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1

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Abstract. For general time-dependent local volatility models, we propose new approximation formulas for the price of call options. This extends previous results of [BGM10b] where stochastic expansions combined with Malliavin calculus were performed to obtain approximation formulas based on the local volatility At The Money. Here, we derive alternative expansions involving the local volatility at strike. Averaging both expansions give even more accurate results. Approximations of the implied volatility are provided as well.

Keywords. Local volatility model, European option, asymptotic expansion, CEV model.

1.1 Introduction

1.1.1 Framework

We consider a linear Brownian motion $(W_t)_{0 \le t \le T}$ defined on a filtered probability space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{0 \le t \le T}, \mathbb{P})$ where $T > 0$ is a fixed terminal time. Here, $(\mathcal{F}_t)_{0 \le t \le T}$ is the completion of the natural filtration of W. This is used to model the dynamics of a risky asset S (e.g. a stock or an index), which price process is $(S_t)_{0 \le t \le T}$. We are mainly interested in valuing European-style financial contracts written on S, exercised at maturity T, which related payoff is of the form $\Phi(S_T)$. We especially pay attention to vanilla options, i.e. $\Phi(S) = (S - K)_+$ (call options) and $\Phi(S) = (K - S)_+$ (put options).

We consider the standard framework of complete market (see for instance [MR98]), and more specifically, we assume that

1. the short-term interest rate $(r_t)_{0 \le t \le T}$ is deterministic and bounded;

2. the risky asset pays a continuous dividend $(q_t)_{0 \leq t \leq T}$, which is deterministic and bounded;

3. S follows a local volatility model, which dynamics is defined by the solution of the following stochastic differential equation:

$$\frac{dS_t}{S_t} = (r_t - q_t)dt + \sigma(t, S_t)dW_t. \qquad (1.1)$$

We denote the compound factor by

$$C_t = \exp\Big(\int_0^t (r_s - q_s)ds\Big). \qquad (1.2)$$

Thus, we have

$$S_t = C_t \exp(X_t), \qquad (1.3)$$

$$X_t = \log(S_0) + \int_0^t \sigma(s, S_s)dW_s - \frac{1}{2}\int_0^t \sigma^2(s, S_s)ds. \qquad (1.4)$$

Note that the above dynamics are directly given under the risk-neutral measure, since we only focus on pricing formulas. Then, the option price at time 0 is given by $\mathbb{E}(e^{-\int_0^T r_s ds}\Phi(S_T))$. Of course, due to the general form of the local volatility function $\sigma(t, S)$, it is hopeless to derive exact closed formulas for such option prices. The aim of this work is to obtain accurate approximations.

1.1.2 Literature background

The interest in local volatility models probably dates back to the work by Dupire [Dup94] among others, who shows that such models are able to fit all call and put option prices at a given observation date (the calibration date). However, except in a few cases, analytical pricing formulas are available (for instance, in the CEV model $\sigma(t, S) = \nu S^{\beta-1}$, see [Sch89]). As alternative numerical methods, one could use a PDE approach but to achieve real-time pricing and calibration routines, it is better to search for approximative formulas, quicker to evaluate. Hagan etal. [HW99] use singular perturbation techniques to obtain an implied volatility expansion, in the case of separable volatility $\sigma(t, S) = \alpha(t)A(S)$. Henry-Labordère [Hen08] transfers heat kernel expansions on price expansions. To tackle the case of non-separable volatility, Piterbarg [Pit05] suggests the use of parameter averaging for some choices of $\sigma(t, S)$. A different approach has been developed in [BGM09]: first a model proxy is chosen, then a smart expansion around this proxy is performed, involving Malliavin calculus to determine explicitly the expansion terms. This approach appears to be quite flexible since it naturally handles time-dependent coefficients and various modeling situations including so far jumps, discrete dividends or stochastic interest rates. More precisely, applications to local volatility model including jumps have been developed in [BGM09] and deeply

investigated along further directions in [BGM10b]. Allowing the interest rates to be stochastic is achieved in [BGM10a], while [BGM10c] considers the case of time-dependent Heston model. In [EG10], the authors investigate the case of assets paying discrete dividends. Within this approach, we are able to prove explicit error estimates that depend on σ and its derivatives, on the maturity and on the payoff. It helps to better understand the roles of each parameter. In addition, the regularity of the payoff is crucial in order to design the expansion and to establish error estimates. These features are extensively discussed in [BGM10b] and [BGM10a].

Nevertheless, regarding the results in [BGM10b], one could legitimately formulate the criticism that we use the local volatility only At The Money (ATM in short) when we take the model proxy as Black-Scholes model and when we compute the expansions. For arbitrary payoffs, this is natural, but for call/put options, this may be strange since the spot and strike variables play somewhat symmetric roles.

Here, we correct this drawback by providing new expansion formulas based on the local volatility at strike (and we even mix the expansions). This article is organized as follows: in the next paragraphs, we define the assumptions and notations used throughout the paper. Then, in the next section, main results are stated. The main proofs are postponed in Appendix. Numerical experiments are presented in Section 1.3.

1.1.3 Standing assumptions for the approximations

Throughout the paper, we assume the following:

- **Assumption** (E). The function σ is bounded and positive ($\sigma_{inf} = \inf\limits_{(t,x)\in[0,T]\times\mathbb{R}^+} \sigma(t,S) > 0$). We denote by $c_E \geq 1$ the smallest constant such that

$$\sup_{(t,S)\in[0,T]\times\mathbb{R}^+} \sigma(t,S) \leq c_E \inf_{(t,S)\in[0,T]\times\mathbb{R}^+} \sigma(t,S).$$

- **Assumption** (R). The function σ is seven-times continuously differentiable in the S-variable and

$$M_1 = \max_{1\leq i\leq 7} \sup_{(t,x)\in[0,T]\times\mathbb{R}} \left| \frac{\partial^i}{\partial x^i}[\sigma(t,\exp(x))] \right| < \infty, \qquad (1.5)$$

$$M_0 = \max\left(M_1, \sup_{(t,S)\in[0,T]\times\mathbb{R}^+} \sigma(t,S) \right) < \infty. \qquad (1.6)$$

The assumption (R) is used in at least two respects: it allows for differentiating coefficients to obtain an expansion formula; it is used to derive error estimates. The assumption (E) is an ellipticity-type condition that enables us to handle the error analysis for non-smooth payoffs Φ (such as call/put options). This is the standard framework developed in [BGM09].

Note that for deterministic volatility functions, one has $M_1 = 0$.

1.1.4 Definitions and other notations

In the representation of the expansion formulas, we repeatedly use the following integral operator.

Definition 1 (Integral Operator). *The integral operator ω^T is defined as follows: for any integrable function l, we set*

$$\omega(l)_t^T = \int_t^T l_u du$$

for $t \in [0, T]$. Similarly, for integrable functions (l_1, l_2), we put for $t \in [0, T]$

$$\omega(l_1, l_2)_t^T = \omega(l_1 \omega(l_2)_\cdot^T)_t^T = \int_t^T l_{1,r} \Big(\int_r^T l_{2,s} ds \Big) dr.$$

The n-times iteration is defined analogously: for any integrable functions $(l_1, \cdots, l_n)$, we set

$$\omega(l_1, \cdots, l_n)_t^T = \omega(l_1 \omega(l_2, \cdots, l_n)_\cdot^T)_t^T$$

for $t \in [0, T]$.

We also use a short notation for Greeks.

Definition 2 (Greeks). *Let Z be a random variable and let h be a payoff function. We define the ith Greek for the variable Z by the quantity (if it has a meaning):*

$$\mathrm{Greek}_i^h(Z) = \frac{\partial^i \mathbb{E}[h(Z + x)]}{\partial x^i}\Big|_{x=0}.$$

Definition 3 (Black-Scholes formula and related Greeks). *Using usual notation, the Black-Scholes formula for call option and constant parameters (σ, r, q) writes*

$$\mathrm{Call}^{\mathrm{BS}}(t, S; T, K; \sigma, r, q) = Se^{-q(T-t)}\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2),$$

where $\mathcal{N}(d) = \int_{-\infty}^d \frac{e^{-u^2/2}}{\sqrt{2\pi}} du$ and

$$d_1 = d_1(t, S; T, K; \sigma, r, q) = \frac{1}{\sigma\sqrt{T-t}} \log\Big(\frac{Se^{-q(T-t)}}{Ke^{-r(T-t)}}\Big) + \frac{1}{2}\sigma\sqrt{T-t},$$

$$d_2 = d_2(t, S; T, K; \sigma, r, q) = d_1 - \sigma\sqrt{T-t}.$$

For time dependent coefficients $(\sigma_s, r_s, q_s)_{0 \leq s \leq T}$, the call price formula is deduced from the Black-Scholes formula by replacing the arguments σ^2, r and q by their time-average on the interval $[t, T]$. The resulting formula is denoted by $\mathrm{Call}^{\mathrm{BS}}(t, S; T, K; (\sigma_s)_s, (r_s)_s, (q_s)_s)$.

For $t < T$ and $\sigma > 0$, the function $(S, K) \mapsto \mathrm{Call}^{\mathrm{BS}}(t, S; T, K; \sigma, r, q)$ is smooth and its sensitivities $\frac{\partial^i}{\partial S^i}\mathrm{Call}^{\mathrm{BS}}(t, S; T, K; \sigma, r, q)$ and $\frac{\partial^i}{\partial K^i}\mathrm{Call}^{\mathrm{BS}}(t, S; T, K; \sigma, r, q)$ are given explicitly in Proposition 1 (see Appendix C), for $i = 1, \ldots, 6$. They will be used in our expansion formulas (see Theorems 2 and 3).

1.2 Expansion formulas

In this section, we give several expansion formulas, with a second and third order accuracy. The general principle for deriving such approximations is to choose a relevant proxy and to expand the quantities of interest around this proxy. First, we recall the general results from [BGM09], where the proxy is obtained by freezing the local volatility at the initial spot value (ATM). Second we apply these expansions to call options. Third, using the Dupire forward PDE satisfied by the call price as a function of maturity and strike, we propose a new proxy where the volatility is frozen at the strike value K (instead of S_0). We then derive new second and third approximation formulas. Finally, some expansions of implied volatility are provided.

