The ”strange term” in the periodic homogenization for multivalued Leray-Lions operators in perforated domains

Alain Damlamian, Nicolas Meunier

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Résumé

Nous étudions par la méthode de l'éclatement priodique de Cioranescu, Damlamian et Griso, l'homogénéisation des équations de la forme

$$-\text{div } d_\varepsilon = f, \text{ with } \big(\nabla u_{\varepsilon,\delta}(x), d_{\varepsilon,\delta}(x)\big) \in A_\varepsilon(x)$$

où A_ε est une fonction dont les valeurs sont des opérateurs maximaux monotones et le domaine est perforé composé de trous de taille $\varepsilon\delta$ périodiquement répartis dans le domaine. Sous des hypothèses de croissance et de coercivité sur A_ε, si les deux suites d'opérateurs maximaux monotone éclatés (suivant deux opérations distinctes) convergent au sens des graphes vers des opérateurs maximaux monotones $A(x, y)$ et $A_0(x, z)$ pour presque tout $(x, y, z) \in \Omega \times Y \times \mathbf{R}^N$, quand $\varepsilon \to 0$, alors toute valeur d'adhérence (u_0, d_0) de la suite $(u_{\varepsilon,\delta}, d_{\varepsilon,\delta})$ pour la topologie faible dans l'espace de Sobolev naturellement associ est une solution du probleme homogénéisé qui s'exprime en terme de la fonction u_0 seule. Ce rsultat s'applique au cas o $A_\varepsilon(x)$ est de la forme $B(x/\varepsilon)$ o $B(y)$ est priodique et continu en $y = 0$, et en particulier au cas du p-Laplacien oscillant.

Abstract

Using the periodic unfolding method of Cioranescu, Damlamian and Griso, we study the homogenization for equations of the form

$$-\text{div } d_\varepsilon = f, \text{ with } \big(\nabla u_{\varepsilon,\delta}(x), d_{\varepsilon,\delta}(x)\big) \in A_\varepsilon(x)$$

in a perforated domain with holes of size $\varepsilon\delta$ periodically distributed in the domain, where A_ε is a function whose values are maximal monotone

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graphs (on $\mathbf{R}^N$). Two different unfolding operators are involved in such a geometric situation. Under appropriate growth and coercivity assumptions, if the corresponding two sequences of unfolded maximal monotone graphs converge in the graph sense to the maximal monotone graphs $A(x,y)$ and $A_0(x,z)$ for almost every $(x,y,z) \in \Omega \times Y \times \mathbf{R}^N$, as $\varepsilon \to 0$, then every cluster point (u_0, d_0) of the sequence $(u_{\varepsilon,\delta}, d_{\varepsilon,\delta})$ for the weak topology in the naturally associated Sobolev space is a solution of the homogenized problem which is expressed in terms of u_0 alone. This result applies to the case where $A_\varepsilon(x)$ is of the form $B(x/\varepsilon)$ where $B(y)$ is periodic and continuous at $y = 0$, and, in particular, to the oscillating p-Laplacian.

1 Introduction

This article is devoted to periodic homogenization for nonlinear partial differential equations with oscillating coefficients in domains perforated by small holes of size $\varepsilon\delta$ and periodically distributed with period ε. The homogeneous Dirichlet condition is imposed on the boundaries of these holes. This type of equation models various physical problems arising in media with holes or heterogeneous materials with various competing length-scales.

Our starting point is the following theorem, where $\mathfrak{LL}(\mathcal{O}, \mathbf{R}^N, p, \alpha, m)$ denotes the measurable maps from the open domain $\mathcal{O}$ in $\mathbf{R}^N$ to the set $\mathfrak{M}(\mathbf{R}^N \times \mathbf{R}^N)$ of all maximal monotone maps from $\mathbf{R}^N$ into itself such that

$$\alpha\Big(\frac{\|\xi\|^p}{p} + \frac{\|\eta\|^{p'}}{p'}\Big) \le \langle \eta, \xi \rangle + m(x). \tag{1.1}$$

The function m is assumed to be $L^1(\mathcal{O})$.

Theorem 1.1. *Let $\mathcal{O}$ be a bounded domain in $\mathbf{R}^N$, $p \in (1,\infty)$, $m \in L^1(\mathcal{O})$, and $f \in L^{p'}(\mathcal{O})$. Let $(A_n)_{n\in\mathbf{N}}$ be a sequence in $\mathfrak{LL}(\mathcal{O}, \mathbf{R}^N, p, \alpha, m)$ and suppose that for each $n \in \mathbf{N}$, u_n is a solution of*

$$\begin{cases} -\mathrm{div}\ d_n = f \ \text{ in } \mathcal{D}'(\mathcal{O}), \\ \big(\nabla u_n(x), d_n(x)\big) \in A_n(x), \\ u_n \in W_0^{1,p}(\mathcal{O}). \end{cases} \tag{1.2}$$

Suppose furthermore that for a.e. $x \in \mathcal{O}$, the sequence $A_n(x)$ converges in the sense of multivalued operators to $A_0(x)$.
Then, A_0 belongs to $\mathfrak{LL}(\mathcal{O}, \mathbf{R}^N, p, \alpha, m)$, the sequence $(u_n, d_n)_{n\in\mathbf{N}}$ is bounded in the space $W_0^{1,p}(\mathcal{O}) \times L^{p'}(\Omega)$ and every one of its weak limit-points is a solution of

$$\begin{cases} -\text{div } d = f \ \ in \ \mathcal{D}'(\mathcal{O}), \\ \big(\nabla u(x), d(x)\big) \in A(x), \\ u \in W_0^{1,p}(\mathcal{O}). \end{cases} \tag{1.3}$$

Actually, the well-known result of J. Leray and J.-L. Lions ([18]) implies that problem (1.2) has a unique solution in the case where $A_\varepsilon(x)$ is single-valued and strictly monotone for a.e. $x \in \Omega$. Theorem 1.1 itself allows to obtain solutions of the same problem in the general case (by approximating in the sense of graphs a multivalued element of $\mathfrak{L}\mathfrak{L}(\Omega, \mathbf{R}^N, p, \alpha, m)$ by a sequence of such single-valued elements) (see also [9, 17]). Such solutions need not be unique.

In the present, work we consider the homogenization (when the small parameters ε, δ go to 0) of the same problem in a domain $\Omega^*_{\varepsilon,\delta}$ with small Dirichlet holes:

$$\begin{cases} -\text{div } d_{\varepsilon,\delta} = f_\varepsilon \ \text{ in } \mathcal{D}'(\Omega^*_{\varepsilon,\delta}), \\ \big(\nabla u_{\varepsilon,\delta}(x), d_{\varepsilon,\delta}(x)\big) \in A_\varepsilon(x), \\ u_{\varepsilon,\delta} \in W_0^{1,p}(\Omega^*_{\varepsilon,\delta}). \end{cases} \tag{1.4}$$

The domain $\Omega^*_{\varepsilon,\delta}$ is constructed as follows. Y is a reference period in $\mathbf{R}^N$ associated to the set of periods b (which is a basis of $\mathbf{R}^N$). Then B is a (not empty) bounded open subset of $\mathbf{R}^N$ and, for δ small enough, Y^*_δ is $Y \setminus \delta\overline{B}$. Finally

$$\Omega^*_{\varepsilon,\delta} = \Big\{ x \in \Omega, \ \text{ such that } x \in \bigcup_{k \in \mathbf{Z}^N} \varepsilon\{Y^*_\delta + k \cdot b\} \Big\}.$$

In problem (1.4), it is assumed that $1 < p < \infty$, $p^{-1} + p^{'-1} = 1$, f_ε belongs to $W^{-1,p'}(\Omega)$ and A_ε to $\mathfrak{L}\mathfrak{L}(\Omega, \mathbf{R}^N, p, \alpha, m)$. The solutions $u_{\varepsilon,\delta}$ are naturally extended by 0 in the holes and as such belong to $W_0^{1,p}(\Omega)$. One can also extend $d_{\varepsilon,\delta}$ measurably in the holes by an element of $A_\varepsilon(x, 0)$ so as to belong to $L^{p'}(\Omega)$.

Let $(u_{\varepsilon,\delta}, d_{\varepsilon,\delta}) \in W_0^{1,p}(\Omega) \times L^p(\Omega; \mathbf{R}^N)$ be solutions of problem (1.4). Suppose furthermore that the two sequences of unfolded graphs $\mathcal{T}_\varepsilon(A_\varepsilon)(x, y)$ and $A_{\varepsilon,\delta}(x, z)$ converge in the sense of maximal monotone graphs, where $A_{\varepsilon,\delta}(x, z)$ is given by

$$(\xi, \eta) \in A_{\varepsilon,\delta}(x, z) \Leftrightarrow (\delta^{-N/p}\xi, \delta^{-N/p'}\eta) \in \mathcal{T}_{\varepsilon,\delta}(A_\varepsilon)(x, z). \tag{1.5}$$

Then we show that, provided ε and δ converge to 0 in a controlled way, every weak cluster point (u_0, d_0) in $W_0^{1,p}(\Omega) \times L^p(\Omega, \mathbf{R}^N)$ of a sequence $(u_{\varepsilon,\delta}, d_{\varepsilon,\delta})$ of solutions of problem (1.4) is itself a solution of the homogenized equation:

$$\begin{cases} -\text{div } d - k_1 \boldsymbol{\Theta}^{-1}(-k_1 u) \ni f, \\ \big(\nabla_x u, d\big) \in A_{hom}(x) u \in W_0^{1,p}(\Omega), \end{cases} \tag{1.6}$$

where $A_{hom} \in \mathfrak{L}\mathfrak{L}(\Omega, \mathbf{R}^N, p, \alpha, \bar{m})$ is defined in terms of the limit of the sequence of unfolded graphs $\mathcal{T}_\varepsilon(A_\varepsilon)(x, y)$ alone, while $\mathbf{\Theta}$ is the maximal monotone graph (on $\mathbf{R} \times \mathbf{R}$) obtained from the limit A_0 of the sequence of second unfolded graphs $A_{\varepsilon,\delta}(x, z)$ alone. The constant k_1 is given as $\lim_{(\varepsilon.\delta)\to(0,0)} \dfrac{\delta^{\frac{N}{p-1}}}{\varepsilon}$, which, without loss of generality, can be assumed to exist.

The outline of the paper is as follows. Section 2 is devoted to the definition of the unfolding operators and their main properties. In subsection 2.1, we recall the definition of the unfolding operator $\mathcal{T}_\varepsilon$ for the periodic case in fixed domains (see [11] and [15] as well as [19] and [20]) and in subsection 2.2, we recall the unfolding operator $\mathcal{T}_{\varepsilon,\delta}$ depending of two small parameters ε and δ corresponding to the scales ε and $\varepsilon\delta$ (see [12] for the proofs). It was first introduced in a similar form in [6] and [7].

In section 3 we consider the homogenization problem and we state our main result in Theorem 3.4. For simplicity, we assume a homogeneous Dirichlet boundary condition on the outer boundary of the domain, but more general boundary conditions can be assumed there, provided the outer boundary is Lipschitz and the perforations do not intersect it. In each case, we obtain both the unfolded and the classical (standard) form for the limit problem. The operator $\mathcal{T}_\varepsilon$ allows to homogenize the coefficients of the differential operators, whereas the operator $\mathcal{T}_{\varepsilon,\delta}$ generates the "strange term" $\mathbf{\Theta}$ in the limit in the spirit of [10].

In the Appendix, we recall the definition of maximal monotone operators and of the notion of convergence of maximal monotone graphs (see [16] for the proofs). We also consider sequences of maximal monotone-valued measurable functions, their canonical extensions and we prove key results about their convergence (including a new generalization of Minty's method in Proposition 4.10).

Note that there are many papers in the litterature which study the homogenization problem for perforated domains in the non-linear case. We only refer to [13], [6] and the bibliographies therein. The present paper is the first making use of the unfolding method

2 The periodic unfolding operator

In this section we recall, the general properties of the periodic unfolding operator introduced in [11] and of its variants and generalizations introduced in [12].

Let Y be a set containing the origin and having the paving property in $\mathbf{R}^N$ (cf. [15]) with respect to a set of N periods $(b_1, \cdots, b_N)$. The classical example is the unit cube of $\mathbf{R}^N$ centered in the origin, $Y = \left]-\frac{1}{2}, \frac{1}{2}\right[^N$. We consider the subgroup generated by the periods in $\mathbf{R}^N$ and all the corresponding translates of Y. To each $x \in \mathbf{R}^N$ we can associate its integer part, $[x]_Y$ belonging to the subgroup of periods, such that $x - [x]_Y \in Y$, the latter being defined as its fractional part, i.e, $\{x\}_Y = x - [x]_Y$. Therefore we have $x = \varepsilon \left\{\frac{x}{\varepsilon}\right\}_Y + \varepsilon \left[\frac{x}{\varepsilon}\right]_Y$ for any $x \in \mathbf{R}^N$. Without ambiguity, we will drop the subscript Y for the integer part and fractional part functions in the sequel.

Remark 1. This definition is ambiguous, but only on a set of zero measure, which is enough for our purpose.

