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Rate of decay to 0 of the solutions to a nonlinear parabolic equation

I.Ben Arbi¹

Résumé: On étudie l'ordre de convergence vers 0, quand $t \longrightarrow +\infty$ de la solution de l'équation $\psi_t - \Delta\psi + |\psi|^{p-1}\psi = 0$ avec les conditions aux limites de Neumann dans un ouvert connexe borné de $\mathbb{R}^n$ où $p > 1$. On montre que soit $\psi(t, \cdot)$ converge vers 0 exponentiellement, soit $\psi(t, \cdot)$ décroît comme $t^{-\frac{1}{(p-1)}}$.

Abstract: We study the decay rate to 0, as $t \longrightarrow +\infty$ of the solution of equation $\psi_t - \Delta\psi + |\psi|^{p-1}\psi = 0$ with Neumann boundary conditions in a bounded smooth open connected domain of $\mathbb{R}^n$ where $p > 1$. We show that either $\psi(t, \cdot)$ converges to 0 exponentially fast or $\psi(t, \cdot)$ decreases like $t^{-\frac{1}{(p-1)}}$.

Keywords: rate of decay, parabolic equation

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1 Introduction and main results.

In this paper we consider the following nonlinear parabolic equation

$$\begin{cases} \psi_t - \Delta\psi + g(\psi) = 0 & \text{in } \mathbb{R}^+ \times \Omega, \\ \frac{\partial\psi}{\partial n} = 0 & \text{on } \mathbb{R}^+ \times \partial\Omega. \end{cases} \quad (1.1)$$

where Ω is a bounded smooth open connected domain of $\mathbb{R}^n$ and $g \in C^1(\mathbb{R})$ satisfies

$$g(0) = 0 \quad (1.2)$$

and for some $p > 1$

$$\exists c > 0, \forall s \in \mathbb{R}, \quad 0 \leq g'(s) \leq c|s|^{p-1}. \quad (1.3)$$

From (1.2)-(1.3) we deduce that $g(s)$ has the sign of s and

$$\forall s \in \mathbb{R}, \quad |g(s)| \leq \frac{c}{p}|s|^p. \quad (1.4)$$

We define the operator A by

$$D(A) = \{\psi \in H^2(\Omega), \frac{\partial\psi}{\partial n} = 0 \text{ on } \partial\Omega\}$$

and

$$\forall \psi \in D(A), \quad A\psi = -\Delta\psi$$

On the other hand the operator B defined by

$$D(B) = \{\psi \in L^2(\Omega), -\Delta\psi + g(\psi) \in L^2(\Omega) \text{ and } \frac{\partial\psi}{\partial n} = 0 \text{ on } \partial\Omega\}$$

and

$$\forall \psi \in D(B), \quad B\psi = -\Delta\psi + g(\psi)$$

is well-known to be maximal monotone in $L^2(\Omega)$. As a consequence of [1, 2] for any $\psi_0 \in L^2(\Omega)$ there exists a unique weak solution of the equation

$$\psi' + B\psi = 0 \text{ on } \mathbb{R}^+; \quad \psi(0, x) = \psi_0. \quad (1.5)$$

In addition it is well known that if $\psi_0 \in L^\infty(\Omega)$, $\psi(t, \cdot)$ remains in $L^\infty(\Omega)$ for all $t > 0$. Finally [8] contains an estimate of the solution in $C(\bar{\Omega})$ and $C^1(\bar{\Omega})$ for $t > 0$, which is valid for any sufficiently regular domain.

Concerning the behaviour for t large, in [5], A.Haraux established in the case of a pure power nonlinearity the exponential convergence to 0 of the projection on the range of A of the solution of equation (1.1). Moreover in [4], the study of the equation $u'' + u' - \Delta u + g(u) = 0$ with Neumann boundary conditions and where g satisfies $\exists C, c > 0, \forall s \in \mathbb{R}, c|s|^{p-1} \leq g'(s) \leq C|s|^{p-1}$ for some $p > 1$, showed that either $u(t)$ converges to 0 exponentially fast, or $\|u(t)\|_{H_0^1(\Omega)} \geq \gamma t^{-1/(p-1)}$ with $\gamma > 1$ for $t \geq 1$.

Several authors have treated some variants of equation (1.1), for example in [6] with $g(u) = c|u|^{p-1}u - \lambda_1 u$ and with Dirichlet boundary conditions, was studied the decay rate at the infinity of solutions to (1.1), where $\lambda_1 > 0$ is the first eigenvalue of $-\Delta$ in $H_0^1(\Omega)$. The result obtained there is optimal for positive solutions.