1.2.1 A general result

We first state two expansion results in a quite general form, so that we can apply it later to various situations. Let $(Y_t)_{0 \leq t \leq T}$ the solution of

$$dY_t = -\frac{1}{2}a^2(t, Y_t)dt + a(t, Y_t)dW_t, \qquad Y_0 \text{ given.} \qquad (1.7)$$

Theorem 1 (Second and third order approximations [BGM10b, Theorems 2.1 and 2.3]). *Assume that*

- *the function a is bounded and positive ($a_{inf} = \inf\limits_{(t,y) \in [0,T] \times \mathbb{R}} a(t, y) > 0$). We denote by $c_E \geq 1$ the smallest constant such that*

$$\sup_{(t,y) \in [0,T] \times \mathbb{R}} a(t, y) \leq c_E \inf_{(t,y) \in [0,T] \times \mathbb{R}} a(t, y).$$

- *the function a is seven-times continuously differentiable in the y-variable and*

$$M_{Y,1} = \max_{1 \leq i \leq 7} \sup_{(t,y) \in [0,T] \times \mathbb{R}} \left| \partial_y^i a(t, y) \right| < \infty, \qquad (1.8)$$

$$M_{Y,0} = \max \left(M_{Y,1}, \sup_{(t,y) \in [0,T] \times \mathbb{R}} a(t, y) \right) < \infty. \qquad (1.9)$$

- *the function $h : \mathbb{R} \mapsto \mathbb{R}$ is a.e. differentiable. In addition, h and h' have at most an exponential growth: $|h(x)| + |h'(x)| \leq c_h e^{c_h|x|}$ for any x, for a constant c_h.*

Define

- *the Gaussian process $(Y_t^P)_{0 \leq t \leq T}$ by*

$$Y_t^P = Y_0 - \frac{1}{2} \int_0^t a^2(s, Y_0)ds + \int_0^t a(s, Y_0)dW_s;$$

- $a(t) := a(t, Y_0)$, $a^{(1)}(t) := \partial_y^1 a(t, Y_0)$ and $a^{(2)}(t) := \partial_y^2 a(t, Y_0)$;
- *the expansion coefficients computed using the function $a(t, .)$ at Y_0:*

$$c_{1,T} = \omega(a^2, aa^{(1)})_0^T, \qquad\qquad c_{2,T} = \omega(a^2, (a^{(1)})^2)_0^T,$$
$$c_{3,T} = \omega(a^2, aa^{(2)})_0^T, \qquad\qquad c_{4,T} = \omega(a^2, a^2, (a^{(1)})^2)_0^T,$$
$$c_{5,T} = \omega(a^2, a^2, aa^{(2)})_0^T, \qquad\qquad c_{6,T} = \omega(a^2, aa^{(1)}, aa^{(1)})_0^T,$$
$$c_{7,T} = \omega(a^2, a^2, aa^{(1)}, aa^{(1)})_0^T, \qquad c_{8,T} = \omega(a^2, aa^{(1)}, a^2, aa^{(1)})_0^T.$$

Then, the following expansion formulas hold.

a) Second order approximation. *One has*

$$\mathbb{E}[h(Y_T)] = \mathbb{E}[h(Y_T^P)] + c_{1,T}\left(\frac{1}{2}\text{Greek}_1^h(Y_T^P) - \frac{3}{2}\text{Greek}_2^h(Y_T^P) + \text{Greek}_3^h(Y_T^P)\right)$$
$$+ \text{Error}_2, \qquad\qquad (1.10)$$

where

$$|\text{Error}_2| \leq C \sup_{v \in [0,1]} \|h^{(1)}(vY_T + (1-v)Y_T^P)\|_2 \left(\frac{M_{Y,0}}{a_{inf}}\right) M_{Y,1} M_{Y,0}^2 T^{3/2}$$

and the constant C depends (in an increasing way) only on the upper bounds of the model parameters, on c_E and on the maturity.

b) Third order approximation. *One has*

$$\mathbb{E}[h(Y_T)] = \mathbb{E}[h(Y_T^P)] + \sum_{i=1}^{6} \eta_{i,T}\text{Greek}_i^h(Y_T^P) + \text{Error}_3, \qquad (1.11)$$

where

$$\eta_{1,T} = \frac{c_{1,T}}{2} - \frac{c_{2,T}}{2} - \frac{c_{3,T}}{2} - \frac{c_{4,T}}{4} - \frac{c_{5,T}}{4} - \frac{c_{6,T}}{2},$$
$$\eta_{2,T} = -\frac{3c_{1,T}}{2} + \frac{c_{2,T}}{2} + \frac{c_{3,T}}{2} + \frac{5c_{4,T}}{4} + \frac{5c_{5,T}}{4} + \frac{7c_{6,T}}{2} + \frac{c_{7,T}}{2} + \frac{c_{8,T}}{4},$$
$$\eta_{3,T} = c_{1,T} - 2c_{4,T} - 2c_{5,T} - 6c_{6,T} - 3c_{7,T} - \frac{3c_{8,T}}{2},$$
$$\eta_{4,T} = c_{4,T} + c_{5,T} + 3c_{6,T} + \frac{13c_{7,T}}{2} + \frac{13c_{8,T}}{4},$$
$$\eta_{5,T} = -6c_{7,T} - 3c_{8,T},$$
$$\eta_{6,T} = 2c_{7,T} + c_{8,T},$$

and

$$|\text{Error}_3| \leq C \sup_{v \in [0,1]} \|h^{(1)}(vY_T + (1-v)Y_T^P)\|_2 \left(\frac{M_{Y,0}}{a_{inf}}\right)^2 M_{Y,1} M_{Y,0}^3 T^2.$$

As before, the constant C depends (in an increasing way) only on the upper bounds of the model parameters, on c_E and on the maturity.

As explained in [BGM10b], the approximation order is related to the power m in the error upper bounds $M_{Y,1}M_{Y,0}^m(\sqrt{T})^{m+1}$. The smaller the volatility ($M_{Y,0} \to 0$) or its variations ($M_{Y,1} \to 0$) or the maturity ($T \to 0$), the more accurate the approximations. Since the proxy is Gaussian, the computation of $\mathbb{E}[h(Y_T^P)]$ and $\mathrm{Greek}_i^h(Y_T^P)$ can be performed in closed forms for usual functions h (such as call/put payoffs), or by using efficient numerical integration techniques in other cases.

An interesting property of these expansion formulas is that they are exact for $h(x) = e^x$ (indeed $\mathbb{E}[h(Y_T)] = \mathbb{E}[h(Y_T^P)] = \mathrm{Greek}_i^h(Y_T^P) = e^{Y_0}$, and the sum of expansion coefficients is equal to 0). In particular, when further applied to the local volatility model (1.1), this implies that the call/put parity will be preserved within these approximations.

When the function $(t,x) \mapsto a(t,x)$ is piecewise constant w.r.t. the time variable, the coefficients $(c_{i,T})_{1 \le i \le 8}$ can be quickly and simultaneously computed for different maturities T, using recursion (see [BGM09, Proposition 4.1]). In other situations, numerical integration is likely needed.

1.2.2 Application to expansion formulas for call price

We go back to the local volatility model (1.1) and to the evaluation of call options. In view of (1.4), the call price at time 0 is equal to

$$\mathrm{Call}(T,K) = \mathbb{E}(e^{-\int_0^T r_s ds}(S_T - K)_+) = \mathbb{E}(h(X_T))$$

where $h(x) = e^{-\int_0^T r_s ds}(C_T e^x - K)_+$. In order to apply previous expansion results, it remains to identify the function $a(\cdot)$ in the dynamics of $dX_t = a(t,X_t)dW_t - \frac{1}{2}a^2(t,X_t)dt$. Comparing with (1.4), it follows that

$$a(t,x) = \sigma(t, C_t \exp(x)).$$

Owing to the assumptions (R) and (E) on σ, one can apply Theorem 1 to $Y = X$ and to $h(x) = e^{-\int_0^T r_s ds}(C_T e^x - K)_+$, in order to obtain expansion formulas for call prices in local volatility models. The next step consists in transforming the Greeks in the X-variable in the (usual) Greeks in the S-variable, and in expressing the coefficients $c_{i,T}$ using the derivatives of σ. These computations are detailed in Appendix A. We obtain the following

Theorem 2 (Second and third order approximations for call options, based on the ATM local volatility).
Assume (E) and (R). Set $\sigma_t := \sigma(t, C_t S_0)$, $\sigma_t^{(1)} := \partial_S \sigma(t, C_t S_0)$, $\sigma_t^{(2)} := \partial_S^2 \sigma(t, C_t S_0)$ and

$$\alpha_{1,T} = \omega(\sigma^2, S_0 C \sigma \sigma^{(1)})_0^T, \qquad \alpha_{2,T} = \omega(\sigma^2, (S_0 C \sigma \sigma^{(1)})^2)_0^T,$$
$$\alpha_{3,T} = \omega(\sigma^2, S_0^2 C^2 \sigma \sigma^{(2)} + S_0 C \sigma \sigma^{(1)})_0^T, \quad \alpha_{4,T} = \omega(\sigma^2, \sigma^2, (S_0 C \sigma \sigma^{(1)})^2)_0^T,$$
$$\alpha_{5,T} = \omega(\sigma^2, \sigma^2, S_0^2 C^2 \sigma \sigma^{(2)} + S_0 C \sigma \sigma^{(1)})_0^T, \quad \alpha_{6,T} = \omega(\sigma^2, S_0 C \sigma \sigma^{(1)}, S_0 C \sigma \sigma^{(1)})_0^T,$$
$$\alpha_{7,T} = \omega(\sigma^2, \sigma^2, S_0 C \sigma \sigma^{(1)}, S_0 C \sigma \sigma^{(1)})_0^T, \quad \alpha_{8,T} = \omega(\sigma^2, S_0 C \sigma \sigma^{(1)}, \sigma^2, S_0 C \sigma \sigma^{(1)})_0^T.$$

a) Second order approximation. *One has*

$$\text{Call}(T, K) = \text{Call}^{\text{BS}}(0, S_0; T, K) \tag{1.12}$$

$$+ \alpha_{1,T} \left(\frac{3}{2} S_0^2 \partial_S^2 \text{Call}^{\text{BS}}(0, S_0; T, K) + S_0^3 \partial_S^3 \text{Call}^{\text{BS}}(0, S_0; T, K) \right) + \text{Error}_2,$$

$$|\text{Error}_2| \leq CS_0 \exp\left(- \frac{[\log(S_0 C_T/K)]^2}{8|\sigma|_\infty^2 T} \right) \left(\frac{M_0}{\sigma_{inf}} \right) M_1 M_0^2 T^{3/2}$$

where the Black-Scholes price and greeks are computed using the time dependent parameters $(\sigma_t, r_t, q_t)_{0 \leq t \leq T}$.

b) Third order approximation. *One has*

$$\text{Call}(T, K) = \text{Call}^{\text{BS}}(0, S_0; T, K) \tag{1.13}$$

$$+ \sum_{i=2}^{6} \pi_{i,T} S_0^i \partial_S^i \text{Call}^{\text{BS}}(0, S_0; T, K) + \text{Error}_3,$$

where

$$\pi_{2,T} = \frac{3}{2}\alpha_{1,T} + \frac{1}{2}\alpha_{2,T} + \frac{1}{2}\alpha_{3,T} + \frac{9}{4}\alpha_{4,T} + \frac{9}{4}\alpha_{5,T} + \frac{13}{2}\alpha_{6,T} + 9\alpha_{7,T} + \frac{9}{2}\alpha_{8,T},$$

$$\pi_{3,T} = \alpha_{1,T} + 4\alpha_{4,T} + 4\alpha_{5,T} + 12\alpha_{6,T} + 66\alpha_{7,T} + 33\alpha_{8,T},$$

$$\pi_{4,T} = \alpha_{4,T} + \alpha_{5,T} + 3\alpha_{6,T} + \frac{153}{2}\alpha_{7,T} + \frac{153}{4}\alpha_{8,T},$$

$$\pi_{5,T} = 24\alpha_{7,T} + 12\alpha_{8,T},$$

$$\pi_{6,T} = 2\alpha_{7,T} + \alpha_{8,T},$$

$$|\text{Error}_3| \leq CS_0 \exp\left(- \frac{[\log(S_0 C_T/K)]^2}{8|\sigma|_\infty^2 T} \right) \left(\frac{M_0}{\sigma_{inf}} \right)^2 M_1 M_0^3 T^2.$$

In the above expansions, the constant C *depends (in an increasing way) only on the upper bounds of the model parameters, on* c_E *and on the maturity.*

Note that the local volatility and its derivatives are computed along the ATM forward curve $(S_0 C_t)_{0 \leq t \leq T}$.

