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SIAM J. CONTROL OPTIM.
Vol. 0, No. 0, pp. 000–000

CONTROLLABILITY OF NETWORKS OF SPATIALLY ONE-DIMENSIONAL SECOND ORDER PDEs—AN ALGEBRAIC APPROACH*

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Abstract. We discuss controllability of systems that are initially given by boundary coupled PDEs of second order. These systems may be described by modules over particular rings of distributions and ultradistributions with compact support arising from the solution of the Cauchy problem of the PDE under consideration with data on the time axis. We show that those rings are Bézout domains. This property is utilized in order to derive algebraic and trajectory related controllability results.

Key words. partial differential equation, controllability, division algorithm, trigonometric ring

AMS subject classifications. 93B25, 93C20, 93B05

DOI. 10.1137/08072437X

1. Introduction.

The solution of control design problems is, in general, preceded by a controllability analysis of the system under consideration. While for linear finite dimensional systems both algebraic and analytic controllability notions are used in parallel, the analysis of infinite dimensional systems is dominated by (functional) analytic methods [8]. The latter approach has proven to be useful, in particular, for the analysis of state space controllability, i.e., the possibility of steering the system under consideration from a given initial state to a desired final state. For example, controllability of the same class of systems as considered in the present contribution has been analyzed this way in [26, 9]. However, the behavioral controllability notion due to Willems [51] is a conceptually interesting alternative to the classical notion of state space controllability, since it directly refers to the concatenation of solutions without the need for a state space.

The connections between the behavioral and the algebraic system properties have been pointed out by Fliess for linear finite dimensional systems: From the algebraic (module theoretic) viewpoint, a linear system is a finitely generated module [11]. For finite dimensional systems, torsion freeness, i.e., the absence of autonomous subsystems, is equivalent to freeness, that is, the existence of a basis of the module.[1] All these algebraic system properties are equivalent to the behavioral controllability of the system [12]. More generally, systems described by linear PDEs have been considered in [32]. This latter contribution, which introduced methods of algebraic analysis [33] into control theory, gave birth to increasing research activities in the field of so-called multidimensional systems theory—see, e.g., [39, 57, 56, 54] and the references

*Received by the editors May 16, 2008; accepted for publication (in revised form) January 25, 2010; published electronically DATE.
http://www.siam.org/journals/sicon/x-x/72437.html

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[1]Note that freeness of the module corresponds to the flatness of the system under consideration in the sense of the theory of nonlinear finite dimensional systems, while its basis corresponds to a flat output [13].

therein. Again, it turned out that torsion freeness of the system module is equivalent to controllability in the behavioral sense, while freeness is no longer a necessary condition. However, despite its theoretical elegance, there are some drawbacks of the multidimensional systems approach, impeding its application, in particular, to boundary controlled evolution equations. In its standard definition introduced in [55, 37] the associated behavioral controllability notion puts no special emphasis to a particular variable and is, therefore, inappropriate for the analysis of evolution equations. In order to overcome this limitation, the alternative notion of time controllability has been introduced in [47, 48]. However, as in the previous references, there is no possibility to consider boundary conditions within the multidimensional framework.

A rather different approach to the control of distributed parameter systems goes back to [31, 16, 30], where the general solution of the wave equation is used in order to obtain a differential delay system. For other spatially one-dimensional evolution equations similar techniques lead to more general systems of convolution equations [1, 35, 34, 53]. In [52, 46, 45] the approach has been formulated in a systematic way, making it applicable to a wide range of hyperbolic and parabolic equations for which the Cauchy problem with data on the time axis is well posed within a suitable space of generalized functions: The desired convolutional system is obtained by first solving the Cauchy problem and plugging its solution into the boundary conditions, i.e., the equations imposed by the boundary conditions further restrict the Cauchy data. Unfortunately, the structure of these coefficient rings, i.e., rings of compactly supported distributions and ultradistributions, is in general more involved than in the finite dimensional case. In particular, they are neither principal ideal nor Noetherian domains. For this reason, as in the multidimensional case, the two basic controllability related module properties, torsion freeness and freeness, are not necessarily equivalent.[2] An approach to circumvent the problems caused by this "lack of structure" is the concept of π-freeness, which relies on localization and was at first developed for linear delay systems [14]. This way a basis can be introduced at least within an appropriate extension of the module under consideration. The approach has proven to be very useful for both trajectory planning and open loop control design [52, 46, 43, 44, 34, 27, 53]. Nevertheless it seems to be difficult to compare such purely algebraic controllability notions to the behavioral ones. For this reason, within the present contribution, we do not use localization.

Instead, we restrict ourselves to a particular class of boundary value problems, i.e., to networks of spatially one-dimensional parabolic and hyperbolic constant coefficient PDEs of second order. Here, by a network, we understand a system consisting of several branches, each of which is governed by a system of PDEs and which are coupled via the boundary conditions. Such models may occur in various fields of physical and technological processes. Simple models of diffusion processes in tubular reactors as well as heat conduction phenomena may be described by parabolic equations. Hyperbolic equations are used in order to describe wave propagation phenomena, occurring, for example, in the modeling of elastic strings, flexible rods, or electric transmission lines [7, 6]. However, in many applications one is faced not only with a single device of this type but also with plants consisting of several interconnected components. Think of networks of transmission lines or elastic strings. Even devices with interior actuation points (or intervals), such as multiple zone ovens in the steel and glass industries, may be modeled using sequentially interconnected boundary

[2]See [21, 2] for results on the ideal structure of distribution rings and [49, 50] for a discussion of systems defined over rings of compactly supported distributions without referring to PDEs.

value problems. Through algebraic properties of adequate coefficient rings obtained for the considered class of PDEs, we investigate the related controllabilities of the associated system module and establish some controllability results including module theoretic and behavioral ones. In accordance with [52, 46, 45], we use the general solution of the Cauchy problem w.r.t. space in order to rewrite the given model as a linear system of convolutional equations. The latter are regarded as the defining relations of a finitely presented module. The coefficient ring of this module is a subring of the ring of compactly supported distributions or ultradistributions depending on the PDE under consideration. It turns out that this ring is a Bézout domain; i.e., every finitely generated ideal is principal. An algorithm enabling us to calculate the generator of a given finitely generated ideal is presented within this paper. This latter result is strongly inspired by those derived in [3, 17] for particular rings of distributed delay operators which in our setting may arise from the wave equation. The derived properties of the coefficient ring allow us to decompose the system module into a free module and a torsion module. Finally, from these algebraic results, we deduce the trajectorian controllability of the free submodule in the sense of [15] and its behavioral controllability in the sense of [51].

The paper is organized as follows. In section 2 we introduce the class of models considered as systems of PDEs which are coupled via their boundary conditions. We show how to pass from this model to a system of convolution equations giving rise to our module theoretic setting. Section 3 is devoted to the study of the coefficient ring of this module. In section 4 we obtain several controllability results for the systems under consideration. Finally, in section 5, we apply the method to a system example of two boundary coupled PDEs.

2. Boundary value problems as convolutional systems.

2.1. Models considered. We assume that the model equations for the distributed variables in $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_l$ and the lumped variables in $\boldsymbol{u} = (u_1, \ldots, u_m)$ are given by

$$\begin{aligned} \partial_x \boldsymbol{w}_i = A_i \boldsymbol{w}_i + B_i \boldsymbol{u}, \quad \boldsymbol{w}_i \in \mathcal{W}_i^2, \quad \boldsymbol{u} \in (\mathscr{F}(\mathbb{R}))^m, \\ A_i \in (\mathbb{R}[\partial_t])^{2\times 2}, \quad B_i \in (\mathbb{R}[\partial_t])^{2\times m}, \end{aligned} \tag{2.1a}$$

where $\mathscr{F}(\mathbb{R})$ denotes either a suitable space $\mathscr{E}_*(\mathbb{R})$ of (complex-valued) infinitely differentiable functions to be specified in section 2.2 or a corresponding space $\mathscr{D}'_*(\mathbb{R})$ of (ultra-)distributions. Moreover, $\mathcal{W}_i$ stands either for $\mathscr{E}_*(\Omega_i \times \mathbb{R})$ or $C^0(\Omega_i, \mathscr{D}'_*(\mathbb{R}))$, the latter of which is defined in section B.2 in Appendix B.

The assumptions which are crucial for the applicability of our approach are twofold. First, we assume that all the matrices $A_1, \ldots, A_l$ give rise to the same characteristic polynomial, namely,

$$\det(\lambda I - A_i) = \lambda^2 - \sigma, \quad \sigma = a\partial_t^2 + b\partial_t + c \neq 0, \quad a, b, c \in \mathbb{R}, \quad a \geqslant 0. \tag{2.1b}$$

Additionally, we require the intervals $\Omega_1, \ldots, \Omega_l$ of the definition of the above differential equations to be rationally dependent. More precisely, we assume the Ω_i $(i = 1, \ldots, l)$ to be given by an open neighborhood of

$$\tilde{\Omega}_i = [x_{i,0}, x_{i,1}], \quad \ell_i = x_{i,1} - x_{i,0} = q_i \ell, \quad q_i \in \mathbb{Q}, \quad \ell \in \mathbb{R}. \tag{2.1c}$$

In the following, and without further loss of generality, we assume $x_{i,0} = 0$. The model is completed by the boundary conditions

$$\sum_{i=1}^{l} L_i \boldsymbol{w}_i(0) + R_i \boldsymbol{w}_i(\ell_i) + D\boldsymbol{u} = 0, \tag{2.1d}$$

where $D \in (\mathbb{R}[\partial_t])^{q\times m}$ and $L_i, R_i \in (\mathbb{R}[\partial_t])^{q\times 2}$.

Remark 2.1. In a more general setting, instead of the boundary conditions (2.1d), one could consider auxiliary conditions of the form

$$\sum_{i=1}^{l} Q_i(\boldsymbol{w}_i) + D\boldsymbol{u} = 0.$$

Here,

$$Q_i(\boldsymbol{w}_i) = \sum_{j=0}^{\nu} L_{i,j}\boldsymbol{w}_i(\alpha_{i,j}\ell) + \sum_{j=1}^{\mu} \int_{\Omega_{i,j}} Q^{\star}_{i,j}(x)\boldsymbol{w}_i(x)dx,$$

with $L_{i,j} \in (\mathbb{R}[\partial_t])^{q\times 2}$, $Q^{\star}_{i,j} \in (\mathbb{R}[\partial_t, x])^{q\times 2}$, $\Omega_i \supset \Omega_{i,j} = [\beta_{i,j,1}\ell, \beta_{i,j,2}\ell]$, $\alpha_{i,j}, \beta_{i,j,k} \in \mathbb{Q} \cap \Omega_i$, and $\mu, \nu \in \mathbb{N}$.