According to La Salle's invariance principle, cf. [3, 7], any solution ψ of (1.1), having a precompact range on $\mathbb{R}^+$ with values to $L^\infty(\Omega)$, converges to a continuum of stationary solutions of equation (1.1), which reduces here to the constants of some sub-interval of $g^{-1}(0)$. By monotonicity, it is in fact known that ψ converges to some constant $a \in g^{-1}(0)$.

Our first result is valid without any additional hypothesis on g

Theorem 1.1. *Let g satisfy (1.2) and (1.3). Then any solution ψ of (1.1) satisfies the following alternative as $t \rightarrow \infty$: either*

$$\|\psi(t, \cdot)\|_\infty \leq C e^{-\lambda_2 t}, \quad (1.6)$$

or

$$\exists c' > 0, \forall t \geq 1, \quad \left| \int_\Omega \psi(t, x) dx \right| \geq c' t^{-\frac{1}{p-1}}, \quad (1.7)$$

where $\lambda_2 > 0$ is the second eigenvalue of A in $D(A)$.

Our second result provides a more accurate estimate when $g(\psi) = |\psi|^{p-1}\psi$

Theorem 1.2. *Let us consider the nonlinear parabolic problem (1.1) with $g(\psi) = |\psi|^{p-1}\psi$, then any solution ψ of (1.1) satisfies the following alternative as $t \rightarrow \infty$: either*

$$\|\psi(t, \cdot)\|_\infty \leq C e^{-\lambda_2 t}, \quad (1.8)$$

or

$$\forall t \geq 1, \quad \left\| |\psi(t, \cdot)| - ((p-1)t)^{-\frac{1}{p-1}} \right\|_\infty \leq K t^{-\frac{1}{(p-1)-1}}, \quad (1.9)$$

where $K, C > 0$, $p > 1$ and $\lambda_2 > 0$ is the second eigenvalue of A in $D(A)$.

In the following Proposition, we consider two special cases showing that both possibilities in the second result in the Theorem 1.1 can actually happen.

Proposition 1.3. *Let g satisfy (1.2) and (1.3). Then we have*

- (i) *If Ω is symmetric around 0, g is odd and $\psi(0, \cdot)$ is an odd function in Ω , then any solution of (1.1) satisfies (1.6).*
- (ii) *If $\psi(0, \cdot) \geq 0$ and ψ does not vanish a.e in Ω , then any solution of (1.1) satisfies (1.7).*

Finally, our last result shows that the second possibility is sharp for a class of functions g more general than the pure power

Proposition 1.4. *Under the additional hypothesis*

$$\exists k_1 > 0, \forall s \in \mathbb{R}, |g(s)| \geq k_1 |s|^p \quad (1.10)$$

for any solution ψ of (1.1), we have

$$\forall t \geq 1, \quad \|\psi(t, \cdot)\|_\infty \leq \left\{ \frac{1}{k_1(p-1)} \right\}^{\frac{1}{p-1}} t^{-\frac{1}{p-1}} \quad (1.11)$$

2 Proof of Proposition 1.4.

Proof. If $\psi(0, \cdot) = 0$, we have $\psi(t, \cdot) \equiv 0$ and the result is obvious. Otherwise let z be defined by

$$z(t) = \left\{ \frac{1}{\|\psi(0, \cdot)\|_\infty^{1-p} + k_1(p-1)t} \right\}^{\frac{1}{p-1}}$$

Then z is a solution of the following nonlinear ODE problem

$$\begin{cases} z' + k_1 z^p = 0, \\ z(0) = \|\psi(0, \cdot)\|_\infty. \end{cases} \quad (2.1)$$

Under the additional condition(1.10), we will show that z is a super solution of (1.1). Indeed, we have

$$\begin{aligned} z_t - \Delta z + g(z) &= -k_1(z(0)^{1-p} + k_1(p-1)t)^{-\frac{p}{p-1}} + g(z) \\ &\geq -k_1(z(0)^{1-p} + k_1(p-1)t)^{-\frac{p}{p-1}} + k_1|z|^p \\ &\geq 0. \end{aligned}$$

Since $\psi(0, \cdot) \leq z(0)$ we deduce, by standard comparison principle, that $\psi(t, \cdot) \leq z(t) \forall t \geq 1$.

