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A posteriori error estimates for a finite element approximation of transmission problems with sign changing coefficients

Serge Nicaise and Juliette Venel *

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Abstract

We perform the a posteriori error analysis of residual type of a transmission problem with sign changing coefficients. According to [6] if the contrast is large enough, the continuous problem can be transformed into a coercive one. We further show that a similar property holds for the discrete problem for any regular meshes, extending the framework from [6]. The reliability and efficiency of the proposed estimator is confirmed by some numerical tests.

Key Words A posteriori estimator, non positive definite diffusion problems.

AMS (MOS) subject classification 65N30; 65N15, 65N50,

1 Introduction

Recent years have witnessed a growing interest in the study of diffusion problems with a sign changing coefficient. These problems appear in several areas of physics, for example in electromagnetism [12, 15, 16, 18, 19]. Thus some mathematical investigations have been performed and concern existence results [7, 19] and numerical approximations by the finite element methods [19, 4, 5, 6], with some a priori error analyses. But for such problems the regularity of the solution may be poor and/or unknown and consequently an a posteriori error analysis would be more appropriate. This analysis is the aim of the present paper.

For continuous Galerkin finite element methods, there now exists a large amount of literature on a posteriori error estimations for (positive definite) problems in mechanics or electromagnetism. Usually locally defined a posteriori error estimators are designed. We refer the reader to the monographs [2, 3, 17, 21] for a good overview on this topic.

In contrast to the recent paper [6] we will not use quasi-uniform meshes that are not realistic for an a posteriori error analysis. That is why we improve their finite element analysis in order to allow only regular meshes in Ciarlet’s sense [8].

The paper is structured as follows: We recall in Section 2 the ”diffusion” problem and the technique from [6] that allows to establish its well-posedness for sufficiently large contrast. In Section 3, we prove that the discrete approximation is well-posed by introducing an ad-hoc discrete lifting operator. The a posteriori error analysis is performed in Section 4, where upper and lower bounds are obtained. Finally in Section 5 some numerical tests are presented that confirm the reliability and efficiency of our estimator.

Let us finish this introduction with some notations used in the remainder of the paper: On \( D \), the \( L^2(D) \)-norm will be denoted by \( \| \cdot \|_D \). The usual norm and semi-norm of \( H^s(D) \) \((s \geq 0)\) are denoted by \( \| \cdot \|_{s,D} \) and \( | \cdot |_{s,D} \), respectively. In the case \( D = \Omega \), the index \( \Omega \) will be omitted. Finally, the notations \( a \lesssim b \) and \( a \sim b \) mean the existence of positive constants \( C_1 \) and \( C_2 \), which are independent of the mesh size and of the considered quantities \( a \) and \( b \) such that \( a \leq C_2 b \) and \( C_1 b \leq a \leq C_2 b \), respectively. In other words, the constants may depend on the aspect ratio of the mesh and the diffusion coefficient (see below).

## 2 The boundary value problem

Let \( \Omega \) be a bounded open domain of \( \mathbb{R}^2 \) with boundary \( \Gamma \). We suppose that \( \Omega \) is split up into two sub-domains \( \Omega_+ \) and \( \Omega_- \) with a Lipschitz boundary that we suppose to be polygonal in such a way that
\[
\bar{\Omega} = \bar{\Omega}_+ \cup \bar{\Omega}_-, \quad \Omega_+ \cap \Omega_- = \emptyset,
\]
see Figure 1 for an example.

We now assume that the diffusion coefficient \( a \) belongs to \( L^\infty(\Omega) \) and is positive (resp. negative) on \( \Omega_+ \) (resp. \( \Omega_- \)). Namely there exists \( \epsilon_0 > 0 \) such that
\[
\begin{align*}
a(x) &\geq \epsilon_0, \text{ for a. e. } x \in \Omega_+, \quad (1) \\
a(x) &\leq -\epsilon_0, \text{ for a. e. } x \in \Omega_-.
\end{align*}
\]