2.2. Solution of the Cauchy problem. This section recalls some properties of the solution of a single Cauchy problem of the form (2.1a) with initial conditions given at $x = \xi$, i.e.,

$$\partial_x \boldsymbol{w} = A\boldsymbol{w} + B\boldsymbol{u}, \quad \boldsymbol{w} \in \mathcal{W}^2, \quad \boldsymbol{w}(\xi) = \boldsymbol{w}_\xi \in (\mathscr{F}(\mathbb{R}))^2, \tag{2.2}$$

with A, B having the same properties as A_i, B_i ($i = 1, \ldots, l$) introduced within the previous section and $\mathcal{W}$ standing either for $C^0(\Omega, \mathscr{D}'_*(\mathbb{R}))$ or $\mathscr{E}_*(\Omega \times \mathbb{R})$. To this end, we start with the initial value problem

$$(\partial_x^2 - \sigma)v(x) = 0, \quad v(0) = v_0, \quad (\partial_x v)(0) = v_1, \tag{2.3}$$

associated with the characteristic equation (2.1b). It is easy to verify that the line $x = 0$ is not a characteristic of $\partial_x^2 - \sigma^2$. More precisely, for $a > 0$ the operator $\partial_x^2 - \sigma$ is hyperbolic w.r.t. the line $x = 0$ in the sense of [23, Def. 12.3.3], while in the (parabolic) case $a = 0$ this statement is true only for its principal part ∂_x^2.

According to [22, Cor. 8.6.9] the Cauchy problem (2.3) possesses at most one solution belonging to the space $\mathscr{E}(\Omega \times \mathbb{R})$ of infinitely differentiable functions on $\Omega \times \mathbb{R}$. However, existence of such a solution for arbitrary $v_0, v_1 \in \mathscr{E}(\mathbb{R})$ and continuous dependence on the initial data are ensured in the hyperbolic case only [23, Cor. 12.5.7]. In the parabolic case, existence and continuous dependence on the initial data are guaranteed only as long as the data belongs to the *small Gevrey class* $\mathscr{E}_{(2)}(\mathbb{R})$ defined in[3] [23, Def. 12.7.3, p. 137] (see [23, Thm. 12.5.6] and [41, Thm. 2.5.2, Prop. 2.5.6] and the explicit formulas in Appendix B).

Consequently, $\mathscr{E}_*$ (resp., $\mathscr{D}_*$) stands for the spaces $\mathscr{E}$ (resp., $\mathscr{D}$) in the hyperbolic case and for $\mathscr{E}_{(2)}$ (resp., $\mathscr{D}_{(2)}$) in the parabolic case, where $\mathscr{D}$ (resp., $\mathscr{D}_{(2)}$) denotes the space of smooth functions with compact support (resp., the space of compactly supported elements of $\mathscr{E}_{(2)}$). Due to the linearity of the PDE under consideration, the solution of (2.2) can be written as

$$v = \widetilde{C}[v_0] + \widetilde{S}[v_1], \quad \partial_x v = \sigma\widetilde{S}[v_0] + \widetilde{C}[v_1],$$

[3] See also [41, 24, 25] and Definition B.1 in Appendix B.

where the continuous mappings $\widetilde{C}, \widetilde{S} : \mathscr{E}_*(\mathbb{R}) \to \mathscr{E}_*(\mathbb{R}^2)$ are defined by the initial value problems

$$(\partial_x^2 - \sigma)\widetilde{C}[v_0] = 0, \quad \widetilde{C}[v_0](0) = v_0, \quad \partial_x \widetilde{C}[v_0](0) = 0, \tag{2.4a}$$

$$(\partial_x^2 - \sigma)\widetilde{S}[v_1] = 0, \quad \widetilde{S}[v_1](0) = 0, \quad \partial_x \widetilde{S}[v_1](0) = v_1. \tag{2.4b}$$

Therein, the formula for $\partial_x v$ is a simple consequence of the fact that this function is the solution of (2.3) with data $v_1, \sigma v_0$. Since the coefficients of the PDE are constant, those mappings commute with time shifts and can, thus, be identified with compactly supported (ultra-)distributions (see, e.g., [10, p. 121], [22, Thm. 4.2.1] for the hyperbolic case) acting by convolution.[4] Their support satisfies (cf. the above cited references and the explicit representations in Appendix A)

$$\operatorname{ch}\operatorname{supp}(C(x), S(x)) = \left\{t : -|x|\sqrt{a} \le t \le |x|\sqrt{a}\right\}, \tag{2.5}$$

with $\operatorname{ch}\operatorname{supp}(C(x), S(x))$ denoting the complex hull of $\operatorname{supp}(C(x), S(x))$. Since, for $\mathscr{E}_*(\mathbb{R})$-data, $C(x)v_0$ and $S(x)v_1$ belong to $\mathscr{E}(\mathbb{R}^2)$ and due to the continuous dependence of the solution on the initial conditions, both functions, C and S, belong to $C^\infty(\Omega, \mathscr{E}'_*(\mathbb{R}))$ defined in section B.2 in Appendix B. They can, therefore, be defined by

$$\partial_x^2 C(x) = \sigma C(x), \quad C(0) = 1, \quad (\partial_x C)(0) = 0, \tag{2.6a}$$

$$\partial_x^2 S(x) = \sigma S(x), \quad S(0) = 0, \quad (\partial_x S)(0) = 1, \tag{2.6b}$$

where 1 is the identity element in $\mathscr{E}'_*(\mathbb{R})$ w.r.t. convolution, i.e., the Dirac distribution.[5] Convolving these equations with (ultra-)distributions $v_1, v_2 \in \mathscr{D}'_*(\mathbb{R})$ yields the solution $v = Cv_1 + Sv_2 \in C^\infty(\Omega, \mathscr{E}'_*(\mathbb{R}))$ for the Cauchy problem (2.3) with $\mathscr{D}'_*(\mathbb{R})$-data.

Let $\mathscr{F}$ stand either for $\mathscr{D}'_*$ or $\mathscr{E}_*$, and consider the Cauchy problem (2.2) with data $\boldsymbol{w}_\xi \in (\mathscr{F}(\mathbb{R}))^2$. Using the above defined $\mathscr{E}'_*$-valued functions C and S, one easily verifies that the (unique) solution $\boldsymbol{w}$ of this problem, given by[6]

$$\boldsymbol{w}(x) = \Phi(x, \xi)\boldsymbol{w}_\xi + \Psi(x, \xi)\boldsymbol{u}, \tag{2.7}$$

with

$$\Phi(x, \xi) = AS(x - \xi) + IC(x - \xi), \quad \Psi(x, \xi) = \int_\xi^x \Phi(x, \zeta) B d\zeta, \tag{2.8}$$

belongs to $\mathcal{W}^2$. Here, I denotes the identity in $(\mathscr{E}'_*(\mathbb{R}))^2$.

Since $\Phi(x, \xi)B$ vanishes on $\{t : |t| > \sqrt{a}|x - \xi|\}$, the restriction of the kernel of the above integral to $([\xi, x] \times (\sqrt{a}|\xi - x|, \infty)) \cup ([\xi, x] \times (-\infty, -\sqrt{a}|\xi - x|))$ equals zero. Hence, $x \mapsto \Psi(x, \xi) \in (\mathscr{E}'_*(\mathbb{R}))^{2\times m}$ vanishes outside $\{(x, t) : |t| \le \sqrt{a}|\xi - x|\}$.

From the uniqueness of this solution one deduces the addition formula

$$\Phi(x, \xi)\Phi(\xi, \zeta) = \Phi(x, \zeta).$$

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[4] Throughout this paper convolution always means partial convolution w.r.t. t. This operation will be continuously denoted by juxtaposition.

[5] Differentiation and integration are defined in section B.2 in Appendix B.

[6] That $\boldsymbol{w}$ is indeed a solution can be checked by plugging it into the PDE. Uniqueness follows as usual by assuming the existence of two different solutions or, equivalently, of a nonzero solution of the homogeneous PDE with zero data. Differentiating w.r.t. x and using the Cayley–Hamilton theorem, one observes that both components of the solution satisfy (2.3) with zero data.

For A the companion matrix of the characteristic polynomial, i.e.,

$$A = \begin{pmatrix} 0 & 1 \\ \sigma & 0 \end{pmatrix}, \quad \Phi(x,\xi) = \begin{pmatrix} C(x-\xi) & S(x-\xi) \\ \sigma S(x-\xi) & C(x-\xi) \end{pmatrix}, \tag{2.9}$$

this yields, in particular,

$$C(x+y) = C(x)C(y) + \sigma S(x)S(y), \quad S(x+y) = C(x)S(y) + S(x)C(y). \tag{2.10}$$

According to (2.8) the entries of the matrix $\Phi(x,\xi)$ are linear combinations of $C(x-\xi)$, $S(x-\xi)$ while those of $\Psi(x,\xi)$ may also contain the integrals of S and C, which may be easily obtained by integrating (2.6a):

$$\int_0^x C(\zeta)dx = S(x), \quad \int_0^x S(\zeta)dx = (C(x)-1)/\sigma. \tag{2.11}$$

Later, the latter equations will essentially ease our controllability analysis.

Remark 2.2. Two alternative starting points for the above considerations could be the fundamental solution of (2.3) vanishing in a half-space (cf. [23, Thm. 12.5.1] in the hyperbolic case) or a spectral approach using the partial Fourier–Laplace transform (w.r.t. t) in connection with the corresponding Paley–Wiener theorems for distributions (see, e.g., [22, Thm. 7.3.1]) and ultradistributions (see, e.g., [25, Thm. 1.1]).

2.3. A module presented by a system of convolution equations. In the previous section we have discussed the solutions of the initial value problems associated with the equations (2.1a). In what follows, these results are used in order to define an algebraic structure representing the model under consideration, i.e., a module over a suitable subring of $\mathscr{E}'_*$ admitting the same solutions as (2.1).