A similar calculation shows that $\psi(t, \cdot) \geq -z(t)$, $\forall t \geq 1$ which concludes the proof. $\blacksquare$

3 A general result on the range component.

Defining the orthogonal projection $P : H \longrightarrow N$, where

$$H = L^2(\Omega), N = \ker(A) \text{ and } P\psi(t, \cdot) = \frac{1}{|\Omega|} \int_{\Omega} \psi(t, x) dx,$$

as already mentioned in the introduction, [5] showed that for $g(\psi) = |\psi|^{p-1}\psi$, the following estimate holds

$$\|\psi(t) - P\psi(t)\|_{L^2(\Omega)} \leq K e^{-\lambda_2 t},$$

for some constant $K > 0$. In this section, we will show that we have the same result for any function g satisfying (1.3).

Proposition 3.1. *Let $\psi \in C(\mathbb{R}^+, L^\infty)$ be any solution of (1.1). Assume that g is a locally Lipschitz non-decreasing function. Then we have ,*

$$\|\psi(t) - P\psi(t)\|_2 \leq \|\psi(0) - P\psi(0)\|_2 e^{-\lambda_2 t}, \quad (3.1)$$

where $\|\cdot\|_2$ denotes the norm in $L^2(\Omega)$ and $\lambda_2 > 0$ is the second eigenvalue of $-\Delta$ in $L^2(\Omega)$ with Neumann boundary conditions .

Proof. We denote by (u, v) the inner product of two functions u, v of $L^2(\Omega)$. Since g is a nondecreasing function, for all $\psi \in L^\infty(\Omega)$, we have a.e. in $x \in \Omega$

$$(g(\psi) - g(P\psi))(\psi - P\psi) \geq 0$$

and then by integrating over Ω

$$(g(\psi), \psi - P\psi) - (g(P\psi), \psi - P\psi) \geq 0. \quad (3.2)$$

Since $g(P\psi)$ is a constant and $(\psi - P\psi) \in N^\perp$, we deduce that $(g(P\psi), \psi - P\psi) = 0$. Hence from (3.2),

$$(g(\psi), \psi - P\psi) \geq 0.$$

Setting

$$w = \psi - P\psi,$$

we have since $\Delta P\psi = P\Delta\psi = 0$

$$w' - \Delta w = \psi' - \Delta\psi - P\psi' + \Delta P\psi = (I - P)(\psi' - \Delta\psi) = -(I - P)g(\psi).$$

Thus, since $(w, (I - P)g(\psi)) = ((I - P)w, g(\psi)) = (w, g(\psi)) = (g(\psi), \psi - P\psi)$ we find

$$\frac{1}{2} \frac{d}{dt} \|w(t)\|_2^2 = (w, \Delta w) - (g(\psi), \psi - P\psi) \leq -\lambda_2 |w|^2.$$

By integrating we obtain (3.1). ■

4 Proof of Theorem 1.1.

We set $\psi = u + w$, where $u = P\psi$ and $w = (I - P)\psi$. By projecting (1.1) on N we obtain

$$u' + P(g(\psi)) = 0, \tag{4.1}$$

where we have used that $P(A\psi) = 0$, since $R(A) \subset N^\perp$. Noticing that

$$u' + P(g(u)) + P(g(\psi) - g(u)) = u' + g(u) + P(g(\psi) - g(u)),$$

we can rewrite the equation (4.1) as

$$u' + g(u) = -P(g(\psi) - g(u)).$$

By the assumption (1.3), we deduce that

$$|P(g(\psi) - g(u))| \leq \frac{1}{|\Omega|} \|g(\psi) - g(u)\|_1 \leq \frac{c}{|\Omega|} (\|\psi\|_{2p-2}^{p-1} + \|u\|_{2p-2}^{p-1}) \|w\|_2.$$

But ψ and u are uniformly bounded and from Proposition 3.1 we have the estimate $\|w(t)\|_2 \leq K e^{-\lambda_2 t}$. Therefore

$$|P(g(\psi) - g(u))| \leq K' e^{-\lambda_2 t},$$

with $K' > 0$. Which leads us to study the equation:

$$u' + g(u) = f(t) \quad \text{in} \quad \mathbb{R}^+, \tag{4.2}$$

where

$$f(t) = P(g(\psi) - g(u))$$

and

$$|f(t)| \leq K' e^{-\lambda_2 t}.$$

Using the same method as in [5], we show the following result:

Lemma 4.1. *Let $c > 0$, $\gamma > 0$, $p > 1$ and g satisfying (1.2) and (1.3) Let $M > 0$ such that*

$$M \leq \left(\frac{\gamma}{2c}\right)^{\frac{1}{p-1}}, \quad (4.3)$$

$c_1 > 0$ with

$$c_1 \leq \frac{\gamma}{2}M,$$

then for every function f satisfying

$$|f(t)| \leq c_1 e^{-\gamma t},$$

there exists a unique function $v \in C^1(\mathbb{R}^+)$ satisfying

$$\forall t \geq 0, \quad v' + g(v) = f(t) \quad (4.4)$$

and

$$\sup_{t \in (0, +\infty)} \{e^{\gamma t} |v(t)|\} \leq M. \quad (4.5)$$

Proof. Since any solution of (4.4).(4.5) satisfies the integral equation

$$v(t) = - \int_t^{+\infty} (f(s) - g(v(s))) ds. \quad (4.6)$$

We look for a solution of (4.6). It is then natural to introduce the following function space :

$$X = \{v \in C(0, +\infty); \sup_{t \in (0, +\infty)} e^{\gamma t} |v(t)| \leq M\},$$

equipped with the distance associated to the norm

$$\|v\|_\gamma = \sup_{t \in (0, +\infty)} e^{\gamma t} |v(t)|. \quad (4.7)$$

We consider the operator $\mathcal{T} : X \rightarrow C(0, +\infty)$ defined by

$$\mathcal{T}v(t) = - \int_t^{+\infty} (f(s) - g(v(s))) ds.$$

From (1.4), we have the estimate

$$\forall s \in \mathbb{R}^+; \quad |g(v(s))| \leq \frac{c}{p} |v(s)|^p.$$

First we will show that $\mathcal{T}(X) \subset X$. Let $v \in X$, then for all $t \geq 0$,

$$\begin{aligned}
|\mathcal{T}v(t)| &\leq \int_t^{+\infty} |f(s)|ds + \int_t^{+\infty} |g(v(s))|ds \\
&\leq \int_t^{+\infty} |f(s)|ds + \frac{c}{p} \int_t^{+\infty} |v(s)|^p ds \\
&\leq \frac{c_1}{\gamma} e^{-\gamma t} + \frac{c}{p} M^p \int_t^{+\infty} e^{-p\gamma s} ds \\
&\leq \left(\frac{c_1}{\gamma} + \frac{cM^p}{p^2\gamma} \right) e^{-\gamma t} \\
&\leq \left(\frac{M}{2} + Mc \frac{M^{p-1}}{p^2\gamma} \right) e^{-\gamma t} \\
&\leq \left(\frac{M}{2} + \frac{M}{2p^2} \right) e^{-\gamma t}.
\end{aligned}$$

Since $p > 1$, it follows that

$$|\mathcal{T}v(t)| \leq M e^{-\gamma t}.$$

Hence by (4.5), we obtain that $\mathcal{T}v \in X$, with

$$\|\mathcal{T}v(t)\|_\gamma \leq M.$$

Secondly, we will prove that $\mathcal{T}$ is a contraction on X . In fact, for $x, \bar{x} \in X$ and for all $t \geq 0$

$$\begin{aligned}
|\mathcal{T}x(t) - \mathcal{T}\bar{x}(t)| &\leq \int_t^{+\infty} |g(x(s)) - g(\bar{x}(s))|ds \\
&\leq cM^{p-1} \int_t^{+\infty} e^{-p\gamma s} e^{\gamma s} |x(s) - \bar{x}(s)|ds \\
&\leq \frac{cM^{p-1}}{p\gamma} \|x - \bar{x}\|_\gamma e^{-\gamma t}.
\end{aligned}$$

Then we have

$$|\mathcal{T}x(t) - \mathcal{T}\bar{x}(t)| e^{\gamma t} \leq \frac{cM^{p-1}}{p\gamma} \|x - \bar{x}\|_\gamma.$$

Therefore, since M^{p-1} satisfies (4.3), we conclude that $\forall x, \bar{x} \in X$

$$\|\mathcal{T}x - \mathcal{T}\bar{x}\|_\gamma \leq \frac{1}{2} \|x - \bar{x}\|_\gamma.$$

