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H₂ for HIFOO

Denis Arzelier¹, Georgia Deaconu², Suat Gumussoy³, Didier Henrion⁴

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Abstract

HIFOO is a public-domain Matlab package initially designed for $H_\infty$ fixed-order controller synthesis, using nonsmooth nonconvex optimization techniques. It was later on extended to multi-objective synthesis, including strong and simultaneous stabilization under $H_\infty$ constraints. In this paper we describe a further extension of HIFOO to $H_2$ performance criteria, making it possible to address mixed $H_2/H_\infty$ synthesis. We give implementation details and report our extensive benchmark results.

Keywords: fixed-order controller design, $H_2$ control, mixed $H_2/H_\infty$ control, optimization.

1 Introduction

HIFOO is a public-domain Matlab package originally conceived during a stay of Michael Overton at the Czech Technical University in Prague, Czech Republic, in the summer of 2005. HIFOO relies upon HANSO, a general purpose implementation of an hybrid algorithm for nonsmooth optimization, mixing standard quasi-Newton (BFGS) and gradient sampling techniques. The acronym HIFOO (pronounce [haıfu:]) stands for H-infinity Fixed-Order Optimization, and the package is aimed at designing a stabilizing linear controller of fixed-order for a linear plant in standard state-space configuration while minimizing the $H_\infty$ norm of the closed-loop transfer function.

The first version of HIFOO was released and presented during the IFAC Symposium on Robust Control Design in Toulouse, France in the summer of 2006, see [6], based on the theoretical achievements reported in [5]. HIFOO was later on extended to cope

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with multiple plant stabilization and multiple conflicting objectives and the second major release of HIFOO was announced during the IFAC Symposium on Robust Control Design in Haifa, Israel, in the summer of 2009, see [7]

Since then HIFOO has been used by various scholars and engineers. Benefiting from feedback from users, we feel that it is now timely to extend HIFOO to $H_2$ norm specifications. Indeed, $H_2$ optimal design, a generalization of the well-known linear quadratic regulator design, is traditionally used in modern control theory jointly with $H_\infty$ optimal design, see [11]. In particular, the versatile framework of mixed $H_2/H_\infty$ design described e.g. in [9] is frequently used when designing high-performance control laws for example in aerospace systems, see [2]. See also [10] for an application of the $H_2$ norm for smoothening $H_\infty$ optimization.

The objective is this paper is to describe the extension of HIFOO to $H_2$ norm specifications in such a way that users understand the basic mechanisms underlying the package, and may be able to implement their own extensions to fit their needs for their target applications. For example, the algorithms of HIFOO can also be extended to cope with discrete-time systems, pole placement specifications or time-delay systems. On the HIFOO webpage

\[
\text{www.cs.nyu.edu/overton/software/hifoo}
\]

we are maintaining a list of publications reporting such extensions and applications in engineering. The HIFOO and HANSO packages can also be downloaded there.

2 $H_2$ and $H_2/H_\infty$ synthesis

2.1 $H_2$ synthesis

We use the standard state-space setup

\[
\begin{align*}
\dot{x} &= Ax + B_1w + B_2u \\
z &= C_1x + D_{11}w + D_{12}u \\
y &= C_2x + D_{21}w + D_{22}u
\end{align*}
\]

where $x$ contains the states, $u$ the physical (control) inputs, $y$ the physical (measured) outputs, $w$ the performance inputs and $z$ the performance outputs. Without loss of generality, we assume that $D_{22} = 0$

otherwise we can use a linear change of variables on the system inputs and outputs, see e.g. [11].