To this end we substitute the general solutions of the initial value problems into the boundary conditions (2.1d). This way, one obtains the following linear system of equations for $\boldsymbol{u}$ and the values of $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_l$:

$$\boldsymbol{w}_i(x) = \Phi_i(x,\xi_i)\boldsymbol{w}_i(\xi_i) + \Psi_i(x,\xi_i)\boldsymbol{u}, \quad x \in \Omega_i, \tag{2.12a}$$

$$P_{\boldsymbol{\xi}} \boldsymbol{c}_{\boldsymbol{\xi}} = 0. \tag{2.12b}$$

Here, $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_n)$, with $\xi_i \in \Omega_i$ arbitrary but fixed, $\boldsymbol{c}_{\boldsymbol{\xi}}^T = (\boldsymbol{w}_1^T(\xi_1) \cdots \boldsymbol{w}_l^T(\xi_l), \boldsymbol{u}^T)$, and

$$P_{\boldsymbol{\xi}} = \big(P_{\boldsymbol{\xi},1} \cdots P_{\boldsymbol{\xi},l+1}\big),$$

with

$$P_{\boldsymbol{\xi},i} = L_i\Phi_i(0,\xi_i) + R_i\Phi_i(\ell_i,\xi_i), \quad i = 1,\ldots,l,$$
$$P_{\boldsymbol{\xi},l+1} = D + \sum_{i=1}^{l} L_i\Psi_i(0,\xi_i) + R_i\Psi_i(\ell_i,\xi_i).$$

By section 2.2 the above equations are equivalent to (2.3) in the sense that they admit the same solutions. Moreover, the entries of $\Phi_i(x,\xi_i)$, $\Psi_i(x,\xi_i)$ $(i = 1,\ldots,l)$ and those of $P_{\boldsymbol{\xi}}$ are composed of the values of the functions $S, C : \mathbb{R} \to \mathscr{E}'_*$ and, additionally, values of the spatial integrals of C and S. Thus, they can be read over the ring $\mathcal{R}^I_{\mathbb{R}} \subset \mathscr{E}'_*$ which, for any $\mathbb{X} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$, is defined as $\mathcal{R}^I_{\mathbb{X}} = \mathbb{C}[\partial_t, \mathfrak{S}_{\mathbb{X}}, \mathfrak{S}^I_{\mathbb{X}}]$, with

$$\mathfrak{S}_{\mathbb{X}} = \{C(z\ell), S(z\ell) | z \in \mathbb{X}\}, \quad \mathfrak{S}^I_{\mathbb{X}} = \{C^I(z\ell), S^I(z\ell) | z \in \mathbb{X}\},$$

ℓ given as in (2.1c), and, according to (2.11),

$$S^I(x) = \int_0^x S(\zeta)d\zeta = \frac{C(x)-1}{\sigma} \in \mathscr{E}'_*(\mathbb{R}), \quad C^I(x) = \int_0^x C(\zeta)d\zeta = S(x) \in \mathscr{E}'_*(\mathbb{R}).$$

Inspired by the results given in [3, 17, 30], and in view of the simplification of the following algebraic considerations, instead of the ring $\mathcal{R}^I_{\mathbb{X}}$ we will use a slightly larger ring, given by $\mathcal{R}_{\mathbb{X}} = \mathbb{C}(\partial_t)[\mathfrak{S}_{\mathbb{X}}] \cap \mathscr{E}'_*(\mathbb{R})$. We are now in position to define the system module Σ that will represent all the equations to be satisfied by the lumped variables in $\boldsymbol{u}$ and the values in $\boldsymbol{w}(x)$, $x \in \mathbb{X}$, of the distributed variables in $\boldsymbol{w}$. This module contains the variables in $\tilde{\boldsymbol{u}}$, $\tilde{\boldsymbol{w}}(x)$, $x \in \mathbb{X}$, such that the solutions of (2.12) are just the $\mathcal{R}_{\mathbb{X}}$-homomorphisms to the solution space $\mathscr{F}(\mathbb{R})$; i.e., for any $\mathfrak{t} \in \mathrm{Hom}_{\mathcal{R}_{\mathbb{X}}}(\Sigma_{\mathbb{X}}, \mathscr{F}(\mathbb{R}))$,

$$c_i = \mathfrak{t}(\tilde{c}_i), \quad i = 1, \ldots, 2l+m.$$

Definition 2.1. *Let* $\Sigma_{\mathbb{X}} = \mathcal{R}^{2l+m}_{\mathbb{X}} / P_{\boldsymbol{\xi}} \mathcal{R}^{2l+m}_{\mathbb{X}}$, $\boldsymbol{\xi} \in \mathbb{X}^l$, $\mathbb{X} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$. *The* convolutional system *associated with the boundary value problem* (2.1) *is the module* $\Sigma = \Sigma_{\mathbb{R}}$.

One easily verifies that $\Sigma_{\mathbb{X}}$ does not depend on the choice of $\boldsymbol{\xi} \in \mathbb{X}^l$ (cf. [52, sect. 3.3.] and [46, Rem. 4]). In view of the assumed mutual rational dependence of the lengths $\ell_1, \ldots, \ell_l$ for the analysis of the system properties, it is useful to start with the system $\Sigma_{\mathbb{Q}}$, i.e., a system containing only the values of the distributed variables at rational multiples of ℓ. However, having analyzed the properties of $\Sigma_{\mathbb{Q}}$, we may pass to Σ by an extension of scalars, i.e., $\Sigma \cong \mathcal{R}_{\mathbb{R}} \otimes_{\mathcal{R}_{\mathbb{Q}}} \Sigma_{\mathbb{Q}}$.

Remark 2.3. Note that the procedure described in section 2 can be applied to any network of spatially one-dimensional systems of PDEs with constant or spatially dependent coefficients provided the Cauchy problem is well posed in some space of ultradifferentiable functions with compact support. However, in the general case, the coefficient ring of the resulting module is generated by the values of more than just two functions (such as S and C). In particular for a pth order system of PDEs with constant coefficients, the above described procedure can be performed in a manner completely analogous to that described above if the principal part of the operator $\det(\partial_x - A)$ is hyperbolic w.r.t. the line $x = 0$. The only differences will be the need for p linear independent functions $S_1, \ldots, S_p$ taking values in $\mathscr{E}'_{((p/(p-1)))}$ instead of just S and C. Moreover, the formula (2.8) has to be adapted.

3. The ring $\mathcal{R}_{\mathbb{Q}}$ is a Bézout domain.

In this section we study the structures of the ideals within the ring $\mathcal{R}_{\mathbb{Q}}$. To this end, we first establish some results on the ideals in $\mathbb{C}(\partial_t)[\mathfrak{S}_{\mathbb{Q}}]$ and $\mathbb{C}(\partial_t)[\mathfrak{S}_{\mathbb{Z}}]$.

3.1. Ideals in $\mathbb{C}(\partial_t)[\mathfrak{S}_{\mathbb{Q}}]$ and $\mathbb{C}(\partial_t)[\mathfrak{S}_{\mathbb{Z}}]$.

In the following, we will replace $\mathbb{C}(\partial_t)$ by any field k. Moreover, the rings $\mathbb{C}(\partial_t)[\mathfrak{S}_{\mathbb{X}}]$ are replaced by arbitrary rings in which the addition formulas derived in section 2.2 hold. More precisely, for an arbitrary field k and an additive subgroup $\mathbb{X}$ of $\mathbb{R}$ we consider the ring $\widetilde{\mathcal{R}}_{\mathbb{X}} := k[C^*_a, S^*_a; a \in \mathbb{X}]/\mathfrak{a}$, where, for fixed $\sigma \in k$, the ideal $\mathfrak{a}$ is generated by

$$C^*_a C^*_b \pm \sigma S^*_a S^*_b - C^*_{a\pm b},\ S^*_a C^*_b \pm C^*_a S^*_b - S^*_{a\pm b},\ C^*_0 - 1,\ S^*_0, \quad a, b \in \mathbb{X}.$$

Denoting the canonical images of C^*_a and S^*_a in $\widetilde{\mathcal{R}}_{\mathbb{X}}$ by C_a and S_a, one deduces the relations

$$C_a C_b \pm \sigma S_a S_b = C_{a\pm b}, \quad S_a C_b \pm C_a S_b = S_{a\pm b}, \tag{3.1a}$$

$$C_0 = 1, \quad S_0 = 0, \quad C_a = C_{-a}, \quad S_a = -S_{-a}, \tag{3.1b}$$

$$2C_a C_b = C_{a+b} + C_{a-b}, \quad 2\sigma S_a S_b = C_{a+b} - C_{a-b}, \quad 2C_a S_b = S_{a+b} - S_{a-b}. \tag{3.1c}$$

Moreover, any element $r \in \widetilde{\mathcal{R}}_{\mathbb{X}}$ can be written in the form

$$r = \sum_{i=0}^{n} a_{\alpha_i} C_{\alpha_i} + b_{\alpha_i} S_{\alpha_i}, \quad n \in \mathbb{N}, \quad a_{\alpha_i}, b_{\alpha_i} \in k, \quad \alpha_i \in \mathbb{X}^+, \tag{3.2}$$

where $\mathbb{X}^+ = \{|\alpha| : \alpha \in \mathbb{X}\}$. Finally, the units in $\widetilde{\mathcal{R}}_{\mathbb{X}}$ belong to k.

In the following, it is necessary to distinguish the cases where the equation $\lambda^2 - \sigma = 0$ either has a solution over k or does not. For our application, this is clearly equivalent to the question of whether the roots of the characteristic equation (2.1b) belong to $\mathbb{R}[\partial_t]$. The necessity to distinguish these cases is explained by the following simple example, which, in addition, shows that the cases $\mathbb{X} = \mathbb{Z}$ and $\mathbb{X} = \mathbb{Q}$ need to be analyzed separately.

Example 3.1. Consider the ideal $\mathfrak{I} = (a, b)$, $a = S_1$, $b = C_1 + 1$. Over $\widetilde{\mathcal{R}}_{\mathbb{Q}}$ we have

$$a = S_1 = 2C_{1/2}S_{1/2}, \quad b = C_1 + 1 = 2C_{1/2}^2.$$

Thus, both generators belong to $(C_{1/2})$, which, conversely, belongs to $\mathfrak{I}$ since $2C_{1/2} = -\sigma S_{1/2} a + C_{1/2} b$. The ideal $\mathfrak{I}$ is, therefore, generated by $C_{1/2}$, which does not belong to $\widetilde{\mathcal{R}}_{\mathbb{Z}}$ if $\lambda^2 - \sigma$ is irreducible over k. However, the situation is different if $\sqrt{\sigma}$ belongs to k. From the relations given in (3.1), it follows immediately that

$$(C_{1/2} + \sqrt{\sigma}\, S_{1/2})(C_{1/2} - \sqrt{\sigma}\, S_{1/2}) = 1.$$

As a consequence, over $\widetilde{\mathcal{R}}_{\mathbb{Q}}$, $C_{1/2}$ can be factorized as

$$C_{1/2} = (C_{1/2} + \sqrt{\sigma} S_{1/2})(C_{1/2} - \sqrt{\sigma} S_{1/2}) C_{1/2} = (C_{1/2} + \sqrt{\sigma} S_{1/2})(1 + C_1 - \sqrt{\sigma} S_1)/2.$$

The element $C_{1/2}$ is, thus, associated with $1 + C_1 - \sqrt{\sigma} S_1$, which indeed belongs to $\widetilde{\mathcal{R}}_{\mathbb{Z}}$.

3.1.1. The polynomial $\lambda^2 - \sigma$ is reducible over k.

PROPOSITION 3.1. *The ring $\widetilde{\mathcal{R}}_{\mathbb{Z}}$ is a principal ideal domain (PID).*

Proof. From the addition formulas given in (3.1), it follows that $\widetilde{\mathcal{R}}_{\mathbb{Z}}$ is isomorphic to $k[S_1, C_1]$, which, in turn, is isomorphic to $k[z^{-1}, z]$ by

$$S_1 \mapsto \frac{z^{-1} - z^1}{\lambda}, \quad C_1 \mapsto z^{-1} + z^1.$$

The latter ring is Euclidean with the norm function given by the difference of the degrees of the monomials of maximal and minimal degrees w.r.t. z. ∎

COROLLARY 3.2. *The ring $\widetilde{\mathcal{R}}_{\mathbb{Q}}$ is a Bézout domain.*

Remark 3.1. Note that, for the same reason as given in Remark 3.3 below, $\widetilde{\mathcal{R}}_{\mathbb{Q}}$ is not a PID.

