18th ICPR paper: ANALYSIS OF STANDARD ORDERING POLICIES WITHIN THE FRAMEWORK OF MRP THEORY

Robert W Grubbstrom, Thi Thu Thuy Huynh

18th ICPR paper: ANALYSIS OF STANDARD ORDERING POLICIES WITHIN THE FRAMEWORK OF MRP THEORY

Journal:	*International Journal of Production Research*
Manuscript ID:	TPRS-2005-IJPR-0467
Manuscript Type:	Original Manuscript
Date Submitted by the Author:	23-Nov-2005
Complete List of Authors:	Grubbstrom, Robert; Linkoping Institute of Technology, Dept of Production Economics Huynh, Thi Thu Thuy; Linköping Institute of Technology, Dept of Production Economics
Keywords:	PRODUCTION PLANNING, MRP
Keywords (user):	

ANALYSIS OF STANDARD ORDERING POLICIES WITHIN THE FRAMEWORK OF MRP THEORY

Robert W. Grubbström* and Thi Thu Thuy Huynh

Department of Production Economics

Linköping Institute of Technology

S-581 83 Linköping, Sweden

rwg@ipe.liu.se and thuy.huynh@ipe.liu.se

Abstract

A number of different standard ordering policies are presented within the methodology of Material Requirements Planning (MRP), such as Lot-For-Lot (L4L), Fixed Order Quantity (FOQ), Fixed Period Requirements (FPR), etc.

In MRP Theory the time development of the production-inventory system is determined by a set of fundamental equations for available inventory, total inventory and backlogs using Input-Output Analysis for capturing the Bill of Materials and Laplace transforms for describing the advanced timing requirements.

This paper aims at formally introducing standard ordering policies into the fundamental equations of MRP Theory in order to analyse the possibility to obtain closed-form expressions for the time development of the system, when such rules are applied.

Keywords

Material requirements planning, ordering policy, lot-for-lot, fixed order quantity, fixed period requirements, Laplace transform, input-output analysis.

1 INTRODUCTION

The ordering policies in an MRP system have a direct impact on the outcome data. In for instance Orlicky (1975), some standard ordering policies such as Lot-For-Lot (L4L), Fixed Order Quantity (FOQ), Fixed Period Requirements (FPR) are defined. The MRP-calculation in Orlicky (1975) for operations management and production economics in tables are described in formulas in Segerstedt (1996).

The fundamental equations of MRP Theory have been developed in several earlier papers, beginning with Grubbström and Ovrin (1992) and in some earlier unpublished studies. These equations are balance equations in the frequency domain explaining the development of total inventory, available inventory, backlogs and allocations. Input-Output Analysis is used for capturing the Bill of Materials and Laplace transforms for describing the advanced timing requirements (lead times). In Grubbström and Tang (2000) an overview is presented.

The current paper analyses the question of developing closed-form expressions for production when applying basic ordering rules. Order sizes are to be decided in time when inventories are zero or near to zero.

In Grubbström and Ovrin (1992) the problem treated here was touched upon. In that paper the processes took place in discrete time, and the z-transformation was applied. Grubbström and Molinder (1994) and Molinder (1995) also followed up some research into this issue in the continuous time case.

Below, we will limit our attention to the three ordering policies L4L, FOQ, and FPR. We will also be limiting our attention to deterministic situations.

The L4L rule, being the simplest (also called "as required") involves an order to be placed exactly large enough to cover requirements. This means that available inventory is kept at a zero level (assuming no initial inventories). In L4L, both the size of the order and the interval between orders will vary over time in a general case. Production, by this rule, is therefore adapted directly to requirements.

With the FOQ policy, the order size is always the same. An order is placed as soon as there is not enough available inventory to cover requirements. If demand fluctuates, the interval between orders also must fluctuate. In general, available inventory will only reach a zero level on occasion.

Applying the FPR policy, the interval during which total demand is generated and which the order should cover is constant. An order is placed just large enough to cover the total requirements during each such interval. At the end of these intervals, available inventory will have dropped to zero.

This paper is structured as follows. We present a brief overview of the fundamental equations of MRP Theory in Section 2. This is followed by deriving the basic properties of production for the three ordering policies in relation to this theory in Section 3. Section 4 and 5 are devoted to solving for the production development.

The following basic notation is used:

s	complex Laplace frequency.
$D(t)$	external demand
$P(t)$	Production
d_j	coefficient of Laurent expansion, $j = \ldots -2,-1, 0, 1, 2,\ldots$
$R(t)$	available inventory
τ	lead time
$\mathbf{H}$	input matrix (Bill-Of-Materials)

$\hat{D}(t)$	requirements (external + internal demand)
$\hat{n}$	number of production batches during period considered
t_j, T_j	time of requirements event or production event

Tildes or the operator $\pounds\{\cdot\}$ are used to denote Laplace transforms of the corresponding time functions, i. e. $\tilde{f}(s)=\pounds\{f(t)\}=\int_0^\infty f(t)e^{-st}dt=\sum_{n=0}^{\infty} f_n e^{-st_n}$ for a continuous time function $f(t)$, and bars are used for denoting cumulative functions, like $\bar{f}(t)=\int_0^t f(\alpha)d\alpha$. Boldface characters represent vectors and matrices.