Let Ω be an open and bounded set in $\mathbf{R}^N$, the sets $\widehat{\Omega}_\varepsilon$ and Λ_ε are defined respectively as the largest finite union of εY cells contained in Ω and as the subset of Ω consisting of the the parts of εY cells intersecting the boundary $\partial\Omega$. More precisely, we have:

$$\widehat{\Omega}_\varepsilon = \left\{ x \in \Omega, \left(\varepsilon \left[\frac{x}{\varepsilon}\right] + \varepsilon Y \right) \subset \Omega \right\} \quad \text{and} \quad \Lambda_\varepsilon = \Omega \setminus \widehat{\Omega}_\varepsilon. \tag{2.1}$$

2.1 The case of fixed domains: the operator $\mathcal{T}_\varepsilon$

We give here the definition of the unfolding operator and its main properties (for details and proofs, the reader is referred to [11] and [15]).

Definition 2.1. The unfolding operator $\mathcal{T}_\varepsilon : L^p(\Omega) \to L^p(\Omega \times Y)$ is defined as follows for $\phi \in L^p(\Omega)$:

$$\mathcal{T}_\varepsilon(\phi)(x, y) = \begin{cases} \phi\Big(\varepsilon \left[\frac{x}{\varepsilon}\right] + \varepsilon y\Big) & \text{if } (x, y) \in \widehat{\Omega}_\varepsilon \times Y, \\ 0 & \text{if } (x, y) \in \Lambda_\varepsilon \times Y. \end{cases}$$

Theorem 2.2. (Properties of the operator $\mathcal{T}_\varepsilon$)
1. For any $v, w \in L^p(\Omega)$, $\mathcal{T}_\varepsilon(vw) = \mathcal{T}_\varepsilon(v)\mathcal{T}_\varepsilon(w)$.
2. The "exact integration" formula. For every $w \in L^p(\Omega)$,

$$\frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}_\varepsilon(w)(x, y) \, \mathrm{d}x \, \mathrm{d}y = \int_\Omega w(x) \, \mathrm{d}x - \int_{\Lambda_\varepsilon} w(x) \, \mathrm{d}x = \int_{\widehat{\Omega}_\varepsilon} w(x) \, \mathrm{d}x.$$

3. The L^1 control. For every $u \in L^1(\Omega)$,

$$\frac{1}{|Y|} \int_{\Omega \times Y} |\mathcal{T}_\varepsilon(u)| \, \mathrm{d}x \, \mathrm{d}y \leq \int_\Omega |u| \, \mathrm{d}x.$$

4. For every $u \in L^1(\Omega)$,

$$\left| \int_\Omega u \, \mathrm{d}x - \frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}_\varepsilon(u) \, \mathrm{d}x \, \mathrm{d}y \right| \leq \int_{\Lambda_\varepsilon} |u| \, \mathrm{d}x. \tag{2.2}$$

5. Let $\{w_\varepsilon\}$ be a sequence which converges strongly in $L^p(\Omega)$ to some w, (for $1 \le p < +\infty$). Then,

$$\mathcal{T}_\varepsilon(w_\varepsilon) \to w \text{ strongly in } L^p(\Omega \times Y).$$

6. Let $\{w_\varepsilon\}$ be a sequence which converges weakly in $W^{1,p}(\Omega)$ to some w ($1 \le p < +\infty$). Then, there exists a subsequence and $\widehat{w} \in L^p\left(\Omega; W^{1,p}_{per}(Y)\right)$ such that

$$\mathcal{T}_\varepsilon(\nabla w_\varepsilon) \rightharpoonup \nabla_x w + \nabla_y \widehat{w} \quad \text{weakly in } L^p(\Omega \times Y).$$

Remark 2. If $u : \mathbf{R}^N \to S$ and $f : S \to S'$, then

$$\mathcal{T}_\varepsilon(f \circ u) = f \circ \mathcal{T}_\varepsilon(u).$$

In particular if $u : \mathbf{R}^N \to S$ and $v : \mathbf{R}^N \to T$, the preceding property applied to the projections $P : (u, v) \mapsto u$ and $Q : (u, v) \mapsto v$ yields

$$\mathcal{T}_\varepsilon\big((u, v)\big) = \big(\mathcal{T}_\varepsilon(u), \mathcal{T}_\varepsilon(v)\big).$$

Therefore, if $F : S \times T \to R$,

$$\mathcal{T}_\varepsilon\big(F(u, v)\big) = F\big(\mathcal{T}_\varepsilon(u), \mathcal{T}_\varepsilon(v)\big). \tag{2.3}$$

Useful particular cases are when $S = \mathbf{R}$, $T = \mathbf{R}$ and $F : (s, t) \to st$ and when $S = \mathbf{R}^N$, $T = \mathbf{R}^N$ and F is the dot product.
The previous formulas apply also for functions defined on an open set Ω, with the obvious modifications on the set Λ_ε.

Remark 3. Property 4 shows that every integral of a function w on Ω, is "almost equivalent" to the integral of its unfolded on $\Omega \times Y$, the "integration defect" arises only from the cells intersecting the boundary $\partial\Omega$ and is controlled by the right hand side integral in (2.2). Hence we deduce the following result.

Proposition 2.3. *If $\{w_\varepsilon\}$ is a sequence in $L^1(\Omega)$ satisfying*

$$\int_{\Lambda_\varepsilon} |w_\varepsilon| \;\mathrm{d}x \to 0,$$

then

$$\int_\Omega w_\varepsilon \;\mathrm{d}x - \frac{1}{|Y|} \int_{\Omega \times Y} \mathcal{T}_\varepsilon(w_\varepsilon) \;\mathrm{d}x\; \mathrm{d}y \to 0.$$

We end this subsection by recalling the notion of local average of a function.

Definition 2.4. The local average $M^\varepsilon_Y : L^p(\Omega) \mapsto L^p(\Omega)$, is defined for any ϕ in $L^p(\Omega)$, $1 \le p < \infty$, by

$$M^\varepsilon_Y(\phi)(x) = \frac{1}{|Y|} \int_Y \mathcal{T}_\varepsilon(\phi)(x, y) \;\mathrm{d}y.$$

Remark 4. The function $M_Y^\varepsilon(\phi)$ is indeed a local average, since

$$M_Y^\varepsilon(\phi)(x) = \frac{1}{|Y|}\int_Y \mathcal{T}_\varepsilon(\phi)(x,y)\ \mathrm{d}y = \begin{cases} \dfrac{1}{\varepsilon^N|Y|}\displaystyle\int_{\varepsilon\left[\frac{x}{\varepsilon}\right]+\varepsilon Y} \phi(\zeta)\ \mathrm{d}\zeta, & \text{if } x\in\widehat{\Omega}_\varepsilon,\\ 0, & \text{if } x\in\Lambda_\varepsilon.\end{cases}$$

Remark 5. Note that $\mathcal{T}_\varepsilon(M_Y^\varepsilon(\phi)) = M_Y^\varepsilon(\phi)$ on the set $\Omega\times Y$.

The next proposition, which is also frequently used, is classical (and also follows from the previous theorem):

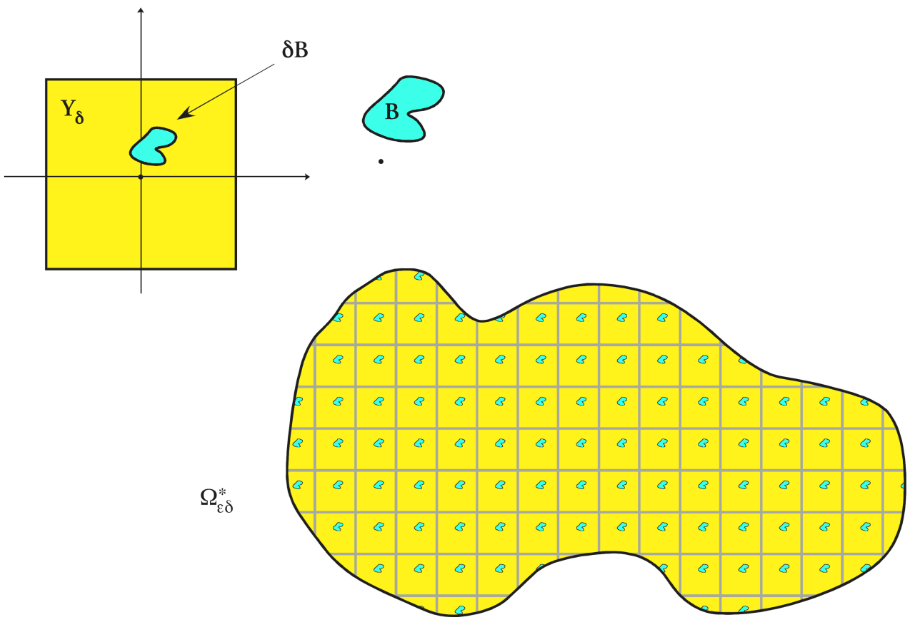
Proposition 2.5. *Let $\{w_\varepsilon\}$ be a sequence such that $w_\varepsilon \to w$ strongly in $L^p(\Omega)$ where $1\le p<\infty$. Then we have*

$$M_Y^\varepsilon(w_\varepsilon)\to w \quad \textit{strongly in } L^p(\Omega).$$

2.2 Unfolding in domains with volume-distributed "small" holes: the operator $\mathcal{T}_{\varepsilon,\delta}$

We consider domains with holes of size $\varepsilon\delta$ (δ will go to 0 with ε) and εY-periodically distributed. More precisely, for a given a given bounded open set B and for δ small enough (so that $\delta B \subset\subset Y$), we denote Y_δ^* the set $Y\setminus\delta\overline{B}$ and define the perforated domain $\Omega^*_{\varepsilon,\delta}$ as

$$\Omega^*_{\varepsilon,\delta} = \Big\{x\in\Omega,\ \text{such that}\Big\{\frac{x}{\varepsilon}\Big\}\in Y_\delta^*\Big\}. \tag{2.4}$$



The sets B and Y_δ^* and the corresponding $\Omega^*_{\varepsilon,\delta}$

This geometry with "small" holes leads to the definition of a new unfolding operator $\mathcal{T}_{\varepsilon,\delta}$ (see [12]), depending on both parameters ε and δ. In the next sections, we will consider functions $v_{\varepsilon,\delta}$ that vanish on the whole boundary of the perforated domain $\Omega^*_{\varepsilon,\delta}$, namely belonging to the space $W_0^{1,p}(\Omega^*_{\varepsilon,\delta})$. They are naturally extended by zero to the whole of Ω and these extensions still denoted $v_{\varepsilon,\delta}$ are functions in $W_0^{1,p}(\Omega)$. This justifies the introduction of $\mathcal{T}_{\varepsilon,\delta}$ on the fix domain Ω (but keeping in mind that our aim will be to apply it to the extensions).

Definition 2.6. For $\phi \in L^p(\Omega)$, $(1 \leq p < \infty)$, the unfolding operator $\mathcal{T}_{\varepsilon,\delta} : L^p(\Omega) \to L^p(\Omega \times \mathbf{R}^N)$ is defined as

$$\mathcal{T}_{\varepsilon,\delta}(\phi)(x,z) = \begin{cases} \mathcal{T}_\varepsilon(x, \delta z) & \text{if } (x,z) \in \widehat{\Omega}_\varepsilon \times \dfrac{1}{\delta}Y, \\ 0 & \text{otherwise.} \end{cases}$$

The next results follow directly from Theorem 2.2 by using the change of variable $z = (1/\delta)y$.

Theorem 2.7. (Properties of the operator $\mathcal{T}_{\varepsilon,\delta}$)
1. For every $v, w \in L^p(\Omega)$, $\mathcal{T}_{\varepsilon,\delta}(vw) = \mathcal{T}_{\varepsilon,\delta}(v)\mathcal{T}_{\varepsilon,\delta}(w)$.
2. For $u \in L^1(\Omega)$,

$$\frac{\delta^N}{|Y|}\int_{\Omega\times\mathbf{R}^N} |\mathcal{T}_{\varepsilon,\delta}(u)| \ \mathrm{d}x \ \mathrm{d}z \leq \int_\Omega |u| \ \mathrm{d}x.$$

3. For every $u \in L^p(\Omega)$, $1 \leq p < +\infty$,

$$\|\mathcal{T}_{\varepsilon,\delta}(u)\|^p_{L^p(\Omega\times\mathbf{R}^N)} \leq \frac{|Y|}{\delta^N}\|u\|^p_{L^p(\Omega)}.$$

4. For every $u \in L^1(\Omega)$,

$$\left|\int_\Omega u \ \mathrm{d}x - \frac{\delta^N}{|Y|}\int_{\Omega\times\mathbf{R}^N} \mathcal{T}_{\varepsilon,\delta}(u) \ \mathrm{d}x \ \mathrm{d}z\right| \leq \int_{\Lambda_\varepsilon} |u| \ \mathrm{d}x.$$

5. Let $u \in W^{1,p}(\Omega)$. Then,

$$\mathcal{T}_{\varepsilon,\delta}(\nabla_x u) = \frac{1}{\varepsilon\delta}\nabla_z\big(\mathcal{T}_{\varepsilon,\delta}(u)\big) \quad \text{in} \quad \Omega \times \frac{1}{\delta}Y.$$