1.2.3 Other expansions based on the local volatility at strike

In the previous approximation formulas, the ATM local volatility plays a central role. This is quite natural for arbitrary functions h, like in the general form of Theorem 1. But when dealing with call-put options, the local volatility at strike presumably plays a similarly important role. The aim of this paragraph is to derive similar expansion formulas, but using the volatility at strike. To achieve this goal, we follow the Dupire approach [Dup94], which writes a PDE satisfied by the call price function $(T, K) \mapsto \text{Call}(T, K) = \mathbb{E}(e^{-\int_0^T r_s ds}(S_T - K)_+)$. Indeed, we know that

$$\frac{\partial \text{Call}(T, K)}{\partial T} = -q_T \text{Call}(T, K) - (r_T - q_T)K\frac{\partial \text{Call}(T, K)}{\partial K} + \frac{1}{2}\sigma^2(T, K)K^2\frac{\partial^2 \text{Call}(T, K)}{\partial K^2},$$
$$\text{Call}(0, K) = (S_0 - K)_+.$$

In other words, instead of handling a PDE in the backward variables (t, S) with a call payoff as a terminal condition, we now deal with a PDE in the forward variables (T, K), with a put payoff as an initial condition. This latter has a probabilistic Feynman-Kac representation

$$\text{Call}(T, K) = e^{-\int_0^T q_{T-t}dt}\mathbb{E}[(S_0 - K_T)_+].$$

using the following diffusion process $(K_t)_{0 \le t \le T}$:

$$\frac{dK_t}{K_t} = -(r_{T-t} - q_{T-t})dt + \sigma(T - t, K_t)dW_t, \quad K_0 = K. \tag{1.14}$$

Define the process $(Y_t)_{0 \le t \le T}$ as follows:

$$K_t = e^{-\int_0^t (r_{T-s} - q_{T-s})ds}e^{Y_t} = \frac{C_{T-t}}{C_T}e^{Y_t}. \tag{1.15}$$

Then, Y has a dynamics of the form (1.7) with $a(t, y) = \sigma(T - t, \frac{C_{T-t}}{C_T}e^y)$. Thus, we are in a position to apply the general Theorem 1, to Y and to the function $h(y) = e^{-\int_0^T q_{T-t}dt}(S_0 - \frac{C_0}{C_T}e^y)_+$. Retransforming the Greeks w.r.t. the Y-variable into usual Greeks w.r.t. K, we obtain the new following expansion formulas (see Appendix B for the proof).

Theorem 3 (Second and third order approximations for call options, based on the local volatility at strike).
Assume (E) and (R). Set $\tilde{C}_t = \frac{C_{T-t}}{C_T}$, $\tilde{\sigma}_t := \sigma(T - t, \tilde{C}_t K)$, $\tilde{\sigma}_t^{(1)} := \partial_S \sigma(T - t, \tilde{C}_t K)$, $\tilde{\sigma}_t^{(2)} = \partial_{S^2}^2 \sigma(T - t, \tilde{C}_t K)$ *and*

$$\tilde{\alpha}_{1,T} = \omega(\tilde{\sigma}^2, K\tilde{C}\tilde{\sigma}\tilde{\sigma}^{(1)})_0^T, \qquad \tilde{\alpha}_{2,T} = \omega(\tilde{\sigma}^2, (K\tilde{C}\tilde{\sigma}^{(1)})^2)_0^T,$$
$$\tilde{\alpha}_{3,T} = \omega(\tilde{\sigma}^2, K^2\tilde{C}^2\tilde{\sigma}\tilde{\sigma}^{(2)} + K\tilde{C}\tilde{\sigma}\tilde{\sigma}^{(1)})_0^T, \qquad \tilde{\alpha}_{4,T} = \omega(\tilde{\sigma}^2, \tilde{\sigma}^2, (K\tilde{C}\tilde{\sigma}^{(1)})^2)_0^T,$$
$$\tilde{\alpha}_{5,T} = \omega(\tilde{\sigma}^2, \tilde{\sigma}^2, K^2\tilde{C}^2\tilde{\sigma}\tilde{\sigma}^{(2)} + K\tilde{C}\tilde{\sigma}\tilde{\sigma}^{(1)})_0^T, \qquad \alpha_{6,T} = \omega(\tilde{\sigma}^2, K\tilde{C}\tilde{\sigma}\tilde{\sigma}^{(1)}, K\tilde{C}\tilde{\sigma}\tilde{\sigma}^{(1)})_0^T,$$
$$\tilde{\alpha}_{7,T} = \omega(\tilde{\sigma}^2, \tilde{\sigma}^2, K\tilde{C}\tilde{\sigma}\tilde{\sigma}^{(1)}, K\tilde{C}\tilde{\sigma}\tilde{\sigma}^{(1)})_0^T, \qquad \tilde{\alpha}_{8,T} = \omega(\tilde{\sigma}^2, K\tilde{C}\tilde{\sigma}\tilde{\sigma}^{(1)}, \tilde{\sigma}^2, K\tilde{C}\tilde{\sigma}\tilde{\sigma}^{(1)})_0^T.$$

a) Second order approximation. *One has*

$$\text{Call}(T, K) = \text{Call}^{\text{BS}}(0, S_0; T, K) \tag{1.16}$$
$$+ \tilde{\alpha}_{1,T}\left(\frac{3}{2}K^2\partial_K^2\text{Call}^{\text{BS}}(0, S_0; T, K) + K^3\partial_K^3\text{Call}^{\text{BS}}(0, S_0; T, K)\right) + \text{Error}_2,$$
$$|\text{Error}_2| \le CK\exp\left(-\frac{[\log(S_0 C_T/K)]^2}{8|\sigma|_\infty^2 T}\right)\left(\frac{M_0}{\sigma_{inf}}\right)M_1 M_0^2 T^{3/2}$$

where the Black-Scholes price and greeks are computed using the time dependent parameters $(\tilde{\sigma}_t, r_t, q_t)_{0 \leq t \leq T}$.
b) Third order approximation. *One has*

$$\text{Call}(T, K) = \text{Call}^{\text{BS}}(0, S_0; T, K) \tag{1.17}$$
$$+ \sum_{i=2}^{6} \tilde{\pi}_{i,T} K^i \partial_K^i \text{Call}^{\text{BS}}(0, S_0; T, K) + \text{Error}_3,$$

where

$$\tilde{\pi}_{2,T} = \frac{3}{2}\tilde{\alpha}_{1,T} + \frac{1}{2}\tilde{\alpha}_{2,T} + \frac{1}{2}\tilde{\alpha}_{3,T} + \frac{9}{4}\tilde{\alpha}_{4,T} + \frac{9}{4}\tilde{\alpha}_{5,T} + \frac{13}{2}\tilde{\alpha}_{6,T} + 9\tilde{\alpha}_{7,T} + \frac{9}{2}\tilde{\alpha}_{8,T},$$

$$\tilde{\pi}_{3,T} = \tilde{\alpha}_{1,T} + 4\tilde{\alpha}_{4,T} + 4\tilde{\alpha}_{5,T} + 12\tilde{\alpha}_{6,T} + 66\tilde{\alpha}_{7,T} + 33\tilde{\alpha}_{8,T},$$

$$\tilde{\pi}_{4,T} = \tilde{\alpha}_{4,T} + \tilde{\alpha}_{5,T} + 3\tilde{\alpha}_{6,T} + \frac{153}{2}\tilde{\alpha}_{7,T} + \frac{153}{4}\tilde{\alpha}_{8,T},$$

$$\tilde{\pi}_{5,T} = 24\tilde{\alpha}_{7,T} + 12\tilde{\alpha}_{8,T},$$

$$\tilde{\pi}_{6,T} = 2\tilde{\alpha}_{7,T} + \tilde{\alpha}_{8,T},$$

$$|\text{Error}_3| \leq CK \exp\left(-\frac{[\log(S_0 C_T/K)]^2}{8|\sigma|_\infty^2 T}\right) \left(\frac{M_0}{\sigma_{inf}}\right)^2 M_1 M_0^3 T^2.$$

In the above expansions, the constant C depends (in an increasing way) only on the upper bounds of the model parameters, on c_E and on the maturity.

1.2.4 Expansion formulas for implied volatility

Interestingly, the previous expansions of call price can be turned into expansions of implied volatility $\sigma^I(0, S_0; T, K)$ defined by $\text{Call}(T, K) = \text{Call}^{\text{BS}}(0, S_0; T, K; \sigma^I(0, S_0; T, K), (r_t)_{0 \leq t \leq T}, (q_t)_{0 \leq t \leq T})$. To achieve this, we use nice relations between Greeks (see below and Proposition 1), omitting to indicate all the parameters whenever unambiguous:

$$\text{Vega} = \frac{\partial}{\partial \sigma} \text{Call}^{\text{BS}}(0, S; T, K) = Se^{-qT} \mathcal{N}'(d_1)\sqrt{T} = Ke^{-rT} \mathcal{N}'(d_2)\sqrt{T},$$

$$S^2 \Gamma_S = S^2 \frac{\partial^2}{\partial S^2} \text{Call}^{\text{BS}}(0, S; T, K) = \frac{Se^{-qT} \mathcal{N}'(d_1)}{\sigma\sqrt{T}} = \frac{\text{Vega}}{\sigma T},$$

$$S^3 \text{Speed}_S = S^3 \frac{\partial^3}{\partial S^3} \text{Call}^{\text{BS}}(0, S; T, K) = -S^2 \Gamma_S \left(\frac{d_1}{\sigma\sqrt{T}} + 1\right) = -\frac{\text{Vega}}{\sigma T}\left(\frac{d_1}{\sigma\sqrt{T}} + 1\right),$$

$$K^2 \Gamma_K = \frac{\partial^2}{\partial K^2} \text{Call}^{\text{BS}}(0, S; T, K) = \frac{Ke^{-rT} \mathcal{N}'(d_2)}{\sigma\sqrt{T}} = \frac{\text{Vega}}{\sigma T},$$

$$K^3 \text{Speed}_K = K^3 \frac{\partial^3}{\partial K^3} \text{Call}^{\text{BS}}(0, S; T, K) = -K^2 \Gamma_K \left(1 - \frac{d_2}{\sigma\sqrt{T}}\right) = -\frac{\text{Vega}}{\sigma T}\left(1 - \frac{d_2}{\sigma\sqrt{T}}\right).$$