Thus $\mathcal{T}$ is a $\frac{1}{2}$ -Lipschitz functional on the complete metric space X and the result follows from the Banach fixed point theorem. From (4.6) it follows easily that v satisfies (4.4). Then the uniqueness of v follows from the uniqueness of the solution of (4.6) ($\mathcal{T}$ is a contraction) and the fact that any solution of (4.4) satisfies (4.6). The existence comes from the fact that conversely any solution of (4.6) satisfies (4.4). ■

Proof of Theorem 1.1 continued. Consequently, we have a solution v that satisfies the equation (4.4) and for all $t \geq T_0$

$$|v(t)| \leq Me^{-\lambda_2 t}, \quad (4.8)$$

where $M = M'e^{\gamma T_0}$ and $M' > 0$. If we subtract (4.4) from (4.2) we obtain

$$(u - v)' + g(u) - g(v) = 0.$$

Setting $z = u - v$, we complete the proof analyzing two cases.

1st case: If $z(T_0) = 0$, then for all $t \geq T_0$, $z(t) = 0$. Hence $u \equiv v$ and from (4.8) it follows that

$$|u(t)| \leq Me^{-\lambda_2 t}.$$

Then, using (3.1), we obtain

$$\|\psi(t)\|_2 \leq M'e^{-\lambda_2 t}.$$

Finally by reasoning as in [5], [6] we obtain (1.6).

2nd case: If $z(T_0) \neq 0$ then $\forall t \geq T_0$, $z(t) \neq 0$ and we have

$$z'(t) + \frac{g(u(t)) - g(v(t))}{u(t) - v(t)} z(t) = 0.$$

Since g is a monotonic function,

$$\alpha(t) := \frac{g(u(t)) - g(v(t))}{u(t) - v(t)}$$

is a strictly positive function. Moreover, $\exists \theta \in]0, 1[$,

$$\alpha(t) = g'(\theta u(t) + (1 - \theta)v(t)) \leq c|\theta u(t) + (1 - \theta)v(t)|^{p-1} \quad (4.9)$$

We distinguish 2 cases:

– $p > 2$ then by convexity of the p th power we have

$$|\theta u(t) + (1 - \theta)v(t)|^{p-1} \leq \theta|u|^{p-1} + (1 - \theta)|v|^{p-1} \leq |u|^{p-1} + |v|^{p-1}.$$

– $1 < p < 2$ we study the function $(x+y)^a - x^a$ for $0 < a < 1$ and $x, y > 0$ we prove that $X \longrightarrow (1 + X)^a - X^a$ is a decreasing function on $(0, +\infty)$, we deduce $0 < (1 + X)^a - X^a < 1$ and it follows by homogeneity that $(x + y)^a < x^a + y^a$ by letting $X = \frac{x}{y}$. Then we conclude that $|\theta u(t) + (1 - \theta)v(t)|^{p-1} \leq |u|^{p-1} + |v|^{p-1}$. Consequently we obtain

$$\forall p > 1, \quad \alpha(t) \leq c(|u|^{p-1} + |v|^{p-1}). \quad (4.10)$$

Setting $y = |z|$, we have that

$$y' + \alpha(t)y \geq 0.$$

Then the estimate (4.10) implies that

$$y' \geq -c(|u|^{p-1} + |v|^{p-1})y \geq -c(|z + v|^{p-1} + |v|^{p-1})y.$$

Hence there exists some constants $c_2, c_3 > 0$ such that

$$y' \geq -c_2(|z|^{p-1})y - c_3|v|^{p-1}y.$$

Since $y = |z|$,

$$y' \geq -c_2y^p - c_3|v|^{p-1}y.$$

Putting $a(t) = c_3|v|^{p-1}$, we deduce that

$$y' + a(t)y \geq -c_2y^p. \quad (4.11)$$

We set

$$A(t) = -c_3 \int_t^{+\infty} |v|^{p-1} ds, \quad \omega(t) = e^{A(t)}y$$

and by replacing ω in (4.11), we obtain

$$\omega(t) \geq \left\{ \frac{1}{\omega(0)^{1-p} + (p-1)c_4t} \right\}^{\frac{1}{p-1}}$$

with $c_4 = \left(\int_0^{+\infty} a(s) ds \right)^{-(p-1)}$. Then for t large enough we have

$$\omega(t) \geq Kt^{-\frac{1}{p-1}}.$$

Since $t \longrightarrow e^{A(t)}$ is a bounded function, we conclude (1.7) by observing that $u = z + v$ and v tends to 0 exponentially at infinity.