Remark 3.2. Having in mind that in our application σ is given according to (2.1b) and $k = \mathbb{C}(\partial_t)$, for $\sqrt{\sigma} \in k$, say $\sqrt{\sigma} = \lambda$, the operators $C(x)$ and $S(x)$ introduced in section 2.2 are constructed from point delays: From $\lambda = s\sqrt{a} - \alpha$, $\alpha \in \mathbb{C}$, we obtain $2C(x) = \mathrm{e}^{x(\partial_t\sqrt{a} - \alpha)} + \mathrm{e}^{-x(\partial_t\sqrt{a} - \alpha)}$. Note that in this case our results are simply a restatement of those presented in [3, Thm. 1] and [17, Thm. 3.2].

3.1.2. The polynomial $\lambda^2 - \sigma$ is irreducible over k. As indicated by Example 3.1, the second case, i.e., the equation $\lambda^2 - \sigma = 0$ has no solutions over k, is much more challenging than the first one. There, the ring $\widetilde{\mathcal{R}}_{\mathbb{Z}}$ corresponds basically to the ring $\mathbb{Q}[x,y]/[x^2+y^2-1]$ of trigonometric polynomials which is obtained from $\widetilde{\mathcal{R}}_{\mathbb{Z}}$ for $\sigma = -1$ and $k = \mathbb{Q}$. The latter ring is lacking the pleasing properties of a PID or even the ones of a Bézout domain.[7] However, Example 3.1 suggests that the difficulties can be circumvented when allowing one to halve the argument, i.e., working with $\widetilde{\mathcal{R}}_{\mathbb{Q}}$ instead of $\widetilde{\mathcal{R}}_{\mathbb{Z}}$.

Definition 3.3. *For any nonzero $r \in \widetilde{\mathcal{R}}_{\mathbb{X}}$ the* norm $\nu(r)$ *is defined as the highest $\alpha \in \mathbb{X}^+$ such that at least one of the coefficients a_α and b_α in (3.2) is nonzero.*

Lemma 3.4. *Let S be the multiplicative subset of $\widetilde{\mathcal{R}}_{\mathbb{Z}}$ consisting of all the elements such that either the coefficients with odd or those with even indices vanish. More precisely, any element s of S can be written as*

$$s = \sum_{i \in I_s} a_{s,i} C_i + b_{s,i} S_i, \quad I_s = \left\{ \nu(s) - 2i \,\middle|\, i \in \mathbb{Z},\, 0 \leqslant i \leqslant \frac{\nu(s)}{2} \right\}.$$

Let $p, q \in S$, the norms of which are strictly positive. Without loss of generality assume $\nu(p) \geqslant \nu(q)$. Consider the ideal $\mathfrak{I} = (p,q)$ generated by p and q. Then there exist $\bar{p}, \bar{q} \in S$, with $\mathfrak{I} = (\bar{p}, \bar{q})$ and either $\nu(p) > \nu(\bar{p}) \geqslant \nu(\bar{q})$ or $\bar{q} = 0$.

Proof. In the following, three different cases are considered.

Case 1. If $\nu(p) > \nu(q)$, one can apply a division step similar to that of polynomials. More precisely, we will show that there exist $r, h \in S$ with either $r = 0$ or $\nu(r) < \nu(p)$ such that $p = qh + r$. Then we may set $\bar{p} = q$, $\bar{q} = r$ (or vice versa) to complete the discussion of the first case.

In order to show that r, h with the claimed properties exist, set

$$h = a_h C_\Delta + b_h S_\Delta, \quad \Delta = \nu(p) - \nu(q),$$

where the coefficients $a_h, b_h \in k$ have to be determined appropriately. It follows that

$$\begin{aligned} s = hq &= \sum_{i \in I_q} \Big((a_h a_{q,i} C_i C_\Delta + b_h a_{q,i} C_i S_\Delta) + (a_h b_{q,i} S_i C_\Delta + b_h b_{q,i} S_i S_\Delta) \Big) \\ &= \frac{1}{2\sigma} \sum_{i \in I_q} \Big((\sigma a_h a_{q,i} + b_h b_{q,i}) C_{\Delta+i} + (\sigma a_h a_{q,i} - b_h b_{q,i}) C_{\Delta-i} \\ &\qquad + \sigma (b_h a_{q,i} + a_h b_{q,i}) S_{\Delta+i} + \sigma (b_h a_{q,i} - a_h b_{q,i}) S_{\Delta-i} \Big) \\ &= \sum_{i \in I_p} a_{s,i} C_i + b_{s,i} S_i, \end{aligned}$$

where the leading coefficients are given by

$$a_{s,\nu(p)} = \frac{1}{2\sigma} \big(\sigma a_h a_{q,\nu(q)} + b_h b_{q,\nu(q)} \big), \quad b_{s,\nu(p)} = \frac{1}{2} \big(b_h a_{q,\nu(q)} + a_h b_{q,\nu(q)} \big).$$

From this equation and from $r = hq - p$, the norm of r is smaller than that of p if and only if a_h, b_h satisfy

$$\begin{pmatrix} a_{q,\nu(q)} & \sigma^{-1} b_{q,\nu(q)} \\ b_{q,\nu(q)} & a_{q,\nu(q)} \end{pmatrix} \begin{pmatrix} a_h \\ b_h \end{pmatrix} = 2 \begin{pmatrix} a_{p,\nu(p)} \\ b_{p,\nu(p)} \end{pmatrix}. \tag{3.3}$$

[7] Actually, the trigonometric ring is a Dedekind domain [36, Thm. 3.1].

By the definition of the norm, at least one of the coefficients $a_{q,\nu(q)}$ and $a_{q,\nu(q)}$ is nonzero. Since, additionally, $\sqrt{\sigma} \notin k$, it follows that $\sigma a_{q,\nu(q)}^2 - b_{q,\nu(q)}^2 \neq 0$ and a_h, b_h can be always chosen according to (3.3).

Case 2. If $\nu(p) = \nu(q)$ and for some $c \in k$ the equations $a_{q,\nu(q)} = c a_{p,\nu(p)}$, $b_{q,\nu(q)} = c b_{p,\nu(p)}$ hold, the ideal $\mathfrak{I}$ is generated by $\bar{p} = p$, $\bar{q} = q - cp$, where $\nu(\bar{q}) < \nu(\bar{p})$. If $\bar{q} = 0$, the proof is complete; otherwise we can proceed according to the first case with the pair $\bar{p}, \bar{q}$ instead of p, q.

Case 3. If $\nu(p) = \nu(q)$ but we are not in the second case, set

$$\begin{pmatrix} p & q \end{pmatrix}^T = A_1 \begin{pmatrix} \tilde{p} & \tilde{q} \end{pmatrix}^T, \tag{3.4a}$$

$$\begin{pmatrix} \tilde{p} & \tilde{q} \end{pmatrix}^T = A_2 \begin{pmatrix} \bar{p} & \bar{q} \end{pmatrix}^T, \tag{3.4b}$$

with

$$A_1 = \begin{pmatrix} a_{p,n} & b_{p,n} \\ a_{q,n} & b_{q,n} \end{pmatrix}, \quad A_2 = \begin{pmatrix} C_1 & \sigma S_1 \\ S_1 & C_1 \end{pmatrix}, \quad n = \nu(q) = \nu(p).$$

Obviously, p, q belong to the ideal generated by $\bar{p}, \bar{q}$. Both matrices, A_1 and A_2, are invertible, the first one since otherwise we would be in the second case, the latter one since, by (3.1), its determinant equals 1. Thus, $(\bar{p}, \bar{q}) = (\tilde{p}, \tilde{q}) = (p, q)$.

It remains to show that the norms of $\bar{p}$ and $\bar{q}$ are both smaller than n. From (3.4a), one obtains $\nu(\tilde{p}) = \nu(\tilde{q}) = n$, with $a_{\tilde{p},\nu(q)} = b_{\tilde{q},\nu(q)} = 1$, $b_{\tilde{p},\nu(q)} = a_{\tilde{q},\nu(q)} = 0$. From (3.4b), one has

$$\bar{p} = C_1 C_n - \sigma S_1 S_n + \sum_{i \in I_p^*} C_1 \bigl(a_{\tilde{p},i} C_i + b_{\tilde{p},i} S_i\bigr) - \sigma S_1 \bigl(a_{\tilde{q},i} C_i + b_{\tilde{q},i} S_i\bigr),$$

$$\bar{q} = C_1 S_n - S_1 C_n + \sum_{i \in I_p^*} C_1 \bigl(a_{\tilde{q},i} C_i + b_{\tilde{q},i} S_i\bigr) - S_1 \bigl(a_{\tilde{p},i} C_i + b_{\tilde{p},i} S_i\bigr),$$

with $I_p^* = I_p \setminus \{n\}$. The norms of the sums in the above expression are at most $n-1$, while for the leading terms one obtains, according to (3.1a),

$$C_1 C_n - \sigma S_1 S_n = C_{n-1}, \quad C_1 S_n - S_1 C_n = S_{n-1}.$$

Thus, the norms of $\bar{p}$, $\bar{q}$ cannot exceed $n-1$. □

Lemma 3.5. *Let $p, q \in S \subset \widetilde{\mathcal{R}}_{\mathbb{Z}}$, with $\nu(p) \geqslant \nu(q)$. Then there exist $\bar{p}, \bar{q} \in S \cap (p, q)$ such that $(p, q) = (\bar{p}, \bar{q})$ and $\nu(\bar{q}) < \nu(q)$, $\nu(p) \leqslant \nu(q)$ or $\bar{q} = 0$.*

Proof. By Lemma 3.4, $(p, q) = (p^*, q^*)$, with $\nu(p) > \nu(p^*) \geqslant \nu(q^*)$ or $q^* = 0$. In the latter case the claim has been proved. Otherwise, repeat the above argument p^*, q^* until we are in the claimed situation, which happens after at most $\nu(p) - \nu(q) + 1$ steps. □

Proposition 3.6. *Any ideal $\mathfrak{I}$ in $\widetilde{\mathcal{R}}_{\mathbb{Z}}$ generated by a subset $\mathfrak{G}$ of S is principal.*

Proof. Step 1. We show that up to multiplication with units there is only one element q of lowest norm $\nu(q) = n$ in $S \cap \mathfrak{I}$. To this end, assume there are at least two such elements, say p and q. By Lemma 3.5 there exist $\bar{p}, \bar{q} \in S$, with $(\bar{p}, \bar{q}) = (p, q)$, where $n > \nu(\bar{p}) \geqslant \nu(\bar{q})$ or $n \geqslant \nu(\bar{p})$ and $\bar{q} = 0$. Since n is the lowest possible norm for an element of $\mathfrak{I} \cap S$, only the case $n = \nu(\bar{p})$ and $\bar{q} = 0$ remains. But this can happen only if we are in Case 2 of Lemma 3.4 having $\bar{p} = p$ and $q = cp$, $c \in k^\times$.