Now, consider the second order expansion formula based on the ATM local volatility: it becomes

$$\mathrm{Call}(T,K) = \mathrm{Call}^{\mathrm{BS}}(0, S_0; T, K) - \mathrm{Vega}\frac{\alpha_{1,T}}{\sqrt{T}(\int_0^T \sigma_s^2 ds)^{3/2}} \log\left(\frac{S_0 C_T}{K}\right) + \mathrm{Error}_2.$$

This reads as an expansion of the implied volatility. The derivation is similar for the second order expansion formula based on the local volatility at strike. This proves the following

Theorem 4 (Second order approximations on implied volatilities).
Assume (E) and (R). Using the notations of Theorems 2 and 3, we have

$$\sigma^I(0, S_0; T, K) = \left(\frac{1}{T}\int_0^T \sigma_s^2 ds\right)^{\frac{1}{2}} - \frac{\alpha_{1,T}}{T^{\frac{1}{2}}(\int_0^T \sigma_s^2 ds)^{\frac{3}{2}}} \log\left(\frac{S_0 C_T}{K}\right) \qquad (1.18)$$
$$+ \mathrm{Error}_2^I,$$

$$\sigma^I(0, S_0; T, K) = \left(\frac{1}{T}\int_0^T \tilde{\sigma}_s^2 ds\right)^{\frac{1}{2}} + \frac{\tilde{\alpha}_{1,T}}{T^{\frac{1}{2}}(\int_0^T \tilde{\sigma}_s^2 ds)^{\frac{3}{2}}} \log\left(\frac{S_0 C_T}{K}\right) \qquad (1.19)$$
$$+ \widetilde{\mathrm{Error}}_2^I.$$

Note that in the first case (1.18), the local volatility is computed ATM, while in the second one (1.19), it is computed at strike.

In addition to these direct implied volatility approximations, one can upper bound the residual terms Error_2^I and $\widetilde{\mathrm{Error}}_2^I$, simply applying the error estimates from Theorems 2 and 3. We do not give the details of this derivation. As it can be expected, the error estimates depend on the ratio $\frac{\log(S_0 C_T/K)}{|\sigma|_\infty \sqrt{T}}$, but actually, they are locally uniform w.r.t. this ratio. More precisely, for any $\xi > 0$, there is a constant C_ξ which depends (in an increasing way) on ξ, on the upper bounds of the model parameters, on c_E, on the maturity and on the ratio M_0/σ_{inf} such that for any S_0 and K satisfying $|\log(S_0 C_T/K)| \leq \xi|\sigma|_\infty \sqrt{T}$ we have

$$|\mathrm{Error}_2^I| + |\widetilde{\mathrm{Error}}_2^I| \leq C_\xi M_1 M_0^2 T.$$

Thus, inaccuracies may occur for very small or very large strikes, a feature which is confirmed by the further numerical experiments. In view of the above upper bounds, the relative errors on implied volatility are locally of order $M_1 M_0 T$, justifying the label of second order approximations.

This paves the way for the derivation of a third order expansion of implied volatility, but unfortunately, we have not been able to simplify the computations in order to get a sufficiently nice expression. This will be further investigated.

1.2.5 Applications to time-dependent CEV model

To conclude this section, we specify the results when the volatility has the form

$$\sigma(t, S) = \nu_t S^{\beta_t - 1}, \tag{1.20}$$

i.e. a CEV-type volatility with a time-dependent level $(\nu_t)_{0 \leq t \leq T}$ and a time-dependent skew $(\beta_t)_{0 \leq t \leq T}$. In order to apply Theorems 2 and 3, all what is needed is to give the expressions for the coefficients $(\alpha_{i,T}, \tilde{\alpha}_{i,T})_{1 \leq i \leq 6}$. First, the proxy volatilities are given by $\sigma_t = \nu_t (C_t S_0)^{\beta_t - 1}$ and $\tilde{\sigma}_t = \nu_{T-t}(\tilde{C}_t K)^{\beta_{T-t} - 1}$, where $\tilde{C}_t = C_{T-t}/C_T$; then, we have

$$\alpha_{1,T} = \omega(\sigma^2, (\beta - 1)\sigma^2)_0^T, \qquad\qquad \alpha_{2,T} = \alpha_{3,T} = \omega(\sigma^2, (\beta - 1)^2 \sigma^2)_0^T,$$

$$\alpha_{4,T} = \alpha_{5,T} = \omega(\sigma^2, \sigma^2, (\beta - 1)^2 \sigma^2)_0^T, \quad \alpha_{6,T} = \omega(\sigma^2, (\beta - 1)\sigma^2, (\beta - 1)\sigma^2)_0^T,$$

$$\alpha_{7,T} = \omega(\sigma^2, \sigma^2, (\beta - 1)\sigma^2, (\beta - 1)\sigma^2)_0^T, \quad \alpha_{8,T} = \omega(\sigma^2, (\beta - 1)\sigma^2, \sigma^2, (\beta - 1)\sigma^2)_0^T.$$

The expressions are similar for $(\tilde{\alpha}_{i,T})_{1 \leq i \leq 6}$, by replacing σ_t by $\tilde{\sigma}_t$ and $(\beta_t - 1)$ by $(\beta_{T-t} - 1)$ in the above formulas. In the case of constant parameters $\nu_t = \nu$, $\beta_t = \beta$ and $\mu = r - q$, all the previous quantities can be expressed in closed forms (the values of the integral operator $\omega(.)_0^T$ are given by iterated integrals of exponential functions). We give them in the simple case $\mu = 0$. By setting $\sigma = \nu S_0^{\beta - 1}$ and $\tilde{\sigma} = \nu K^{\beta - 1}$, we obtain

$$\alpha_{1,T} = (\beta - 1)\sigma^4 \frac{T^2}{2}, \qquad\qquad \alpha_{2,T} = \alpha_{3,T} = (\beta - 1)^2 \sigma^4 \frac{T^2}{2},$$

$$\alpha_{4,T} = \alpha_{5,T} = \alpha_{6,T} = (\beta - 1)^2 \sigma^6 \frac{T^3}{6}, \qquad \alpha_{7,T} = \alpha_{8,T} = (\beta - 1)^2 \sigma^8 \frac{T^4}{24}.$$

Replacing σ by $\tilde{\sigma}$ gives the values for $(\tilde{\alpha}_{i,T})_{1 \leq i \leq 6}$.

1.3 Numerical results

In the numerical tests we report here, we take $r = q = 0$ and we consider a CEV model (1.20) for the volatility, with constant parameters ν and β. For additional tests with time-dependent parameters, see [BGM10b]. We choose $S_0 = 1$, $\nu = 25\%$ and we allow β to vary. Actually, we consider two values: $\beta = 0.8$ which is not far from the log-normal case, and $\beta = 0.2$ which is rather different. We test the accuracy of different approximations, for various maturities (3-6 months, 1-1.5-2-3-5-10 years) and various strikes.

<div align="center">INSERT TABLE 1.1 ABOUT HERE</div>

The range of strikes depends on the maturity: the tested values are reported in Table 1.1. Essentially, the strikes are roughly equal to $S_0 \exp(\xi \nu \sqrt{T})$ where ξ is taken as various quantiles of the standard Gaussian law (we take the quantiles $1\% - 5\% - 10\% - 20\% - 30\% - 40\% - 50\% - 60\% - 70\% - 80\% - 90\% - 95\% - 99\%$): this means that the first and last columns of strikes are associated to very ITM options or very OTM options.

For the sake of completeness, in Table 1.2 and 1.3 we report the implied volatilities related to the (exact) call price in CEV model with constant parameters (our computations are based on the work by Schroder [Sch89]).

INSERT TABLES 1.2 and 1.3 ABOUT HERE

We aim at comparing the following different approximations.

1. `ImpVol(AppPrice(2,S0))`: this is the implied volatility of the second order expansion based on the ATM local volatility (see (1.12) in Theorem 2).
2. `AppImpVol(2,S0)`: this is the second order implied volatility expansion based on the ATM local volatility (see (1.18) in Theorem 4).
3. `ImpVol(AppPrice(2,K))`: this is the implied volatility of the second order expansion based on the local volatility at strike (see (1.16) in Theorem 3).
4. `AppImpVol(2,K)`: this is the second order implied volatility expansion based on the local volatility at strike (see (1.19) in Theorem 4).
5. `ImpVol(AppPrice(3,S0))`: this is the implied volatility of the third order expansion based on the ATM local volatility (see (1.13) in Theorem 2).
6. `ImpVol(AppPrice(3,K))`: this is the implied volatility of the third order expansion based on the local volatility at strike (see (1.17) in Theorem 3).
7. `Av.ImpVol(AppPrice(3,.))`: this is the average of `ImpVol(AppPrice(3,S0))` and `ImpVol(AppPrice(3,K))`. The interest in this approximation is explained later.

INSERT TABLES 1.4 and 1.5 ABOUT HERE

In Table 1.4 (resp. Table 1.5), we report the errors on implied volatility using the six first aforementioned approximations, for $\beta = 0.8$ (resp. $\beta = 0.2$). The errors are expressed in bps (basis points): an implied volatility of 25.01% instead of 25% yields 1bp error. For instance, on the first row of Table 1.4, the value -12.3 is associated to the approximation error of `ImpVol(AppPrice(2,S0))` for the first strike of maturity $T = 3M$ (i.e. $K = 0.70$); on the fourth row of Table 1.4, the value -0.9 refers to the approximation error of `AppImpVol(2,K)` for the second strike of maturity $T = 3M$ (i.e. $K = 0.75$), and so one. Sometimes (especially for very small and very large strikes), the price approximation is out of the non-arbitrage interval for call options: in this case, one can not define a value for the implied volatility and we report ND in the tabular. For all these results, a medium (or large) error on implied volatility may yield a small (or reasonable) error on prices: this is especially true for ITM or OTM options, for which the Vega is small (see the discussion in [BGM10c]).

Influence of β and T. Generally speaking, we observe that for $\beta = 0.8$, the errors are smaller compared to $\beta = 0.2$: it is not surprising since the lognormal proxy suits better in the first case. This can also be explained by our error estimates, since M_1 is essentially proportional to $|\beta - 1|$. Errors are increasing w.r.t. T, which is also coherent with our error estimates.

Influence of K. For usual values of strike (essentially in the Gaussian quantile range $[10\%, 90\%]$), errors are small (or very small, depending on the approximation that is used), usually smaller than 10bps for $\beta = 0.8$ up to 10Y

maturity, and smaller than 20bps for $\beta = 0.2$ up to maturity 5Y. Error approximations on implied volatility are much larger for very ITM or very OTM options. For these situations, it may be a good idea to incorporate known asymptotic on the implied volatility (see for instance [Lee04]).

Influence of the type of approximation. Regarding the second order approximations, within this model it gives lower bounds on implied volatility (and on price). This systematic underestimation is a drawback of these approximations. Notice that it is usually much better to use the direct approximation on implied volatility (Theorem 4) compared to the implied volatility of the price approximation. However, these implied volatility expansions underestimate the true value as well.