2 THE FUNDAMENTAL EQUATIONS OF MRP THEORY

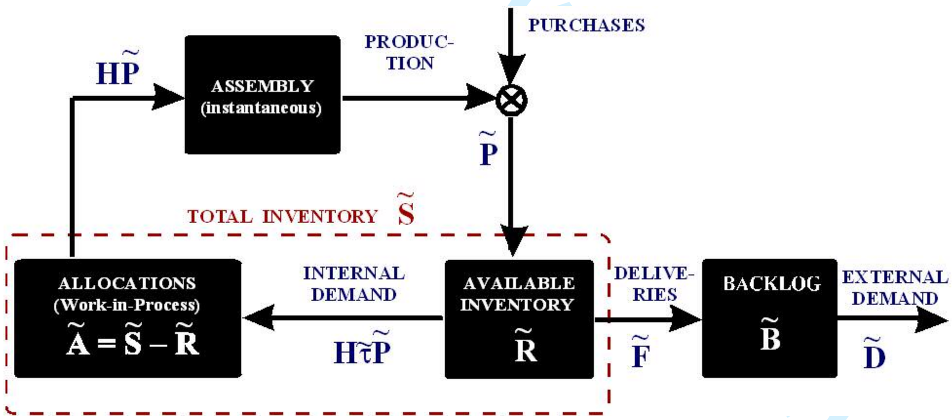

Figure 1. Flowchart of the fundamental equations of MRP Theory (Grubbström and Tang 2000).

The fundamental equations of MRP Theory are balance equations describing the time development of total inventory, available inventory, backlogs, and allocations (see Grubbström and Tang, 2000). With the policies we are studying, backlogs will not occur in deterministic demand cases. In such cases, the fundamental equations for total inventory and available inventory may be written:

$$\tilde{\mathbf{S}}(s) = \frac{\mathbf{S}(0) + \left(\mathbf{I} - \mathbf{H}\right)\tilde{\mathbf{P}}(s) - \tilde{\mathbf{D}}(s)}{s}, \tag{1}$$

$$\tilde{\mathbf{R}}(s) = \frac{\mathbf{R}(0) + \left(\mathbf{I} - \mathbf{H}\tilde{\boldsymbol{\tau}}(s)\right)\tilde{\mathbf{P}}(s) - \tilde{\mathbf{D}}(s)}{s}, \tag{2}$$

where $\tilde{\mathbf{S}}$ is the column vector of items in inventory (including allocations), $\tilde{\mathbf{R}}$ is the column vector of items in available inventory (total inventory less allocations), $\tilde{\mathbf{D}}$ is the vector of external demand, $\tilde{\mathbf{P}}$ is the vector of items produced, $\mathbf{H}$ is the input matrix (the Bill of Materials), and $\mathbf{I}$ is the identity matrix. The lead time matrix $\tilde{\boldsymbol{\tau}}(s)$ is a matrix with lead time operators in its diagonal positions:

$$\tilde{\boldsymbol{\tau}}(s) = \begin{bmatrix} e^{s\tau_1} & 0 & \cdots & 0 \\ 0 & e^{s\tau_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{s\tau_N} \end{bmatrix}. \tag{3}$$

Internal (dependent) demand is given by $\mathbf{H}\tilde{\boldsymbol{\tau}}(s)\tilde{\mathbf{P}}(s)$, since the input matrix and lead time matrix together determine all advanced requirements of sub-components in amount and timing, given a production schedule $\tilde{\mathbf{P}}(s)$.

The main problem treated in this chapter, is how production $\tilde{\mathbf{P}}$ is determined by each of the policies, when facing a given external (independent) demand $\tilde{\mathbf{D}}$.

In one-item systems, there is no internal (dependent) demand and, thus, independent and dependent demand, on the one hand, and available and total inventory, on the other, will coincide. The fundamental equations (1)-(2) then collapse into:

$$\tilde{R}(s) = \frac{R(0) + \tilde{P}(s) - \tilde{D}(s)}{s}. \tag{4}$$

Properties of cumulative requirements

The minimum necessary production to meet external (independent) requirements is always given by

$$(\mathbf{I} - \mathbf{H}\tilde{\boldsymbol{\tau}})^{-1}\tilde{\mathbf{D}}(s) = \left(\mathbf{I} + \mathbf{H}\tilde{\boldsymbol{\tau}} + (\mathbf{H}\tilde{\boldsymbol{\tau}})^2 + (\mathbf{H}\tilde{\boldsymbol{\tau}})^3 + \dots\right)\tilde{\mathbf{D}}(s), \tag{5}$$

where the Neumann expansion has been used (Grubbström, 1999). This expansion is valid as long as the numerical values of all characteristic roots of **H** are less than unity. For assembly systems, in which **H** is triangular with zeros along its main diagonal and above, this is indeed so. The expansion will converge in at most N terms, where N is the dimension of **H**.

Assuming only one end product at the top level, we have

$$\tilde{\mathbf{D}}(s) = \begin{bmatrix} \tilde{D}_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \tag{6}$$

and minimum cumulative requirements become

$$\left[(\mathbf{I} - \mathbf{H}\tilde{\boldsymbol{\tau}})^{-1}\right]_{col1} \tilde{D}_1 / s = \left[\mathbf{I} + \mathbf{H}\tilde{\boldsymbol{\tau}} + (\mathbf{H}\tilde{\boldsymbol{\tau}})^2 + (\mathbf{H}\tilde{\boldsymbol{\tau}})^3 + \dots\right]_{col1} \tilde{D}_1 / s \tag{7}$$