In this situation we consider the following second order boundary value problem with Dirichlet boundary conditions:
\[
\begin{align*}
-\text{div} \left( a \nabla u \right) &= f \quad \text{in } \Omega, \\
u &= 0 \quad \text{on } \Gamma.
\end{align*}
\]
The variational formulation of (3) involves the bilinear form

\[ B(u, v) = \int_{\Omega} a \nabla u \cdot \nabla v \]

and the Hilbert space

\[ H^1_0(\Omega) = \{ u \in H^1(\Omega) : u = 0 \text{ on } \Gamma \}. \]

Due to the lack of coercivity of \( B \) on \( H^1_0(\Omega) \) (see \([7, 5, 6]\)), this problem does not fit into a standard framework. In \([5, 6]\), the proposed approach is to use a bijective and continuous linear mapping \( T \) from \( H^1_0(\Omega) \) into itself that allows to come back to the coercive framework. Namely these authors assume that \( B(u, T^*v) \) is coercive in the sense that there exists \( \alpha > 0 \) such that

\[ B(u, T^*u) \geq \alpha \| u \|_{H^1_0(\Omega)}^2, \]

(4)

Hence given \( f \in L^2(\Omega) \), by the Lax-Milgram theorem the problem

\[ B(u, T^*v) = \int_{\Omega} fTv \quad \forall v \in H^1_0(\Omega), \]

(5)

has a unique solution \( u \in H^1_0(\Omega) \). Since \( T \) is an isomorphism, the original problem

\[ B(u, v) = \int_{\Omega} fv \quad \forall v \in H^1_0(\Omega), \]

(6)

has also a unique solution \( u \in H^1_0(\Omega) \).

In \([6]\), the mapping \( T \) is built by using a trace lifting operator \( R \) from \( H^{1/2}_0(\Sigma) \) into \( H^1(\Omega_-) \), where \( \Sigma = \partial \Omega_- \cap \partial \Omega_+ \) is the interface between \( \Omega_- \) and \( \Omega_+ \),

\[ H^1_\pm(\Omega_\pm) = \{ u \in H^1(\Omega_\pm) : u = 0 \text{ on } \partial \Omega_\pm \setminus \Sigma \}, \]

and

\[ H^{1/2}_0(\Sigma) = \{ u|_\Sigma : u \in H^1(\Omega_-) \} = \{ u|_\Sigma : u \in H^1(\Omega_+) \} \]

is the space of the restrictions to \( \Sigma \) of functions in \( H^1(\Omega_-) \) (or in \( H^1(\Omega_+) \)). This last space may be equipped with the norms

\[ \| p \|_{1/2, \pm} = \inf_{u \in H^1_\pm(\Omega_\pm)} \| u \|_{1, \Omega_\pm}. \]

With the help of such a lifting, a possible mapping \( T \) is given by (see \([6]\))

\[ Tv = \begin{cases} v_+ & \text{in } \Omega_+, \\ -v_- + 2R(v_+|_\Sigma) & \text{in } \Omega_-, \end{cases} \]

where \( v_\pm \) denotes the restriction of \( v \) to \( \Omega_\pm \). With this choice, it is shown in Proposition 3.1 of \([6]\) that (4) holds if

\[ K_R = \sup_{\substack{v \in H^1_+(\Omega_+) \\setminus \{0\}}} \frac{|B_-(R(v|_\Sigma), R(v|_\Sigma))|}{B_+(v, v)} < 1, \]

(7)
where \( B_{\pm}(u,v) = \int_{\Omega_{\pm}} a \nabla u \cdot \nabla v. \)

For concrete applications, one can make the following particular choice for \( \mathcal{R} \), that we denote by \( \mathcal{R}_p \): for any \( \varphi \in H^{1/2}_0(\Sigma) \) we define \( \mathcal{R}_p(\varphi) = w \) as the unique solution \( w \in H^1(\Omega_-) \) of

\[ \Delta w = 0 \text{ in } \Omega_-, \quad w = \varphi \text{ on } \Sigma. \]

With this choice, one obtains that \( K_{\mathcal{R}_p} < 1 \) if the contrast

\[ \frac{\min_{\Omega_-} |a|}{\max_{\Omega_+} a} \]

is large enough, we refer to Section 3 of [6] for more details.