As expected, third order approximations are more accurate than second order ones. The improvement is more significant for $\beta = 0.2$. In Figures 1.1 and 1.2, we plot the errors on implied volatility for the maturity $T = 1.5Y$ (this choice is unimportant) for both values of β.

<div align="center">INSERT FIGURES 1.1 AND 1.2 ABOUT HERE</div>

We first observe that `ImpVol(AppPrice(3,S0))` overestimates the true value for $K \gg S_0$ and yields an underestimation for $K \ll S_0$. This is the converse regarding `ImpVol(AppPrice(3,K))`. On Tables 1.4 and 1.5, we can check that this is generally satisfied for any maturity. Thus, an heuristic rule may be to consider the following confidence interval for the exact implied volatility:

$$\sigma^I(0, S_0; T, K) \in \big(\texttt{ImpVol(AppPrice(3,K))}, \texttt{ImpVol(AppPrice(3,S0))}\big).$$

If the width of this interval is too large, it somehow indicates an inaccuracy in our approximations.

Secondly, we observe that the errors using `ImpVol(AppPrice(3,S0))` and `ImpVol(AppPrice(3,K))` have roughly the same magnitude (but with opposite signs). Then, if we define the average

$$
\begin{aligned}
&\texttt{Av.ImpVol(AppPrice(3,.))}\\
&= \frac{1}{2}\big(\texttt{ImpVol(AppPrice(3,S0))} + \texttt{ImpVol(AppPrice(3,K))}\big),
\end{aligned}
\tag{1.21}
$$

we expect to obtain a much better implied volatility estimate. The errors for `Av.ImpVol(AppPrice(3,.))` for $\beta = 0.8$ and $\beta = 0.2$ are reported in Tables 1.6 and 1.7. Observe that for maturities smaller than 5Y, the accuracy is truly excellent (i.e. smaller than few bps) for a widened range of strikes.

<div align="center">INSERT TABLES 1.6 and 1.7 ABOUT HERE</div>

We have compared our approximations with the known implied volatility approximation in the CEV model (with zero interest rates and zero dividend) (see [Hen08, formula (5.41) p.141]):

$$\sigma^I(0, S_0; T, K) \approx \nu \frac{(1-\beta)\ln(K/S_0)}{K^{1-\beta} - S_0^{1-\beta}} \left(1 + \frac{(\beta-1)^2 \nu^2 T}{24} \big(\frac{S_0 + K}{2}\big)^{2\beta-2}\right).$$

This latter approximation yields a slightly better numerical accuracy compared to ours (and it is quicker to evaluate). However, our approximations are also able to deal naturally with general time-dependent local volatility (with piecewise continuity in time), as a difference with [Hen08, Chapter 5] for instance, or with stochastic interest rates [BGM10a]. This may be a significant advantage compared to other approaches, while maintaining tight error estimates.

A Proof of Theorem 2

We apply Theorem 1, by taking $h(x) = e^{-\int_0^T r_s ds}(C_T e^x - K)_+$ and $a(t, x) = \sigma(t, C_t \exp(x))$. The required assumptions on h and a are satisfied owing to assumptions (E) and (R). By simple computations, we easily check that $M_{Y,0} = M_0$ and $M_{Y,1} = M_1$. The proxy of X used in Theorem 1 now writes $X_t^P = \log(S_0) - \frac{1}{2}\int_0^t \sigma_s^2 ds + \int_0^t \sigma_s dW_s$.

Main term and correction terms. From this, we deduce that the main term $\mathbb{E}(h(X_T^P))$ in the expansion is equal to

$$\mathbb{E}(e^{-\int_0^T r_s ds}(C_T e^{X_T^P} - K)_+) = \text{Call}^{\text{BS}}(0, S_0; T, K; (\sigma_t)_{0 \leq t \leq T}, (r_t)_{0 \leq t \leq T}, (q_t)_{0 \leq t \leq T}).$$

In the following, for the sake of brevity, we omit to indicate in the Black-Scholes formula the dependence w.r.t. $(\sigma_t, r_t, q_t)_{0 \leq t \leq T}$. For computing the sensitivities $\text{Greek}_i^h(X_T^P) = \partial_x^i \mathbb{E}(h(X_T^P + x))|_{x=0}$, we proceed similarly to the main term. First, we have $\mathbb{E}(h(X_T^P + x)) = \text{Call}^{\text{BS}}(0, S_0 e^x; T, K)$. By successive differentiations, we obtain (using matrix notation)

$$\begin{pmatrix} \text{Greek}_1^h(X_T^P) \\ \text{Greek}_2^h(X_T^P) \\ \text{Greek}_3^h(X_T^P) \\ \text{Greek}_4^h(X_T^P) \\ \text{Greek}_5^h(X_T^P) \\ \text{Greek}_6^h(X_T^P) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 & 0 \\ 1 & 7 & 6 & 1 & 0 & 0 \\ 1 & 15 & 25 & 10 & 1 & 0 \\ 1 & 31 & 90 & 65 & 15 & 1 \end{pmatrix} \begin{pmatrix} S_0 \partial_S \text{Call}^{\text{BS}}(0, S_0; T, K) \\ S_0^2 \partial_S^2 \text{Call}^{\text{BS}}(0, S_0; T, K) \\ S_0^3 \partial_S^3 \text{Call}^{\text{BS}}(0, S_0; T, K) \\ S_0^4 \partial_S^4 \text{Call}^{\text{BS}}(0, S_0; T, K) \\ S_0^5 \partial_S^5 \text{Call}^{\text{BS}}(0, S_0; T, K) \\ S_0^6 \partial_S^6 \text{Call}^{\text{BS}}(0, S_0; T, K) \end{pmatrix}. \quad (1.22)$$

Regarding the summation of the correction terms, it implies that $\sum_{i=1}^6 \eta_{i,T} \text{Greek}_i^h(X_T^P) = \sum_{i=1}^6 \hat{\eta}_{i,T} S_0^i \partial_S^i \text{Call}^{\text{BS}}(0, S_0; T, K)$ where

$$\hat{\eta}_{1,T} = 0,$$

$$\hat{\eta}_{2,T} = \frac{3}{2}c_{1,T} + \frac{1}{2}c_{2,T} + \frac{1}{2}c_{3,T} + \frac{9}{4}c_{4,T} + \frac{9}{4}c_{5,T} + \frac{13}{2}c_{6,T} + 9c_{7,T} + \frac{9}{2}c_{8,T},$$

$$\hat{\eta}_{3,T} = c_{1,T} + 4c_{4,T} + 4c_{5,T} + 12c_{6,T} + 66c_{7,T} + 33c_{8,T},$$

$$\hat{\eta}_{4,T} = c_{4,T} + c_{5,T} + 3c_{6,T} + \frac{153}{2}c_{7,T} + \frac{153}{4}c_{8,T},$$

$$\hat{\eta}_{5,T} = 24c_{7,T} + 12c_{8,T},$$

$$\hat{\eta}_{6,T} = 2c_{7,T} + c_{8,T}.$$

The expressions of the coefficients $(c_{i,T})_{1\leq i\leq 8}$ are given in Theorem 1, but in order to specify them in the current case $a(t,x)=\sigma(t,C_t\exp(x))$, we denote them by $(\alpha_{i,T})_{1\leq i\leq 8}$ instead of $(c_{i,T})_{1\leq i\leq 8}$. Easy computations show that these definitions coincide with those given in Theorem 2. Then, the second order expansion formula is obtained by keeping only the first coefficient $\alpha_{1,T}$, while all the coefficients are taken for the third order expansion formula.

Error estimates. We have already observed that $M_{Y,0}=M_0$ and $M_{Y,1}=M_1$. It remains to estimate the factor $\|h^{(1)}(vX_T+(1-v)X_T^P)\|_2$ arising in the error bounds of Theorem 1. For $v\in[0,1]$, define $\sigma_t^v:=v\sigma(t,X_t)+(1-v)\sigma_t\in[\sigma_{inf},|\sigma|_\infty]$ and $\sigma_t^{2,v}:=v\sigma^2(t,X_t)+(1-v)\sigma_t^2\in[\sigma_{inf}^2,|\sigma|_\infty^2]$: clearly we have

$$d(vX_t+(1-v)X_t^P)=\sigma_t^v dW_t-\frac{1}{2}\sigma_t^{2,v}dt.$$

We denote by $\mathbb{P}^v$ the probability measure under which $W_t^v=W_t-2\int_0^t\sigma_s^v ds$ is a Brownian motion. Then, putting $d_0=\log(S_0C_T/K)$ and using $h'(x)=e^{-\int_0^T q_s ds}e^x\mathbf{1}_{x-\log(S_0)>-d_0}$, we obtain

$$\mathbb{E}([h'(vX_T+(1-v)X_T^P)]^2)$$
$$=S_0^2 e^{-2\int_0^T q_s ds}\mathbb{E}(e^{2\int_0^T\sigma_s^v dW_s-\int_0^T\sigma_s^{2,v}ds}\mathbf{1}_{\int_0^T\sigma_s^v dW_s-\frac{1}{2}\int_0^T\sigma_s^{2,v}ds>-d_0})$$
$$=S_0^2 e^{-2\int_0^T q_s ds}\mathbb{E}^v(e^{2\int_0^T[\sigma_s^v]^2 ds-\int_0^T\sigma_s^{2,v}ds}\mathbf{1}_{\int_0^T\sigma_s^v dW_s^v+2\int_0^T[\sigma_s^v]^2 ds-\frac{1}{2}\int_0^T\sigma_s^{2,v}ds>-d_0})$$
$$\leq S_0^2 e^{-2\int_0^T q_s ds+2|\sigma|_\infty^2 T}\mathbb{P}^v(\int_0^T\sigma_s^v dW_s^v+2|\sigma|_\infty^2 T>-d_0). \tag{1.23}$$

If $-d_0>2|\sigma|_\infty^2 T$, one can apply the Bernstein exponential inequality to show that the above probability is bounded by $\exp(-\frac{(-d_0-2|\sigma|_\infty^2 T)^2}{2|\sigma|_\infty^2 T})$. Using the inequality $(a-b)^2\geq\frac{1}{2}a^2-b^2$, it follows that

$$\sup_{v\in[0,1]}\mathbb{E}([h'(vX_T+(1-v)X_T^P)]^2)\leq CS_0^2\exp(-\frac{d_0^2}{4|\sigma|_\infty^2 T}) \tag{1.24}$$

where the constant C depends in an increasing way on the bounds on the coefficients and on the maturity. Note that the inequality (1.24) is also valid if $0\leq-d_0\leq2|\sigma|_\infty^2 T$: indeed, from (1.23), we write $\mathbb{E}([h'(vX_T+(1-v)X_T^P)]^2)\leq S_0^2 e^{-2\int_0^T q_s ds+2|\sigma|_\infty^2 T}\leq S_0^2 e^{-2\int_0^T q_s ds+2|\sigma|_\infty^2 T}\exp(\frac{(2|\sigma|_\infty^2 T)^2}{4|\sigma|_\infty^2 T})\exp(-\frac{d_0^2}{4|\sigma|_\infty^2 T})\leq CS_0^2\exp(-\frac{d_0^2}{4|\sigma|_\infty^2 T})$. To sum up we obtain

$$\sup_{v\in[0,1]}\|[h'(vX_T+(1-v)X_T^P)]\|_2\leq CS_0\exp(-\frac{[\log(S_0C_T/K)]^2}{8|\sigma|_\infty^2 T}) \tag{1.25}$$

for any $d_0\leq0$, or equivalently for any $K\geq S_0C_T$. Thus, the announced estimates on Error₂ and Error₃ are valid for any Out of The Money calls. Using a similar analysis, the same estimates hold for Out of The Money puts ($K\leq$

$S_0 C_T$). But, since the call/put parity is preserved within these expansions, error estimates are equal for call/put with the same characteristics. Thus, estimates for Out of The Money puts transfer to In The Money calls. This completes the proof.   □

A careful inspection of the current proof and that of Theorem 1 reveals that the factor 8 in the exponential (1.25) can be improved and actually, it can be taken strictly larger than 2: this gives presumably better error estimates for $K \ll S_0$ or $K \gg S_0$.