■

5 Proof of Theorem 1.2.

Considering $g(\psi) = |\psi|^{p-1}\psi$, g satisfies (1.2) and (1.3) with $c = p$. Hence Lemma 4.1 is applicable with $c = p$, therefore we assume

$$M \leq \left(\frac{\lambda_2}{2p} \right)^{\frac{1}{p-1}}.$$

In (4.2), we replace $g(\psi) = |\psi|^{p-1}\psi$, we can subtract (4.4) from (4.2), we deduce:

$$(u - v)' + |u|^{p-1}u - |v|^{p-1}v = 0. \quad (5.1)$$

We will study two cases.

1st case: If $z(T_0) = 0$ then for all $t \geq T_0$, $z(t) = 0$. Hence $u \equiv v$ and from (4.8) it follows that

$$|u(t)| \leq M e^{-\lambda_2 t}.$$

Moreover, using (3.1), we obtain (1.8).

2nd case: if $z(T_0) \neq 0$ then for all $t \geq T_0$ $z(t) \neq 0$ and we have

$$z'(t) + |u(t)|^{p-1}u(t) - |v(t)|^{p-1}v(t) = z'(t) + \alpha(t)z(t) = 0.$$

with

$$\begin{aligned} \alpha(t) &= \frac{|u(t)|^{p-1}u(t) - |v(t)|^{p-1}v(t)}{u(t) - v(t)} \\ &= \frac{|z(t) + v(t)|^{p-1}(z(t) + v(t)) - |v(t)|^{p-1}v(t)}{z(t)} \\ &= \frac{|z(t)|^{p-1} \left| 1 + \frac{v(t)}{z(t)} \right|^{p-1} (z(t) + v(t)) - |v(t)|^{p-1}v(t)}{z(t)}. \end{aligned}$$

In that case $\alpha(t) > 0$, indeed $t \mapsto |u(t)|^{p-1}u(t)$ is non decreasing function. By applying Taylor's formula, we obtain

$$\alpha(t) = |z(t)|^{p-1} + \beta(t), \quad (5.2)$$

with $|\beta(t)| \leq B e^{-\eta t}$, where $\eta > 0$ is any positive number smaller than $(p-1)\lambda_2$ and $B > 0$. Replacing α by its expression in (5.2), equation (5.1) becomes

$$z' + |z|^{p-1}z + \beta(t)z = 0. \quad (5.3)$$

Let $y = |z|$, we obtain

$$y' + y^p + \beta(t)y = 0. \quad (5.4)$$

Setting $\xi(t) = e^{A(t)}y(t)$, with $A(t) = -\int_t^{+\infty} \beta(s)ds$ we find

$$|A(t)| \leq \int_t^{+\infty} |\beta(s)|ds \leq \frac{B}{\eta}e^{-\eta t}. \quad (5.5)$$

By Taylor's formula we have

$$\forall h \in [-1, 1], \quad |e^h - 1| \leq 2|h|, \quad (5.6)$$

we can give the following estimate

$$|\xi(t) - y(t)| \leq y(t)|e^{A(t)} - 1| \leq 2|A(t)|y(t) \leq \frac{2B}{\eta}\|y\|_{\infty}e^{-\eta t},$$

we conclude

$$|\xi(t) - y(t)| \leq ke^{-\delta t}, \quad (5.7)$$

where $\delta = \eta$ and $k = \frac{2B}{\eta}\|y\|_{\infty}$.

Replacing $\xi(t)$ in (5.3), we have

$$\begin{aligned} -e^{-A(t)}\xi'(t) &= e^{-pA(t)}\xi^p(t) \\ \Leftrightarrow -\frac{\xi'(t)}{\xi^p(t)} &= e^{(1-p)A(t)} \\ \Leftrightarrow \left(\frac{1}{\xi^{p-1}(t)}\right)' &= (p-1)e^{(1-p)A(t)}. \end{aligned}$$

$t \mapsto e^{-(p-1)A(t)}$ is bounded and tends to 1 at infinity, ξ is given by

$$\xi(t)^{p-1} = \frac{1}{\xi(t_0)^{1-p} + (p-1) \int_{t_0}^t e^{-(p-1)A(s)}ds}.$$

we set

$$D(t) = \xi(t_0)^{1-p} + (p-1) \int_{t_0}^t e^{-(p-1)A(s)}ds \quad \text{and} \quad h(t) = -(p-1)A(t),$$

then we show that $D(t) - (p-1)t$ is bounded.