We want to design a controller with state-space representation

\[
\begin{align*}
\dot{x}_K &= A_Kx_K + B_Ky \\
u &= C_Kx_K + DKy
\end{align*}
\]

2
so that the closed-loop system equations become
\[
\begin{align*}
\dot{x} &= A(k)x + B(k)w \\
z &= C(k)x + D(k)w
\end{align*}
\]
in the extended state vector \( x = [x^T \ x_K^T]^T \) with matrices
\[
\begin{align*}
A(k) &= \begin{bmatrix} A + B_2D_KC_2 & B_2C_K \\ B_KC_2 & A_K \end{bmatrix} \\
B(k) &= \begin{bmatrix} B_1 + B_2D_KD_{21} \\ B_KD_{21} \end{bmatrix} \\
C(k) &= \begin{bmatrix} C_1 + D_{12}D_KC_2 & B_KD_{21} \end{bmatrix} \\
D(k) &= D_{11} + D_{12}D_KD_{21}
\end{align*}
\]
depending affinely on the vector \( k \) containing all parameters in the controller matrices.

The \( H_2 \) norm of the closed-loop transfer function \( T(s) \) between input \( w \) and output \( z \) is finite only if matrix \( A \) is asymptotically stable and if \( D \) is zero (no direct feedthrough). This enforces the following affine constraint on the \( D_K \) controller matrix:
\[
D_{11} + D_{12}D_KD_{21} = 0. \tag{1}
\]

We use the singular value decomposition to rewrite this affine constraint in an explicit parametric vector form, therefore reducing the number of parameters in controller vector \( k \). If the above system of equations has no solution, then there is no controller achieving a finite \( H_2 \) norm.

In order to use the quasi-Newton optimization algorithms of HANSO, we must provide a function evaluating the \( H_2 \) norm in closed-loop and its gradient, given controller parameters. Formulas can already be found in the technical literature [8], but they are reproduced here for the reader’s convenience. The (square of the) norm of the transfer function \( T(s) \) is given by
\[
f(k) = \|T(s)\|_2^2 = \text{trace } (CX(k)C^T) = \text{trace } (B^T Y(k)B)
\]
where matrices \( X(k) \) and \( Y(k) \) solve the Lyapunov equations
\[
\begin{align*}
A^T(k)X(k) + X(k)A(k) + C^T(k)C(k) &= 0, \\
A(k)Y(k) + A^T(k)Y(k) + B(k)B(k)^T &= 0
\end{align*}
\]
and hence depend rationally on \( K \). The gradient of the \( H_2 \) norm with respect to controller parameters \( K \) is given by:
\[
\nabla_K f(k) = 2(B_2^TX(k) + D_{12}^TC(k))Y(k)C_2 + 2B_2^TX(k)B(k)D_{21}^T
\]
upon defining the augmented system matrices
\[
\begin{align*}
B_2 &= \begin{bmatrix} 0 & B_2 \\ 1 & 0 \end{bmatrix}, & C_2 &= \begin{bmatrix} 0 & 1 \\ C_2 & 0 \end{bmatrix}, \\
D_{12} &= [0 \ D_{12}], & D_{21} &= \begin{bmatrix} 0 \\ D_{21} \end{bmatrix}
\end{align*}
\]
As an academic example for which the $H_2$ optimal controller can be computed analytically, consider the system

\[
\begin{align*}
\dot{x} &= -1 + w + u \\
z &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
y &= x
\end{align*}
\]

with a static controller

\[u = ky\]

with $k$ a real scalar to be found. Closed-loop system matrices are

\[
A = -1 + k, \quad B = 1, \quad C = \begin{bmatrix} 1 \\ k \end{bmatrix}.
\]

The first Lyapunov equation in (2) reads

\[2(-1 + k)X(k) + 1 + k^2 = 0\]

so the square of the $H_2$ norm is equal to

\[f(k) = \frac{1 + k^2}{2(1 - k)}.\]

For the gradient computation, we have to solve the second Lyapunov equation in (2)

\[2(-1 + k)Y(k) + 1 = 0\]

and hence

\[\nabla f(k) = \frac{1 + 2k - k^2}{2(1 - k)^2}.\]

This gradient vanishes at two points, one of which violating the closed-loop stability condition $-1 + k < 0$. The other point yields the optimal feedback gain

\[k^* = 1 - \sqrt{2} \approx -0.4142\]

see Figure [I].