**Remark 2.1** Note that in [5, 6] the authors consider sub-domains \( \Omega_+ \) and \( \Omega_- \) with a pseudo-Lipschitz boundary. However the previous arguments from [6] (shortly summarized above) are not valid in this case since the space \( H^1(\Omega_+) \) equipped with the norm \( | \cdot |_{1,\Omega_+} \) is not complete.

### 3 The discrete approximated problem

Here we consider the following standard Galerkin approximation of our continuous problem. We consider a triangulation \( T \) of \( \Omega \), that is a “partition” of \( \Omega \) made of triangles \( T \) (closed subsets of \( \bar{\Omega} \)) whose edges are denoted by \( e \). We assume that this triangulation is regular, i.e., for any element \( T \), the ratio \( h_T/\rho_T \) is bounded by a constant \( \sigma > 0 \) independent of \( T \) and of the mesh size \( h = \max_{T \in T} h_T \), where \( h_T \) is the diameter of \( T \) and \( \rho_T \) the diameter of its largest inscribed ball. We further assume that \( T \) is conforming with the partition of \( \Omega \), i.e., each triangle is assumed to be either included into \( \Omega_+ \) or into \( \Omega_- \). With each edge \( e \) of the triangulation, we denote by \( h_e \) its length and \( n_e \) a unit normal vector (whose orientation can be arbitrary chosen) and the so-called patch \( \omega_e = \bigcup_{e \subset T} T \), the union of triangles having \( e \) as edge. We similarly associate with each vertex \( x \), a patch \( \omega_x = \bigcup_{x \in T} T \).

For a triangle \( T \), \( n_T \) stands for the outer unit normal vector of \( T \). \( \mathcal{E} \) (resp. \( \mathcal{N} \)) represents the set of edges (resp. vertices) of the triangulation. In the sequel, we need to distinguish between edges (or vertices) included into \( \bar{\Omega} \) or into \( \Gamma \); in other words, we set

\[ \mathcal{E}_{\text{int}} = \{ e \in \mathcal{E} : e \subset \Omega \}, \]

\[ \mathcal{E}_T = \{ e \in \mathcal{E} : e \subset \Gamma \}, \]

\[ \mathcal{N}_{\text{int}} = \{ x \in \mathcal{N} : x \in \Omega \}. \]

Problem (6) is approximated by the continuous finite element space:

\[ V_h = \left\{ v_h \in H^1_0(\Omega) : v_{h|T} \in \mathbb{P}_l(T), \forall T \in \mathcal{T} \right\}, \quad (8) \]
where $\ell$ is a fixed positive integer and the space $\mathbb{P}_\ell(T)$ consists of polynomials of degree at most $\ell$.

The Galerkin approximation of problem (6) reads now: Find $u_h \in V_h$, such that

$$B(u_h, v_h) = \int_\Omega f v_h \quad \forall v_h \in V_h.$$  \hspace{1cm} (9)

Since there is no reason that the bilinear form would be coercive on $V_h$, as in [6] we need to use a discrete mapping $T_h$ from $V_h$ into itself defined by (see [6])

$$T_h v_h = \begin{cases} v_h^+ \text{ in } \Omega_+, \\ -v_h^- + 2\mathcal{R}_h(v_{h+|\Sigma}) \text{ in } \Omega_-, \end{cases}$$

where $\mathcal{R}_h$ is a discrete version of the operator $\mathcal{R}$. Here contrary to [6] and in order to avoid the use of quasi-uniform meshes (meaningless in an a posteriori error analysis), we take

$$\mathcal{R}_h = I_h \mathcal{R},$$ \hspace{1cm} (10)

where $I_h$ is a sort of Clément interpolation operator [9] and $\mathcal{R}$ is any trace lifting operator from $H^{1/2}_0(\Sigma)$ into $H^1(\Omega_-)$ (see the previous section). More precisely for $\varphi_h \in H_h(\Sigma) = \{v_{h|\Sigma} : v_h \in V_h\}$, we set