B Proof of Theorem 3

The derivation of the expansion is obtained following the same lines as those for Theorem 2. We detail only the main arguments. The proxy for the process $(Y_t)_{0 \le t \le T}$ is defined by $Y_t^P = \log(K) + \int_0^t \tilde{\sigma}_s dW_s - \frac{1}{2} \int_0^t \tilde{\sigma}_s^2 ds$. We interpret $e^{Y_T^P}/K$ as the Radon-Nikodym derivative of a new measure $\tilde{\mathbb{P}}$ w.r.t. $\mathbb{P}$ on $\mathcal{F}_T$, under which $\tilde{W}_t = W_t - \int_0^t \tilde{\sigma}_s ds$ is a standard BM; then we obtain

$$
\mathbb{E}(e^{-\int_0^T q_{T-t} dt}(S_0 - \frac{C_0}{C_T} e^{Y_T^P})_+) = \mathbb{E}(e^{-\int_0^T r_s ds} \frac{e^{Y_T^P}}{K}(S_0 C_T \frac{K}{e^{Y_T^P}} - K)_+)
$$
$$
= \tilde{\mathbb{E}}(e^{-\int_0^T r_s ds}(S_0 e^{\int_0^T (r_s - q_s) ds} e^{-\int_0^T \tilde{\sigma}_s d\tilde{W}_s - \frac{1}{2}\int_0^T \tilde{\sigma}_s^2 ds} - K)_+)
$$
$$
= \text{Call}^{\text{BS}}(0, S_0; T, K; (\tilde{\sigma}_t)_{0 \le t \le T}, (r_t)_{0 \le t \le T}, (q_t)_{0 \le t \le T}).
$$

This gives the main term in the expansion. Regarding the computation of the sensitivities $\text{Greek}_i^h(Y_T^P)$, observe that $\mathbb{E}(h(Y_T^P + x)) = \text{Call}^{\text{BS}}(0, S_0; T, Ke^x)$, omitting the last parameters $(\tilde{\sigma}_t, r_t, q_t)_{0 \le t \le T}$. Thus, we easily relate the sensitivities $\text{Greek}_i^h(Y_T^P)$ to the Greeks of $\text{Call}^{\text{BS}}(0, S_0; T, K)$ w.r.t. K (instead of S_0 in the Theorem 2). The relation is affine and is similar to (1.22). The other steps of the proof are analogous to that of Theorem 2, replacing S_0 and σ by K and $\tilde{\sigma}$ in most places.   □

C Computations of derivatives of the Black-Scholes price function w.r.t. S and K

In the following proposition, we make explicit the formulas for the six first derivatives of $\text{Call}^{\text{BS}}(0, S; T, K; \sigma, r, q)$ (in short $\text{Call}^{\text{BS}}(0, S; T, K)$) w.r.t. S and K, leaving the proofs to the reader. These formulas are necessary to implement the expansions of Theorems 2 and 3.

Proposition 1 (Black-Scholes Greeks). *Using the notation from Definition 3, the sensitivities w.r.t. S are given by*

$$\Delta_{\mathrm{S}}(t,S;T,K) = \frac{\partial}{\partial S}\mathrm{Call}^{\mathrm{BS}}(t,S;T,K) = e^{-q(T-t)}\mathcal{N}(d_1),$$

$$\Gamma_{\mathrm{S}}(t,S;T,K) = \frac{\partial^2}{\partial S^2}\mathrm{Call}^{\mathrm{BS}}(t,S;T,K) = e^{-q(T-t)}\frac{\mathcal{N}'(d_1)}{S\sigma\sqrt{T-t}},$$

$$\mathrm{Speed}_{\mathrm{S}}(t,S;T,K) = \frac{\partial^3}{\partial S^3}\mathrm{Call}^{\mathrm{BS}}(t,S;T,K) = -\frac{\Gamma_{\mathrm{S}}}{S}\Big(\frac{d_1}{\sigma\sqrt{T-t}} + 1\Big),$$

$$\frac{\partial^4}{\partial S^4}\mathrm{Call}^{\mathrm{BS}}(t,S;T,K) = \frac{\Gamma_{\mathrm{S}}}{S^2}\Big(2 + \frac{3d_1}{\sigma\sqrt{T-t}} + \frac{d_1^2-1}{\sigma^2(T-t)}\Big),$$

$$\frac{\partial^5}{\partial S^5}\mathrm{Call}^{\mathrm{BS}}(t,S;T,K) = \frac{\Gamma_{\mathrm{S}}}{S^3}\Big(-6 - \frac{11d_1}{\sigma\sqrt{T-t}} + \frac{6(1-d_1^2)}{\sigma^2(T-t)} + \frac{3d_1-d_1^3}{\sigma^3(T-t)^{\frac{3}{2}}}\Big),$$

$$\frac{\partial^6}{\partial S^6}\mathrm{Call}^{\mathrm{BS}}(t,S;T,K) = \frac{\Gamma_{\mathrm{S}}}{S^4}\Big(24 + \frac{50d_1}{\sigma\sqrt{T-t}} + \frac{35(d_1^2-1)}{\sigma^2(T-t)} + \frac{10d_1(d_1^2-3)}{\sigma^3(T-t)^{\frac{3}{2}}} + \frac{3(1-2d_1^2)+d_1^4}{\sigma^4(T-t)^2}\Big).$$

The sensitivities w.r.t. K are given by

$$\Delta_{\mathrm{K}}(t,S;T,K) = \frac{\partial}{\partial K}\mathrm{Call}^{\mathrm{BS}}(t,S;T,K) = -e^{-r(T-t)}\mathcal{N}(d_2),$$

$$\Gamma_{\mathrm{K}}(t,S;T,K) = \frac{\partial^2}{\partial K^2}\mathrm{Call}^{\mathrm{BS}}(t,S;T,K) = e^{-r(T-t)}\frac{\mathcal{N}'(d_2)}{K\sigma\sqrt{T-t}},$$

$$\mathrm{Speed}_{\mathrm{K}}(t,S;T,K) = \frac{\partial^3}{\partial K^3}\mathrm{Call}^{\mathrm{BS}}(t,S;T,K) = -\frac{\Gamma_{\mathrm{K}}}{K}\Big(1 - \frac{d_2}{\sigma\sqrt{T-t}}\Big),$$

$$\frac{\partial^4}{\partial K^4}\mathrm{Call}^{\mathrm{BS}}(t,S;T,K) = \frac{\Gamma_{\mathrm{K}}}{K^2}\Big(2 - \frac{3d_2}{\sigma\sqrt{T-t}} + \frac{d_2^2-1}{\sigma^2(T-t)}\Big),$$

$$\frac{\partial^5}{\partial K^5}\mathrm{Call}^{\mathrm{BS}}(t,S;T,K) = \frac{\Gamma_{\mathrm{K}}}{K^3}\Big(-6 + \frac{11d_2}{\sigma\sqrt{T-t}} + \frac{6(1-d_2^2)}{\sigma^2(T-t)} + \frac{d_2^3-3d_2}{\sigma^3(T-t)^{\frac{3}{2}}}\Big),$$

$$\frac{\partial^6}{\partial K^6}\mathrm{Call}^{\mathrm{BS}}(t,S;T,K) = \frac{\Gamma_{\mathrm{K}}}{K^4}\Big(24 - \frac{50d_2}{\sigma\sqrt{T-t}} + \frac{35(d_2^2-1)}{\sigma^2(T-t)} - \frac{10d_2(d_2^2-3)}{\sigma^3(T-t)^{\frac{3}{2}}} + \frac{3(1-2d_2^2)+d_2^4}{\sigma^4(T-t)^2}\Big).$$

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T	K												
3M	0.70	0.75	0.80	0.85	0.90	0.95	1.00	1.05	1.10	1.20	1.25	1.30	1.35
6M	0.65	0.75	0.80	0.85	0.90	0.95	1.00	1.05	1.10	1.15	1.25	1.35	1.50
1Y	0.55	0.65	0.75	0.80	0.90	0.95	1.00	1.05	1.15	1.25	1.40	1.50	1.80
1.5Y	0.50	0.60	0.70	0.75	0.85	0.95	1.00	1.10	1.15	1.30	1.50	1.65	2.00
2Y	0.45	0.55	0.65	0.75	0.85	0.90	1.00	1.10	1.20	1.35	1.55	1.80	2.30
3Y	0.35	0.50	0.55	0.70	0.80	0.90	1.00	1.10	1.25	1.45	1.75	2.05	2.70
5Y	0.25	0.40	0.50	0.60	0.75	0.85	1.00	1.15	1.35	1.60	2.05	2.50	3.60
10Y	0.15	0.25	0.35	0.50	0.65	0.80	1.00	1.20	1.50	1.95	2.75	3.65	6.30

Table 1.1. Set of maturities and strikes used for the numerical tests.

3M	25.908	25.728	25.563	25.409	25.265	25.129	25.001	24.879	24.763	24.548	24.447	24.350	24.258
6M	26.096	25.728	25.564	25.410	25.266	25.130	25.001	24.880	24.764	24.654	24.448	24.258	24.001
1Y	26.530	26.096	25.729	25.565	25.267	25.131	25.003	24.881	24.655	24.449	24.171	24.002	23.562
1.5Y	26.780	26.304	25.907	25.731	25.413	25.133	25.004	24.766	24.656	24.353	24.003	23.772	23.311
2Y	27.058	26.531	26.099	25.732	25.414	25.270	25.005	24.768	24.552	24.262	23.925	23.564	22.980
3Y	27.729	26.783	26.534	25.911	25.570	25.272	25.008	24.770	24.453	24.089	23.633	23.254	22.605
5Y	28.646	27.377	26.788	26.313	25.739	25.421	25.012	24.664	24.268	23.854	23.258	22.788	21.943
10Y	30.079	28.658	27.746	26.800	26.118	25.586	25.022	24.568	24.020	23.386	22.573	21.918	20.694

Table 1.2. CEV model ($\beta = 0.8$): implied volatilities in %.