Step 2. We now show that any element of $\mathfrak{G}$ belongs to (q), where q is defined as in the first step. To this end, choose any element p from $\mathfrak{G}$. Applying Case 1 of

Lemma 3.4 several times, one gets $p = hq + r$, $\nu(r) \leqslant n$, $r \in S$. Since, by assumption, q has the smallest possible norm, it follows that $\nu(r) = n$ or $r = 0$. This, in turn, yields $r = cq$, $c \in k$, according to Step 1. Finally, we have $p = (h + c)q$ and, therefore, $\mathfrak{I} = (q)$. □

PROPOSITION 3.7. *Any finitely generated ideal $\mathfrak{I}$ in $\widetilde{\mathcal{R}}_{\mathbb{Q}}$ is principal; i.e., $\widetilde{\mathcal{R}}_{\mathbb{Q}}$ is a Bézout domain.*

Proof. Let $\mathfrak{I} = (r_1, \dots, r_m)$ for some $m \in \mathbb{N}$. Write the generators according to (3.2), i.e.,

$$r_j = \sum_{i=0}^{n_j} a_{\alpha_{j,i}} C_{\alpha_{j,i}} + b_{\alpha_{j,i}} S_{\alpha_{j,i}}, \quad n_j \in \mathbb{N}, \quad a_{\alpha_{i,j}}, b_{\alpha_{i,j}} \in k, \quad \alpha_{i,j} \in Q^+. \tag{3.5}$$

Let d be a common denominator of all the $\alpha_{i,j}$ occurring in these equations. Then the generators of $\mathfrak{I}$ can be identified with elements of the subset S defined in Lemma 3.4 of the ring $\widetilde{\mathcal{R}}_{\mathbb{Z}}$ via the embedding $E : \widetilde{\mathcal{R}}_{\mathbb{Z}} \to \widetilde{\mathcal{R}}_{\mathbb{Q}}$ which is defined by $\tilde{C}_2 \mapsto C_{1/d}$, $\tilde{S}_2 \mapsto S_{1/d}$. Let $\tilde{r}_1, \dots, \tilde{r}_m$ be elements of $\widetilde{\mathcal{R}}_{\mathbb{Z}}$, the images of which are $r_1, \dots, r_m$. The ideal $\tilde{\mathfrak{I}}$ generated by $\tilde{r}_1, \dots, \tilde{r}_m$ is principal by Proposition 3.6. Consequently, $\mathfrak{I}$ is generated by the image of the generator of $\tilde{\mathfrak{I}}$ under E. □

Remark 3.3. Note that neither $\widetilde{\mathcal{R}}_{\mathbb{Q}}$ nor $\widetilde{\mathcal{R}}_{\mathbb{Z}}$ is a PID. The first is not Noetherian: As an example, for an ideal that is not finitely generated take $(\{S_{1/2^n} | n \in \mathbb{N}\})$. Moreover, $\widetilde{\mathcal{R}}_{\mathbb{Z}}$ is not a PID since there are finitely generated ideals that cannot be generated by one single element: The ideal $(S_1, C_1 + 1)$, viewed as an element of $\widetilde{\mathcal{R}}_{\mathbb{Q}}$, is generated by $C_{1/2}$, which does not belong to $\widetilde{\mathcal{R}}_{\mathbb{Z}}$.

3.2. $\mathcal{R}_{\mathbb{Q}}$ is a Bézout domain. We are now in position to prove that $\mathcal{R}_{\mathbb{Q}}$ is a Bézout domain. After the preparation done in the previous subsection, the remaining steps are very similar to those given in [3, 17]. In particular, the proof of Lemma 3.8 which prepares Theorem 3.9 is strongly inspired by [3, Thm. 1].

LEMMA 3.8. *For two coprime elements $p, q \in \mathcal{R}_{\mathbb{Q}}$ there exist $a, b \in \mathcal{R}_{\mathbb{Q}}$ such that $ap + bq = 1$.*

Proof. By Proposition 3.7 (resp., Corollary 3.2), $\mathbb{C}(\partial_t)[\mathfrak{S}_{\mathbb{Q}}]$ is a Bézout domain. Thus, there exist $a, b \in \mathbb{C}[\partial_t, \mathfrak{S}] \subset \mathcal{R}_{\mathbb{Q}}$ such that $ap + bq = h$, where $h \in \mathbb{C}[\partial_t]$. Write h as product $h = \prod_{i=1}^{N} (\partial_t - s_i)$, and proceed by induction (we do not assume $s_i \neq s_j$).

Assume there exist $a, b \in \mathcal{R}_{\mathbb{Q}}$, with $ap + bq = \prod_{i=1}^{N} (\partial_t - s_i)$. In the following, for any $\gamma \in \mathscr{E}'_*$ we set $\bar{\gamma} = \mathscr{L}(\gamma)(s_N)$, with the entire function $\mathscr{L}(\gamma)$ being the Laplace transform of γ. By the coprimeness of p, q

$$a^\star = \begin{cases} \dfrac{\bar{q}a - q\bar{a}}{\bar{q}(\partial_t - s_N)}, & (\partial_t - s_N) \nmid q, \\ \dfrac{a}{s - s_N}, & (\partial_t - s_N) \mid q, \end{cases} \qquad b^\star = \begin{cases} \dfrac{\bar{p}b - p\bar{b}}{\bar{p}(\partial_t - s_N)}, & (\partial_t - s_N) \nmid p, \\ \dfrac{b}{s - s_N}, & (\partial_t - s_N) \mid p \end{cases}$$

belong to $\mathcal{R}_{\mathbb{Q}}$. Here, we used the fact that a differential operator from $\mathbb{C}[\partial_t]$ divides an element of $\mathscr{E}'_*$ if and only if the quotient of the respective Laplace transforms is an entire function.[8] One easily verifies that $pa^\star + qb^\star = \prod_{i=1}^{N-1} (\partial_t - s_i)$.

Applying this step N times completes the proof. □

[8] This is a simple corollary of the Paley–Wiener theorems [22, Thm. 7.3.1]) and [25, Thm. 1.1]. See [22, Thm. 7.3.2] for a detailed proof in the case $\mathscr{E}'_* = \mathscr{E}'$.

THEOREM 3.9. *The ring $\mathcal{R}_{\mathbb{Q}}$ is a Bézout domain; i.e., any finitely generated ideal is principal.*

Proof. We show that any two elements $p, q \in \mathcal{R}_{\mathbb{Q}}$ possess a common divisor $\tilde{c}$ that can be written as a linear combination of p, q. (It is then the unique greatest common divisor (gcd) of p and q.)

According to section 3.1 the ring $\mathbb{C}(\partial_t)[\mathfrak{S}_{\mathbb{Q}}]$ is a Bézout domain. Consequently, there are elements $a, b \in \mathbb{C}[\partial_t, \mathfrak{S}_{\mathbb{Q}}]$ such that

$$c = ap + bq \in \mathbb{C}[\partial_t, \mathfrak{S}_{\mathbb{Q}}] \tag{3.6}$$

is a gcd in $\mathbb{C}(\partial_t)[\mathfrak{S}_{\mathbb{Q}}]$. Hence, p/c and q/c belong to $\mathbb{C}(\partial_t)[\mathfrak{S}_{\mathbb{Q}}]$. In particular, there are $n_i \in \mathcal{R}_{\mathbb{Q}}$, $d_i \in \mathbb{C}[\partial_t]$, with $\gcd_{\mathcal{R}_{\mathbb{Q}}}(n_i, d_i) = 1$ $(i = 1, 2)$ such that $p/c = n_1/d_1$ and $q/c = n_2/d_2$. It follows that $pd_1 = cn_1$, $qd_2 = cn_2$. Consequently, both d_1 and d_2 divide c in $\mathcal{R}_{\mathbb{Q}}$. Since d_1 and d_2 are polynomials, they possess a least common multiple $h = d_1 d_2/\gcd(d_1, d_2)$, and it follows that $\tilde{c} = c/h \in \mathcal{R}_{\mathbb{Q}}$. Clearly, $\tilde{c}$ divides both p and q. Dividing (3.6) by $\tilde{c}$ yields the equation

$$a \underbrace{n_1 d_2/\gcd(d_1, d_2)}_{\tilde{p}} + b \underbrace{n_2 d_1/\gcd(d_1, d_2)}_{\tilde{q}} = \underbrace{d_1 d_2/\gcd(d_1, d_2)}_{h}. \tag{3.7}$$

By the coprimeness of n_1 and d_1 (resp., n_2 and d_2), it follows that $\gcd(\tilde{p}, h) = d_2/\gcd(d_1, d_2)$ (resp., $\gcd(\tilde{q}, h) = d_1/\gcd(d_1, d_2)$). Thus, $\gcd(\tilde{q}, h)$ and $\gcd(\tilde{p}, h)$ are coprime and, since by (3.7) any common divisor of $\tilde{p}$ and $\tilde{q}$ divides h, we can finally conclude the coprimeness of $\tilde{p}$ and $\tilde{q}$. Thus, by Lemma 3.8, there are $a^\star, b^\star \in \mathcal{R}_{\mathbb{Q}}$ such that $a^\star \tilde{p} + b^\star \tilde{q} = 1$. The claim follows directly by multiplying this equation by $\tilde{c}$. ∎

4. Controllability analysis.

4.1. System controllabilities. In this section we emphasize several controllability notions which are defined directly on the basis of the above system definition without referring to a solution space. For the latter we refer to the next subsection. Let us start with some purely algebraic definitions.

DEFINITION 4.1. *An* R-system Λ*, or a* system over R*, is an R-module. A* presentation matrix *of a finitely presented R-system Σ is a matrix P such that $\Sigma \cong [v]/[Pv]$, where $[v]$ is free with basis v. An* output $\boldsymbol{y}$ *is a subset, which may be empty, of* Λ*. An* input-output R-system*, or an* input-output system over R*, is an R-dynamics equipped with an output.*

DEFINITION 4.2 (see, e.g., [14, Def. 2.4.]). *Let A be an R-algebra. An R-system Λ is said to be* A-torsion-free controllable *(resp.,* A-projective controllable*,* A-free controllable*) if the A-module $A \otimes_R \Lambda$ is torsion-free (resp., projective, free). An* R-torsion-free *(resp.,* R-projective*,* R-free*)* controllable *R-system is simply called* torsion-free *(resp.,* projective*,* free*)* controllable*.*

Elementary homological algebra (see, e.g., [42]) yields the following proposition.