$$I_h \mathcal{R}(\varphi_h) = \sum_{x \in N_-} \alpha_x \lambda_x,$$

where $N_- = N_{int} \cap \bar{\Omega}_-$, $\lambda_x$ is the standard hat function (defined by $\lambda_x \in V_h$ and satisfying $\lambda_x(y) = \delta_{xy}$) and $\alpha_x \in \mathbb{R}$ are defined by

$$\alpha_x = \begin{cases} |\omega_x|^{-1} \int_{\omega_x} \mathcal{R}(\varphi_h) \quad & \text{if } x \in N_{int} \cap \Omega_-, \\ \varphi_h(x) \quad & \text{if } x \in N_{int} \cap \Sigma, \end{cases}$$

where we recall that $\omega_x$ is the patch associated with $x$, which is simply the support of $\lambda_x$. Note that $I_h$ coincides with the Clément interpolation operator $I_{Cl}$ for the nodes in $\Omega_-$ and only differs on the nodes on $\Sigma$. Indeed let us recall the definition of $I_{Cl} \mathcal{R}(\varphi_h)$ (defined in a Scott-Zhang manner [20] for the points belonging to $\Sigma$):

$$I_{Cl} \mathcal{R}(\varphi_h) = \sum_{x \in N_-} \beta_x \lambda_x$$

with

$$\beta_x = \begin{cases} |\omega_x|^{-1} \int_{\omega_x} \mathcal{R}(\varphi_h) \quad & \text{if } x \in N_{int} \cap \Omega_-, \\ |e_x|^{-1} \int_{e_x} \mathcal{R}(\varphi_h) d\sigma \quad & \text{if } x \in N_{int} \cap \Sigma \text{ with } e_x = \omega_x \cap \Sigma. \end{cases}$$

The definition of $I_h$ aims at ensuring that

$$I_h \mathcal{R}(\varphi_h) = \varphi_h \text{ on } \Sigma.$$ 

Let us now prove that $\mathcal{R}_h$ is uniformly bounded.
Theorem 3.1 For all $h > 0$ and $\varphi_h \in H_h(\Sigma)$, one has

$$|R_h(\varphi_h)|_{1, \Omega_-} \lesssim \|\varphi_h\|_{1/2, \Omega_-}.$$  

Proof: For the sake of simplicity we make the proof in the case $\ell = 1$, the general case is treated in the same manner by using modified Clément interpolation operator.
Since $R$ is bounded from $H^{1/2}_0(\Sigma)$ into $H^1(\Omega_-)$, one has

$$|R(\varphi_h)|_{1, \Omega_-} \lesssim \|\varphi_h\|_{1/2, \Omega_-}. \quad (11)$$

Hence it suffices to show that

$$|(I - I_h)R(\varphi_h)|_{1, \Omega_-} \lesssim \|\varphi_h\|_{1/2, \Omega_-}. \quad (12)$$

For that purpose, we distinguish the triangles $T$ that have no nodes in $N_{\text{int}} \cap \Sigma$ to the other ones:

1. If $T$ has no nodes in $N_{\text{int}} \cap \Sigma$, then $I_h R(\varphi_h)$ coincides with $I_{\text{Cl}} R(\varphi_h)$ on $T$ and therefore by a standard property of the Clément interpolation operator, we have

$$|(I - I_h)R(\varphi_h)|_{1, T} = |(I - I_{\text{Cl}})R(\varphi_h)|_{1, T} \lesssim \|R(\varphi_h)\|_{1, \omega_T}, \quad (13)$$

where the patch $\omega_T$ is given by $\omega_T = \bigcup T'$. 