Using HIFOO with the input sequence

\[
P=\text{struct}('A',-1,'B1',1,'B2',1,...
'\text{C1}',[1;0],'\text{C2}',1,'\text{D11}',[0;0],...$
'D12',[0;1],'D21',0,'D22',0);$
\text{options.prtlevel}=2;$
K=hifoo(P,'t',options)
\]

we generate the 3 sequences of optimized $H_2$ norms displayed on Figure [2], yielding an optimal $H_2$ norm of 0.6436 consistent with the analytic global minimum $\sqrt{\sqrt{2} - 1}$. Note the use of the optional third input parameter specifying a verbose printing level. Note also that the sequences generated on your own computer may differ since random starting points are used.
2.2 Mixed $H_2/H_\infty$ synthesis

One interesting feature of adding $H_2$ performance in the HIFOO package is the possibility to address the general mixed $H_2/H_\infty$ synthesis problem depicted on Figure 3 where the open-loop plant is denoted by $P$ and the controller is denoted by $K$. A minimal state-space realization of the plant is given by

$$P(s) := \begin{bmatrix} A & B_\infty & B_2 & B \\ C_\infty & D_\infty & 0 & D_{\infty u} \\ C_2 & 0 & 0 & D_{2 u} \\ C & D_{y \infty} & 0 & 0 \end{bmatrix}.$$
Figure 3: Standard feedback configuration for mixed $H_2/H_\infty$ synthesis.

The optimization problem reads

$$\min_K \|P_2(s)\|_2$$

s.t. $$\|P_\infty(s)\|_\infty \leq \gamma_\infty$$

where $P_2(s)$ is the transfer function between $H_2$ performance signals $w_2$ and $z_2$, and $P_\infty(s)$ is the transfer function between $H_\infty$ signals $w_\infty$ and $z_\infty$.

$$P_2(s) := \begin{bmatrix} A & B_2 & B \\ C_2 & 0 & D_{2u} \\ 0 & 0 & 0 \end{bmatrix}$$

$$P_\infty(s) := \begin{bmatrix} A & B_\infty & B \\ C_\infty & D_\infty & D_{\infty u} \\ C & D_{y\infty} & 0 \end{bmatrix}.$$  \hspace{1cm} (3)

An academic example for which the global optimal solution has been calculated in [3] is used as an illustration for the mixed $H_2/H_\infty$ synthesis problem. Data for the model are given by

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B_2 = 1_2 \quad D_{2u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C_\infty = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad B_\infty = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad D_{\infty u} = 0$$

$$D_\infty = 0 \quad D_{y\infty} = 0 \quad D_{y2} = 0_{1\times2}.$$  

The analytical solution may be found by solving the following mathematical programming problem as in [3]

$$\min_k J(k)$$

s.t. $k < 0$

$$f(k) \leq \gamma_\infty.$$  \hspace{1cm} (4)
For a non redundant mixed $H_2/H_\infty$ ($1 < \gamma_\infty < \frac{3}{\sqrt{5}}$), the global optimal solution is

$$k^* = -\sqrt{2 - 2\sqrt{1 - 1/\gamma^2}}$$

$$\|P_2\|_2 = \alpha^* = \sqrt{\frac{4-3\sqrt{1-1/\gamma^2}}{\sqrt{2-2\sqrt{1-1/\gamma^2}}}}$$

(5)