3M	28.755	28.003	27.312	26.673	26.080	25.528	25.010	24.535	24.074	23.232	22.845	22.477	22.128
6M	29.590	28.017	27.325	26.686	26.092	25.539	25.021	24.535	24.078	23.646	22.851	22.133	21.177
1Y	31.537	29.624	28.046	27.352	26.116	25.561	25.042	24.555	23.664	22.867	21.814	21.189	19.602
1.5Y	32.706	30.568	28.831	28.075	26.736	25.583	25.062	24.115	23.681	22.513	21.202	20.359	18.733
2Y	34.034	31.618	29.692	28.103	26.761	26.163	25.083	24.133	23.288	22.177	20.921	19.621	17.619
3Y	37.339	32.840	31.698	28.924	27.459	26.209	25.124	24.170	22.930	21.547	19.882	18.555	16.406
5Y	42.069	35.797	33.000	30.816	28.271	26.908	25.205	23.802	22.262	20.709	18.589	17.011	14.382
10Y	47.850	41.604	37.460	33.144	30.082	27.758	25.378	23.535	21.407	19.089	16.346	14.325	10.993

Table 1.3. CEV model ($\beta = 0.2$): implied volatilities in %.
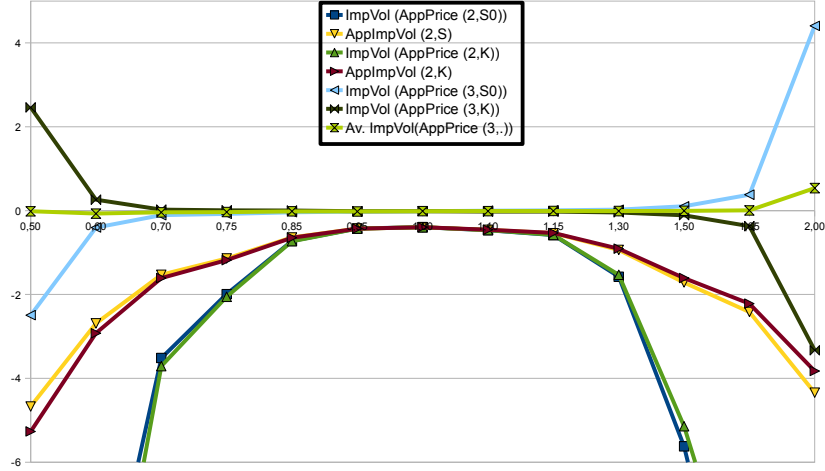
Fig. 1.1. CEV model ($\beta = 0.8$): errors in bps on the implied volatility using the 7 approximations `ImpVol(AppPrice(2,S0))`, `AppImpVol(2,S0)`, `ImpVol(AppPrice(2,K))`, `AppImpVol(2,K)`, `ImpVol(AppPrice(3,S0))`, `ImpVol(AppPrice(3,K))` and `Av. ImpVol(AppPrice(3,.))`

3M	-12.3	-5.8	-2.4	-0.9	-0.3	-0.1	-0.1	-0.1	-0.2	-1.3	-2.6	-4.9	-8.5
	-1.7	-0.9	-0.5	-0.3	-0.2	-0.1	-0.1	-0.1	-0.1	-0.3	-0.5	-0.6	-0.8
	-17.1	-6.8	-2.6	-0.9	-0.3	-0.1	-0.1	-0.1	-0.2	-1.2	-2.4	-4.2	-6.8
	-1.7	-0.9	-0.5	-0.3	-0.2	-0.1	-0.1	-0.1	-0.1	-0.3	-0.5	-0.6	-0.8
	-1.4	-0.4	-0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.1	0.4
	0.6	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.1	-0.4
6M	-13.3	-3.4	-1.5	-0.6	-0.3	-0.2	-0.1	-0.2	-0.2	-0.4	-1.6	-4.4	-14.8
	-1.9	-0.9	-0.6	-0.4	-0.2	-0.2	-0.1	-0.2	-0.2	-0.3	-0.5	-0.9	-1.5
	-17.7	-3.7	-1.6	-0.7	-0.3	-0.2	-0.1	-0.2	-0.2	-0.4	-1.5	-4.0	-11.2
	-2.1	-0.9	-0.6	-0.4	-0.2	-0.2	-0.1	-0.2	-0.2	-0.3	-0.5	-0.8	-1.4
	-1.1	-0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.8
	0.7	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.1	-0.7
1Y	-23.5	-8.0	-2.3	-1.2	-0.4	-0.3	-0.3	-0.3	-0.5	-1.2	-3.9	-7.7	-36.8
	-3.5	-1.9	-1.0	-0.7	-0.4	-0.3	-0.3	-0.3	-0.4	-0.7	-1.2	-1.6	-3.1
	-34.1	-9.2	-2.4	-1.2	-0.4	-0.3	-0.3	-0.3	-0.5	-1.2	-3.6	-6.7	-23.2
	-3.9	-2.0	-1.0	-0.7	-0.4	-0.3	-0.3	-0.3	-0.4	-0.6	-1.1	-1.5	-2.8
	-2.1	-0.3	-0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.2	3.0
	2.0	0.2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.1	-0.2	-2.3
1.5Y	-28.4	-10.6	-3.5	-2.0	-0.7	-0.4	-0.4	-0.5	-0.6	-1.6	-5.6	-12.1	-50.0
	-4.7	-2.7	-1.5	-1.1	-0.6	-0.4	-0.4	-0.5	-0.5	-0.9	-1.7	-2.4	-4.3
	-41.3	-12.2	-3.7	-2.1	-0.7	-0.4	-0.4	-0.5	-0.6	-1.5	-5.1	-10.2	-29.7
	-5.3	-2.9	-1.6	-1.2	-0.6	-0.4	-0.4	-0.5	-0.5	-0.9	-1.6	-2.2	-3.8
	-2.5	-0.4	-0.1	-0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.4	4.4
	2.5	0.3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	-0.1	-0.4	-3.3
2Y	-36.5	-14.5	-5.3	-1.9	-0.8	-0.6	-0.5	-0.6	-0.9	-2.0	-6.0	-17.6	-91.0
	-6.2	-3.7	-2.2	-1.3	-0.8	-0.6	-0.5	-0.6	-0.8	-1.2	-2.1	-3.3	-6.2
	-55.7	-17.2	-5.6	-1.9	-0.8	-0.6	-0.5	-0.6	-0.9	-2.0	-5.5	-14.2	-43.6
	-7.1	-4.0	-2.3	-1.3	-0.8	-0.6	-0.5	-0.6	-0.8	-1.2	-1.9	-3.0	-5.3
	-3.5	-0.6	-0.2	-0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.1	0.7	10.0
	3.6	0.5	0.1	0.0	0.0	0.0	0.0	0.0	0.0	-0.1	-0.1	-0.6	-6.6
3Y	-64.7	-17.8	-11.3	-2.9	-1.4	-0.9	-0.8	-0.8	-1.3	-3.1	-10.5	-27.0	-140.9
	-10.5	-5.0	-3.9	-1.9	-1.2	-0.9	-0.8	-0.8	-1.1	-1.8	-3.2	-4.9	-8.8
	-122.7	-21.1	-12.6	-3.0	-1.4	-0.9	-0.8	-0.8	-1.2	-3.0	-9.3	-20.7	-57.5
	-12.6	-5.6	-4.3	-2.0	-1.3	-0.9	-0.8	-0.8	-1.1	-1.8	-3.0	-4.3	-7.3
	-8.9	-0.8	-0.4	-0.1	-0.1	0.0	0.0	0.0	0.0	0.1	0.3	1.2	16.0
	10.7	0.6	0.3	0.1	0.0	0.0	0.0	-0.1	-0.1	-0.3	-0.3	-1.2	-10.1
5Y	-106.7	-30.6	-13.2	-5.9	-2.2	-1.5	-1.2	-1.3	-2.1	-4.8	-17.3	-45.5	-471.9
	-18.1	-8.6	-5.5	-3.6	-2.0	-1.5	-1.2	-1.3	-1.9	-2.9	-5.3	-7.9	-14.5
	-256.0	-38.2	-14.5	-6.1	-2.2	-1.5	-1.2	-1.3	-2.1	-4.6	-14.9	-32.2	-88.5
	-23.2	-10.0	-6.1	-3.8	-2.1	-1.5	-1.2	-1.3	-1.8	-2.7	-4.7	-6.7	-11.5
	-18.8	-1.6	-0.5	-0.3	-0.1	-0.1	0.0	0.0	0.1	0.1	0.5	2.4	38.9
	23.1	1.3	0.3	0.2	0.1	0.0	0.0	-0.1	-0.1	-0.2	-0.5	-2.4	-20.9
10Y	-172.3	-69.5	-30.2	-10.0	-4.4	-2.7	-2.2	-2.4	-3.6	-9.1	-34.6	-103.0	ND
	-33.7	-19.2	-12.1	-6.7	-4.1	-2.8	-2.2	-2.4	-3.4	-5.6	-10.2	-15.5	-29.6
	-472.8	-94.7	-34.0	-10.3	-4.4	-2.8	-2.2	-2.4	-3.6	-8.7	-28.2	-60.5	-159.3
	-47.5	-24.3	-14.3	-7.3	-4.2	-2.8	-2.2	-2.4	-3.3	-5.1	-8.7	-12.3	-20.9
	-33.9	-5.0	-1.1	-0.5	-0.3	-0.2	0.0	0.1	0.2	0.3	1.1	6.7	146.5
	27.4	2.5	0.6	0.4	0.2	0.1	0.0	-0.1	-0.2	-0.3	-1.3	-7.0	-58.7

Table 1.4. CEV model ($\beta = 0.8$): errors in bps on the implied volatility using the 6 approximations `ImpVol(AppPrice(2,S0))`, `AppImpVol(2,S0)`, `ImpVol(AppPrice(2,K))`, `AppImpVol(2,K)`, `ImpVol(AppPrice(3,S0))` and `ImpVol(AppPrice(3,K))`.