The triangular nature of **H** for an assembly system creates the following first elements

$$\left[(\mathbf{I} - \mathbf{H}\tilde{\boldsymbol{\tau}})^{-1}\right]_{row1} \tilde{\mathbf{D}} / s = \tilde{D}_1 / s \tag{8}$$

$$\left[(\mathbf{I} - \mathbf{H}\tilde{\boldsymbol{\tau}})^{-1}\right]_{row2} \tilde{\mathbf{D}} / s = e^{s\tau_1} H_{21} \tilde{D}_1 / s \tag{9}$$

$$\left[\left(\mathbf{I}-\mathbf{H}\tilde{\boldsymbol{\tau}}\right)^{-1}\right]_{row\,3}\tilde{\mathbf{D}}/s = e^{s\tau_1}\left(H_{13}+H_{32}H_{21}e^{s\tau_2}\right)\tilde{D}_1/s \tag{10}$$

$$\left[\left(\mathbf{I}-\mathbf{H}\tilde{\boldsymbol{\tau}}\right)^{-1}\right]_{row\,4}\tilde{\mathbf{D}}/s = e^{s\tau_1}\left(H_{14}+H_{42}H_{21}e^{s\tau_2}+H_{43}H_{31}e^{s\tau_3}+H_{43}H_{32}H_{21}e^{s(\tau_2+\tau_3)}\right)\tilde{D}_1/s \tag{11}$$

This reveals that requirements occur at times in advance of external demand in such a way that steps are generated sums of lead times ahead of top-level requirements. The further down in the product structure tree, the more opportunities for additional steps exist. For instance, on level 4, if all relevant H_{ij} are non-zero, then, generated by an external demand event at t, items will be required at the points in time $t-\tau_1$, $t-\tau_1-\tau_2$, $t-\tau_1-\tau_3$, and $t-\tau_1-\tau_2-\tau_3$, whereas on level 3, items may be required only at $t-\tau_1$ and $t-\tau_1-\tau_2$. The total number of possible times increases geometrically, but all need not exist. One may also note that all possible times at a higher level are repeated again at all lower levels. Therefore the set of possible times for the entire system can be found by studying the lowest level in the system. But zeros in the matrix below the diagonal will rule out some combinations.

A sequence of external demand events together with the set of lead times will thus generate a sequence of possible internal demand events. On levels above the lowest level, the sequence will be a subset of the sequence at the bottom level. Certain points in time may be covered more than once at least two reasons, (i) if external demand includes events at distances equal to combinations of lead times, and (ii) if combinations of lead times happen to be equal (such as if τ_2 were equal to τ_3 above).

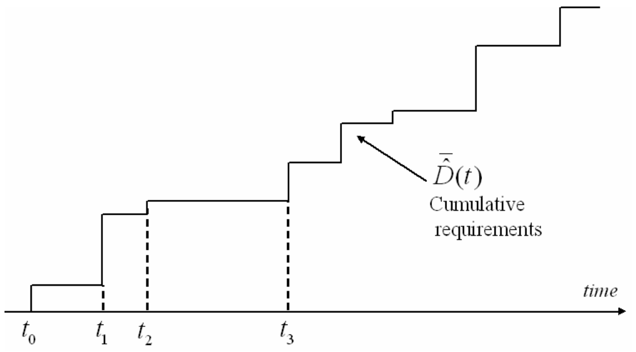

Figure 2. Cumulative requirements as a staircase.

In total, we can therefore regard cumulative requirements on any level (including the top level) to be made up of a staircase of steps occurring at the times t_0, t_1, t_2, … , of which several steps may have a zero height.

These requirements are given by

$$\tilde{\tilde{\hat{\mathbf{D}}}}(s) = \left(\mathbf{H}\tilde{\boldsymbol{\tau}}(s)\tilde{\mathbf{P}}(s) + \tilde{\mathbf{D}}\right)/s\,, \tag{12}$$

for a general production policy, and in the general L4L case by (see Section 3.1 below)

$$\tilde{\tilde{\hat{\mathbf{D}}}}(s) = \left(\mathbf{I} - \mathbf{H}\tilde{\boldsymbol{\tau}}(s)\right)^{-1}\tilde{\mathbf{D}}/s \tag{13}$$

When there is only one end item in the L4L case, we have

$$\tilde{\tilde{\hat{\mathbf{D}}}}(s) = \left[\left(\mathbf{I} - \mathbf{H}\tilde{\boldsymbol{\tau}}(s)\right)^{-1}\right]_{col1}\tilde{D}_1/s\,. \tag{14}$$

The staircase is illustrated in Figure 3.2 for an individual item.

3 TIME DEVELOPMENT OF PRODUCTION FOR BASIC ORDERING POLICIES

3.1 Lot-For-Lot (L4L) policy

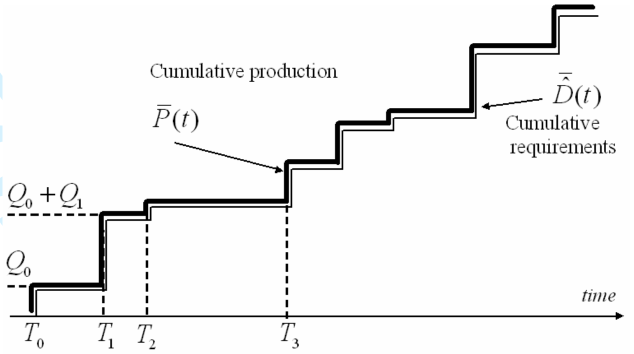

Figure 3. The Lot-for-Lot Policy. Production of an item (thick staircase) follows exactly total requirements (sum of independent and dependent demand, thin staircase).

In the L4L case, with zero initial available inventory **R**(0)=**0**, the solution is perfectly simple, also in the multi-item case with non-zero lead times. It is then just a matter of keeping available inventory at a zero level. This means

$$\tilde{\mathbf{R}}(s) = \frac{\mathbf{R}(0) + \left(\mathbf{I} - \mathbf{H}\tilde{\boldsymbol{\tau}}(s)\right)\tilde{\mathbf{P}}(s) - \tilde{\mathbf{D}}(s)}{s} = \mathbf{0}\,, \tag{15}$$

from which production is determined as

$$\tilde{\mathbf{P}}(s) = \left(\mathbf{I} - \mathbf{H}\tilde{\boldsymbol{\tau}}(s)\right)^{-1}\tilde{\mathbf{D}}(s) = \hat{\mathbf{D}}(s)\,, \tag{16}$$

as is illustrated in Figure 3. If there are non-zero initial available inventories, the expression needs to be adjusted slightly.