For $\gamma = 1.2$, HIFOO gives the global optimal solution

$$k^* = -0.9458$$

$$\|P_2\|_2 = 1.5735 \quad \|P_\infty\|_\infty = 1.2$$

(6)

with the input sequence

$$P_2=\text{struct('A', [0 1; -1 0], 'B1', eye(2), 'B2', [0;1],...\}
\text{'C1', [1 0; 0 0], 'C2', [0 1], 'D11', zeros(2,2),...\}
\text{'D12', [0;1], 'D21', [0 0], 'D22', 0);\}
\text{Pinf = struct('A', [0 1; -1 0], 'B1', [1;0],...\}
\text{'B2', [0;1], 'C1', [0 1], 'C2', [0 1], 'D11', 0,...\}
\text{'D12', 0, 'D21', 0, 'D22', 0);\}
\text{K=hifoo({P2,Pinf},'th', [Inf,1.2])}\}

3 Implementation details

3.1 Implementing $H_2$ norm into HIFOO

In the main HIFOO function hifoomain.m, we added an option 't' for $H_2$ norm specification, without affecting the existing features. Proceeding this way, the $H_2$ norm can enter the objective function or a performance constraint. The $H_2$ synthesis works in the same way as the $H_\infty$ synthesis, using a stabilization phase followed by an optimization phase.

A typical call of HIFOO for H2 static output feedback design is as follows:

$$K = \text{hifoo('AC1','t')}$$

where AC1 refers to a problem of the COMPlib database, see [3, 7].

We added the function htwo.m computing the $H_2$ norm and its gradient, given the controller parameters. We had to pay special attention to the linear system of equations arising from constraint (3). When the user also specifies the controller structure, we have added this constraint to the existing $H_2$ constraints. To do this we had to change the way the controller structure was treated by HIFOO.
The above formulae for the computation of the $H_2$ norm and its gradient are given for $D_{22} = 0$, i.e. zero feedthrough matrix. When this matrix is nonzero we can use the same functions for synthesis, with some precautions. By considering the shifted output $\tilde{y} = y - D_{22}u$, we recover the initial case with zero feedthrough matrix. We compute the controller matrices $\hat{A}_K$, $\hat{B}_K$, $\hat{C}_K$, $\hat{D}_K$ based on the shifted output and then we apply a transformation on the controller matrix. We obtain the final solution:

$$
\begin{align*}
A_K &= \hat{A}_K - \hat{B}_K D_{22}(1 + \hat{D}_K D_{22})^{-1}\hat{C}_K \\
B_K &= \hat{B}_K(1 - D_{22}(1 + \hat{D}_K D_{22})^{-1}D_K) \\
C_K &= (1 + \hat{D}_K D_{22})^{-1}\hat{C}_K \\
D_K &= (1 + \hat{D}_K D_{22})^{-1}\hat{D}_K.
\end{align*}
$$

(7)

Given the way this case is treated, multiple plant optimization only works if the plants have the same feedthrough matrix. Note however that the case of nonzero feedthrough matrix and imposed controller structure cannot be treated by the current version of the program but could be the object of further development.

### 3.2 Numerical linear algebra

For $H_\infty$ norm optimization, HIFOO calls Matlab’s function `eig` to check stability, returns `inf` if unstable, and otherwise calls the Control System Toolbox function `norm`, which proceeds by bisection on successive computations of spectra of Hamiltonian matrices. This latter function relies heavily on system matrix scaling, on SLICOT routines, and its is regularly updated and improved by The MathWorks Inc. We observe experimentally that calling `eig` once before calling `norm` is negligible (less than 5%) in terms of total computational cost.

For $H_2$ norm optimization, HIFOO calls `eig` to check stability, returns `inf` if unstable, and otherwise calls Matlab’s `lyap` function to compute the norm and its gradient. Experimentally, we observe that the time spent by `eig` to check stability is approximately 20% of the time spent to solve the two Lyapunov functions.

So a priori stability check is negligible for $H_\infty$ optimization and comparatively small but not negligible for $H_2$ optimization. In this latter case there is some room for improvement, but since the overall objective of the HIFOO project is not performance and speed but reliability, we decided to keep the stability check for $H_2$ optimization.