Suppose $N > p \geq 1$, set $p^* = pN/(N-p)$ and denote the Sobolev-Poincaré-Wirtinger constant for $W^{1,p}(Y)$ by C.
6. Let ω be open and bounded in $\mathbf{R}^N$. Then, the following estimates hold:

$$\|\nabla_z\big(\mathcal{T}_{\varepsilon,\delta}(u)\big)\|^p_{L^p(\Omega\times\frac{1}{\delta}Y)} \leq \frac{\varepsilon^p|Y|}{\delta^{N-p}}\|\nabla u\|^p_{L^p(\Omega)}, \tag{2.5}$$

$$\|\mathcal{T}_{\varepsilon,\delta}\big(u - M_Y^\varepsilon(u)\big)\|^p_{L^p(\Omega;L^{p^*}(\mathbf{R}^N))} \le \frac{C\varepsilon^p|Y|}{\delta^{N-p}}\|\nabla u\|^p_{L^p(\Omega)}, \tag{2.6}$$

and

$$\|\mathcal{T}_{\varepsilon,\delta}(u)\|^p_{L^p(\Omega\times\omega)} \le \frac{2C\varepsilon^p|Y|}{\delta^{N-p}}\|\nabla u\|^p_{L^p(\Omega)} + 2|\,\omega\,|\;\|u\|^p_{L^p(\Omega)}.$$

7. Let $\{v_{\varepsilon,\delta}\}$ be a sequence of functions in $W^{1,p}(\Omega)$ which converges weakly to some v_0 when both ε and δ go to zero. Suppose furthermore that $\dfrac{\varepsilon^p}{\delta^{N-p}}$ remains bounded. Then, up to a subsequence, there exist a function U in $L^p(\Omega; L^p_{loc}(\mathbf{R}^N))$ and W in $L^p(\Omega; L^{p^*}(\mathbf{R}^N))$ with $\nabla_z W$ in $L^p(\Omega\times\mathbf{R}^N)$ such that

$$\big(M_Y^\varepsilon(v_{\varepsilon,\delta})1_{\frac{1}{\delta}Y} - \mathcal{T}_{\varepsilon,\delta}(v_{\varepsilon,\delta})\big) \rightharpoonup W \quad \text{weakly in } L^p(\Omega; L^{p^*}(\mathbf{R}^N)),$$

$$\nabla_z\big(\mathcal{T}_{\varepsilon,\delta}(v_{\varepsilon,\delta})\big)\,1_{\frac{1}{\delta}Y} \rightharpoonup -\nabla_z W \quad \text{weakly in } L^p(\Omega\times\mathbf{R}^N),$$

$$\mathcal{T}_{\varepsilon,\delta}(v_{\varepsilon,\delta}) \rightharpoonup U \quad \text{weakly in } L^p(\Omega; L^p_{loc}(\mathbf{R}^N)) \text{ and}$$

$$U(x,z) + W(x,z) = v_0(x) \text{ for a.e. } x \text{ and } z \text{ in } \Omega\times\mathbf{R}^n.$$

Remark 2 above applies in an analogous way to the definition of the unfolded of a dot product .

Concerning the integral formulas, we have the following results, similar to those of the previous subsection which follow from Property 4 in Theorem 2.7:

Proposition 2.8. *If $\{w_\varepsilon\}$ is a sequence in $L^1(\Omega)$ satisfying*

$$\int_{\Lambda_\varepsilon} |w_\varepsilon| \;\, \mathrm{d}x \to 0,$$

then

$$\int_\Omega w_\varepsilon \;\, \mathrm{d}x - \frac{\delta^N}{|Y|}\int_{\Omega\times\mathbf{R}^N} \mathcal{T}_{\varepsilon,\delta}(w_\varepsilon)\;\, \mathrm{d}x\;\mathrm{d}y \to 0.$$

3 Homogenization in domains with small holes which are periodically distributed in volume

In this section, we state our main homogenization result. First, we give the definitions of some functional spaces, then we give a convergence result and its proof. Finally, we study the form of the limit problem. Our proofs use the maximal monotone graph theory. For the convenience of the reader we only give the definition of the graphs we will consider while their properties are given in the appendix.

3.1 The functional setting

We begin with some definitions.
Let X be a locally uniformly convex reflexive separable Banach space and let X' be its dual. The duality product in $X' \times X$ is denoted by $\langle \cdot, \cdot \rangle$. The set of maximal monotone operators from X to X' is denoted by $\mathfrak{M}(X \times X')$. Let Ω be an open set of $\mathbf{R}^N$.

Definition 3.1. For a measurable function m from Ω to $\overline{\mathbf{R}^+}$, for $\alpha > 0$ and $1 < p < +\infty$, $\mathfrak{L}\mathfrak{L}(\Omega, X, p, \alpha, m)$ is the set of $A : \Omega \to \mathfrak{M}(X \times X')$ such that A is measurable and such that for almost every $t \in \Omega$, for every $(\xi, \eta) \in A(t)$,

$$\alpha\Big(\frac{\|\xi\|^p}{p} + \frac{\|\eta\|^{p'}}{p'}\Big) \leq \langle \eta, \xi \rangle + m(t), \tag{3.1}$$

where $p^{-1} + p'^{-1} = 1$.

Remark 6. The definition of the measurability of A is understood in the sense of [8]. Let $(\Omega, \mathcal{T}, \mu)$ will be a σ-finite μ-complete measure-space. A function $A : \Omega \to \mathfrak{M}(X \times X')$ is *measurable* if and only if for every open set $U \subset X \times X'$ (resp closed set, Borel set, open ball, closed ball),

$$\big\{t \in \Omega : A(t) \cap U \neq \emptyset\big\}$$

is measurable in Ω.

Remark 7. This condition is equivalent to the fact that the maximal monotone graphs are coercive with respect to $\|\cdot\|^p$ and that they satisfy a growth condition of the type $\|\eta\|^{p'} \leq C\|\xi\|^p + m_1(t)$. It also has the advantage of being symmetrical with respect to ξ and η.

Remark 8. If $\alpha > 1$, then $\mathfrak{L}\mathfrak{L}(\Omega, X, p, \alpha, m)$ is empty. Indeed, by Young's inequality, the graph $A(t)$ should be bounded in $X \times X'$ for almost every $t \in \Omega$, in contradiction with the maximality of almost every $A(t)$.

Remark 9. From now on, " $\rightarrowtail$ " will denote the convergence of maximal monotone graphs (this is explained in the appendix).

When $\Omega \subsetneq \mathbf{R}^N$, $A : \Omega \to \mathfrak{M}(X \times X')$ is unfolded as follows:

Definition 3.2. Let $\Omega \subsetneq \mathbf{R}^N$ and $A : \Omega \to \mathfrak{M}(X \times X')$. First A is extended to $\mathbf{R}^N$ by $\alpha\|\xi\|^{p-2}F(\xi)$, where F is the duality mapping of X. This extension is still denoted by A. The unfolded graph $\mathcal{T}_\varepsilon(A)$ is now defined on $\mathbf{R}^N \times Y$, but only its restriction to $\Omega \times Y$ will be used.

The perforated domain $\Omega^*_{\varepsilon,\delta}$ is defined by (2.4). Assume that the operator A_ε belongs to $\mathfrak{L}\mathfrak{L}(\Omega, X, p, \alpha, m)$. For $f_\varepsilon \in W^{-1,p'}(\Omega)$, consider the following problem denoted $\mathcal{P}_{\varepsilon,\delta}$:

$$\begin{cases} \text{Find } u_{\varepsilon,\delta} \in W_0^{1,p}(\Omega^*_{\varepsilon,\delta}) \text{ such that} \\ \int_\Omega \langle d_{\varepsilon,\delta}, \nabla\varphi\rangle \; \mathrm{d}x = \langle f_\varepsilon, \varphi\rangle_{W^{-1,p'}(\Omega), W_0^{1,p}(\Omega)} \\ \big(\nabla u_{\varepsilon,\delta}(x), d_{\varepsilon,\delta}(x)\big) \in A_\varepsilon(x), \\ \forall\varphi \in W_0^{1,p'}(\Omega^*_{\varepsilon,\delta}). \end{cases} \tag{3.2}$$

Remark 10. The previous variational problem can be expressed as follows:

$$\begin{cases} -\mathrm{div}\; d_{\varepsilon,\delta} = f_\varepsilon \text{ in } \mathcal{D}'(\Omega^*_{\varepsilon,\delta}), \\ \big(\nabla u_{\varepsilon,\delta}(x), d_{\varepsilon,\delta}(x)\big) \in A_\varepsilon(x), \\ u_{\varepsilon,\delta} \in W_0^{1,p}(\Omega^*_{\varepsilon,\delta}). \end{cases} \tag{3.3}$$

The following result is a direct application of Theorem 1.1 above.

Theorem 3.3. *There exists at least one solution* (u, d) *in* $W_0^{1,p}(\Omega^*_{\varepsilon,\delta}) \times L^{p'}(\Omega; \mathbf{R}^N)$ *for problem* $\mathcal{P}_{\varepsilon,\delta}$*. Moreover, if* A_ε *satisfies some strict monotonicity condition for a.e.* $x \in \Omega^*_{\varepsilon,\delta}$*, the solution is unique.*

Remark 11. We still denote $u_{\varepsilon,\delta}$ and $d_{\varepsilon,\delta}$ the extension by zero of $u_{\varepsilon,\delta}$ and $d_{\varepsilon,\delta}$ to the whole Ω.

In the sequel, we will suppose that $N > p$ and we will study the asymptotic behavior of problem $\mathcal{P}_{\varepsilon,\delta}$, as ε and $\delta = \delta(\varepsilon)$ are such that there exists a positive constant k_1 satisfying

$$k_1 = \lim_{\varepsilon\to 0} \frac{\delta^{\frac{N}{p}-1}}{\varepsilon}, \quad \text{with } 0 \leq k_1 \leq \infty. \tag{3.4}$$

Remark 12. Since ε and δ satisfy (3.4), for simplicity we will say that ε goes to zero instead of (ε, δ) goes to $(0, 0)$.

Remark 13. The cases $k_1 = 0$ and $k_1 = \infty$ are simpler and will be briefly discussed later.

We introduce the following functional spaces which play a crucial role in the statement of our main result:

$$K_B = \{\Phi \in L^{p^*}(\mathbf{R}^N)\;;\; \nabla\Phi \in L^p(\mathbf{R}^N),\; \Phi|_B \text{ is constant }\} \tag{3.5}$$

$$K_B^0 = \{\Phi \in L^{p^*}(\mathbf{R}^N)\;;\; \nabla\Phi \in L^p(\mathbf{R}^N),\; \Phi|_B = 0\}. \tag{3.6}$$

We denote the constant value which an element Φ of K_B takes on B by $\Phi(B)$. It is known that K_B is a Banach space when endowed with the norm

$$\|\Phi\|_{K_B} \doteq \Big(|\Phi(B)|^p + \|\nabla\Phi\|^p_{L^p(\mathbf{R}^N)}\Big)^{\frac{1}{p}},$$

and that K_B^0 is a closed hyperplane of K_B.

We also denote by $\widetilde{K}_B$:

$$\widetilde{K}_B \doteq \{\Phi - \Phi(B); \Phi \in K_B\} = \{\Psi \in L^{p^*}_{\text{loc}}(\mathbf{R}^N), \Psi = 0 \text{ on } B, \nabla\Psi \in L^p(\mathbf{R}^N)\}.$$

For $\Psi = \Phi - \Phi(B)$, let $\ell(\Psi)$ denote the real number $-\Phi(B)$. Then, $\widetilde{K}_B$ is also a Banach space for the norm

$$\|\Psi\|_{\widetilde{K}_B} \doteq \Big(|\ell(\Psi)|^p + \|\nabla\Psi\|^p_{L^p(\mathbf{R}^N)}\Big)^{\frac{1}{p}},$$

Note that ℓ is a continuous linear form on $\widetilde{K}_B$. The number $\ell(\Psi)$ represents the limit at infinity for Ψ (in the sense that $\Psi - \ell(\Psi) \in L^{p*}(\mathbf{R}^N)$). Its kernel is exactly K^0_B.

3.2 The unfolded homogenization result

We derive here the unfolded formulation of the limit problem for $\mathcal{P}_{\varepsilon,\delta}$. In the limit we will observe the contribution of the periodic oscillations as well as the contribution of the perforations.

Here is the main result of this section.

Theorem 3.4. *Assume $1 < p < \infty$, $p^{-1} + p'^{-1} = 1$, m_ε is in $L^1(\Omega)$, $\alpha > 0$ and let $\Omega^*_{\varepsilon,\delta}$ be given by (2.4). Let $A_\varepsilon \in \mathfrak{LL}(\Omega, \mathbf{R}^N, p, \alpha, m_\varepsilon)$ for every $\varepsilon > 0$.*

Assume that there exists measurable maps $m_1 \in L^1(\Omega\times Y)$, $m_2 \in L^1(\Omega\times \mathbf{R}^N)$, $A \in \mathfrak{LL}(\Omega\times Y, \mathbf{R}^N, p, \alpha, m_1)$ and $A_0 \in \mathfrak{LL}(\Omega\times\mathbf{R}^N, \mathbf{R}^N p, \alpha, m_2)$ such that as ε goes to 0, the following convergences hold

$$\begin{aligned} \mathcal{T}_\varepsilon(m_\varepsilon) &\rightarrow m_1 \text{ strongly in } L^1(\Omega\times Y) && (3.7)\\ \delta^N \mathcal{T}_{\varepsilon,\delta}(m_\varepsilon) &\rightarrow m_2 \text{ strongly in } L^1(\Omega\times\mathbf{R}^N) && (3.8)\\ \mathcal{T}_\varepsilon(A_\varepsilon)(x,y) &\rightarrowtail A(x,y), && (3.9)\\ A_{\varepsilon,\delta}(x,z) &\rightarrowtail A_0(x,z), && (3.10) \end{aligned}$$

with

$$(\xi,\eta) \in A_{\varepsilon,\delta}(x,z) \Leftrightarrow (\delta^{-N/p}\xi, \delta^{-N/p'}\eta) \in \mathcal{T}_{\varepsilon,\delta}(A_\varepsilon)(x,z). \qquad (3.11)$$

Assume $f_\varepsilon \to f$ strongly in $W^{-1,p'}(\Omega)$ as $\varepsilon \to 0$.