2. If $T$ has at least one node in $N_{\text{int}} \cap \Sigma$, by the triangle inequality we may write

$$|(I - I_h)R(\varphi_h)|_{1, T} \leq |(I - I_{\text{Cl}})R(\varphi_h)|_{1, T} + |(I_{\text{Cl}} - I_h)R(\varphi_h)|_{1, T}.$$ 

For the first term of this right-hand side we can still use (13) and therefore it remains to estimate the second term. For that one, we notice that

$$(I_{\text{Cl}} - I_h)R(\varphi_h) = \sum_{x \in T \cap \Sigma} (\alpha_x - \beta_x) \lambda_x \text{ on } T.$$ 

Hence

$$|(I_{\text{Cl}} - I_h)R(\varphi_h)|_{1, T} \lesssim \sum_{x \in T \cap \Sigma} |\alpha_x - \beta_x|.$$ 

Since $R(\varphi_h) = \varphi_h$ on $\Sigma$ and due to the definition of $I_{\text{Cl}}$, it follows that for $x \in T \cap \Sigma$,

$$|\alpha_x - \beta_x| = \left| \varphi_h(x) - |e_x|^{-1} \int_{e_x} \varphi_h d\sigma \right|.$$ 

Since all norms are equivalent in finite dimensional spaces, we have for all $v_h \in P_1(e_x)$,

$$|v_h(x)| \lesssim |e_x|^{-1/2} \|v_h\|_{e_x}. \quad (14)$$
Moreover,
\[ |e_x|^{-1/2} \left\| \varphi_h - |e_x|^{-1} \int_{e_x} \varphi_h d\sigma \right\|_{e_x} \lesssim |\varphi_h|_{1/2,e_x}, \]  
(15)
where here $| \cdot |_{1/2,e_x}$ means the standard $H^{1/2}(e_x)$-seminorm. Thus Inequalities (14) with $v_h = \varphi_h - |e_x|^{-1} \int_{e_x} \varphi_h d\sigma$ and (15) imply that
\[ |\alpha_x - \beta_x| \lesssim |\varphi_h|_{1/2,e_x}. \]

All together we have shown that
\[ |(I - I_h)R(\varphi_h)|_{1,T} \lesssim \|R(\varphi_h)|_{1,\omega_T \cap \Omega_-} + |\varphi_h|_{1/2,\omega_T \cap \Omega_-}. \]  
(16)
Taking the sum of the square of (13) and of (16), we obtain that
\[ |(I - I_h)R(\varphi_h)|_{2,\Omega_-}^2 \lesssim \|R(\varphi_h)|_{2,\Omega_-}^2 + |\varphi_h|_{1/2,\Sigma}^2. \]
We conclude thanks to (11) and to the fact that
\[ |\varphi_h|_{1/2,\Sigma} \lesssim \|\varphi_h\|_{1/2,\Sigma}. \]

This Theorem and Proposition 4.2 of [6] allow to conclude that (9) has a unique solution provided that (7) holds, in particular if the contrast is large enough.

Note that the advantage of our construction of $R_h$ is that we no more need the quasi-uniform property of the meshes imposed in [6].

4 The a posteriori error analysis

Error estimators can be constructed in many different ways as, for example, using residual type error estimators which measure locally the jump of the discrete flux [21]. A different method, based on equilibrated fluxes, consists in solving local Neumann boundary value problems [2] or in using Raviart-Thomas interpolant [1, 10, 13, 14]. Here since the coercivity constant is not explicitly known, we chose the simplest approach of residual type.

The residual estimators are denoted by
\[ \eta^2_R = \sum_{T \in \mathcal{T}} \eta^2_{R,T}, \quad \eta^2_J = \sum_{T \in \mathcal{T}} \eta^2_{J,T}, \]  
(17)
where the indicators $\eta_{R,T}$ and $\eta_{J,T}$ are defined by
\[ \eta_{R,T} = h_T \|f_T + \text{div} (a \nabla u_h)\|_T, \]
\[ \eta_{J,T} = \sum_{e \in E_{int} : e \subset T} h_e^{1/2} \|a \nabla u_h \cdot n_e\|_e, \]
when $f_T$ is an approximation of $f$, for instance
\[ f_T = |T|^{-1} \int_T f. \]
Note that $\eta^2_{R,T}$ is meaningful if $a_{|T} \in W^{1,1}(T)$, for all $T \in \mathcal{T}$. 