### 4 Benchmarking

#### 4.1 $H_2$ synthesis

We have extensively benchmarked HIFOO on problem instances studied already in [4] with an LMI/randomized algorithm. Since random starting points are used in HIFOO we kept the best results over 10 attempts each with 3 starting points, with no computation time limit. We ran the algorithms only on systems which are not open-loop stable. For
comparison, we took the best results obtained in [4]. In Table 1, we use the following notations:

•: linear system (1) has a unique solution which is not stabilizing
•: linear system (1) has no solution
+ : algorithm initialized with a stabilizing controller
†: no stabilizing controller was found
r: rank assumptions on problem data are violated.

Also $n_x$, $n_u$, $n_y$ denote the number of states, inputs and outputs. In addition to $H_2$ norms obtained by the LMI algorithm and HIFOO, we also report for information the $H_2$ norm achievable by full-order controller design with HIFOO. Numerical values are reported to three significant digits for space reasons.

In some cases (e.g. IH and CSE2) we observe that the norms achieved with a full-order controller are greater than the norms achieved with a static output feedback controller. This is due to the difficulty of finding a good initial point in the full-order case. A more practical approach, not pursued here, consists in gradually increasing the order of the controller, using the lower order controller found at the previous step.

For the considered examples, HIFOO generally gives better results than the randomized/LMI method of [4]. We also report the performance achievable with a full-order controller designed with HIFOO. We could not use the $H_2$ optimal synthesis functions of the Control System Toolbox for Matlab as the technical assumptions (rank conditions on systems data) under which these functions are guaranteed to work are most of the time violated.

4.2 $H_2$ synthesis for larger order systems

For larger order systems, we compared our results with those of [1] which are also based on nonsmooth optimization (labeled NSO). In Table 2 the column $n_k$ indicates the order of the designed controller. We observe that HIFOO yields better results, except for example CM4 in the static output feedback case.

4.3 Mixed $H_2/H_\infty$ synthesis

In this section we compare the results achieved with HIFOO with those of [4] in the case of the mixed $H_2/H_\infty$ synthesis problem, depicted on Figure 3.

In Table 3 $n_k$ is the order of the controller and $\gamma_\infty$ is the level of $H_\infty$ performance (a constraint). For problem dimensions refer to Table 4.

We observe that HIFOO returns better or similar results than the non-smooth optimization (NSO) method of [4], except for problem CM4 in the static output feedback case.
Based on the CPU times of the NSO method gracefully provided to us by Aude Rondepierre (not reported here), we must however mention that HIFOO is typically much slower. This is not surprising however since HIFOO is Matlab interpreted, contrary to the NSO method which is compiled.

5 Conclusion

This paper documents the extension of HIFOO to $H_2$ performance. The resulting new version 3.0 of HIFOO has been extensively benchmarked on $H_2$ and $H_2/H_\infty$ minimization problems. We illustrated that HIFOO gives better results than alternative methods for most of the considered benchmark problems.

HIFOO is an open-source public-domain software that can be downloaded at

www.cs.nyu.edu/overton/software/hifoo

Feedback from users is welcome and significantly helps us improve the software and our understanding of nonsmooth nonconvex optimization methods applied to systems control. Just before the completion of this work, Pierre Apkarian informed us that several algorithms of nonsmooth optimization have now been implemented by The MathWorks Inc. and will be released in the next version of the Robust Control Toolbox for Matlab. Extensive comparison with HIFOO will therefore be an interesting further research topic.

Acknowledgments

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References


Table 1: $H_2$ norm achieved for SOF controller design with LMI/randomized methods and HIFOO, and full-order controller design with HIFOO.

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<th></th>
<th>$n_x$</th>
<th>$n_u$</th>
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Table 2: $H_2$ norm achieved with HIFOO compared with the nonsmooth optimization method of [1].

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Table 3: Mixed $H_2/H_\infty$ design with HIFOO compared with the nonsmooth optimization method of [1].

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