From (5.5), we know that $t \mapsto h(t)$ is an integrable function, then by (5.6), $(e^{-(p-1)A(t)} - 1)$ is also integrable. In order to show (1.9), we proceed

as follows

$$\begin{aligned}
|D(t) - (p-1)t| &= |\xi(t_0)^{1-p} + (p-1) \int_{t_0}^t e^{-(p-1)A(s)} ds - (p-1)t| \\
&= |\xi(t_0)^{1-p} + (p-1) \int_{t_0}^t (e^{-(p-1)A(s)} - 1) ds - (p-1)t_0|
\end{aligned}$$

Using (5.6) we obtain

$$\begin{aligned}
|D(t) - (p-1)t| &\leq |K| + 2(p-1)^2 \int_{t_0}^t |A(s)| ds \\
&\leq M_1 = |K| + 2(p-1)^2 \frac{B}{\eta^2}
\end{aligned}$$

with $K = \xi(t_0)^{1-p} - (p-1)t_0$. Setting $d(t) = D(t) - (p-1)t$, we obtain

$$\begin{aligned}
\left| \xi(t) - \left(\frac{1}{(p-1)t} \right)^{\frac{1}{(p-1)}} \right| &= \left| \left(\frac{1}{D(t)} \right)^{\frac{1}{(p-1)}} - \left(\frac{1}{(p-1)t} \right)^{\frac{1}{(p-1)}} \right| \\
&= \left| \left(\frac{1}{D(t) - (p-1)t + (p-1)t} \right)^{\frac{1}{(p-1)}} - \left(\frac{1}{(p-1)t} \right)^{\frac{1}{(p-1)}} \right| \\
&= \left| \left(\frac{1}{d(t) + (p-1)t} \right)^{\frac{1}{(p-1)}} - \left(\frac{1}{(p-1)t} \right)^{\frac{1}{(p-1)}} \right| \\
&= \left| \left(\frac{1}{(p-1)t} \right)^{\frac{1}{(p-1)}} \left(\frac{1}{\frac{d(t)}{(p-1)t} + 1} \right)^{\frac{1}{(p-1)}} - \left(\frac{1}{(p-1)t} \right)^{\frac{1}{(p-1)}} \right| \\
&= \left(\frac{1}{(p-1)t} \right)^{\frac{1}{(p-1)}} \left| \left(\frac{d(t)}{(p-1)t} + 1 \right)^{-\frac{1}{(p-1)}} - 1 \right| \\
&= \left(\frac{1}{(p-1)t} \right)^{\frac{1}{(p-1)}} \left(1 - \left(\frac{d(t)}{(p-1)t} + 1 \right)^{-\frac{1}{(p-1)}} \right).
\end{aligned}$$

Since $\left| \frac{d(t)}{(p-1)t} \right| < 1$ for t large enough. Let $\eta(t) = \left(1 + \frac{d(t)}{(p-1)t} \right)^{-\frac{1}{(p-1)}}$, by the mean value Theorem and if we suppose that $\left| \frac{d(t)}{(p-1)t} \right| \leq \frac{1}{2}$, we obtain

$$|\eta'(t)| \leq \frac{1}{p-1} \times 2^{1+\frac{1}{p-1}}.$$

Therefore

$$\left| 1 - \left(1 + \frac{d(t)}{(p-1)t} \right)^{-\frac{1}{(p-1)}} \right| \leq \frac{1}{p-1} \times 2^{1+\frac{1}{p-1}} \left| \frac{d(t)}{(p-1)t} \right|.$$

As we have seen above, $d(t)$ is bounded by M_1 then we conclude

$$\left| \xi(t) - \left(\frac{1}{(p-1)t} \right)^{\frac{1}{(p-1)}} \right| \leq C t^{-1-\frac{1}{p-1}}.$$

With $C = \left(\frac{1}{p-1} \right)^{\frac{p}{p-1}} \times 2^{1+\frac{1}{p-1}} M_1$.