Then, the set $(u_{\varepsilon,\delta}, d_{\varepsilon,\delta})_{\varepsilon>0}$ of solutions of all the problems $\mathcal{P}_{\varepsilon,\delta}$ is weakly compact in $W^{1,p}_0(\Omega) \times L^{p'}(\Omega;\mathbf{R}^N)$.
For every weak limit point $(u_0, d_0) \in W^{1,p}_0(\Omega) \times L^{p'}(\Omega;\mathbf{R}^N)$, there exists $(\hat{u}, \hat{d}) \in W^{1,p}_{per}(Y) \times L^{p'}(Y;\mathbf{R}^N)$ and $(U, \xi_0) \in L^p(\Omega;\widetilde{K}_B) \times L^{p'}(\Omega;\mathbf{R}^N)$ such that there is a subsequence $(\varepsilon_n, \delta_n)_{n\geq 1}$ with $\varepsilon_n \to 0$, $\delta_n \to 0$ satisfying (3.4)

and

$$
\begin{cases}
u_{\varepsilon_n,\delta_n} \rightharpoonup u_0 \quad \text{weakly in } W_0^{1,p}(\Omega),\\
\mathcal{T}_{\varepsilon_n}(\nabla u_{\varepsilon_n,\delta_n}) \rightharpoonup \nabla u_0 + \nabla_y \hat{u} \quad \text{weakly in } L^p(\Omega \times Y),\\
d_{\varepsilon_n,\delta_n} \rightharpoonup d_0 \quad \text{weakly in } L^{p'}(\Omega; \mathbf{R}^N),\\
\mathcal{T}_{\varepsilon_n}(d_{\varepsilon_n,\delta_n}) \rightharpoonup d_0 + \hat{d} \text{ weakly in } L^{p'}(\Omega \times Y),\\
\mathcal{T}_{\varepsilon_n,\delta_n}(u_{\varepsilon_n,\delta_n}) \rightharpoonup U \quad \text{weakly in } L^p(\Omega; L^p_{loc}(\mathbf{R}^N)),\\
\ell(U) = u_0 \text{ for a.e. } x \in \Omega,\\
\nabla_z\big(\mathcal{T}_{\varepsilon_n,\delta_n}(u_{\varepsilon_n,\delta_n})\big)\, 1_{\frac{1}{\delta}Y} \rightharpoonup \nabla_z U \quad \text{weakly in } L^p(\Omega \times \mathbf{R}^N),\\
\delta_n^{N/p'} \mathcal{T}_{\varepsilon_n,\delta_n}(d_{\varepsilon_n,\delta_n}) \rightharpoonup \xi_0 \quad \text{weakly in } L^{p'}(\Omega \times \mathbf{R}^N),
\end{cases}
\tag{3.12}
$$

as $n \to \infty$, with

$$
\int_Y \hat{d}\ \mathrm{d}y = 0 \text{ and } \int_Y \langle \hat{d}, \nabla_y \varphi \rangle\ \mathrm{d}y = 0 \ \ \forall \varphi \in W^{1,p}_{per}(Y), \tag{3.13}
$$

$$
\int_{\Omega\times(\mathbf{R}^N\setminus\overline{B})} \langle \xi_0, \varphi \nabla_z v \rangle\ \mathrm{d}z = 0 \qquad \forall (\varphi, v) \in W_0^{1,p}(\Omega \times \mathbf{R}^N) \times K_B^0. \tag{3.14}
$$

Moreover,

$$
\int_\Omega \langle d_0, \nabla_x \psi \rangle\ \mathrm{d}x - \frac{k_1}{|Y|} \int_{\Omega\times\partial B} (\xi_0 \cdot \nu_B)\psi\ \mathrm{d}x\ \mathrm{d}\sigma_z = \int_\Omega f\psi\ \mathrm{d}x, \tag{3.15}
$$

for all $\psi \in W_0^{1,p}(\Omega)$, where ν_B is the inward normal vector on ∂B and $\mathrm{d}\sigma_z$ *is its surface measure.*

Furthermore,

$$
\big(\nabla u_0 + \nabla_y \hat{u}, d_0 + \hat{d}\big) \in A(x,y) \text{ for a.e. } (x,y) \in \Omega \times Y,
$$

$$
(k_1 \nabla_z U, \xi_0) \in A_0(x,z) \text{ for a.e. } (x,z) \in \Omega \times (\mathbf{R}^N \setminus \overline{B}).
$$

Remark 14. For the linear case, the proof of theorem 3.4 was given in [12].

Remark 15. The case of the periodic homogenization for similar problems involving pseudo-monotone operators was treated in [6] using a result of [2] which immediately implies that the sequence $(\nabla u_{\varepsilon,\delta})$ converges almost everywhere. In the case of Theorem 3.4, this is not true, and the **crux of the proof** is to overcome this difficulty, using a non-standard form of Minty's method (see Lemma 4.2).

Remark 16. Note that $A_{\varepsilon,\delta}$ belongs to $\mathfrak{LL}(\hat{\Omega}_\varepsilon \times \frac{1}{\delta}Y, \mathbf{R}^N, p, \alpha, \delta^N \mathcal{T}_{\varepsilon,\delta}(m))$. Indeed, first recalling the definition (3.11) of $A_{\varepsilon,\delta}$ we have that for almost every $(x,z) \in \hat{\Omega}_\varepsilon \times \frac{1}{\delta}Y$, $(\xi,\eta) \in A_{\varepsilon,\delta}(x,z)$ iff $(\delta^{-N/p}\xi, \delta^{-N/p'}\eta) \in \mathcal{T}_{\varepsilon,\delta}(A_\varepsilon)(x,z)$. Next, from relation (3.1) in the definition of $\mathfrak{LL}(\Omega, \mathbf{R}^N, p, \alpha, m)$ together with the fact that $\mathcal{T}_{\varepsilon,\delta}(A_\varepsilon)$ belongs to $\mathfrak{LL}(\hat{\Omega}_\varepsilon \times \frac{1}{\delta}Y, \mathbf{R}^N, p, \alpha, \delta^N \mathcal{T}_{\varepsilon,\delta}(m_\varepsilon))$, we deduce that for almost every $(t,u) \in \hat{\Omega}_\varepsilon \times \frac{1}{\delta} Y$, for every $(\xi,\eta) \in A_{\varepsilon,\delta}(t,u)$, we have

$$\alpha\Big(\frac{\|\delta^{-N/p}\xi\|^p}{p} + \frac{\|\delta^{-N/p'}\eta\|^{p'}}{p'}\Big) \leq \langle \delta^{-N/p'}\eta, \delta^{-N/p}\xi\rangle + \mathcal{T}_{\varepsilon,\delta}(m_\varepsilon)(t,u),$$

hence the result by multiplying this inequality by δ^N.
A similar (albeit simpler) result holds for $\mathcal{T}_\varepsilon(A_\varepsilon)$.

Remark 17. Note that if m_ε converges strongly to m in L^1 then convergences (3.7) and (3.8) hold.

Remark 18. For simplicity, we will not relabel the subsequences and denote them all as (ε,δ).

Example 1. **The case of periodic** A_ε. Suppose that A_ε is of the form $A_\varepsilon(x) = B(\frac{x}{\varepsilon})$, with B in $\mathcal{LL}(Y, \mathbf{R}^N, p, \alpha, m_0)$ and extended by Y-periodicity. Then, it easily follows that condition (3.9) is satisfied with $A(x,y) \doteq B(y)$. If furthermore, $B(y)$ is continuous at $y=0$ (in the sense of maximal monotone graphs), then condition (3.10) is satisfied, A_0 being the following graph associated with $B(0)$:

$$(\xi,\eta) \in A_0 \Longleftrightarrow$$
$$\text{there is a sequence } (\delta^{\frac{-N}{p}}\xi_\delta, \eta_\delta) \in B(0) \text{ such that } \lim_{\delta\to 0}(\xi_\delta, \delta^{\frac{N}{p'}}\eta_\delta) = (\xi,\eta). \tag{3.16}$$

This is the normalized asymptote graph of $B(0)$ because the factors $\delta^{\frac{-N}{p}}$ and $\delta^{\frac{N}{p'}}$ take into account the growth condition satisfied by B.

Example 2. The case of the **oscillating p-Laplacian** corresponds to the single-valued $B(y)$ given by $\eta = b(y)|\xi|^{p-2}\xi$, where b is measurable, Y-periodic, bounded below by α and above by $\alpha^{1-p'}$. Then, one easily checks that $A(x,y)(\xi) \doteq b(y)|\xi|^{p-2}\xi$. If b is continuous at $y=0$, the maximal monotone graph A_0 is also single-valued and given by $A_0(\xi) \doteq b(0)|\xi|^{p-2}\xi$.

We begin the proof of theorem 3.4 with two elementary lemmas.

Lemma 3.5. *Let $\delta_0 > 0$. Then, for $N > p$, the set*

$$\bigcup_{0<\delta<\delta_0} \{\phi \in W^{1,p}_{per}(Y); \phi = 0 \ \ on \ \ \delta B\}$$

is dense in $W^{1,p}_{per}(Y)$.

Proof. Let $\psi \in C^\infty_{per}(\overline{Y})$ be fixed. Consider $\phi \in W^{1,p}_{per}(Y)$ smooth such that $supp(\phi) \subset B(O,2)$, $\phi = 1$ on $B(O,1)$. Let ϕ_δ be such that $\phi_\delta(x) = \phi(\frac{x}{\delta})$, then $|\nabla\phi_\delta| \leq \frac{C}{\delta}$. Define $\Phi_\delta = (1-\phi_\delta)\,\psi$, we see that

$$||\Phi_\delta - \psi||_{L^p(Y)} + ||\nabla\Phi_\delta - \nabla\psi||_{L^p(Y)} \leq \int_{2\delta B(O,1)} |\psi|^p \, \mathrm{d}y + \int_{2\delta B(O,1)} |\nabla\psi|^p \, \mathrm{d}y$$

$$+ \int_{2\delta B(O,1)} |\nabla \phi_\delta|^p |\psi|^p \, \mathrm{d}y.$$

Using the definition of ϕ_δ, one gets

$$\int_{2\delta B(O,1)} |\nabla \phi_\delta|^p |\psi|^p \, \mathrm{d}y \leq C\delta^{N-p} \|\psi\|^p_{L^\infty(Y)},$$

hence, we deduce that

$$\Phi_\delta \to \psi \quad \text{strongly in } W^{1,p}_{per}(Y) \text{ as } \delta \to 0.$$

Since $W^{1,p}_{per}(Y)$ is the closure of $C^\infty_{per}(\overline{Y})$ in the $W^{1,p}$-norm, a density argument completes the proof. □ □

Lemma 3.6. *Assume* (3.4), *and let v be in K_B. For δ small enough, set*

$$w_{\varepsilon,\delta}(x) = v(B) - v\Big(\frac{1}{\delta}\Big\{\frac{x}{\varepsilon}\Big\}_Y\Big) \quad \textit{for } x \in \mathbb{R}^N.$$

Then, for $0 < k_1 < \infty$,

$$w_{\varepsilon,\delta} \rightharpoonup v(B) \quad \textit{weakly in } W^{1,p}_{loc}(\mathbb{R}^N) \textit{ (hence weakly in } W^{1,p}(\Omega)). \tag{3.17}$$

For $k_1 = 0$, $w_{\varepsilon,\delta} \to v(B)$ strongly in $W^{1,p}_{loc}(\mathbb{R}^N)$ (hence strongly in $W^{1,p}(\Omega)$).

Furthermore, $w_{\varepsilon,\delta}$ vanishes outside $\Omega^*_{\varepsilon,\delta}$ and

$$\mathcal{T}_{\varepsilon,\delta}(\nabla w_{\varepsilon,\delta}) = -\frac{1}{\varepsilon\delta}\nabla_z v \quad \textit{in} \quad \widehat{\Omega}_\varepsilon \times \frac{1}{\delta}Y. \tag{3.18}$$

The proof of this lemma is similar to the proof of Lemma 4.2 from [12].

The proof of Theorem 3.4 uses several intermediate results. Throughout the proof, to simplify the notations, we only write ε, δ instead of ε_n, δ_n.