3.2 Fixed order quantity (FOQ) case

A fixed order quantity policy may be specified for any item in an MRP system. However, in practice it would be applicable to items with ordering cost sufficiently high to rule out ordering in net requirement quantities, period by period. The replenishments occur as available inventory approaches to zero.

With the FOQ policy, having Q as the fixed order size, production for an individual item will behave according to

$$\tilde{P}(s) = Q\sum_{n=0}^{\hat{n}-1} e^{-sT_n} \ , \qquad (17)$$

where T_n is the time when the nth batch is completed (made available), see Figure 4. These are the variables that need be determined by the policy. When production is other than L4L, the requirements on lower levels will depend on production on levels above. This implies, in the general case, that new sets of possible times of requirement events might be introduced.

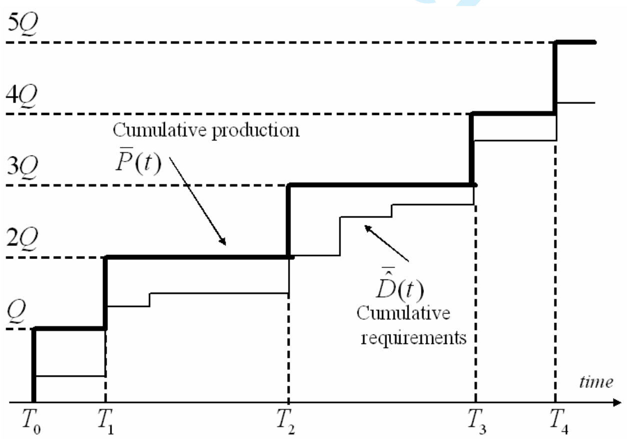

Figure 4. The Fixed Order Quantity Policy (FOQ). Production of an item is made in equally-sized batches as late as possible without avoiding negative available inventory.

Available inventory of this individual item will then be

$$\tilde{R}(s)=\left(R(0)+Q\sum_{n=0}^{\hat{n}-1}e^{-sT_n}-\tilde{\hat{D}}(s)\right)s^{-1}, \tag{18}$$

and the policy is to make all T_n as late as possible without causing $R(t)$ to become negative.

3.2 Fixed period requirements (FPR) case

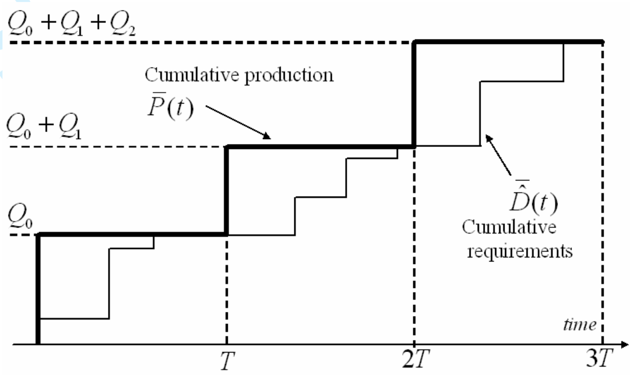

Figure 5. The Fixed Period Requirements Policy (FPR). Production of an item is made in as small batches as possible to cover future requirements during a constant period, without creating a negative available inventory.

Under the FPR policy, the ordering interval is constant and the quantities are allowed to vary. Production of a certain item will thus obey

$$\tilde{P}(s)=\sum_{n=0}^{\hat{n}-1}Q_n e^{-snT} \tag{19}$$

where T denotes the constant time interval between orders and Q_n is the batch size at time $T_n = nT$. The FPR policy is demonstrated in Figure 5.

The FPR policy requires the Q_n to be made as small as possible without violating the non-negativity condition for available inventory. This implies that available inventory is likely to take on a zero value during finite time intervals, which is not the case, other than by chance (or at the end), with the FOQ policy, as seen when comparing Figures 4 and 5.

4 SOLUTIONS TO NON-NEGATIVITY CONDITIONS FOR AVAILABLE INVENTORY WITH REQUIREMENTS AS DISCRETE EVENTS

As shown above, the L4L policy provides an immediate explicit expression for the production on all levels. For the other two policies, this is not equally simple.

In the FOQ case we need to solve for the latest as possible batch times T_0, T_1, T_2, … , such that available inventory $R(t)$ is kept non-negative. The solution in the time domain is simple for the individual item, given the requirements $\bar{\hat{\mathbf{D}}}(t_i)$ at the times when there are steps t_0, t_1, t_2,…

Examining, for successive values of n, $\arg\max_{t_i}\left(\bar{\hat{D}}(t_{i-1}) \le nQ < \bar{\hat{D}}(t_i)\right) \ge 0$, there will be a unique index i assigned to each n, which we denote i_n. So, $T_n = t_{i_n}$ will be the latest time that batch n can be produced. Hence, the solution to the FOQ production staircase is:

$$\tilde{\bar{P}}(s) = Q\sum_{n=0}^{\hat{n}-1} e^{-sT_{i_n}} / s \ . \tag{20}$$

In the FPR case instead, we need to solve for the smallest possible batch size at times 0, T, $2T$, … , not violating the non-negativity of available inventory. The sequence of batches Q_n generated by

$$Q_n = \bar{\hat{D}}((n+1)T) - \sum_{i=0}^{n-1} Q_i \ , \tag{21}$$

or, equivalently,

$$Q_n = \bar{\hat{D}}((n+1)T) - \bar{\hat{D}}(nT), \tag{22}$$

for successive values of $n = 0, 1, 2, \ldots$, will uniquely determine the production staircase satisfying the conditions. Hence, the production staircase becomes:

$$\tilde{\bar{P}}(s) = \sum_{n=0}^{\hat{n}-1} \left(\bar{\hat{D}}((n+1)T) - \bar{\hat{D}}(nT) \right) e^{-snT} / s. \tag{23}$$

Until now, we have investigated consequences of the non-negativity requirements in the time domain.