7
4.1 Upper bound

**Theorem 4.1** Assume that \( a \in L^\infty(\Omega) \) satisfies (1)-(2) and that \( a|_T \in W^{1,1}(T) \), for all \( T \in \mathcal{T} \). Assume further that (7) holds. Let \( u \in H^1_0(\Omega) \) be the unique solution of Problem (6) and let \( u_h \) be its Galerkin approximation, i.e. \( u_h \in V_h \) a solution of (9). Then one has

\[
\|\nabla (u - u_h)\| \lesssim \eta_R + \eta_I + \text{osc}(f),
\]

where

\[
\text{osc}(f) = \left( \sum_{T \in \mathcal{T}} h_T^2 \| f - f_T \|^2 \right)^{\frac{1}{2}}.
\]

**Proof:** By the coerciveness assumption (4), we may write

\[
\|\nabla (u - u_h)\|^2 \lesssim B(u - u_h, \mathbb{T}(u - u_h)).
\]

But we notice that the Galerkin relation

\[ B(u - u_h, v_h) = 0 \quad \forall v_h \in V_h \]

holds. Hence by taking \( v_h = I_{CI} \mathbb{T}(u - u_h) \), (19) may be written

\[
\|\nabla (u - u_h)\|^2 \lesssim B(u - u_h, (I - I_{CI}) \mathbb{T}(u - u_h)).
\]

Now we apply standard arguments, see for instance [21]. Namely applying element-wise Green’s formula and writing for shortness \( w = (I - I_{CI}) \mathbb{T}(u - u_h) \), we get

\[
\|\nabla (u - u_h)\|^2 \lesssim - \sum_{T \in \mathcal{T}} \int_T \text{div} (a\nabla (u - u_h)) w
\]

\[+ \sum_{e \in \mathcal{E}_{int}} \int_e [a\nabla (u - u_h) \cdot n] w d\sigma, \]

reminding that \( w = 0 \) on \( \Gamma \). By Cauchy-Schwarz’s inequality we directly obtain

\[
\|\nabla (u - u_h)\|^2 \lesssim \sum_{T \in \mathcal{T}} \| f + \text{div} (a\nabla u_h) \|_T \| w \|_T
\]

\[+ \sum_{e \in \mathcal{E}_{int}} \|[a\nabla u_h \cdot n] \|_e \| w \|_e. \]

By standard interpolation error estimates, we get

\[
\|\nabla (u - u_h)\|^2 \lesssim \left( \sum_{T \in \mathcal{T}} h_T^2 \| f + \text{div} (a\nabla u_h) \|^2_T \right)^{\frac{1}{2}} \| \mathbb{T}(u - u_h) \|_{1,\Omega}.
\]
Since $T$ is an isomorphism, we conclude that
\[
\|\nabla (u - u_h)\| \lesssim \left( \sum_{T \in T} h_T^2 \| f + \text{div} (a \nabla u_h) \|_T^2 \right.
\]
\[
+ \sum_{e \in \mathcal{E}_{\text{int}}} h_e \| [a \nabla u_h \cdot n]_e \|_e^2 \right)^{1/2}.
\]
This leads to the conclusion due to the triangle inequality.

\section*{4.2 Lower bound}

The lower bound is fully standard since by a careful reading of the proof of Proposition 1.5 of \cite{21}, we see that it does not use the positiveness of the diffusion coefficient $a$. Hence we can state the

\begin{theorem}
Let the assumptions of Theorems 4.1 be satisfied. Assume furthermore that $a_{|T}$ is constant for all $T \in T$. Then for each element $T \in T$ the following estimate holds
\[
\eta_{R,T} + \eta_{I,T} \lesssim \|u - u_h\|_{1,\omega_T} + \text{osc}(f, \omega_T),
\]
where
\[
\text{osc}(f, \omega_T)^2 \equiv \sum_{T' \subseteq \omega_T} h_{T'}^2 \|f - f_{T'}\|_{T'}^2.
\]
\end{theorem}