The first step is

Proposition 3.7. *Under assumptions of Theorem 3.4, the set of solutions $(u_{\varepsilon,\delta}, d_{\varepsilon,\delta})_{\varepsilon>0}$ of the problem $\mathcal{P}_{\varepsilon,\delta}$ is weakly compact in $W^{1,p}_0(\Omega) \times L^{p'}(\Omega; \mathbf{R}^N)$. For every weak limit point $(u_0, d_0) \in W^{1,p}_0(\Omega) \times L^{p'}(\Omega; \mathbf{R}^N)$, there exists $(\hat{u}, \hat{d}) \in W^{1,p}_{per}(Y) \times L^{p'}(Y; \mathbf{R}^N)$ and $(U, \xi_0) \in L^p(\Omega; \widetilde{K}_B) \times L^{p'}(\Omega; \mathbf{R}^N)$ and a subsequence $(\varepsilon_n, \delta_n)_{n\geq 1}$ going to $(0,0)$ and satisfying (3.4) such that, as $n \to \infty$,* (3.12), (3.13), (3.14) *and* (3.15) *are satisfied.*

Proof. The proof (given here for the case $k_1 > 0$) is broken down in four steps.

– **Step one**: weak compactness of the unfolded sequences $\big(\mathcal{T}_\varepsilon(\nabla u_{\varepsilon,\delta}), \mathcal{T}_\varepsilon(d_{\varepsilon,\delta})\big)$.

From inequality (3.1) and taking $u_{\varepsilon,\delta}$ as a test function in equation (3.2), we deduce that the sequence $(u_{\varepsilon,\delta})_\varepsilon$ is bounded in $W^{1,p}(\Omega)$ and the sequence $(d_{\varepsilon,\delta})_\varepsilon$ is bounded in $L^{p'}(\Omega;\mathbf{R}^N)$. Hence they are weakly compact.

Assume now $u_{\varepsilon,\delta} \rightharpoonup u_0$ weakly in $W_0^{1,p}(\Omega)$ and $d_{\varepsilon_n,\delta} \rightharpoonup d_0$ in $L^{p'}(\Omega;\mathbf{R}^N)$. By Theorem 2.2 (6), up to a subsequence, there is some $\hat{u} \in L^p\big(\Omega; W^{1,p}_{\rm per}(Y)\big)$ such that

$$\mathcal{T}_\varepsilon(\nabla u_{\varepsilon,\delta}) \rightharpoonup \nabla_x u_0 + \nabla_y \hat{u}$$

weakly in $L^p(\mathbf{R}^N \times Y;\mathbf{R}^N)$. Consequently, the sequence $\mathcal{T}_\varepsilon(d_{\varepsilon,\delta})$ is bounded in the space $L^{p'}(\mathbf{R}^N \times Y)$. Using Theorem 2.2 (3), again up to a subsequence, there exists some $\eta \in L^{p'}(\mathbf{R}^N \times Y)$, so that $\mathcal{T}_\varepsilon(d_{\varepsilon,\delta}) \rightharpoonup \eta$ and $d_0 = \frac{1}{|Y|}\int_Y \eta(y)\ {\rm d}y$. So, setting $\hat{d} = \eta - d_0$, one has $\displaystyle\int_Y \hat{d}\ {\rm d}y = 0$ and

$$\mathcal{T}_\varepsilon(d_{\varepsilon,\delta}) \rightharpoonup d_0 + \hat{d} \text{ weakly in } L^{p'}(\mathbf{R}^N \times Y).$$

– **Step two**: the equation satisfied by the corresponding weak limits.

Let $\varphi \in \mathcal{D}(\Omega)$ and $\psi \in C^\infty(\mathbf{R}^N)$, such that ψ is Y-periodic and vanishes on a neighborhood of the origin. The function $\varphi_\varepsilon(x) = \varepsilon\varphi(x)\psi(x/\varepsilon)$ is a valid test function for (3.2). As $\varepsilon \to 0$, by Theorem 2.2, $\varphi_\varepsilon \rightharpoonup 0$ weakly in $W^{1,p}(\Omega)$ and $\mathcal{T}_\varepsilon(\nabla\varphi_\varepsilon) \to \varphi(x)\nabla\psi(y)$ strongly in $L^p(\Omega; W^{1,p}_{\rm per}(Y))$, so that going to the limit in

$$\frac{1}{|Y|}\int_{\mathbf{R}^N\times Y} \langle \mathcal{T}_\varepsilon(d_{\varepsilon,\delta}), \mathcal{T}_\varepsilon(\nabla\varphi_\varepsilon)\rangle\ {\rm d}x\ {\rm d}y = \int_\Omega f_{\varepsilon,\delta}\varphi_\varepsilon\ {\rm d}x.$$

gives

$$\frac{1}{|Y|}\int_{\Omega\times Y} \langle d_0(x) + \hat{d}(x,y), \varphi(x)\nabla\psi(y)\rangle\ {\rm d}x\ {\rm d}y = 0.$$

Since φ is arbitrary and $N > p$ and making use of Lemma 3.5, we deduce that for almost every $x \in \Omega$, for every $\psi \in W^{1,p}_{\rm per}(Y)$,

$$\frac{1}{|Y|}\int_Y \langle \hat{d}(x,y), \nabla_y\psi(y)\rangle\ {\rm d}y = 0,$$

i.e. $-{\rm div}\ \hat{d}(x,\cdot) = 0$ in $(\mathcal{C}^\infty_{\rm per})'(Y)$. This is (3.13), the first equation of the unfolded formulation for the limit problem.

– **Step three**: weak convergence (up to a subsequence) of the unfolded sequences $(\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta})), (\nabla_z\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}))$ and $(\delta^{N/p'}\mathcal{T}_{\varepsilon,\delta}(d_{\varepsilon,\delta}))$.

Using property 7 of Theorem 2.7, we know that there exists a U in $L^p(\Omega; L^p_{loc}(\mathbf{R}^N))$ such that (again, up to a subsequence),

$$\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) \rightharpoonup U \quad \text{weakly in } L^p(\Omega; L^p_{loc}(\mathbf{R}^N)). \tag{3.19}$$

Furthermore, Proposition 2.5 gives

$$M_Y^\varepsilon(u_{\varepsilon,\delta})1_{\frac{1}{\delta}Y} \to u_0 \ \text{ strongly in } L^p(\Omega; L^p_{loc}(\mathbf{R}^N)). \tag{3.20}$$

On the other hand, using property 7) of Theorem 2.7 we deduce that there exists a W in $L^p(\Omega; L^{p^*}(\mathbf{R}^N))$ with $\nabla_z W$ in $L^p(\Omega \times \mathbf{R}^N)$ with

$$M_Y^\varepsilon(u_{\varepsilon,\delta})1_{\frac{1}{\delta}Y} - \mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) \rightharpoonup W \tag{3.21}$$

weakly in $L^p(\Omega; L^{p^*}(\mathbf{R}^N))$. From (3.19), (3.20) and (3.21), it follows that

$$U + W = u_0 \quad \text{and } \nabla_z U = -\nabla_z W.$$

Finally, by Theorem 2.7 (7) again, the following weak convergence in $L^p(\Omega \times \mathbf{R}^N)$ holds:

$$\nabla_z\big(\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta})\big)\, 1_{\frac{1}{\delta}Y} = \varepsilon\delta\mathcal{T}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta}) \rightharpoonup \nabla_z U. \tag{3.22}$$

Recalling Definition 2.6, we know that $\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) = 0 \quad \text{in } \Omega \times B$, so that using (3.19), we obtain that

$$U = 0 \ \text{ on } \ \Omega \times B.$$

Therefore, $W = u_0 - U$ belongs to $L^p(\Omega; K_B)$ and this implies that

$$W(\cdot, B) = \ell(U) = u_0 \text{ in } L^p(\Omega).$$

Now set

$$\xi_{\varepsilon,\delta} = \delta^{N/p'}\mathcal{T}_{\varepsilon,\delta}(d_{\varepsilon,\delta}). \tag{3.23}$$

By definition,

$$\xi_{\varepsilon,\delta} \in A_{\varepsilon,\delta}(\delta^{\frac{N}{p}}\mathcal{T}_{\varepsilon,\delta}(\nabla u_{\varepsilon,\delta})). \tag{3.24}$$

Since the sequence $(d_{\varepsilon,\delta})$ is bounded in $L^{p'}(\Omega\times\mathbf{R}^N)$, Theorem 2.7 (3) implies that there exists ξ_0 such that (up to a subsequence),

$$\xi_{\varepsilon,\delta} \rightharpoonup \xi_0 \text{ weakly in } L^{p'}(\Omega \times \mathbf{R}^N). \tag{3.25}$$

Furthermore, using (3.24) together with Theorem 2.7 (5), it follows that

$$\xi_{\varepsilon,\delta} \in A_{\varepsilon,\delta}\Big(\frac{\delta^{N/p-1}}{\varepsilon}\nabla_z\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta})\Big). \tag{3.26}$$

– **Step four**: the equations satisfied by the corresponding weak limits.
In order to describe the contribution of the perforations, we use the functions $w_{\varepsilon,\delta}$ introduced in Lemma 3.6 associated with $v \in K_B \cap \mathcal{D}(\mathbf{R}^N)$. Recall that these functions vanish on the holes. For ψ in $\mathcal{D}(\Omega)$, using $w_{\varepsilon,\delta}\,\psi$ as a test function in (3.2), one has

$$\int_{\Omega^*_{\varepsilon,\delta}} \langle d_{\varepsilon,\delta}, \psi\ \nabla w_{\varepsilon,\delta} + w_{\varepsilon,\delta}\ \nabla\psi\rangle \ \mathrm{d}x = \int_{\Omega^*_{\varepsilon,\delta}} f_\varepsilon\ w_{\varepsilon,\delta}\ \psi\ \mathrm{d}x.$$

For simplicity, let us denote by E^1_ε and E^2_ε the terms:

$$\begin{aligned} E^1_\varepsilon &= \int_{\Omega^*_{\varepsilon,\delta}} \langle d_{\varepsilon,\delta}, \nabla w_{\varepsilon,\delta}\rangle \, \psi \ \mathrm{d}x, \\ E^2_\varepsilon &= \int_{\Omega^*_{\varepsilon,\delta}} \langle d_{\varepsilon,\delta}, \nabla \psi\rangle \, w_{\varepsilon,\delta} \ \mathrm{d}x. \end{aligned}$$

First, recalling convergence (3.17) together with the strong convergence of the sequence (f_ε), we know that

$$\lim_{\varepsilon\to 0} \left(E^1_\varepsilon + E^2_\varepsilon\right) = v(B) \int_\Omega f\psi \ \mathrm{d}x. \tag{3.27}$$

We study separately each term of the left-hand side of the previous expression. Let us start with the second one. From the strong convergence of $(w_{\varepsilon,\delta})$ in L^p_{loc} to $v(B)$ (which follows from (3.17)), together with the weak convergence of $(d_{\varepsilon,\delta})$ in $L^{p'}$, we deduce that

$$\lim_{\varepsilon\to 0} E^2_\varepsilon = v(B) \int_\Omega \langle d_0, \nabla\psi\rangle \ \mathrm{d}x. \tag{3.28}$$

We now apply the unfolding operator $\mathcal{T}_{\varepsilon,\delta}$ to the the first term (equality follows from the fact that ψ has a compact support in Ω):

$$\begin{aligned} E^1_\varepsilon &= \frac{\delta^N}{|Y|} \int_{\Omega\times\frac{1}{\delta}Y} \langle \mathcal{T}_{\varepsilon,\delta}(d_{\varepsilon,\delta}), \mathcal{T}_{\varepsilon,\delta}(\nabla w_{\varepsilon,\delta})\rangle \ \mathcal{T}_{\varepsilon,\delta}(\psi) \ \mathrm{d}x \ \mathrm{d}z \\ &= \frac{1}{|Y|} \int_{\Omega\times\frac{1}{\delta}Y} \langle \delta^{N/p'} \mathcal{T}_{\varepsilon,\delta}(d_{\varepsilon,\delta}), \delta^{N/p}\mathcal{T}_{\varepsilon,\delta}(\nabla w_{\varepsilon,\delta})\rangle \ \mathcal{T}_{\varepsilon,\delta}(\psi) \ \mathrm{d}x \ \mathrm{d}z. \end{aligned}$$

With our notations, E^1_ε can be rewritten as

$$E^1_\varepsilon = \frac{1}{|Y|} \int_{\Omega\times\frac{1}{\delta}Y} \langle \xi_{\varepsilon,\delta}, \frac{\delta^{N/p-1}}{\varepsilon}(-\nabla_z v)\rangle \ \mathcal{T}_{\varepsilon,\delta}(\psi) \ \mathrm{d}x \ \mathrm{d}z.$$

On the other hand, using the inequality

$$||\mathcal{T}_{\varepsilon,\delta}(\psi) - \psi||_{L^\infty(\widehat{\Omega}_\varepsilon\times\frac{1}{\delta}Y)} \leq C\,\varepsilon ||\nabla\psi||_{L^\infty(\Omega)},$$

we deduce that the following convergence holds uniformly

$$\mathcal{T}_{\varepsilon,\delta}(\psi) \to \psi. \tag{3.29}$$

Putting now (3.4), (3.25) and (3.29) together, we obtain that

$$\lim_{\varepsilon\to 0} E^1_\varepsilon = \frac{k_1}{|Y|} \int_{\Omega\times\mathbf{R}^N} \langle \xi_0, (-\nabla_z v)\rangle \ \psi \ \mathrm{d}x \ \mathrm{d}z. \tag{3.30}$$