A corresponding set of non-negativity conditions in the frequency domain is given in Feller (1971) as the following provisions. If a time function $f(t)$ having the transform $\tilde{f}(s)$ is non-negative in the time domain, then the following property must hold

$$(-1)^j \frac{d^j \tilde{f}(s)}{ds^j} = (-1)^j \tilde{f}^{(j)}(s) \geq 0, \tag{24}$$

for all integers $j > 0$ and all real s.

In our case, we are looking at available inventories which for the FOQ policy are given by (assuming initial available inventory to be zero)

$$R(t) = £^{-1}\left\{\tilde{R}(s)\right\} = £^{-1}\left\{\left(Q\sum_{n=0}^{\hat{n}-1} e^{-sT_n} - \tilde{\hat{D}}(s) \right) s^{-1} \right\} =$$

$$= £^{-1}\left\{\left(Q\sum_{n=0}^{\hat{n}-1} e^{-sT_n} - \sum_{l=1}\left(\left(\bar{\hat{D}}(t_l) - \bar{\hat{D}}(t_{l-1}) \right) e^{-st_l} \right) \right) s^{-1} \right\} \geq 0 \tag{25}$$

Choosing the situation for the interval of the ith requirements step and mth batch, and writing

$$\tilde{f}(s) = Q\sum_{n=0}^{m-1} e^{-sT_n} - \sum_{l=1}^{i}\left(\left(\bar{\hat{D}}(t_l) - \bar{\hat{D}}(t_{l-1}) \right) e^{-st_l} \right), \tag{26}$$

$$\tilde{g}(s) = s^{-1}, \tag{27}$$

we apply Euler's formula

$$\left(\tilde{f}(s)\tilde{g}(s)\right)^{(j)} = \sum_{k=0}^{j} \binom{j}{k} \tilde{f}(s)^{(k)} \tilde{g}(s)^{(j-k)} \tag{28}$$

to Eq. (26). Differentiating $\tilde{f}(s)$ and $\tilde{g}(s)$ the number of times required, we obtain

$$\tilde{f}(s)^{(k)} = Q\sum_{n=0}^{m-1}\left(-T_n\right)^k e^{-sT_n} - \sum_{l=1}^{i}\left(-t_l\right)^k \left(\bar{\hat{D}}(t_l) - \bar{\hat{D}}(t_{l-1})\right)e^{-st_l}\,, \tag{29}$$

$$\tilde{g}(s)^{j-k} = (-1)^{j-k}(j-k)!\,s^{-1-j+k}\,, \tag{30}$$

and thus

$$(-1)^j \left(\tilde{f}(s)\tilde{g}(s)\right)^{(j)} = (-1)^j \sum_{k=0}^{j} \frac{j!}{k!(j-k)!} \tilde{f}(s)^{(k)} \tilde{g}(s)^{(j-k)} =$$

$$= \frac{j!}{s^{j+1}}\left(Q\sum_{n=0}^{m-1} e^{-sT_n} \sum_{k=0}^{j} \frac{\left(sT_n\right)^k}{k!} - \sum_{l=1}^{i}\left(\bar{\hat{D}}(t_l) - \bar{\hat{D}}(t_{l-1})\right)e^{-st_l} \sum_{k=0}^{j} \frac{\left(st_l\right)^k}{k!} \right)$$

$$\to \frac{j!}{s^{j+1}}\left(mQ - \bar{\hat{D}}(t_l)\right) \geq 0\,, \tag{31}$$

for large values of j. This again provides the result in (20).

5 SOLUTIONS TO NON-NEGATIVITY CONDITIONS FOR AVAILABLE INVENTORY WITH REQUIREMENTS AS CONTINUOUS EVENTS

In cases when the cumulative requirements are assumed to be a continuous time function given by an analytical expression, we may apply Cauchy's Residue Theorem for solving for production in the FOQ and FPR policy cases. A residue is the coefficient of the first negative power in a *Laurent expansion* around a pole, i. e. where the numerator of an expression evaluates to zero.

For the FOQ policy, we need to solve for the points in time when available inventory drops to zero.

$$R(t)=\pounds\left\{\tilde{R}(s)\right\}^{-1}=\pounds\left\{\left(Q\sum_{n=0}^{\hat{n}-1}e^{-sT_n}-\tilde{\hat{D}}(s)\right)s^{-1}\right\}^{-1}. \tag{32}$$