3M	-131.8	-71.5	-33.0	-12.7	-4.2	-1.5	-1.0	-2.3	-3.9	-22.8	-52.9	-124.1	-525.3
	-18.8	-12.6	-8.0	-4.8	-2.7	-1.5	-1.0	-2.3	-2.7	-5.5	-7.6	-10.1	-12.9
	ND	-134.0	-43.9	-14.2	-4.3	-1.6	-1.0	-2.3	-3.7	-18.5	-34.8	-57.1	-84.0
	-24.4	-15.4	-9.3	-5.3	-2.8	-1.5	-1.0	-2.2	-2.7	-4.9	-6.5	-8.4	-10.3
	-31.8	-9.5	-2.1	-0.5	-0.2	-0.1	0.0	-0.9	-0.5	0.6	3.4	12.8	40.5
	57.2	11.7	2.1	0.5	0.2	0.1	0.0	-0.7	-1.1	-3.3	-9.1	-20.3	
6M	-152.3	-47.5	-22.8	-10.1	-4.6	-2.6	-2.1	-2.3	-3.6	-7.0	-27.8	-97.9	ND
	-28.2	-14.0	-9.4	-6.1	-3.8	-2.6	-2.1	-2.3	-3.1	-4.4	-8.2	-13.4	-23.2
	-466.9	-63.4	-26.3	-10.8	-4.7	-2.6	-2.1	-2.3	-3.5	-6.6	-22.7	-56.0	-130.1
	-38.4	-16.8	-10.7	-6.6	-4.0	-2.6	-2.1	-2.3	-3.0	-4.1	-7.1	-10.9	-17.1
	-31.0	-3.3	-1.1	-0.6	-0.3	-0.2	0.0	0.1	0.2	0.4	1.2	7.8	94.9
	41.7	3.4	1.2	0.6	0.3	0.1	0.0	-0.1	-0.2	-0.3	-1.1	-6.7	-41.0
1Y	-257.5	-105.8	-34.8	-18.9	-6.5	-4.8	-4.2	-4.3	-7.4	-18.9	-77.7	-203.6	ND
	-55.8	-31.6	-16.9	-12.1	-6.2	-4.8	-4.2	-4.3	-6.1	-9.8	-17.8	-24.4	-48.0
	ND	-164.6	-40.5	-20.7	-6.7	-4.8	-4.2	-4.3	-7.1	-16.9	-52.8	-90.9	-227.4
	-84.9	-41.8	-19.7	-13.4	-6.3	-4.8	-4.2	-4.2	-5.8	-8.7	-14.3	-18.3	-30.8
	-63.4	-11.2	-2.4	-1.5	-0.6	-0.3	-0.1	0.2	0.6	1.2	4.9	18.0	343.5
	77.5	12.0	3.0	1.8	0.6	0.2	-0.1	-0.3	-0.6	-1.0	-4.7	-15.0	-111.1
1.5Y	-313.2	-139.6	-53.3	-31.8	-12.3	-7.0	-6.2	-6.9	-8.6	-25.4	-118.7	-574.7	ND
	-77.5	-46.0	-26.4	-19.8	-11.1	-7.0	-6.2	-6.8	-7.9	-13.7	-25.7	-36.7	-66.4
	ND	-221.0	-63.9	-35.7	-12.8	-7.0	-6.2	-6.8	-8.4	-22.4	-73.0	-131.2	-273.3
	-124.7	-63.5	-32.0	-22.6	-11.6	-7.0	-6.2	-6.7	-7.6	-12.0	-19.6	-25.7	-39.3
	-76.9	-16.4	-4.4	-3.0	-1.5	-0.5	-0.1	0.6	0.9	2.0	8.6	39.6	470.0
	81.1	18.1	6.0	3.8	1.5	0.3	-0.1	-0.7	-0.9	-1.6	-8.1	-29.8	-150.5
2Y	-395.3	-187.6	-79.0	-31.6	-14.5	-11.0	-8.3	-8.7	-12.8	-33.0	-125.5	ND	ND
	-104.9	-63.9	-38.4	-22.7	-13.6	-10.9	-8.3	-8.6	-11.1	-17.8	-30.4	-49.9	-94.8
	ND	-314.7	-98.3	-34.7	-14.9	-11.1	-8.3	-8.6	-12.2	-28.6	-78.5	-172.3	-342.2
	-180.4	-93.1	-48.5	-25.5	-14.1	-11.0	-8.3	-8.5	-10.5	-15.2	-22.8	-32.7	-50.2
	-105.8	-25.5	-7.6	-3.8	-2.0	-1.4	-0.2	0.7	1.5	2.9	9.3	76.9	770.3
	99.9	28.0	10.5	4.6	1.8	0.9	-0.2	-1.0	-1.5	-2.2	-8.7	-50.8	-228.6
3Y	-651.4	-234.4	-160.1	-48.9	-24.7	-15.6	-12.4	-12.3	-18.6	-51.2	-279.0	ND	ND
	-184.1	-90.8	-72.0	-35.7	-22.8	-15.5	-12.4	-12.3	-16.1	-26.3	-47.8	-73.4	-133.8
	ND	-375.7	-219.9	-54.3	-25.7	-15.7	-12.4	-12.2	-17.6	-42.6	-125.7	-231.1	-397.4
	-375.3	-138.1	-101.1	-41.3	-24.0	-15.7	-12.4	-12.2	-15.0	-21.5	-32.8	-43.5	-62.4
	-228.9	-33.9	-19.4	-7.2	-4.3	-2.1	-0.5	0.8	2.6	5.1	22.6	156.9	1012.9
	129.8	37.9	25.9	8.8	3.9	1.4	-0.5	-1.5	-2.5	-3.7	-21.6	-89.8	-295.9
5Y	ND	-392.5	-198.4	-100.5	-42.0	-28.2	-20.5	-19.9	-30.9	-80.9	ND	ND	ND
	-320.6	-163.4	-106.9	-70.8	-39.5	-28.2	-20.5	-19.9	-26.3	-40.9	-76.7	-117.4	-219.1
	ND	-618.2	-247.5	-113.0	-43.8	-28.6	-20.5	-19.7	-28.7	-64.3	-188.0	-314.1	-468.3
	-830.8	-283.4	-154.1	-88.3	-42.3	-28.7	-20.5	-19.7	-23.7	-31.7	-46.8	-59.8	-81.3
	-414.2	-68.3	-27.9	-17.0	-9.3	-5.5	-1.3	1.8	5.2	9.4	49.9	341.0	1498.8
	-666.6	40.4	35.9	21.7	7.9	2.8	-1.3	-3.4	-4.8	-6.7	-47.8	-165.5	-400.1
10Y	ND	-731.7	-386.6	-160.2	-80.6	-51.7	-37.8	-35.2	-51.8	-169.5	ND	ND	ND
	-387.9	-274.1	-196.2	-121.2	-77.4	-52.6	-37.8	-35.8	-46.2	-76.8	-146.2	-227.2	-439.9
	1545.1	-250.4	-303.5	-155.1	-81.7	-52.1	-37.8	-34.9	-48.1	-118.9	-304.8	-431.9	-406.7
	ND	-784.3	-387.4	-168.5	-87.5	-53.9	-37.8	-35.2	-40.1	-52.3	-71.3	-85.6	-103.8
	-447.8	-67.5	-13.5	-17.4	-16.2	-10.3	-2.6	3.5	10.6	19.8	151.4	855.1	2397.6
	ND	-1411.2	-136.9	34.6	23.9	8.4	-2.6	-7.4	-9.8	-15.8	-135.4	-323.5	-406.7

Table 1.5. CEV model (β = 0.2): errors in bps on the implied volatility using the 6 approximations `ImpVol(AppPrice(2,S0))`, `AppImpVol(2,S0)`, `ImpVol(AppPrice(2,K))`, `AppImpVol(2,K)`, `ImpVol(AppPrice(3,S0))` and `ImpVol(AppPrice(3,K))`.
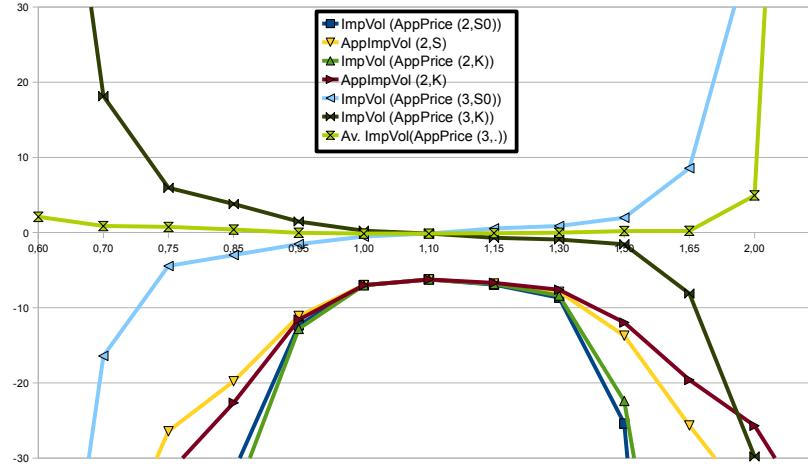
Fig. 1.2. CEV model ($\beta = 0.2$): errors in bps on the implied volatility using the 7 approximations `ImpVol(AppPrice(2,S0))`, `AppImpVol(2,S0)`, `ImpVol(AppPrice(2,K))`, `AppImpVol(2,K)`, `ImpVol(AppPrice(3,S0))`, `ImpVol(AppPrice(3,K))` and `Av. ImpVol(AppPrice(3,.))`

3M	-0.41	-0.13	-0.04	-0.02	-0.02	-0.01	0.00	0.00	0.00	0.00	0.01	0.00	-0.01
6M	-0.16	-0.04	-0.03	-0.02	-0.01	-0.01	0.00	-0.01	0.00	-0.01	0.00	0.01	0.04
1Y	-0.05	-0.06	-0.04	-0.03	-0.01	-0.02	-0.01	-0.01	0.00	-0.01	0.00	0.00	0.34
1.5Y	-0.02	-0.07	-0.04	-0.03	-0.02	-0.02	-0.01	-0.01	-0.01	-0.01	0.00	0.01	0.54
2Y	0.06	-0.08	-0.05	-0.03	-0.02	-0.02	-0.02	-0.01	-0.01	-0.01	-0.01	0.01	1.69
3Y	0.89	-0.09	-0.06	-0.03	-0.02	-0.02	-0.02	-0.02	-0.01	-0.01	-0.01	0.02	2.98
5Y	2.17	-0.16	-0.06	-0.04	-0.03	-0.03	-0.03	-0.02	-0.01	-0.01	-0.01	0.01	8.99
10Y	-3.24	-1.24	-0.23	-0.05	-0.05	-0.05	-0.04	-0.03	-0.02	-0.01	-0.12	-0.13	43.89

Table 1.6. CEV model ($\beta = 0.8$): errors in bps on the implied volatility using `Av. ImpVol(AppPrice(3,.))`

3M	12.69	1.08	0.01	0.00	-0.01	-0.01	0.00	-0.95	-0.60	-0.27	0.01	1.86	10.08
6M	5.36	0.07	0.07	0.03	0.00	-0.02	-0.02	-0.02	0.00	0.02	0.04	0.54	26.98
1Y	7.01	0.37	0.31	0.14	-0.04	-0.07	-0.06	-0.06	0.01	0.11	0.13	1.52	116.22
1.5Y	2.09	0.87	0.76	0.40	-0.03	-0.14	-0.14	-0.07	-0.02	0.21	0.23	4.94	159.77
2Y	-2.93	1.24	1.46	0.40	-0.12	-0.21	-0.23	-0.14	0.02	0.34	0.28	13.05	270.84
3Y	-49.57	2.02	3.24	0.82	-0.19	-0.50	-0.50	-0.34	0.05	0.69	0.49	33.54	358.51
5Y	-540.41	-13.94	3.99	2.33	-0.69	-1.31	-1.27	-0.77	0.22	1.37	1.04	87.77	549.35
10Y	ND	-739.31	-75.21	8.58	3.84	-0.94	-2.64	-1.96	0.38	2.00	7.99	265.78	995.47

Table 1.7. CEV model ($\beta = 0.2$): errors in bps on the implied volatility using `Av. ImpVol(AppPrice(3,.))`