From (3.27), (3.28) and (3.30), the limit equation becomes:

$$v(B)\int_{\Omega\times Y}\langle d_0,\nabla\psi\rangle\ \mathrm{d}x-\frac{k_1}{|Y|}\int_{\Omega\times\mathbf{R}^N\setminus B}\langle\xi_0,\psi\nabla_z v\rangle\ \mathrm{d}x\ \mathrm{d}z$$
$$=v(B)\int_\Omega f\psi\ \mathrm{d}x, \tag{3.31}$$

which, by density, holds true for all $\psi\in W_0^{1,p}(\Omega)$ and $v\in K_B$. Choosing $v(B)=0$ in (3.31) gives (3.14). This implies that

$$-\mathrm{div}_z\xi_0=0 \text{ in } \mathcal{D}'(\Omega\times(\mathbf{R}^N\setminus\overline{B})). \tag{3.32}$$

Now, Stoke's formula associated with (3.32), gives for every v in K_B

$$\int_{\mathbf{R}^N\setminus\overline{B}}\langle\xi_0,\nabla_z v\rangle\ \mathrm{d}z=v(B)\int_{\partial B}(\xi_0\ \cdot\ \nu_B)\ \mathrm{d}\sigma_z. \tag{3.33}$$

Multiplying (3.33) by ψ, integrating over Ω and combining with (3.31) gives (3.15). □

□

We introduce a new functional space $K^p(\Omega;B)$ to which the function W obtained in (3.21) already belongs:

$$K^p(\Omega;B)\ =\ \{V\in L^p(\Omega;K_B)\text{ such that }V(x,B)\in W_0^{1,p}(\Omega)\}, \tag{3.34}$$

with the norm:

$$\|V\|^p_{K^p(\Omega;B)}=\|V\|^p_{L^p(\Omega\times\mathbf{R}^N)}+\|\nabla_z V\|^p_{L^p(\Omega;L^{p^*}(\mathbf{R}^N\setminus\overline{B}))}+\|\nabla_x V(.,B)\|^p_{L^p(\Omega)}. \tag{3.35}$$

Proposition 3.8. *For all $V\in K^p(\Omega,B)$,*

$$\frac{1}{|Y|}\int_{\Omega\times Y}\langle d_0,\nabla_x V(x,B)\rangle\ \mathrm{d}x\ \mathrm{d}y-\frac{k_1}{|Y|}\int_{\Omega\times\mathbf{R}^N\setminus\overline{B}}\langle\xi_0,\nabla_z V(x,B)\rangle\ \mathrm{d}x\ \mathrm{d}z$$
$$=\int_\Omega fV(x,B)\ \mathrm{d}x. \tag{3.36}$$

Proof. For almost every $x\in\Omega$, $V(x,\cdot)$ belongs to K_B, so by (3.33), one gets

$$\int_{\mathbf{R}^N\setminus\overline{B}}\langle\xi_0,\nabla_z V(x,z)\rangle\ \mathrm{d}z=V(x,B)\int_{\partial B}(\xi_0\ \cdot\ \nu_B)\ \mathrm{d}\sigma_z. \tag{3.37}$$

Denote by ψ the function $V(x,B)$ and integrate (3.37) over Ω. Combining the result with (3.15) gives the required conclusion (3.36). □

Proposition 3.9 ("convergence of the energy")**.** *Under the previous hypotheses, the following convergence holds (this extends the convergence of the energy of the linear symetric case). For every ϕ in $\mathcal{D}(\Omega)$,*

$$\lim_{\varepsilon\to\infty}\int_\Omega \langle d_{\varepsilon,\delta}, \nabla(u_{\varepsilon,\delta}\phi)\rangle \ \mathrm{d}x = \int_\Omega \langle d_0, \nabla\phi\rangle \ u_0 \ \mathrm{d}x + \int_\Omega \langle d_0, \nabla u_0\rangle \ \phi \ \mathrm{d}x + \frac{k_1}{|Y|}\int_{\Omega\times\mathbf{R}^N\setminus\overline{B}} \langle \xi_0, \nabla_z U_0\rangle \ \phi \ \mathrm{d}x \ \mathrm{d}z. \tag{3.38}$$

Proof. : Let ϕ be arbitrary in $\mathcal{D}(\Omega)$ and apply Proposition 3.8 for the function $\phi(x)W(x,z)$ (for W given by (3.21)), which belongs to $K^p(\Omega, B)$ (since W does). Recall that by (3.2), $W(x,B) = u_0(x)$ and that $\nabla_z W = -\nabla_z U$. It follows

$$\int_\Omega \langle d_0, \nabla\phi\rangle \ u_0 \ \mathrm{d}x + \int_\Omega \langle d_0, \nabla u_0\rangle \ \phi \ \mathrm{d}x + \frac{k_1}{|Y|}\int_{\Omega\times\mathbf{R}^N\setminus\overline{B}} \langle \xi_0, \nabla_z U_0\rangle \ \phi \ \mathrm{d}x \ \mathrm{d}z = \int_\Omega f u_0 \phi \ \mathrm{d}x. \tag{3.39}$$

On the other hand, using $(u_{\varepsilon,\delta}\phi)$ as a test function in (3.2) yields

$$\int_\Omega \langle d_{\varepsilon,\delta}, \nabla(u_{\varepsilon,\delta}\phi)\rangle \ \mathrm{d}x = \int_\Omega f_\varepsilon u_{\varepsilon,\delta}\phi \ \mathrm{d}x. \tag{3.40}$$

Since $u_{\varepsilon,\delta} \rightharpoonup u_0$ weakly in $W^{1,p}$ and $(f_\varepsilon) \to f$ strongly in $W^{-1,p'}$, equality (3.39) implies

$$\lim_{n\to\infty}\int_\Omega \langle d_{\varepsilon,\delta}, \nabla(u_{\varepsilon,\delta}\phi)\rangle \ \mathrm{d}x = \int_\Omega \langle d_0, \nabla\phi\rangle \ u_0 \ \mathrm{d}x + \int_\Omega \langle d_0, \nabla u_0\rangle \ \phi \ \mathrm{d}x + \frac{k_1}{|Y|}\int_{\Omega\times\mathbf{R}^N\setminus\overline{B}} \langle \xi_0, \nabla_z U_0\rangle \ \phi \ \mathrm{d}x \ \mathrm{d}z.$$

Moreover, expanding the left-hand side of (3.40), we have

$$\int_\Omega \langle d_{\varepsilon,\delta}, \nabla(u_{\varepsilon,\delta}\phi)\rangle \ \mathrm{d}x = \int_\Omega \langle d_{\varepsilon,\delta}, \nabla u_{\varepsilon,\delta}\rangle \ \phi \ \mathrm{d}x + \int_\Omega \langle d_{\varepsilon,\delta}, \nabla\phi\rangle \ u_{\varepsilon,\delta} \ \mathrm{d}x. \tag{3.41}$$

Since $(d_{\varepsilon,\delta})$ converges weakly in $L^{p'}$ and $(u_{\varepsilon,\delta})$ converges strongly in L^p (by Rellich's theorem), one obtains

$$\lim_{\varepsilon\to 0}\int_\Omega \langle d_{\varepsilon,\delta}, \nabla\phi\rangle \ u_{\varepsilon,\delta} \ \mathrm{d}x = \int_\Omega \langle d_0, \nabla\phi\rangle \ u_0 \ \mathrm{d}x. \tag{3.42}$$

Combining with (3.39), (3.41) and (3.42), gives (3.38) and concludes the proof. □ □

Equality (3.38) is **the key** to showing that the limit functions in Theorem 3.4 belong to the appropriate limit maximal monotone graphs. This is stated in the next proposition.

Proposition 3.10. *Under the assumptions of Theorem 3.4, the following inclusions hold:*

$$\big(\nabla u_0 + \nabla_y \hat{u}, d_0 + \hat{d}\big) \in A(x, y) \text{ for a.e. } (x, y) \in \Omega \times Y, \tag{3.43}$$

$$(k_1 \nabla_z U_0, \xi_0) \in A_0(x, z) \text{ for a.e. } (x, z) \in \Omega \times (\mathbf{R}^N \setminus \overline{B}).$$

Proof. The idea of the proof is to separate the contributions in the variables y and z. This is achieved by splitting the constant 1 in two functions as follows.

For each δ, let (v_δ) be a function in $K_B \cap \mathcal{D}(\mathbf{R}^N)$ such that $v_\delta \geq 0$, $v_\delta(B) \equiv 1$ and which increases to 1 as δ decreases to 0. Assume furthermore that the support of each v_δ is included in $\frac{1}{\sqrt{\delta}} Y$. For δ small enough, set

$$\tilde{v}_{\varepsilon,\delta}(x) = v_\delta\Big(\frac{1}{\delta}\Big\{\frac{x}{\varepsilon}\Big\}_Y\Big) \quad \text{for } x \in \mathbb{R}^N, \text{ and } \tilde{w}_{\varepsilon,\delta} = 1 - v_{\varepsilon,\delta}.$$

It is easy to check that

$$\tilde{v}_{\varepsilon,\delta} \to 0 \quad \text{ and } \tilde{w}_{\varepsilon,\delta} \to 1 \text{ a.e. in } \Omega.$$

(this follows from the fact that the measure of support of v_δ is a $o(1/\delta^N)$)). Furthermore, $\tilde{w}_{\varepsilon,\delta}$ vanishes in the holes $\Omega \setminus \Omega^*_{\varepsilon,\delta}$ and

$$\mathcal{T}_{\varepsilon,\delta}(\tilde{v}_{\varepsilon,\delta}) = v_\delta, \tag{3.44}$$

which therefore increases to 1 in $\mathbf{R}^N$.

Choose some non negative ϕ in $\mathcal{D}(\Omega)$ (i.e. $\phi \in \mathcal{D}^+(\Omega)$), and write

$$\begin{aligned}\int_\Omega \langle d_{\varepsilon,\delta}, \nabla u_{\varepsilon,\delta}\rangle \, \phi \, \mathrm{d}x \;&=\; \int_\Omega \langle d_{\varepsilon,\delta}, \tilde{w}_{\varepsilon,\delta} \nabla u_{\varepsilon,\delta}\rangle \, \phi \, \mathrm{d}x \\ &\quad + \int_\Omega \langle d_{\varepsilon,\delta}, \tilde{v}_{\varepsilon,\delta} \nabla u_{\varepsilon,\delta}\rangle \, \phi \, \mathrm{d}x.\end{aligned} \tag{3.45}$$

Consider each term of the right-hand side of the previous equality. The first one is unfolded with $\mathcal{T}_\varepsilon$. Since ϕ has a compact support in Ω, for ε small enough this gives the equality

$$\begin{aligned}&\int_\Omega \langle d_{\varepsilon,\delta}, \nabla u_{\varepsilon,\delta}\rangle \, \phi \, \tilde{w}_{\varepsilon,\delta} \, \mathrm{d}x = \\ &\frac{1}{|Y|} \int_{\Omega \times Y} \langle \mathcal{T}_\varepsilon(d_{\varepsilon,\delta}), \mathcal{T}_\varepsilon(\nabla u_{\varepsilon,\delta})\rangle \, \mathcal{T}_\varepsilon(\phi) \mathcal{T}_\varepsilon(\tilde{w}_{\varepsilon,\delta}) \, \mathrm{d}x \, \mathrm{d}y.\end{aligned} \tag{3.46}$$

Similarly, the second term of the right-hand side of (3.45) is unfolded with $\mathcal{T}_{\varepsilon,\delta}$. Recalling the definition (3.23) of $\xi_{\varepsilon,\delta}$ together with (3.44) we get:

$$\int_\Omega \langle d_{\varepsilon,\delta}, \nabla u_{\varepsilon,\delta}\rangle \; \phi \, \tilde{v}_{\varepsilon,\delta} \; \mathrm{d}x =$$
$$\frac{1}{|Y|}\int_{\Omega\times\frac{1}{\delta}Y} \langle \xi_{\varepsilon,\delta}, \frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\nabla_z \mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta})\rangle \; \mathcal{T}_{\varepsilon,\delta}(\phi) v_\delta \; \mathrm{d}x \; \mathrm{d}z. \tag{3.47}$$

We can now rewrite (3.45) as

$$\int_\Omega \langle d_{\varepsilon,\delta}, \nabla u_{\varepsilon,\delta}\rangle \; \phi \; \mathrm{d}x = \tag{3.48}$$
$$\frac{1}{|Y|}\int_{\Omega\times Y} \langle \mathcal{T}_\varepsilon(d_{\varepsilon,\delta}), \mathcal{T}_\varepsilon(\nabla u_{\varepsilon,\delta})\rangle \; \mathcal{T}_\varepsilon(\phi)\mathcal{T}_\varepsilon(\tilde{w}_{\varepsilon,\delta}) \; \mathrm{d}x \; \mathrm{d}y \; +$$
$$\frac{1}{|Y|}\int_{\Omega\times\mathbf{R}^N} \langle \xi_{\varepsilon,\delta}, \frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\nabla_z \mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta})\rangle \; \mathcal{T}_{\varepsilon,\delta}(\phi) v_\delta \; \mathrm{d}x \; \mathrm{d}z.$$

Consider the measure space $\mathcal{O}$ which is the disjoint union of $\Omega\times Y$ and of $\Omega\times\mathbf{R}^N$ with the standard Lebesgue measure M (in theory, we should use two distinct copies of Ω!). In the space $L^p(O;\mathbf{R}^N)$ and its dual space $L^{p'}(O;\mathbf{R}^N)$, consider the sequences

$$\begin{aligned}\alpha_{\varepsilon,\delta} &= \Big(\mathcal{T}_\varepsilon(\nabla u_{\varepsilon,\delta}) \;,\; \frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\nabla_z \mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta})\Big),\\ \beta_{\varepsilon,\delta} &= \Big(\mathcal{T}_\varepsilon(d_{\varepsilon,\delta}), \xi_{\varepsilon,\delta}\Big).\end{aligned}$$