Assuming zero initial inventories and writing the Laurent expansion of $\tilde{\hat{D}}(s)$ as

$$\tilde{\tilde{\hat{D}}}(s)=\sum_{j=-\infty}^{\infty}d_j s^j \tag{33}$$

for the nth batch, we have

$$R(t)=\frac{1}{2\pi i}\int_{w=\beta-i\infty}^{\beta+i\infty}\left(nQ/w-\tilde{\tilde{\hat{D}}}(w)\right)e^{wt}dw, \tag{34}$$

which evaluated by the Residue Theorem will be

$$R(t)=\frac{1}{2\pi i}\int_{w=\beta-i\infty}^{\beta+i\infty}\left(nQ/w-\tilde{\tilde{\hat{D}}}(w)\right)e^{wt}dw=$$

$$=\operatorname*{Res}_{w=0}\left(\frac{nQ}{w}\right)-\sum_{residues}\left(\tilde{\tilde{\hat{D}}}(w)e^{wt}\right)=$$

$$=nQ-\sum_{residues}\left(\sum_{j=-\infty}^{\infty}d_j w^j\sum_{k=0}^{\infty}\frac{(wt)^k}{k!}\right)=0. \tag{35}$$

As an example, when requirements increase linearly and cumulative requirements therefore increase quadratically, cumulative requirements behave according to

$$\tilde{\tilde{\hat{D}}}(s)=as^{-3},$$

where a is the slope of the linearly increasing requirements. Then, the Laurent expansion collapses into

$$\sum_{j=-\infty}^{\infty}d_j w^j=aw^{-3}, \tag{36}$$

with the only non-zero coefficient $d_{-3}=a$. In this case, the only pole is at $w = 0$, so

$$\operatorname*{Res}_{w=0}\left(\sum_{j=-\infty}^{\infty}d_j w^j\sum_{k=0}^{\infty}\frac{(wt)^k}{k!}\right)=\operatorname*{Res}_{w=0}\left(aw^{-3}\sum_{k=0}^{\infty}\frac{(wt)^k}{k!}\right)=at^2/2!, \tag{37}$$

and

$$R(T) = nQ - aT^2/2 = 0\,. \tag{38}$$

The time of the nth batch will be $T = \sqrt{2nQ/a}$. Hence, cumulative production in the FOQ case will follow

$$\tilde{\tilde{P}}(s) = Q\sum_{n} e^{-s\sqrt{2nQ/a}}/s\,. \tag{39}$$

In the FPR case instead, production has the structure

$$\tilde{P}(s) = \sum_{n=0}^{\hat{n}-1} Q_n e^{-snT}\,, \tag{40}$$

and available inventory will be

$$R(t) = \pounds\left\{\tilde{R}(s)\right\}^{-1} = \pounds\left\{\left(\sum_{n=0}^{\hat{n}-1} Q_n e^{-snT} - \tilde{\hat{D}}(s)\right)s^{-1}\right\}^{-1}. \tag{41}$$

At the end of the nth step of the production staircase (at $t = nT$), we have

$$R(nT) = \frac{1}{2\pi i}\int_{w=\beta-i\infty}^{\beta+i\infty}\left(\sum_{m=0}^{n-1} Q_m/w - \tilde{\tilde{\hat{D}}}(w)\right)e^{wnT}dw\,, \tag{42}$$

which evaluated by the Residue Theorem will be

$$R(nT) = \frac{1}{2\pi i}\int_{w=\beta-i\infty}^{\beta+i\infty}\left(\sum_{m=0}^{n-1} Q_m/w - \tilde{\tilde{\hat{D}}}(w)\right)e^{wnT}dw =$$

$$= \operatorname*{Res}_{w=0}\left(\sum_{m=0}^{n-1} Q_m/w\right) - \sum_{residues}\left(\tilde{\tilde{\hat{D}}}(w)e^{wnT}\right) =$$

$$= \sum_{m=0}^{n-1} Q_m - \sum_{residues}\left(\sum_{j=-\infty}^{\infty} d_j w^j \sum_{k=0}^{\infty}\frac{(wt)^k}{k!}\right) = 0\,. \tag{43}$$

In the quadratically increasing requirement case with cumulative requirements $\tilde{\tilde{\hat{D}}}(s) = as^{-3}$, we thus obtain

$$R(nT) = \sum_{m=0}^{n-1} Q_m - \operatorname*{Res}_{w=0}\left(d_{-3} w^3 \frac{(wnT)^2}{2!} \right) =$$

$$= \sum_{m=0}^{n-1} Q_m - a\frac{(nT)^2}{2!} = 0. \tag{44}$$

The size of the nth batch is therefore

$$Q_n = a\frac{\left((n+1)T\right)^2 - \left(nT\right)^2}{2} = \left(n + 1/2\right) aT^2, \tag{45}$$

and the cumulative production staircase becomes:

$$\tilde{\tilde{P}}(s) = \sum_{n=0}^{\hat{n}-1} Q_n e^{-snT} = aT^2 \sum_{n=0}^{\hat{n}-1} \left(n + 1/2\right) e^{-snT}. \tag{46}$$

6 OPTIMAL FOQ AND OPTIMAL FRP WHEN EXTERNAL DEMAND IS STOCHASTIC

We now assume that external demand is a stochastic process $D(t)$ of the renewal type, i.e.

$$D(t) = \sum_{j=1}^{\infty} \delta(t - \sum_{k=1}^{j} \tau_k), \tag{47}$$

which is made up of sequence of unit impulses $\delta(\cdot)$, i.e. *Dirac delta functions*. Here τ_k is the stochastic interval between the $(k\text{-}1)^{\text{th}}$ and k^{th} demand event, $\tau_k \geq 0$, k = 1, 2, 3... These are considered stochastically independent for different values of k.