PROPOSITION 4.3. *A-free (resp., A-projective) controllability implies A-projective (resp., A-torsion-free) controllability.*

The importance of the notions of torsion-free and free controllability is intuitively clear: While the first one refers to the absence of a nontrivial subsystem which is governed by an autonomous system of equations, the latter refers to the possibility to freely express all system variables in terms of a basis of the system module. For this reason and, secondarily, in reminiscence to the theory of nonlinear finite dimensional systems, we have the following definition.

DEFINITION 4.4. *Take an A-free controllable R-system Λ with a finite output $\boldsymbol{y}$. This output is said to be* A-flat*, or* A-basic*, if $\boldsymbol{y}$ is a basis of $A \otimes_R \Lambda$. If $A \cong R$, then $\boldsymbol{y}$ is simply called* flat*, or* basic*.*

In finite dimensional linear systems theory, the so-called Hautus criterion is a quite popular tool for checking controllability. This criterion has been generalized to delay systems (see, e.g., [30, Def. 5.1]) and to the more general convolutional systems defined over $\mathscr{E}'$ [50, Def. 10] and the ring $\mathcal{M}_0$ of compactly supported Mikusiński operators [52, Def. 4.3]. All those rings may be embedded into the ring of entire functions via the Laplace transform. This motivates the following quite general definition.

DEFINITION AND PROPOSITION 4.1. *Let R be any ring that is isomorphic to a subring of the ring $\mathscr{O}$ of entire functions with pointwise defined multiplication. Denote the embedding $R \to \mathscr{O}$ by $\mathscr{L}$. A finitely presented R-system with presentation matrix P is said to be spectrally controllable if one of the following equivalent conditions holds:*

(i) *The $\mathscr{O}$-matrix $\hat{P} = \mathscr{L}(P)$ satisfies $\exists k \in \mathbb{N} : \forall \sigma \in \mathbb{C} : \operatorname{rank} \hat{P}(\sigma) = k$.*

(ii) *The module $\Sigma_{\mathscr{O}} = \mathscr{O} \otimes_R \Sigma$ is torsion-free.*

Proof. The result is a simple consequence of the fact that $\mathscr{O}$ is an elementary divisor domain;[9] i.e., the matrix $\hat{P}$ admits the Smith normal form. ∎

PROPOSITION 4.5. *Let R be any Bézout domain that is isomorphic to a subring of $\mathscr{O}$ with the embedding $R \to \mathscr{O}$ denoted by $\mathscr{L}$. Then the notions of spectral controllability and R-torsion-free controllability are equivalent if and only if $\mathscr{L}$ maps nonunits in R to nonunits in $\mathscr{O}$.*

Proof. Since R is a Bézout domain, torsion freeness of Σ implies freeness. Tensoring with the free module $\mathscr{O}$ yields another free module $\Sigma_{\mathscr{O}}$ and thus, by Definition and Proposition 4.1, spectral controllability. Again, since R is a Bézout domain, any presentation matrix admits a Hermite form. Thus, the torsion submodule $\mathrm{t}\Sigma$ of Σ can be presented by a triangular square matrix $\mathrm{t}P$ of full rank. If Σ is not torsion-free, at least one diagonal entry of this matrix is not a unit in R. If this entry is mapped to a nonunit in $\Sigma_{\mathscr{O}}$ by $\mathscr{L}$, it admits a complex zero σ_0. Thus, $\mathscr{L}(\mathrm{t}P)$ has a loss off rank at $\sigma = \sigma_0$. Conversely, if there is a nonunit $r \in R$ which corresponds to a unit $\hat{r} \in \mathscr{O}$, consider $\Sigma \cong [\tau]/[r\tau]$. Obviously, the image of τ in $\Sigma_{\mathscr{O}}$ is zero. Thus, the trivial module $\Sigma_{\mathscr{O}}$ is torsion-free. ∎

Remark 4.1. Note that, under the additional assumption that Σ admits a presentation matrix of full row rank, the assumption of R being a Bézout domain may be replaced by a less restrictive one. In this case, equivalence of $(\mathcal{Q} \otimes_R R) \cap \mathscr{O}$-torsion-free controllability, with $\mathcal{Q}$ the ring of rational functions in one complex variable, and spectral controllability may be established (see, e.g., [30, 52] for different examples). In [50, Thm. 14] a related result has been presented for systems over $\mathscr{E}'$. Note that this latter result is formulated for the module of solutions $\operatorname{Hom}_{\mathscr{E}'}(\Sigma, \mathscr{E})$ instead of the system module Σ. This is motivated by the viewpoint that systems admitting the same solutions should be described by the same algebraic structure.

We are now able to state the main result of this section.

THEOREM 4.6. *A finitely presented $\mathcal{R}_{\mathbb{Q}}$-system $\Sigma_{\mathbb{Q}}$ is free if and only if it is torsion-free. More generally, $\Sigma_{\mathbb{Q}} = \mathrm{t}\Sigma_{\mathbb{Q}} \oplus \Sigma_{\mathbb{Q}}/\mathrm{t}\Sigma_{\mathbb{Q}}$, where $\mathrm{t}\Sigma_{\mathbb{Q}}$ is torsion and $\Sigma_{\mathbb{Q}}/\mathrm{t}\Sigma_{\mathbb{Q}}$ is free. Moreover, $\Sigma_{\mathbb{Q}}$ is spectrally controllable if and only if it is torsion-free.*

Proof. Since the first assertion holds for finitely presented modules over any Bézout domain, it holds for any $\mathcal{R}_{\mathbb{Q}}$ system. The second assertion follows from Proposition 4.5. (The fact that the Laplace transform maps any nonunit of $\mathcal{R}_{\mathbb{Q}}$ to a nonunit

[9] See [20, p. 226] for this statement, which is a corollary of two results proved in [19].

in $\mathscr{O}$ is obvious.) ☐

Clearly, the above results hold not only for $\mathcal{R}_{\mathbb{Q}}$ systems but also for Σ defined in Definition 2.1, which is obtained from $\Sigma_{\mathbb{Q}}$ by an extension of scalars.

COROLLARY 4.7. *For the convolutional system* Σ *defined in Definition* 2.1 *one has* $\Sigma = \mathrm{t}\Sigma \oplus \Sigma/\mathrm{t}\Sigma$, *where* $\mathrm{t}\Sigma$ *is torsion and* $\Sigma/\mathrm{t}\Sigma$ *is free. Moreover,* Σ *is spectrally controllable if and only if it is torsion-free.*

4.2. Trajectorian controllability. In this section we will give two different interpretations of our algebraic controllability results that directly refer to trajectories of the system. To this end we need to introduce the notions of a *solution space* and a *trajectory* (or solution), which go back at least to [28]. To the authors' knowledge such notions have been used in linear control theory since [32, 12] and became thereafter a standard notion when relating the algebraic structure of a system with its solutions.

DEFINITION 4.8. *Let* Σ *be an* R*-system and* $\mathscr{F}$ *a space of generalized functions. The space* $\mathscr{F}$ *is called a* solution space *of* Σ *if it can be equipped with the structure of an* R*-module. An* $\mathscr{F}$-trajectory *of* Σ *is an element of* $\mathrm{Hom}_R(\Sigma, \mathscr{F})$.

The crux of the first controllability notion (Definition 4.9) is the question of whether it is possible to assign an arbitrary (generalized) function from $\mathscr{F}$ to *any* system variable.

DEFINITION 4.9 (see [15, sect. 2.2.1]). *An* R*-system is called* $\mathscr{F}$-trajectory controllable *if for any element* $a \in \Sigma$ *and any* $b \in \mathscr{F}$ *there exists a trajectory* $\mathfrak{t}$, *with* $\mathfrak{t}(a) = b$.

The following result is borrowed from [15, Thm. 2.2.1] and applies to any torsion-free controllable R-system where R is a ring of functions or (ultra-)distributions with compact or left bounded support. Their quotient fields are subfields of the Mikusiński field $\mathcal{M}$ defined in [29].

PROPOSITION 4.10. *The system* $\Sigma/\mathrm{t}\Sigma$ *is* $\mathcal{M}$*-trajectory controllable.*

Another elegant trajectory related controllability notion is the following, due to [51]. As above it relies on the notion of a trajectory. However, since it refers to the possibility of connecting trajectories, an appropriate solution space should allow the definition of local properties. This is not possible for the field of Mikusiński operators but is possible for the spaces $\mathscr{D}'_*$ and $\mathscr{E}_*$. The controllability criterion in the behavioral framework is the possibility of concatenating trajectories. In our algebraic setting we may formulate this criterion as follows.

DEFINITION 4.11 (cf. [51, Def. V.1] and [40, Def. V.1]). *Let* Σ *be an* R*-system and* $\mathscr{F}$ *a solution space of* Σ *that possesses the structure of a sheaf on* $\mathbb{R}$. *Then* Σ *is called* $\mathscr{F}$-behavioral controllable *if for any* $a \in \Sigma$ *there are* $t_1^a, t_2^a \in \mathbb{R}$ *such that for any two trajectories* $\mathfrak{t}', \mathfrak{t}'' \in \mathrm{Hom}(\Sigma, \mathscr{F})$ *there exists* $\mathfrak{t} \in \mathrm{Hom}(\Sigma, \mathscr{F})$, *with* $\mathfrak{t}(a)|_{(-\infty,t_1^a)} = \mathfrak{t}'(a)|_{(-\infty,t_1^a)}$ *and* $\mathfrak{t}(a)|_{(t_2^a,\infty)} = \mathfrak{t}''(a)|_{(t_2^a,\infty)}$.

THEOREM 4.12. *The system* $\Sigma/\mathrm{t}\Sigma$, *where* Σ *is defined in Definition* 2.1, *is* $\mathscr{D}'_*(\mathbb{R})$*-behavioral controllable (resp.,* $\mathscr{E}_*(\mathbb{R})$*-behavioral controllable). The system* $\mathrm{t}\Sigma$ *is controllable only if it is the zero module.*

Proof. Since $\Sigma/\mathrm{t}\Sigma$ is free, any homomorphism is uniquely determined by the functions assigned to the basis. For the basis $b = b_1, \ldots, b_n$ we may choose $t_1^b < t_2^b$ and $\mathfrak{t}(b) = \varphi\mathfrak{t}'(b) + (1-\varphi)\mathfrak{t}''(b)$, where $\varphi \in \mathscr{E}_*$ is a cutoff function, i.e., $\varphi_{(-\infty,t_1^b)} = 1$, $\varphi_{(t_2^b,\infty)} = 0$, and juxtaposition denotes the (generalized) pointwise multiplication. Any $a \in \Sigma/\mathrm{t}\Sigma$ is given by $a = \sum_{i=0}^n \alpha_i b_i$, where the coefficients $\alpha_i \mathcal{R}_{\mathbb{R}}$ have compact support. Thus, there exist T_1, T_2 such that $\operatorname{supp} \alpha_i \subseteq [T_1, T_2]$, $i = 1, \ldots, n$. The claim follows by an application of the theorem on supports $t_1^a = t_1^b + T_1$, $\quad t_2^a = t_2^b + T_2$ (see [22, Thm. 4.3.3]).