By Lemma 3.7 together with Theorem 2.7, we know that

$$\begin{aligned}\alpha_{\varepsilon,\delta} &\rightharpoonup \alpha_0 \doteq \Big(\nabla_x u_0 + \nabla_y \hat{u}, k_1\nabla_z U\Big) \text{ weakly in } L^p(\Omega\times Y)\times L^p(\Omega\times\mathbf{R}^N),\\ \beta_{\varepsilon,\delta} &\rightharpoonup \beta_0 \doteq \Big(d_0+\hat{d}, \xi_0\Big) \text{ weakly in } L^{p'}(\Omega\times Y)\times L^{p'}(\Omega\times\mathbf{R}^N).\end{aligned}$$

On the other hand, (3.13), and (3.48) read as

$$\lim_{\varepsilon\to 0}\Big(\int_{\Omega\times Y} \langle \mathcal{T}_\varepsilon(d_{\varepsilon,\delta}), \mathcal{T}_\varepsilon(\nabla u_{\varepsilon,\delta})\rangle \; \mathcal{T}_\varepsilon(\phi)\mathcal{T}_\varepsilon(\tilde{w}_{\varepsilon,\delta}) \; \mathrm{d}x \; \mathrm{d}y \; +$$
$$\int_{\Omega\times\mathbf{R}^N} \langle \xi_{\varepsilon,\delta}, \frac{\delta^{\frac{N}{p}-1}}{\varepsilon}\nabla_z \mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta})\rangle \; \mathcal{T}_{\varepsilon,\delta}(\phi) v_\delta \; \mathrm{d}x \; \mathrm{d}z\Big) =$$
$$\lim_{\varepsilon\to 0}\int_{\mathcal{O}} \alpha_{\varepsilon,\delta}\beta_{\varepsilon,\delta}\Phi_{\varepsilon,\delta}\, dM =$$
$$\int_{\mathcal{O}} \alpha_0\beta_0\Phi_0 \, dM = \tag{3.49}$$
$$\int_\Omega \langle d_0+\hat{d}, \nabla_x u_0 + \nabla_y \hat{u}\rangle \; \phi \; \mathrm{d}x \; \mathrm{d}y$$
$$+\int_{\Omega\times\mathbf{R}^N\setminus B} \langle \xi_0, k_1\nabla_z U\rangle \; \phi \; \mathrm{d}x \; \mathrm{d}z,$$

where $\Phi_{\varepsilon,\delta}$ represents the pair $(\mathcal{T}_\varepsilon(\phi)\mathcal{T}_\varepsilon(\tilde{w}_{\varepsilon,\delta}), \mathcal{T}_{\varepsilon,\delta}(\phi)v_\delta)$ and Φ_0 the pair (ϕ,ϕ). Moreover, for a.e. $((x_1,y),(x_2,z))$ in $\mathcal{O}$, $(\alpha_{\varepsilon,\delta},\beta_{\varepsilon,\delta})$ belongs to the multivalued graph $\mathcal{B}_\varepsilon((x_1,y),(x_2,z))$, where

$$\mathcal{B}_\varepsilon = \Big(\mathcal{T}_\varepsilon A_\varepsilon, A_{\varepsilon,\delta}\Big).$$

Since $\mathcal{T}_\varepsilon(\phi)\mathcal{T}_\varepsilon(\tilde{w}_{\varepsilon,\delta})$ is bounded and converges to ϕ a.e. in $\Omega\times Y$ and $\mathcal{T}_{\varepsilon,\delta}(\phi)v_\delta$ is bounded and converges to ϕ a.e. in $\Omega\times\mathbf{R}^N$ respectively, and since the function 1 can be approximated a.e. in Ω by a bounded sequence of functions such as ϕ, applying Proposition 4.10 of the appendix completes the proof. $\square$ $\square$

3.3 The standard form for the limit problem

In this section, we prove that the unfolded problem is well-posed and we give the formulation in terms of the macroscopic solution alone.

But let us first recall some properties of the homogenized graph in the bulk (see [16]).

Proposition 3.11. *For $\alpha>0$, $m\in L^1(\Omega)$ and $B\in\mathfrak{LL}(Y,\mathbf{R}^N,p,\alpha,\tilde{m})$, define*

$$\begin{aligned} B_{hom} = \Big\{ &(\eta,\xi)\in\mathbf{R}^N\times\mathbf{R}^N : \exists(\hat{u},\hat{d})\in W^{1,p}_{\text{per}}(Y)\times L^{p'}(Y;\mathbf{R}^N),\\ &\frac{1}{|Y|}\int_Y \hat{u}(y)\ \mathrm{d}y=0,\ \frac{1}{|Y|}\int_Y \hat{d}(y)\ \mathrm{d}y=0,\\ &\big(\eta+\nabla\hat{u}(y),\xi+\hat{d}(y)\big)\in B(y)\ a.e.\ \ y\in Y,\\ &\int_Y\langle\hat{d},\nabla_y\varphi\rangle\ \mathrm{d}y=0\ \forall\ \varphi\in W^{1,p}_{per}(Y)\Big\}. \end{aligned} \tag{3.50}$$

Then, B_{hom} belongs to $\mathfrak{M}(\mathbf{R}^N\times\mathbf{R}^N)$, and for every $(\eta,\xi)\in B_{hom}$,

$$\alpha\Big(\frac{\|\eta\|^p}{p}+\frac{\|\xi\|^{p'}}{p'}\Big)\le\langle\xi,\eta\rangle+\frac{1}{|Y|}\int_Y\tilde{m}(y)\,y. \tag{3.51}$$

The above definition can be applied for almost every $x\in\Omega$ to the graphs $A(x,\cdot)$, to give maximal monotone graphs $A_{hom}(x)$ which belong to $\mathfrak{LL}(\Omega,\mathbf{R}^N,p,\alpha,\overline{m})$, where

$$\overline{m}(x)=\frac{1}{|Y|}\int_Y m(x,y)\,dy.$$

Remark 19. The map $x\mapsto A_{hom}(x)$ so defined is the H-limit (see [23] and [22]) of the sequence of graphs A_ε. Our result is in accordance with the fact that $\mathfrak{LL}(\Omega,\mathbf{R}^N,p,\alpha,m)$ is closed under H-limit (see [9]).

Proposition 3.12. $\big(\nabla_x u_0(x), d_0(x)\big)$ *belongs to* $A_{hom}(x)$ *for a.e.* $x \in \Omega$ *.*

Proof. It follows directly from Definition (3.50) in view of (3.13) and (3.43). □ □

To express the second integral of (3.15) in terms of u_0 alone, we need another "local problem", which, in this case, is posed on $\mathbf{R}^N \setminus \overline{B}$.

Proposition 3.13. *For every* $\rho \in \mathbf{R}$*, the following problem admits a solution (not necessarly unique):*

$$\begin{cases} \dfrac{1}{|Y|}\displaystyle\int_{\Omega\times\mathbf{R}^N\setminus\overline{B}} \langle \chi_\rho, \nabla_z v\rangle \; \mathrm{d}x \; \mathrm{d}z = \rho v(B), \\ \forall v \in K_B, \\ \big(\nabla_z V_\rho, \chi_\rho\big) \in A_0(x,z), \\ V_\rho \in K_B, \end{cases} \tag{3.52}$$

where A_0 *is defined in the statement of Theorem 3.4.*

Proof. A variant of the Leray-Lions Theorem 1.1 applies for this problem on the space K_B (in a slightly different form for an unbounded domain) since the right hand side is a continuous linear form on that space. □ □

Definition 3.14. Let θ be the possibly multivalued mapping defined by

$$\theta \;:\; \rho \to \{V_\rho(B)\}, \tag{3.53}$$

where the V_ρ's are all the solutions of (3.52).

Lemma 3.15. *The mapping* θ *is monotone and satisfies*

$$\rho\,\theta(\rho) \geq C|\rho|^p. \tag{3.54}$$

It can be extended to a unique maximal monotone operator defined on $\mathbf{R}$ *which will be denoted* $\boldsymbol{\Theta}$*.*

Proof. Let us first prove the monotonicity. Let $(\nabla_z V_\rho, \chi_\rho)$ and $(\nabla_z V_{\rho'}, \chi_{\rho'})$ belong to A_0 so that $V_\rho(B) \in \theta(\rho)$ and $V_\rho(B) \in \theta(\rho')$. Since A_0 is monotone, we have

$$\langle \chi_\rho - \chi_{\rho'}, \nabla_z V_\rho - \nabla_z V_{\rho'}\rangle \geq 0,$$

which, by using (3.52) (twice!) implies

$$\frac{1}{|Y|}\int\limits_{\Omega\times\mathbf{R}^N\setminus\overline{B}} \langle \chi_\rho - \chi_{\rho'}, \nabla_z V_\rho - \nabla_z V_{\rho'}\rangle \; \mathrm{d}x \; \mathrm{d}z = (\rho-\rho')(V_\rho(B) - V_{\rho'}(B) \geq 0.$$

Let us now establish (3.54). Recalling the growth condition 3.1 satisfied by A_0, a similar computation gives

$$\rho\, V_\rho(B) = \frac{1}{|Y|}\int_{\Omega\times\mathbf{R}^N\setminus\overline{B}} \langle \chi_\rho, \nabla_z V_\rho\rangle \ \mathrm{d}x\ \mathrm{d}z \geq \alpha\Big(\frac{\|\nabla_z V_\rho\|^p_{L^p}}{p} + \frac{\|\chi_\rho\|^{p'}_{L^{p'}}}{p'}\Big).$$

On the other hand by definition of V_ρ, we know that there exists $C > 0$ such that

$$|V_\rho(B)| \leq C\|\nabla_z V_\rho\|_{L^p},$$

hence the result.

Under monotonicity assumptions together with ((3.54)), the extension of θ to a unique maximal monotone operator $\mathbf{\Theta}$ which is everywhere defined is standard (see [3]). □ □

We can now state the limit equation which follows from the previous lemmas together with (3.36):

Theorem 3.16. *Under assumptions of Theorem 3.4, u_0 is a solution (not necessarly unique) of the homogenized equation:*

$$\begin{cases} -\mathrm{div}\ d - k_1\mathbf{\Theta}^{-1}(-k_1 u) \ni f, \\ \big(\nabla_x u, d\big) \in A_{hom}(x) \\ u \in W^{1,p}_0(\Omega). \end{cases} \tag{3.55}$$

Proof. Let u_0 and d_0, U and ξ_0 be given by Theorem 3.4, set

$$\rho_0 = \frac{1}{|Y|}\int_{\Omega\times\partial B} (\xi_0 \cdot \nu_B)\ \mathrm{d}x\ \mathrm{d}\sigma_z.$$

By (3.33) applied with $W \doteq u_0 - U$ substituted for v,

$$\frac{1}{|Y|}\int_{\mathbf{R}^N\setminus\overline{B}} \langle \xi_0, \nabla_z W\rangle\ \mathrm{d}x\ \mathrm{d}z = \rho_0 W(B) \text{ for a.e. } x \in \Omega.$$

By (3.43), $(k_1\nabla_z U(x,z), \xi_0(x,z)) \in A_0(x,z)$ so that

$$\frac{1}{|Y|}\int_{\mathbf{R}^N\setminus\overline{B}} \langle \xi_0, \nabla_z(-k_1 W)\rangle\ \mathrm{d}x\ \mathrm{d}z = -k_1\rho_0 u_0 \text{ for a.e. } x \in \Omega.$$

$$-k_1 W(B) = -k_1 u_0 \in \mathbf{\Theta}(\rho_0).$$

Since $\mathbf{\Theta}$ is a maximal monotone graph (defined on $\mathbf{R}$), $\mathbf{\Theta}^{-1}$ is also a maximal monotone graph and we conclude that

$$\rho_0 {=} \frac{1}{|Y|}\int_{\partial B} (\xi_0 \cdot \nu_B)\ \mathrm{d}x\ \mathrm{d}\sigma_z \in \mathbf{\Theta}^{-1}(-k_1 u_0)).$$

Recalling the limit formulation (3.15)

$$\int_\Omega \langle d_0, \nabla_x \psi \rangle \, \mathrm{d}x - \frac{k_1}{|Y|} \int_{\Omega \times \partial B} (\xi_0 \nu_B) \psi \, \mathrm{d}\sigma_z = \int_\Omega f \psi \, \mathrm{d}x,$$

for all $\psi \in W_0^{1,p}(\Omega)$, we deduce that

$$\int_\Omega \langle d_0, \nabla_x \psi \rangle \, \mathrm{d}x - k_1 \int_\Omega \mathbf{\Theta}^{-1}(-k_1 u_0(x)) \psi \, \mathrm{d}x \ni \int_\Omega f \psi \, \mathrm{d}x.$$

This is the variational formulation of problem (3.55). □ □

Remark 20. The contribution of the oscillations of the matrix A^ε in the homogenized problem are reflected by the presence of A_{hom} on the left hand side in (3.55). The contribution of the perforations is the zero order "strange term" $-k_1 \mathbf{\Theta}^{-1}(-k_1 u)$.