Let $£\{f(t)\} = \tilde{f}(s)$ be the Laplace transform of the probability density function of any individual τ_k. From Grubbström (1996), we then obtain the probability of demand during any given interval t to have the value:

$$\Pr(\sum_{k=1}^{Q}\tau_k = t)dt = \left[£^{-1}\left\{ \tilde{f}^{Q}(s) \right\} \right]_t dt\,. \tag{48}$$

The transform of expected cumulative demand is therefore

$$\mathrm{E}(\tilde{\bar{D}}(s)) = \frac{1}{s}\sum_{j=0}^{\infty} j\; \tilde{f}^{j}(1-\tilde{f}) = \frac{1}{s}\cdot\frac{\tilde{f}}{1-\tilde{f}}. \tag{49}$$

Assuming a zero safety stock, the FOQ policy implies that Q is ordered at T_{i+1}, whenever $Q \geq \bar{D}(T_{i+1}) - \bar{D}(T_i)$.

Total production will now have the transform;

$$\tilde{P}(s) = Q\sum_{i=0}^{\infty} e^{-sT_i} = Q + Qe^{-sT_1} + Qe^{-sT_1-sT_2} + Qe^{-sT_1-sT_2-sT_3} + \ldots\;. \tag{50}$$

Because the T_i are independent, we may drop the index i:

$$\mathrm{E}\left[e^{-sT_j}\right] = \mathrm{E}\left[e^{-sT_k}\right] = \mathrm{E}\left[e^{-sT}\right]. \tag{51}$$

We also have

$$\mathrm{E}\left[e^{-sT}\right] = \int_{T=0}^{\infty} \Pr(\sum_{k=1}^{Q}\tau_k = T)e^{-sT}dT = \int_{T=0}^{\infty}\left[£^{-1}\left\{ \tilde{f}(s)^{Q} \right\} \right]_T e^{-sT}dT = \tilde{f}(s)^{Q}, \tag{52}$$

so that expected total production obeys:

$$\mathrm{E}\left[\tilde{P}(s)\right] = Q\left(1+\mathrm{E}\left[e^{-sT_1}\right] + \mathrm{E}\left[e^{-sT_1}\right]\mathrm{E}\left[e^{-sT_2}\right] + \ldots\right) = \frac{Q}{1-\mathrm{E}\left[e^{-sT}\right]} = \frac{Q}{1-\tilde{f}(s)^{Q}}. \tag{53}$$

Let $\nu(t)$ denote the setup frequency (Molinder, 1995),

$$\nu(t) = \sum_{i=0}^{\infty}\delta\left(t - \sum_{j=0}^{i} T_j\right), \tag{54}$$

where $\delta(\cdot)$ again denotes the Dirac delta function. Then we have the expected setup frequency:

$$\mathrm{E}\left[\tilde{v}(s)\right]=\frac{1}{1-\mathrm{E}\left[e^{-sT}\right]}=\frac{1}{1-\tilde{f}(s)^{Q}}\ . \tag{55}$$

By multiplying $\mathrm{E}\left[\tilde{P}(s)\right]$ by s and taking the limit $s\to 0$, we obtain the long-term average of production:

$$P_{average}=\lim_{s\to 0}s\,\mathrm{E}\left[\tilde{P}(s)\right]=\lim_{s\to 0}\frac{sQ}{1-\tilde{f}(s)^{Q}}=\lim_{s\to 0}\frac{Q}{-\tilde{f}'(s)Q\tilde{f}(s)^{Q-1}}\ . \tag{56}$$

But from the moment generating property of the transform (Grubbström and Tang, 2006), we also have

$$\tilde{f}(0)=1,$$

$$\tilde{f}'(0)=-\lim_{s\to 0}\tilde{f}'(s)=-\lim_{s\to 0}\int_{t=0}^{\infty}(-t)f(t)e^{-st}=-\mathrm{E}\left[\tau_j\right]=-\mu_\tau,$$

$$\tilde{f}''(0)=\lim_{s\to 0}\int_{t=0}^{\infty}(-t)^2 f(t)e^{-st}=\mu_\tau{}^2+\sigma_\tau{}^2,$$

where μ_τ and σ_τ are the mean value and variance of τ respectively. Then, the average production and setup frequency may be written:

$$P_{average}=\frac{1}{\mu_\tau}\ , \tag{57}$$

$$v_{average}=\lim_{s\to 0}s\,\mathrm{E}\left[\tilde{v}(s)\right]=\frac{1}{Q\,\mu_\tau}\ . \tag{58}$$

Assuming the net present value (NPV) of out payments to be minimised, we investigate the optimal value of $Q=FOQ$. The NPV of the cash flow can be written:

$$\text{NPV}=\left[c\text{E}\left[\tilde{P}(s)\right]+K\text{E}\left[\tilde{v}(s)\right]\right]_{s=\rho}=\frac{cQ+K}{1-\text{E}\left[e^{-\rho T}\right]}=\frac{cQ+K}{1-\tilde{f}(\rho)^{Q}}\ , \tag{59}$$

where c is unit production cost and K is setup cost. On differentiating NPV with respect to Q we obtain

$$\frac{\partial\text{NPV}}{\partial Q}=\frac{c}{1-\tilde{f}(\rho)^{Q}}+\frac{\left(cQ+K\right)\tilde{f}(\rho)^{Q}\ln\tilde{f}(\rho)}{\left(1-\tilde{f}(\rho)^{Q}\right)^{2}}=$$

$$=\frac{c\left(1-\tilde{f}(\rho)^{Q}\right)+\left(cQ+K\right)\tilde{f}(\rho)^{Q}\ln\tilde{f}(\rho)}{\left(1-\tilde{f}(\rho)^{Q}\right)^{2}}=0\,. \tag{60}$$

which is the necessary optimisation condition. Using a second-order approximation of $\tilde{f}(\rho)$ provides us with the following optimal order quantity:

$$\text{EOQ}=\sqrt{\frac{2K}{\rho c\mu_{\tau}}}=\sqrt{\frac{2KD_{average}}{\rho c}}\,, \tag{61}$$

which has the standard format.

