Table 3
Iteration counts and varying size of the coarse space N_{smeig} vs. domain decomposition in the skyscraper case.

4. Conclusion & Perspectives

We have considered the linear system arising from the overlapping domain decomposition method and we applied the two-level preconditioner using the Schwarz algorithm and the coarse grid correction. The coarse grid space is based on the low frequency modes of the local DtN map. Its size can be adapted to the difficulty of the problem. With this coarse space, we can obtain fast convergence for problems with large discontinuities (even along the interface) and arbitrary domain decompositions.

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