Remark 21.

1. For the case $k_1 = 0$ the statement of Theorem 3.16 remains valid in its simpler form: the small holes have no influence at the limit (no "strange term"). The proof is actually simpler: by Lemma 3.6, in this case, the sequence $(w_{\varepsilon,\delta})$ converges strongly to $v(B)$ in $W^{1,p}(\Omega)$. Consequently, when going to the limit in (3.2), the contribution of the first term E^1_ε vanishes. There is no need to use the second unfolding, and the result is the same as in [16].

2. The case of $\lim \dfrac{\delta^{\frac{N}{p}-1}}{\varepsilon} = \infty$ is easy to analyse: from Theorem 2.7 (6),

$$\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) \rightharpoonup u_0 \text{ weakly in } L^p(\Omega; L^{p*}_{loc}(\mathbf{R}^N)).$$

On the other hand, since $\mathcal{T}_{\varepsilon,\delta}(u_{\varepsilon,\delta}) = 0$ in $\Omega \times B$, this implies that $u_0 = 0$. Therefore, the influence of the Dirichlet hole is so strong that it forces the convergence of the solutions to 0. In this case, it would be interesting to obtain an equivalent of $u_{\varepsilon\delta}$ if there is any.

4 *Appendix*: Maximal monotone graphs

4.1 Notations

In this section we recall some basics notations about monotone and maximal monotone graphs and functions in a Banach space. For more details see [3, 4, 5].

Let X be a reflexive Banach space and let X' be its dual. The duality pairing on $X' \times X$ is denoted by $\langle \cdot, \cdot \rangle$.

We consider set-valued operators $A : X \to X'$, that is, maps that assign to every point $x \in X$ some set $Ax \subset X'$. These applications are simply called *operators* when no confusions may arise. Similarly when no ambiguities arise, A will also denote the *graph* of the operator A, that is the set $\{(x, \xi) \in X \times X' : \xi \in Ax\}$. The *domain* of a graph A is the set,

$$D(A) = \big\{x \in X \text{ such that } Ax \neq \emptyset\big\}.$$

The operator A is *single-valued* on some set $C \subset X$, if for every $\xi \in C$, $A\xi$ contains at most one element. For operators A, B, we write $A \subseteq B$ whenever $A\xi \subseteq B\xi$ for every $\xi \in X$.

Monotone and maximal monotone graphs can now be defined:

Definition 4.1. The set $A \subset X \times X'$ is a *monotone graph* (or monotone operator) if for every $(\xi_1, \eta_1), (\xi_2, \eta_2) \in A$,

$$\langle \eta_1 - \eta_2, \xi_1 - \xi_2 \rangle \geq 0.$$

The monotone graph A is a *maximal monotone graph* (or maximal monotone operator), if for every monotone graph $B \subset X \times X'$, the inclusion $A \subseteq B$ implies $A = B$. The set of all such maximal monotone operators is denoted $\mathfrak{M}(X \times X')$.

The basic stability result for maximal monotone operators is Minty's original result for the Hibert space setting (see [21]):

Lemma 4.2. *Let $A \subset X \times X'$ be a maximal monotone operator and (ξ_n, η_n) belong to A. Suppose that, as $n \to +\infty$,*

$$\begin{aligned} &\xi_n \rightharpoonup \xi \quad \text{weakly in } X, \\ &\eta_n \rightharpoonup \eta \quad \text{weakly in } X', \\ &\liminf_{n \to +\infty} \langle \eta_n, \xi_n \rangle \leq \langle \eta, \xi \rangle, \end{aligned} \tag{4.1}$$

then $(\xi, \eta) \in A$ and

$$\liminf_{n \to +\infty} \langle \eta_n, \xi_n \rangle = \langle \eta, \xi \rangle.$$

4.2 Convergence of maximal monotone graphs

Following Brezis [3] and Attouch [1], the convergence of maximal monotone graphs is defined as follows:

Definition 4.3. Let $A^n, A \in \mathfrak{M}(X \times X')$ be maximal monotone graphs. The sequence A^n converges to A as $n \to \infty$, ($A^n \rightarrowtail A$), if for every $(\xi, \eta) \in A$ there exists a sequence $(\xi_n, \eta_n) \in A^n$ such that $(\xi_n, \eta_n) \to (\xi, \eta)$ strongly in $X \times X'$ as $n \to \infty$.

The convergence of graphs ensures that weak limits of elements of A^n are in A provided the duality product of the pairs is preserved at the limit. More precisely, one can prove the following theorem in the same way as Lemma 4.2.

Theorem 4.4 (Minty's result). *Let $A^n,\ A \subset X \times X'$ be maximal monotone graphs, and let $(\xi_n, \eta_n) \in A^n$, $(\xi, \eta) \in X \times X'$. If, as $n \to +\infty$,*

$$
\begin{aligned}
& A^n \rightarrowtail A, \\
& \xi_n \rightharpoonup \xi \quad \text{weakly in } X, \\
& \eta_n \rightharpoonup \eta \quad \text{weakly in } X', \\
& \liminf_{n \to +\infty} \langle \eta_n, \xi_n \rangle \leq \langle \eta, \xi \rangle,
\end{aligned} \tag{4.2}
$$

then $(\xi, \eta) \in A$ and

$$\liminf_{n \to +\infty} \langle \eta_n, \xi_n \rangle_n = \langle \eta, \xi \rangle.$$

4.3 Canonical extensions of maximal monotone graphs

Given a $\sigma-$finite measure space O (for simplicity, we denote the measure as $\mathrm{d}x$) it is well-known that if X is reflexive, for $1 \leq p < \infty$, the dual space of $L^p(O;X)$ is $L^{p'}(O;X')$ (with $1/p + 1/p' = 1$). Given a map $A : O \to \mathfrak{M}(X \times X')$, one can define a multivalued operator from $L^p(O;X)$ to $L^{p'}(O;X')$ as follows:

Definition 4.5. Let $A : O \to \mathfrak{M}(X \times X')$, the *canonical extension* of A from $L^p(O;X)$ to $L^{p'}(O;X')$, where $1/p + 1/p' = 1$, is defined by:

$$\mathcal{A} = \big\{(u,v) \in L^p(O;X) \times L^{p'}(O;X') : \big(u(t), v(t)\big) \in A(t) \text{ for a.e. } t \in O\big\}. \tag{4.3}$$

One readily checks that $\mathcal{A}$ is monotone.

Proposition 4.6. *Let $A : O \to \mathfrak{M}(X \times X')$ be measurable. If its canonical extension $\mathcal{A}$ is not empty, then it is maximal monotone.*

Remark 22. The maximality of $A(t)$ for almost every $t \in O$ is not sufficient in order to ensure the maximality of $\mathcal{A}$ as the latter could be empty (see [16] for an example).

Concerning the class $\mathfrak{LL}(O, X, p, \alpha, m)$, the following property is known:

Proposition 4.7 (see [16]). *Let α be strictly positive and m be in $L^1(O)$. If A belongs to $\mathfrak{LL}(O, X, p, \alpha, m)$, then $\mathcal{A}$ is maximal monotone, $D(\mathcal{A}) = L^p(O;X)$ and $\mathcal{A}$ is surjective.*

4.4 Convergence of canonical extensions

Given functions $A,\ A^n : O \to \mathfrak{M}(X \times X')$ and their canonical extensions $\mathcal{A},\ \mathcal{A}^n$ one can wonder whether the a.e. pointwise convergence $A^n(t) \rightarrowtail A(t)$ implies the convergence of the induced graphs $\mathcal{A}^n \rightarrowtail \mathcal{A}$. This is answered as follows (see [16]).

Theorem 4.8. *Let $A, A^n : O \to \mathfrak{M}(X \times X')$ be measurable. Assume*

(i) for almost every $t \in O$, $A^n(t) \rightarrowtail A(t)$ as $n \to \infty$,

(ii) $\mathcal{A}$ and $\mathcal{A}^n$ are maximal monotone,

(iii) there exists $(\alpha_n, \beta_n) \in \mathcal{A}^n$ and $(\alpha, \beta) \in L^p(O;X) \times L^{p'}(O;X')$ such that $(\alpha_n, \beta_n) \to (\alpha, \beta)$ strongly in $L^p(O;X) \times L^{p'}(O;X')$ as $n \to \infty$,

then $\mathcal{A}^n \rightarrowtail \mathcal{A}$.

Remark 23. Assumption (iii) cannot be dropped in general.

Corollary 4.9. *Assume $\alpha > 0$, $m_n \in L^1(O)$, $A^n \in \mathfrak{LL}(O, X, p, \alpha, m_n)$ and $A : O \to \mathfrak{M}(X \times X')$ measurable. If m_n converges strongly to m in $L^1(O)$ and if for almost every t in O, $A^n(t) \rightarrowtail A(t)$ holds, then $A \in \mathfrak{LL}(O, X, p, \alpha, m)$ and $\mathcal{A}^n \rightarrowtail \mathcal{A}$.*

In the case of canonical extensions, there are many possible duality pairings each giving rise to a version of Theorem 4.4. We will show the following generalization of Minty's result (cf. [21]), which is used in the proof of Proposition 3.10 and which involves sequences of such weighted pairings.

Proposition 4.10 (An extension of Minty's result)**.** *Under the hypotheses of Corollary 4.9 suppose that*

$$\begin{aligned} &\xi_n \rightharpoonup \xi \quad \text{weakly in } L^p(O;X), \\ &\eta_n \rightharpoonup \eta \quad \text{weakly in } L^{p'}(O;X') \text{ and} \\ &(\xi_n(x), \eta_n(x)) \in A^n(x) \text{ for a.e. } x \in O. \end{aligned}$$

Suppose furthermore that there is a sequence $(\Phi^k)_{k\in\mathbf{N}}$ of non negative functions of $L^\infty(O)$ which is bounded and converges to 1 a.e. in O. Assume furthermore that for each $k \in \mathbf{N}$ there exists some sequence $(\phi_n^k)_{n\in\mathbf{N}}$ bounded in $L^\infty(O)$ and converging a e. to Φ^k with the property that

$$\liminf_{n\to+\infty} \int_O \langle \eta_n, \xi_n \rangle \phi_n^k \,\mathrm{d}x \le \int_O \langle \eta, \xi \rangle \Phi^k \,\mathrm{d}x. \tag{4.4}$$

Then $(\xi(x), \eta(x)) \in A(x)$ for a.e. $x \in O$.

Proof. By hypothesis, the $\mathcal{A}^n$ as well as $\mathcal{A}$ are defined on the whole of $L^p(O;X)$. Since $\mathcal{A}^n \rightarrowtail \mathcal{A}$, for each $(\alpha, \beta) \in \mathcal{A}$, there exists a sequence $(\alpha_n, \beta_n) \in \mathcal{A}^n$ such that $\alpha_n \to \alpha$ in $L^p(O;X)$ and $\beta_n \to \beta$ in $L^{p'}(O;X')$.

By monotonicity, $\langle \xi_n(x) - \alpha_n(x), \eta_n(x) - \beta_n(x)\rangle \geq 0$ for a.e. $x \in O$. Multiplying by $\phi_n^k \geq 0$ and integrating over O gives

$$\int_O \langle \eta_n, \xi_n\rangle \phi_n^k \,\mathrm{d}x \geq \int_O \langle \eta_n, \beta_n\rangle \phi_n^k \,\mathrm{d}x + \int_O \langle \alpha_n, \xi_n\rangle \phi_n^k \,\mathrm{d}x - \int_O \langle \alpha_n, \beta_n\rangle \phi_n^k \,\mathrm{d}x. \tag{4.5}$$

It is classical (Lebesgue's dominated convergence theorem) that a sequence which is bounded in $L^\infty(O)$ and converges a.e. is a mutiplyier for strong convergence in every $L^q(O)$ for $q \in [1, \infty)$. Consequently, one can pass to the limit in the right hand side of (4.5) to get

$$\liminf_{n\to\infty} \int_O \langle \eta_n, \xi_n\rangle \phi_n^k \,\mathrm{d}x \geq \int_O \langle \eta, \beta\rangle \Phi^k \,\mathrm{d}x + \int_O \langle \alpha, \xi\rangle \Phi^k \,\mathrm{d}x - \int_O \langle \alpha, \beta\rangle \Phi^k \,\mathrm{d}x.$$

By (4.4),

$$\int_O \langle \eta, \xi\rangle \Phi^k \,\mathrm{d}x \geq \int_O \langle \eta, \beta\rangle \Phi^k \,\mathrm{d}x + \int_O \langle \alpha, \xi\rangle \Phi^k \,\mathrm{d}x - \int_O \langle \alpha, \beta\rangle \Phi^k \,\mathrm{d}x,$$

which reads

$$\int_O \langle \xi(x) - \alpha(x), \eta(x) - \beta(x)\rangle \Phi^k(x) \,\mathrm{d}x \geq 0.$$

For $k \to \infty$, the left-hand side of the above inequality converges in a similar way to

$$\int_O \langle \xi(x) - \alpha(x), \eta(x) - \beta(x)\rangle \,\mathrm{d}x.$$

This last integral is therefore bounded below by 0. The maximality of $\mathcal{A}$ now implies $(\xi, \eta) \in \mathcal{A}$, which is equivalent to $\eta(x) \in A(x)(\xi(x))$ a. e. in O, completing the proof. □ □

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