Instead, in the fixed period requirement (FPR) case, the quantity $Q_n=\bar{D}(t_n+T)-\bar{D}(t_n)$ is ordered at the beginning of each interval of length T.

Expected production will then be:

$$\text{E}\left[\tilde{P}(s)\right]=\text{E}\left[\sum_{n=0}^{\infty}\left(\bar{D}(t_n+T)-\bar{D}(t_n)\right)e^{st_n}\right]=$$

$$=\text{E}\left[\bar{D}(T)\right]\left(1+e^{-sT}+e^{-s2T}+\ldots\right)=\frac{T}{\mu_{\tau}}\frac{1}{1-e^{-sT}}\ . \tag{62}$$

Average production is obtained as:

$$P_{average}=\lim_{s\to 0}s\,\text{E}\left[\tilde{P}(s)\right]=\lim_{s\to 0}\frac{sT}{\mu_{\tau}}\frac{1}{1-e^{-sT}}=\frac{T}{\mu_{\tau}Te^{-sT}}=\frac{1}{\mu_{\tau}}\,, \tag{63}$$

and the average setup frequency

$$v_{average} = \lim_{s \to 0} s\,\mathrm{E}\left[\tilde{v}(s)\right] = \lim_{s \to 0} \frac{s}{1-e^{-sT}} = \frac{1}{T}. \qquad (64)$$

The net present value of the cash flow will be:

$$\mathrm{NPV} = \left[c\mathrm{E}\left[\tilde{P}(s)\right] + K\mathrm{E}\left[\tilde{v}(s)\right]\right]_{s=\rho} = c\frac{T}{\mu_\tau}\frac{1}{1-e^{-\rho T}} + \frac{K}{1-e^{-\rho T}}. \qquad (65)$$

To find the optimum interval T, we take the derivative of NPV with respect to T:

$$\frac{\partial \mathrm{NPV}}{\partial T} = \frac{\frac{c}{\mu_\tau}\left(1-e^{-\rho T} - T\rho e^{-\rho T}\right) - K\rho e^{-\rho T}}{\left(1-e^{-\rho T}\right)^2} = 0. \qquad (66)$$

Using again a second-order approximation of $e^{-\rho T}$, the following optimal interval is obtained:

$$T^* = \sqrt{\frac{2K\mu_\tau}{\rho c}} = \sqrt{\frac{2K}{\rho c D_{average}}}. \qquad (67)$$

This shows that in both of the cases FOQ and FPR, the optimal policies are obtained as when using the traditional average inventory approach with the inventory holding cost interpreted as ρc, i.e. interest rate times unit production cost.

7 CONCLUSIONS

The objective of this article has been to analyse the fundamental equations of MRP Theory in view of the basic ordering policies Lot-For-Lot (L4L), Fixed Order Quantity (FOQ) and Fixed Period Requirements (FPR). Our aim has been to find closed-form Laplace transform expressions for the time development of production, when given properties of external demand. When leaving the L4L policy, we have shown that such expressions are possible to derive, but they become considerably more complicated.

In future research, other optimisation aspects within the frequency domain should be considered. Important work into these issues has been started by Bogataj and Bogataj (2004).

8 REFERENCES

Bogataj, M., Bogataj, L., The problems of optimal additional investments in reduction of delays in value chain analyzed in frequency domain, *Informatizacija transportno logističnega grozda*, (*Mathematical economics, operational research and logistics*), 2004, **4**, (University of Ljubljana: Faculty of Economics).

Feller, W., *An introduction to probability theory and its applications*, 1971, **II**, p. 439, Second Edition (Wiley: New York).

Grubbström, R.W., Stochastic properties of a production-inventory process with planned production using transform methodology, *International Journal of Production Economics,* 1996, **45***,* 407-419.

Grubbström, R.W., A net present value approach to safety stocks in a multi-level MRP system, *International Journal of Production Economics,* 1999, **59***,* 361-375.

Grubbström, R.W., Ovrin, P., Intertemporal generalization of the relationship between material requirements planning and input-output analysis**,** *International Journal of Production Economics,* 1992, **26***,* 311-318.

Grubbström, R.W., Molinder, A., Further theoretical considerations on the relationship between MRP, input-output analysis and multi-echelon inventory systems**,** *International Journal of Production Economics,* 1994, **35***,* 299-311.

Grubbström, R.W., Tang, O., The moments and central moments of a compound distribution, *European Journal of Operational Research,* 2006, **170**, 106-119.

Grubbström, R.W., Tang, O., An overview of input-output analysis applied to production-inventory systems, *Economics Systems Research*, 2000, **12**, 3-25.

Molinder, A., *Material Requirements Planning Employing, Input-Output Analysis and Laplace Transforms*, 1995, PROFIL 14 (Production-Economic Research in Linköping: Linköping).

Orlicky, J., *Material requirement planning*, 1975 (McGraw-Hill, Inc.: New York).

Segerstedt, A., Formulas of MRP, *International Journal of Production Economics*, **46-47**, 1996, 127-136.

ANALYSIS OF STANDARD ORDERING POLICIES WITHIN THE FRAMEWORK OF MRP THEORY

Robert W. Grubbström* and Thi Thu Thuy Huynh

*Corresponding author

Robert W. Grubbström,

Department of Production Economics

Linköping Institute of Technology

S-581 83 Linköping, Sweden

Tel: +46-13-281504

Fax: +46-13-288975

e-mail: rwg@ipe.liu.se

Thi Thu Thuy Huynh

Department of Production Economics

Linköping Institute of Technology

S-581 83 Linköping, Sweden

Tel: +46-13-281772

Fax: +46-13-288975

Email: thuy.huynh@ipe.liu.se