We recall that z has a constant sign on $[T_0, +\infty[$ and z and u have the same sign. As in Section 4, we set $u = v + z$ and $\psi = u + w$, we distinguish two cases

- If $z > 0$, this implies that for t large enough $u > 0$ and $|\psi| = \psi$.
Then

$$\left| |\psi| - \left(\frac{1}{(p-1)t} \right)^{\frac{1}{(p-1)}} \right| \leq \left| u - \left(\frac{1}{(p-1)t} \right)^{\frac{1}{(p-1)}} \right| + |w| \leq K t^{-1-\frac{1}{p-1}}.$$

Indeed,

$$\begin{aligned} \left| u - \left(\frac{1}{(p-1)t} \right)^{\frac{1}{(p-1)}} \right| &\leq |u - z| + |z - \xi| + \left| \xi - \left(\frac{1}{(p-1)t} \right)^{\frac{1}{(p-1)}} \right| \\ &\leq M e^{-\lambda_2 t} + k e^{-\delta t} + C t^{-1-\frac{1}{p-1}}. \end{aligned}$$

Since we have (5.7), we obtain (1.9).

- If we suppose that $z < 0$, then implies that $u < 0$. By similar calculations we obtain the same result.

Indeed, $|\psi| = -\psi$, and we have

$$\begin{aligned} \left| |\psi| - \left(\frac{1}{(p-1)t} \right)^{\frac{1}{(p-1)}} \right| &\leq \left| -u - \left(\frac{1}{(p-1)t} \right)^{\frac{1}{(p-1)}} \right| + |w| \\ &\leq Kt^{-1-\frac{1}{p-1}} \end{aligned}$$

since

$$\begin{aligned} \left| -u - \left(\frac{1}{(p-1)t} \right)^{\frac{1}{(p-1)}} \right| &\leq |u - z| + |-z - \xi| + \left| \xi - \left(\frac{1}{(p-1)t} \right)^{\frac{1}{(p-1)}} \right| \\ &\leq Me^{-\lambda_2 t} + ke^{-\delta t} + Ct^{-1-\frac{1}{p-1}}. \end{aligned}$$

Also since we have (5.7), finally we obtain (1.9).

6 Proof of Proposition 1.3

(i) If ψ is an odd function, then $\psi(0, -x) = -\psi(0, x)$. It implies for all $t > 0$, $\psi(t, -x) = -\psi(t, x)$.

In that case

$$u(t) = P\psi(t, x) = \frac{1}{|\Omega|} \int_{\Omega} \psi(t, x) dx = 0.$$

Moreover, we know that $\psi = u + w$, where $w(t) \in N^{\perp}$ for all $t \geq 0$ and (cf. Proposition 3.1) we have

$$\|w(t)\|_{L^{\infty}(\Omega)} \leq Ke^{-\lambda_2 t},$$

hence the solution ψ satisfies (1.8).

(ii) If $\psi(0, x) \geq 0$ and ψ does not vanish a.e in Ω , then for all $t \geq 0$ $\psi(t, x) > 0$, it implies that

$$\int_{\Omega} \psi(t, x) dx > 0. \tag{6.1}$$

We suppose that we have $\|\psi(t, \cdot)\|_\infty \leq Ce^{-\lambda_2 t}$ and we consider the problem (1.1) then we integrate on Ω , we obtain

$$\int_{\Omega} \psi_t(t, x) dx = - \int_{\Omega} g(\psi(t, x)) dx. \quad (6.2)$$

An elementary calculation shows that we have

$$\begin{aligned} \int_{\Omega} g(\psi(t, x)) dx &\leq \frac{c}{p} \int_{\Omega} |\psi(t, x)|^p dx \\ &\leq \frac{c}{p} \int_{\Omega} |\psi(t, x)|^{p-1} \psi(t, x) dx \\ &\leq \frac{c}{p} \int_{\Omega} \|\psi(t, x)\|_\infty^{p-1} \psi(t, x) dx \\ &\leq \frac{c}{p} C^{p-1} e^{-(p-1)\lambda_2 t} \int_{\Omega} \psi(t, x) dx \end{aligned}$$

Now we set $y(t) = \int_{\Omega} \psi(t, x) dx$. From (6.2), we deduce

$$y'(t) \geq -Me^{-\delta t} y(t) \quad (6.3)$$

with $M = \frac{c}{p} C^{p-1}$ and $\delta = (p-1)\lambda_2$.

Since $y(t) > 0$ by (6.1), we can integrate in the interval $[0, t]$, we obtain

$$y(t) \geq y(0) \exp \left\{ -M \int_0^t e^{-\delta s} ds \right\} \geq y(0) \exp \left\{ -\frac{M}{\delta} \right\} > 0. \quad (6.4)$$

Hence y does not tend to 0 for t large, this contradicts our hypothesis and we conclude that y satisfies (1.7).

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