In order to show that $\mathrm{t}\Sigma$ is uncontrollable, take a trajectory $\mathfrak{t}''$, the restriction of which to $\mathrm{t}\Sigma$ is nonzero, and let $\mathfrak{t}' = 0$. Choose a torsion element $\tau \notin \ker \mathfrak{t}''$. Since τ is torsion, one has $\beta\tau = 0$ for some $\beta \in \mathcal{R}_{\mathbb{R}}$. Thus, $\beta\mathfrak{t}''(\tau) = 0$ and, by the theorem on supports, the support of $\mathfrak{t}''(\tau)$ is not bounded from the right, which implies $\mathfrak{t}(\tau) \neq 0$. However, $\operatorname{supp} \mathfrak{t}(\tau)$ must be bounded from the left, which is impossible by the theorem on supports. ☐

Behavioral controllability may be defined directly on the basis of the solution set of (2.1) or (2.12) without making reference to the convolutional system Σ (see, e.g., [51]). The set of all such solutions is called the behavior of the system. Clearly, the (lumped) behavior $\mathcal{B}_1 = \ker_{\mathscr{F}(\mathbb{R})} P$ consisting of all the solutions $\boldsymbol{c} \in \mathscr{F}(\mathbb{R})^{m+2l}$ of (2.12b) is isomorphic to $\mathrm{Hom}(\Sigma, \mathscr{F}(\mathbb{R}))$ as a $\mathbb{C}$-vector space. Since $\mathcal{B}_1$ corresponds to the restriction of $\mathrm{Hom}(\Sigma, \mathscr{F}(\mathbb{R}))$ to the generators of the Σ, the concatenability of two elements of $\mathcal{B}_1$ is immediate.

The (distributed) behavior $\mathcal{B}_2$ comprises all the solutions $(\boldsymbol{w}_1, \dots, \boldsymbol{w}_l, \boldsymbol{u}) \in \mathcal{W}$ of (2.1) with $\mathcal{W} = \mathcal{W}_1^2 \times \cdots \times \mathcal{W}_l^2 \times \mathscr{F}(\mathbb{R})^m$ and either $\mathcal{W}_i = C^0(\Omega_i, \mathscr{E}'_*(\mathbb{R}))$, $\mathscr{F} = \mathscr{E}'_*$ or $\mathcal{W}_i = \mathscr{E}_*(\Omega_i \times \mathbb{R})$, $\mathscr{F} = \mathscr{E}_*$ of (2.1). By the existence and uniqueness of the solution of the Cauchy problems (2.2) with data in $\mathscr{F}(\mathbb{R})$, the restriction mapping

$$\mathcal{W} \to \mathscr{F}(\mathbb{R})^{2l+m}, \quad (\boldsymbol{w}_1, \dots, \boldsymbol{w}_l, \boldsymbol{u}) \mapsto (\boldsymbol{w}_1(\xi_1), \dots, \boldsymbol{w}_l(\xi_l), \boldsymbol{u}) = \boldsymbol{c}$$

is a bijection, the inverse of which is given by (2.12a). In other words, the behaviors $\mathcal{B}_1$ and $\mathcal{B}_2$ are isomorphic as $\mathbb{C}$-vector spaces. As the lumped behavior $\mathcal{B}_1$, the distributed behavior $\mathcal{B}_2$ is called controllable if for any two elements $\boldsymbol{w}', \boldsymbol{w}'' \in \mathcal{B}_2$ there exists $\boldsymbol{w} \in \mathcal{B}_2$ such that $\boldsymbol{w}|_{t\in(-\infty,t_1)} = \boldsymbol{w}'|_{t\in(t_2,\infty)}$ and $\boldsymbol{w}|_{t\in(-\infty,t_1)} = \boldsymbol{w}''|_{t\in(t_2,\infty)}$, with appropriate real constants t_1, t_2. By the properties of the support of the entries of the coefficient matrices Φ_i and Ψ_i which can be deduced from (2.5), i.e.,

$$\operatorname{supp}(\Phi_i(\cdot, \xi_i), \Psi_i(\cdot, \xi_i)) \subset \{(x,t) \in \Omega_i \times \mathbb{R} : |t| < |x - \xi|\sqrt{a}\}, \quad i = 1, \dots, l,$$

the controllability result for $\mathrm{Hom}_{\mathcal{R}_{\mathbb{R}}}(\Sigma, \mathscr{F}(\mathbb{R}))$ translates not only to $\mathcal{B}_1$ but also to $\mathcal{B}_2$.

THEOREM 4.13. *The distributed behavior $\mathcal{B}_2$ associated with the boundary value problem* (2.1) *is controllable if and only if the corresponding convolution system Σ is free.*

Proof. The proof is essentially the same as the one of Theorem 4.12. ☐

Remark 4.2. Even if the boundary value problem under consideration is led back to a one-dimensional convolutional system, we would like to point out that the definition of behavioral controllability is not unique in the distributed case. As already mentioned in the introduction, at first glance, the most direct generalization of behavioral controllability to systems described by PDEs seems to be the one given in [37, Def. 2] which refers to the possibility of concatenating the restrictions of solutions to arbitrary disjoint open sets. W.r.t. this notion, the behavior associated with a set of PDEs is controllable if and only if the system module is torsion-free (see, e.g., [37, p. 398], [57, Cor. 2]). Thus, no boundary controlled distributed parameter system would be controllable, even if it would be controllable in the sense of [8, Def. 4.1.3]. This suggests that the described notion is not suitable for this class of systems. (In our opinion the choice of the term "controllability" for this concept is misleading.) Instead, a weaker concept of controllability is required. Such a notion which refers to the concatenation of solutions only in the time direction is introduced in [47, p. 61]. This latter notion is the appropriate one for boundary controlled systems as considered here.

5. An example: Two boundary coupled equations. In order to illustrate our results, in the following we discuss a simple example. Consider the system of two second order equations

$$\partial_x^2 w_i(x) = \sigma w_i(x), \quad i = 1, 2, \tag{5.1a}$$

defined on an open neighborhood Ω_i of $[0, \ell_i] \subset \mathbb{R}$, where $\sigma = \alpha\partial_t^2 + \beta\partial_t + c$. Those equations are coupled via the boundary conditions ($i = 1, 2$)

$$\mu_{i1} w_i(\ell_i) + \mu_{i2} w_i'(\ell_i) = 0, \tag{5.1b}$$

$$w_i(0) = u. \tag{5.1c}$$

According to section 2.2, the general solution of the initial value problems associated with (5.1a) reads ($i = 1, 2$)

$$\begin{pmatrix} w_i(x) \\ w_i'(x) \end{pmatrix} = \begin{pmatrix} C(x-\ell_i) & S(x-\ell_i) \\ \sigma S(x-\ell_i) & C(x-\ell_i) \end{pmatrix} \begin{pmatrix} c_{i1} \\ c_{i2} \end{pmatrix}, \tag{5.2}$$

with $c_{i1} = w_i(\ell_i)$, $c_{i2} = \partial_x w_i(\ell_i)$. The boundary conditions at $x = \ell_i$ yield

$$\mu_{i1} c_{i1} + \mu_{i2} c_{i2} = 0, \tag{5.3a}$$

$$C(\ell_i)c_{i1} - S(\ell_i)c_{i2} = u. \tag{5.3b}$$

Here, the relations $S(-\ell_i) = -S(\ell_i)$ and $C(-\ell_i) = C(\ell_i)$, derived in section 2.2, have already been incorporated.

Thus, according to Definition 2.1, the convolutional system Σ associated with the boundary value problem (5.1) is the $\mathcal{R}_\mathbb{R}$ module $[c_{11}, c_{12}, c_{21}, c_{22}, u]$, the generators of which are subject to the equations (5.3).

In order to reduce the number of equations, we aim to introduce new variables ω_1 and ω_2 such that (5.3a) is satisfied automatically, i.e.,

$$c_{i1} = -\mu_{i2}\omega_i, \quad c_{i2} = \mu_{i1}\omega_i, \quad i = 1, 2.$$

Indeed, since

$$\omega_i = \frac{1}{\mu_{i1}^2 + \mu_{i2}^2}\left(-\mu_{i2}c_{i1} + \mu_{i1}c_{i2}\right), \quad i = 1, 2,$$

the new variables belong to Σ. Using the new generators ω_1, ω_2, and u, (5.3b) may be rewritten to obtain

$$u = -p_i\omega_i, \quad p_i = \mu_{i2}C(\ell_i) + \mu_{i1}S(\ell_i), \quad i = 1, 2.$$

Thus, $p_1\omega_1 - p_2\omega_2 = 0$, and $\Sigma \cong [\tilde{\omega}_1, \tilde{\omega}_2]/[p_1\tilde{\omega}_1 - p_2\tilde{\omega}_2]$.

In accordance with section 2.1 assume that $\ell_i = n_i\ell$, with $n_i \in \mathbb{N}$ and $i = 1, 2$. Thus, by Theorem 4.6, checking spectral, torsion-free, and free controllabilities is equivalent. Since the aim of this section is not the presentation of a general controllability analysis for the boundary value problem (5.1), but rather to give an example for the application of the derived algebraic results, we shall restrict ourselves to particular values for n_1 and n_2. In order to avoid tedious computations, we choose simply $n_1 = 1$, $n_2 = 2$. Apart from that, we discuss only the generic case; i.e., we do not care about singularities which may occur for particular values of the μ_{ij}, $i, j = 1, 2$.

Applying the algorithms of section 3.1 we obtain $p_1 r_1 + p_2 r_2 = \epsilon$, with

$$\begin{aligned} r_1 &= 2\big((\mu_{21}\mu_{11} - \mu_{22}\mu_{12}\sigma)C(\ell) + (\mu_{22}\mu_{11} - \mu_{21}\mu_{12})\,\sigma S(\ell)\big),\\ r_2 &= \mu_{12}^2\sigma - \mu_{11}^2,\\ \epsilon &= -\mu_{22}\mu_{12}^2(\sigma - \bar{\sigma}), \quad \bar{\sigma} = \frac{2\mu_{21}\mu_{11}\mu_{12} - \mu_{22}\mu_{11}^2}{\mu_{22}\mu_{12}^2}. \end{aligned}$$

Following section 3.2, it remains to modify r_1, r_2 in such a way that ϵ is replaced by a constant. This may be done by applying the induction step of Lemma 3.8 once. To this end, let $\bar{r}_1, \bar{r}_2, \bar{p}_1, \bar{p}_2$ be the complex numbers obtained by setting $\sigma = \bar{\sigma}$ in the Laplace transforms of r_1, r_2, p_1, p_2. Assume that neither $\bar{p}_1$ nor $\bar{p}_2$ is zero. Then the variables

$$q_1 = \frac{\bar{p}_2 r_1 - \bar{r}_1 p_2}{\bar{p}_2\epsilon}, \quad q_2 = \frac{L_{\bar{\sigma}}(p_1)r_2 - L_{\bar{\sigma}}(r_2)p_1}{\bar{p}_1\epsilon}$$

belong to $\mathcal{R}_{\mathbb{Q}}$ and, therefore, to $\mathcal{R}_{\mathbb{R}}$. Thus, we have the Bézout equation $p_1q_1 + p_2q_2 = 1$.