\section{5 Numerical results}

\subsection*{5.1 The polynomial solution}

In order to illustrate our theoretical predictions, this first numerical test consists in validating our computations on a simple case, using an uniform refinement process. Let $\Omega$ be the square $(-1, 1)^2$, $\Omega_+ = (0, 1) \times (-1, 1)$ and $\Omega_- = (-1, 0) \times (-1, 1)$. We assume that $a = 1$ on $\Omega_+$ and $a = \mu < 0$ on $\Omega_-$. In such a situation we can take
\[
\mathcal{R}(v_+)(x, y) = v_+(-x, y) \quad \forall (x, y) \in \Omega_-.
\]
With this choice we see that
\[
K_R = |\mu|,
\]
and therefore for $|\mu| < 1$, (4) holds and Problem (6) has a unique solution. We further easily check that the corresponding mapping $T$ is an isomorphism since $(T)^2 = T$. Similarly by exchanging the role of $\Omega_+$ and $\Omega_-$, (4) will also hold if $|\mu| > 1$.

Now we take as exact solution
\[
\begin{align*}
u(x,y) &= \mu x(x+1)(x-1)(y+1)(y-1) \quad \forall (x,y) \in \Omega_+, \\
u(x,y) &= x(x+1)(x-1)(y+1)(y-1) \quad \forall (x,y) \in \Omega_-,
\end{align*}
\]
Let us recall that $u_h$ is the finite element solution, and set $e_{L^2}(u_h) = \|u - u_h\|$ and $e_{H^1}(u_h) = \|u - u_h\|_1$ the $L^2$ and $H^1$ errors. Moreover let us define $\eta(u_h) = \eta_R + \eta_J$ the estimator and $CV_{L^2}(u_h)$ as the experimental convergence rate of the error $e_{L^2}(u_h)$ (resp. $e_{H^1}(u_h)$) with respect to the mesh size defined by $\text{DoF}^{-1/2}$, where the number of degrees of freedom is $\text{DoF}$, computed from one line of the table to the following one.

Computations are performed with $\mu = -3$ using a global mesh refinement process from an initial cartesian grid. First, it can be seen from Table 1 that the convergence rate of the $H^1$ error norm is equal to one, as theoretically expected (see [6]). Furthermore the convergence rate of the $L^2$ error norm is 2, which is a consequence of the Aubin-Nitsche trick and regularity results for Problem (3). Finally, the reliability of the estimator is ensured since the ratio in the last column (the so-called effectivity index), converges towards a constant close to 6.5.

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<th>$e_{L^2}(u_h)$</th>
<th>$CV_{L^2}$</th>
<th>$e_{H^1}(u_h)$</th>
<th>$CV_{H^1}$</th>
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Table 1: The polynomial solution with $\mu = -3$ (uniform refinement).

5.2 A singular solution

Here we analyze an example introduced in [7] and precise some results from [7]. The domain $\Omega = (-1,1)^2$ is decomposed into two sub-domains $\Omega_+ = (0,1) \times (0,1)$, and $\Omega_- = \Omega \setminus \bar{\Omega}_+$, see Figure 1. As before we take $a = 1$ on $\Omega_+$ and $a = \mu < 0$ on $\Omega_-$. According to Section 3 of [7], Problem (6) has a singularity $S$ at $(0,0)$ if $\mu < -3$ or if $\mu \in (-1/3,0)$ given in polar coordinates by

\[
S_+(r, \theta) = r^\lambda(c_1 \sin(\lambda \theta) + c_2 \sin(\lambda(\frac{\pi}{2} - \theta))) \quad \text{for } 0 < \theta < \frac{\pi}{2},
\]

\[
S_-(r, \theta) = r^\lambda(d_1 \sin(\lambda(\theta - \frac{\pi}{2}) + d_2 \sin(\lambda(2\pi - \theta))) \quad \text{for } \frac{\pi}{2} < \theta < 2\pi,
\]

where $\lambda \in (0, 1)$ is given by

\[
\lambda = \frac{2}{\pi} \arccos \left( \frac{1 - \mu}{2|1 + \mu|} \right),
\]
and the constants $c_1, c_2, d_1, d_2$ are appropriately defined. Now we show using the arguments of Section 2 that for $-\frac{1}{3} < \mu < 0$ and $\mu < -3$, the assumption (4) holds. As before we define

$$R(v_+(x, y)) = \begin{cases} v_+(-x, y) & \forall (x, y) \in (-1, 0) \times (0, 1), \\ v_+(-x, -y) & \forall (x, y) \in (-1, 0) \times (-1, 0), \\ v_+(x, -y) & \forall (x, y) \in (0, 1) \times (-1, 0). \end{cases}$$