From the above results, one easily verifies that with

$$y = q_2\omega_1 + q_1\omega_2$$

one has $\omega_1 = p_2 y$ and $\omega_2 = p_1 y$. Hence, y is a basis of the system under consideration.

6. Conclusion. For a class of convolutional systems associated with boundary coupled second order PDEs, we have derived algebraic controllability results which translate directly into trajectory related controllability conditions. These results rely on a division algorithm for particular convolutions rings of distributions and ultradistributions with compact support which are obtained from the solution of the Cauchy problem associated with the given system of PDEs. However, this means that our algebraic setting does not apply directly to the given boundary value problem but rather to a convolutional system arising from the solutions of the associated Cauchy problem and the boundary conditions. A promising approach allowing an algebraic treatment from the very beginning is currently under investigation.

The current work was motivated by previous contributions [3, 17] in which similar results were presented for differential delay systems. These approaches have been shown to be useful not only for controllability analysis but also for the design of closed loop control schemes using the factorization approach or the method of finite spectrum assignment [4, 5, 18]. This suggests the investigation of similar methods for the class of systems considered within this contribution.

Appendix A. Representation of the operators $S(x)$ and $C(x)$. In this section we give explicit expressions for the solution of the Cauchy problem (2.3). To this end, we will use the notation introduced in section 2.2; i.e., we write the solution in the form $w(x) = C(x)v_0 + S(x)v_1$, with v_0, v_1 being the Cauchy data.

If $a > 0$ in (2.1b), we may rewrite σ as

$$\sigma = \bar{a}^2\left((\partial_t + \alpha)^2 - \beta^2\right), \quad \bar{a} = \sqrt{a}, \quad \alpha = \frac{b}{2a}, \quad \beta = \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}.$$

Then we have for $v_0, v_1 \in \mathscr{E}(\mathbb{R})$ (cf. [38, p. 293])

$$(S(x)v_1)(t) = \int_{-x\bar{a}}^{x\bar{a}} \frac{\exp(-\alpha\tau)}{2\bar{a}} J_0\big(\beta\sqrt{\bar{a}^2x^2-\tau^2}\big) v_1(t-\tau)d\tau, \tag{A.1}$$

$$\begin{aligned}(C(x)v_0)(t) &= \frac{1}{2}\exp\left(-\frac{bx}{2\bar{a}}\right)v_0(t-x\bar{a}) + \frac{1}{2}\exp\left(-\frac{bx}{2\bar{a}}\right)v_0(t+x\bar{a})d\tau \\ &\quad - \frac{\beta\bar{a}x}{2}\int_{-x\bar{a}}^{x\bar{a}} \exp(-\alpha\tau)\frac{J_1\big(\beta\sqrt{\bar{a}^2x^2-\tau^2}\big)}{\sqrt{\bar{a}^2x^2-\tau^2}} v_0(t-\tau)d\tau.\end{aligned} \tag{A.2}$$

In contrast, for $a = 0$ in (2.1b) and $v_0, v_1 \in \mathscr{E}_{(2)}(\mathbb{R})$ those convolution products can be written as power series w.r.t. x:

$$C(x)v_0 = \sum_{k=0}^{\infty} \frac{x^{2k}v_{0,k}}{(2k)!}, \qquad v_{0,k+1} = \partial_t v_{0,k} - bv_{0,k}, \quad v_{0,0} = v_0,$$

$$S(x)v_1 = \sum_{k=0}^{\infty} \frac{x^{2k+1}v_{1,k}}{(2k+1)!}, \quad v_{1,k+1} = \partial_t v_{1,k} - bv_{1,k}, \quad v_{1,0} = v_1,$$

where the differential recursion is obtained by plugging the series ansatz into the PDE.

Appendix B. (Ultra-)distributions.

B.1. Gevrey functions and Beurling ultradistributions of Gevrey type. This section recalls some basic definitions about Gevrey functions and the corresponding classes of ultradistributions.

Definition B.1 (see, e.g., [24], [23, Def. 12.7.3, p. 137]). *An infinitely differentiable function $f : \Omega \to \mathbb{C}$ (with $\Omega \subset \mathbb{R}^n$ open) belongs to the small Gevrey class $\mathscr{E}_{(\alpha)}(\Omega)$ (or the space of Beurling ultradifferentiable functions of Gevrey class α) if for all $M \in \mathbb{R}^+$ and all compact sets $K \subset \Omega$ there exists $C_{K,M}$ such that*

$$\sup_{t\in\Omega, k\geq 0} |\partial_t^{(k)} f(t)| \leq C_{K,M} M^k (k!)^\alpha.$$

A sequence (f_n), $n \in \mathbb{N}$, $f_n \in \mathscr{E}_{(\alpha)}(\Omega)$, converges to $f \in \mathscr{E}_{(\alpha)}(\Omega)$ if for all compact $K \subset \Omega$ and all $M \in \mathbb{R}^+$

$$\lim_{n\to\infty} \sup_{t\in\Omega, k\geq 0} \frac{|\partial_t^{(k)}(f_n(t) - f(t))|}{M^k (k!)^\alpha} = 0.$$

The space of compactly supported functions in $\mathscr{E}_{(\alpha)}$ is denoted by $\mathscr{D}_{(\alpha)}(\Omega)$. A sequence (f_n), $f_n \in \mathscr{D}_{(\alpha)}(\Omega)$, $n \in \mathbb{N}$, converges in $f_n \in \mathscr{D}_{(\alpha)}(\Omega)$ if it converges in $\mathscr{E}_{(\alpha)}(\Omega)$ and, moreover, $\cup_{n\in\mathbb{N}} \operatorname{supp} f_n$ is compact. The space $\mathscr{D}'_{(\alpha)}(\mathbb{R})$ (resp., $\mathscr{E}'_{(\alpha)}(\mathbb{R})$) of Beurling ultradistributions (resp., Beurling ultradistributions with compact support) of Gevrey order α is the space of linear continuous functionals on $\mathscr{D}_{(\alpha)}(\mathbb{R})$ (resp., $\mathscr{E}_{(\alpha)}(\mathbb{R})$).

B.2. (Ultra-)distribution-valued functions. In the following, let $\mathscr{F}$ stand either for $\mathscr{E}_*$ or $\mathscr{D}_*$ and $\mathscr{F}'$ for the respective dual space. Moreover, C^* may stand either for C^∞ or for C^n, $n \in \mathbb{N}$.

Definition B.2. *Let $\Omega_1, \Omega_2 \subset \mathbb{R}$. A(n) (ultra-)distribution-valued function $F : \Omega_1 \to \mathscr{F}'(\Omega_2)$ is of class C^* if it defines a continuous linear map $\widetilde{F} : \mathscr{F}(\Omega_2) \to C^*(\Omega_1)$ via $\widetilde{F}[\varphi](x) := F(x)[\varphi]$, $x \in \Omega_1$, $\varphi \in \mathscr{F}(\Omega_2)$.*

The derivative of a function $C^n(\Omega, \mathscr{F}'(\mathbb{R}))$, $n > 0$ (resp., $C^\infty(\Omega, \mathscr{F}'(\mathbb{R}))$), is defined by $\partial_x F(x)[\varphi] := \partial_x(\widetilde{F}[\varphi](x))$ for all $\varphi \in \mathscr{F}(\Omega_2)$. As a composition of the continuous maps $\mathscr{F}(\Omega_2) \to C^n(\Omega_1)$ and $C^n(\Omega_1) \to C^{n-1}(\Omega_1)$ (resp., $C^\infty(\Omega_1) \to C^\infty(\Omega_1)$) the derivative $\partial_x F$ is continuous.

Lemma B.3. *The derivative $\partial_x F$ of $F \in C^n(\Omega_1, \mathscr{F}'(\Omega_2))$, $n > 0$ (resp., $F \in C^\infty(\Omega_1, \mathscr{F}'(\Omega_2))$), belongs to $C^{n-1}(\Omega_1, \mathscr{F}'(\Omega_2))$ (resp., $C^\infty(\Omega_1, \mathscr{F}'(\Omega_2))$).*

The integral $F^{\mathrm{I}}(x) = \int_\xi^x F(\zeta) d\zeta$ of a function $C^n(\Omega_1, \mathscr{F}'(\Omega_2))$, $(C^n(\Omega_1, \mathscr{F}'(\Omega_2))$ is defined by $F^I(x)[\varphi] = \int_\xi^x \widetilde{F}[\varphi](\zeta) d\zeta$. Since integration defines a continuous map $C^n(\Omega_1) \to C^{n+1}(\Omega_1)$ (resp., $C^\infty(\Omega_1) \to C^\infty(\Omega_1)$) the composition with $\widetilde{F}$ is continuous.

Lemma B.4. *The integral F^I of a function $F \in C^n(\Omega_1, \mathscr{F}'(\Omega_2))$, $n \geq 0$ (resp., $F \in C^n(\Omega_1, \mathscr{F}'(\Omega_2)))$), belongs to $C^{n+1}(\Omega_1, \mathscr{F}'(\Omega_2))$ (resp., $C^\infty(\Omega_1, \mathscr{F}'(\Omega_2))$).*

Lemma B.5. *Let $F \in C^*(\Omega_1, \mathscr{E}'_*(\Omega_2))$ and $G \in \mathscr{F}'(\Omega_2)$. Then, the function $H : \Omega \to \mathscr{F}'(\Omega_2)$ defined by $H(x) = G\, F(x)$ belongs to $C^*(\Omega_1, \mathscr{F}'(\Omega_2))$.*

Proof. From the commutativity of the convolution product of two (ultra-)distributions it follows for $\varphi \in \mathscr{F}(\Omega_2)$ that

$$(G\, F(x))[\varphi] = (F(x)\, G\, \bar{\varphi})(0) = F(x)[\overline{G\, \bar{\varphi}}] = F(x)[G\, \varphi],$$

where, for $\psi \in \mathscr{F}(\Omega_2)$, $\bar{\psi}$ is defined by $\bar{\psi}(t) = \psi(-t)$, $t \in \Omega_2$. Since $G\, \varphi \in \mathscr{E}_*(\Omega_2)$, the function $x \mapsto H(x)[\varphi] = F(x)[G\, \varphi]$ belongs to $C^*(\Omega_1)$ by the assumption on F. Thus, $H \in C^*(\Omega_1, \mathscr{F}'(\Omega_2))$. ∎

Acknowledgments. The authors would like to thank Michel Fliess and Joachim Rudolph for helpful discussions.

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