This extension defines an element of $H^1_+(\Omega_-)$ such that

$$R(v_+) = v_+ \quad \text{on } \Sigma.$$

Moreover with this choice we have

$$\sup_{v \in H^1_+(\Omega_-), \ v \neq 0} \frac{|B_-(R(v), R(v))|}{B_+(v, v)} = 3|\mu|,$$

and therefore for

$$3|\mu| < 1,$$

we deduce that (4) holds.

To exchange the role of $\Omega_+$ and $\Omega_-$ we define the following extension from $\Omega_-$ to $\Omega_+$: for $v_- \in H^1(\Omega_-)$, let

$$R(v_-)(x, y) = v_-(-x, y) + v_-(-x, -y) - v_-(-x, -y) \quad \forall (x, y) \in \Omega_+.$$

We readily check that it defines an element of $H^1_+(\Omega_+)$ such that

$$R(v_-) = v_- \quad \text{on } \Sigma.$$

Moreover with this choice we have (using the estimate $(a + b + c)^2 \leq 3(a^2 + b^2 + c^2)$ valid for all real numbers $a, b, c$)

$$\sup_{v \in H^1_+(\Omega_-), \ v \neq 0} \frac{B_+(R(v), R(v))}{|B_-(v, v)|} \leq 3/|\mu|,$$

and therefore for

$$3/|\mu| < 1,$$

we deduce that (4) holds.

For this second test, we take as exact solution the singular function $u(x, y) = S(x, y)$ for $\mu = -5$ and $\mu = -100$, non-homogeneous Dirichlet boundary conditions on $\Gamma$ are fixed accordingly. First, with uniform meshes, we obtain the expected convergence rate of order $\lambda$ (resp. $2\lambda$) for the $H^1$ (resp. $L^2$) error norm, see Tables 2 and 3. There, for sufficiently fine meshes, we may notice that the effectivity index varies between 1 and 0.6 for $\mu = -5$ or between 9 and 6 for $\mu = -100$. From these results we can say that the effectivity index
depends on $\mu$, this is confirmed by the numerical results obtained by an adaptive algorithm (see below).

Secondly, an adaptive mesh refinement strategy is used based on the estimator $\eta_T = \eta_{R,T} + \eta_{J,T}$, the marking procedure

$$\eta_T > 0.5 \max_T \eta_T'$$

and a standard refinement procedure with a limitation on the minimal angle.

For $\mu = -5$ (resp. $\mu = -100$), Table 4 (resp. 5) displays the same quantitative results as before. There we see that the effectivity index is around 3 (resp. 34), which is quite satisfactory and comparable with results from [11, 14]. As before and in these references we notice that it deteriorates as the contrast becomes larger. On these tables we also remark a convergence order of 0.76 (resp. 1) in the $H^1$-norm and mainly the double in the $L^2$-norm. This yields better orders of convergence as for uniform meshes as expected, the case $\mu = -5$ giving less accurate results due to the high singular behavior of the solution (a similar phenomenon occurs in [11] for instance).
$$k \quad DoF \quad e_{L^2}(u_h) \quad CV_{L^2} \quad e_{H^1}(u_h) \quad CV_{H^1} \quad \frac{\eta(u_h)}{e_{H^1}(u_h)}$$

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Table 4: The singular solution, $\mu = -5, \lambda \approx 0.46$ (local refinement).

$$k \quad DoF \quad e_{L^2}(u_h) \quad CV_{L^2} \quad e_{H^1}(u_h) \quad CV_{H^1} \quad \frac{\eta(u_h)}{e_{H^1}(u_h)}$$

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Table 5: The singular solution, $\mu = -100, \lambda \approx 0.66$ (local refinement).
References


