Fock Bargmann space and Feynman propagator of charged harmonic oscillator in a constant magnetic field.

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Abstract

The Feynman propagator of charged harmonic oscillator in a constant magnetic field is found for the first time by a direct calculation using Fock Bargmann space and a new generating function of two dimensional harmonic oscillators. The calculations involve well-known integrals which an undergraduate should understand.

1. Introduction

Recently the method of path integral is subject of intense work in quantum physics [1-10]. Further more, it is well known that the harmonic oscillator and the motion of a charged particle in a magnetic field are parts of the undergraduate courses [11-17] so it was natural that the search of Feynman propagators for these problems is of great interest. But the proposed methods of calculation: the Schwinger's method, the algebraic method and the path integral are difficult to teach at undergraduate level [1, 2].

So we have proposed the generating function method which is a new direct method for the calculation of Feynman propagator [10]. In our method, we use only the generating function of the basis of harmonic oscillator and the Fock Bargmann space for integration. This method is simple and very useful as an introduction to the methods proposed above and moreover it is widely used to study the theory of angular momentum and group theory [10, 17-20].

In our case, we construct a new generating function of the cylindrical basis, eigenfunctions of the Hamiltonian, and using the Bargmann Fock space, we simply find the correct expression of Feynman propagator of the charged harmonic oscillator in a constant magnetic field.

This work is complementary to the paper [10] and it is aimed to undergraduate students.

In part 2 of this work we review the charged harmonic oscillator in a constant magnetic field. We construct the new generating function of the cylindrical basis of harmonic oscillator in part three. Part four is devoted to the calculation of Feynman propagator.

2. Isotropic charged harmonic oscillator

Considering an isotropic charged harmonic oscillator with electric charge q and mass μ moves in a two-dimensional plane under a uniform magnetic field B perpendiculars to the plane and the vector potentials have the following form [11-17]:

$$A_1 = -By/2, \quad A_2 = Bx/2 \tag{2.1}$$

The Hamiltonian of the system is

$$H = \frac{1}{2\mu}\left[\left(p_1 + \frac{qB}{2c}y\right)^2 + \left(p_2 + \frac{qB}{2c}x\right)^2\right] + \frac{1}{2}\mu\omega_0^2(x^2+y^2) = H_0 - \frac{qB}{2\mu c}L_z \tag{2.2}$$

With

$$H_0 = \frac{1}{2\mu}(p_1^2 + p_2^2) + \frac{\mu\omega^2}{2}(x^2+y^2) \tag{2.3}$$

And

$$\omega = \sqrt{\omega_0^2 + \omega_c^2}, \quad \omega_c = \left(\frac{qB}{2\mu c}\right) . \tag{2.4}$$

We have also $[H_0, L_z] = 0$.

To determine the eigenfunctions of H_0, L_z we use the polar coordinates of two dimensions harmonic oscillators [11-17].

We put

$$x = \rho\cos\varphi, \quad y = \rho\sin\varphi \qquad 0 \le \rho \le \infty, \; 0 \le \varphi \le 2\pi \tag{2.5}$$

Using the method of separation of variables we find the solution of Schrödinger Equation which is the cylindrical basis:

$$\Phi_{jm}(\lambda\rho,\varphi) = \left(\frac{\mu\omega}{\pi\hbar}\right) f_{jm}(\lambda\rho)e^{-2im\varphi} = \sqrt{\frac{\lambda^2}{\pi}}\sqrt{\frac{(j-|m|)!}{(j+|m|)!}}\exp\left(-\frac{\lambda^2\rho^2}{2}\right)L_{j+|m|}^{2|m|}(\lambda^2\rho^2)(\lambda\rho)^{2|m|}e^{-2im\varphi} \tag{2.6}$$

With $\quad j = n + |m| \quad$ and $\quad \lambda = \sqrt{\frac{m\omega}{\hbar}}$ (2.7)

The energy of the system can be given as follows:

$$E_{nm} = \hbar\omega(2n + 2|m| + 1) - m\hbar\frac{qB}{\mu c} \tag{2.8}$$

With $\quad n = 0,1,2,...,\quad$ And $|2m| = 0, \pm 1, \pm 2, ...$.

We emphasize that we can build the generating function of the cylindrical basis from the generating function of Laguerre polynomials but the calculation is more simpler with the generating function that we will build in part three.

3. The generating function of the cylindrical harmonic oscillator

In this part we review the construction of the generating function of the cylindrical basis [18] of the harmonic oscillator which is the eigenfunctions of (H_0, L_z).

We know that the Cartesian basis of harmonic oscillator in Dirac notations is

$$\left| n_x, n_y \right\rangle = a_x^{+nx} a_y^{+ny} \left| 0,0 \right\rangle \tag{3.1}$$

This ket is not eigenfunctions of L_z so to obtain the basis which has this property we must take the transformation [15]

$$\begin{gathered} A_1^+ = \frac{\sqrt{2}}{2}(a_x^+ - i a_y^+), \quad A_2^+ = \frac{\sqrt{2}}{2}(a_x^+ + i a_y^+) \\ L_z = (N_1 - N_2), \quad N = N_1 + N_2, \\ N_1 = A_1^+ A_1,\ N_2 = A_2^+ A_2 \end{gathered} \tag{3.2}$$

The new basis $\left| N_1, N_2 \right\rangle$ can be written in the form

$$\left| N_1, N_2 \right\rangle = \left| j+m, j-m \right\rangle = A_1^{+\,j+m} A_2^{+\,j-m} \left| 0,0 \right\rangle . \tag{3.3}$$

This basis is function of L_z and N with the values 2m and 2j [15].

With $$H = \hbar\omega(N+1) - \frac{qB}{2\mu c} L_z$$

And $$E_{nm} = \hbar\omega(2j+1) - m\hbar\frac{qB}{\mu c}$$

The new generating function may be written in the form:

$$\begin{aligned} \left| G(z_1, z_2) \right\rangle &= \exp[z_1 A_1^+ + z_2 A_2^+] \left| 0,0 \right\rangle \\ &= \exp[a_x^+ \sqrt{2}(z_1 + z_2)/2 + i a_x^+ \sqrt{2}(-z_1 + z_2)/2] \left| 0,0 \right\rangle \end{aligned} \tag{3.4}$$

In term of Cartesian coordinates we write the generating function as:

$$\begin{aligned} G(t_1, t_2, \vec{r}) &= \left(\frac{\lambda^2}{\pi}\right)^{1/2} \exp[-\lambda^2 \frac{x^2 + y^2}{2} + \lambda[z_1(x+iy) + z_2(x-iy)] - z_1 z_2] = \\ &\sum_{jm} (-1)^{j-m} \frac{z_1^{(j+m)} z_2^{(j-m)}}{\sqrt{(j+m)!(j-m)!}} \Phi_{jm}(\lambda\rho, \varphi) \end{aligned} \tag{3.5}$$

And $$\varphi_{jm}(z) = \frac{z_1^{(j+m)} z_2^{(j-m)}}{\sqrt{(j+m)!(j-m)!}}$$

Is the basis of Fock-Bargmann space [10, 19].

The measure of integration is:

$$d\mu(z_1,z_2)=d\mu(z_1)d\mu(z_2) \text{ And } d\mu(z)=\tfrac{1}{\pi}\exp(-z\bar{z})dx_z dy_z,\ z=x_z+iy_z. \tag{3.6}$$

We have also the useful formula

$$e^{\alpha\beta}=\int e^{\alpha z}\ e^{\beta\bar{z}}d\mu(z) \tag{3.7}$$

4. The Feynman propagator of two-dimensional isotropic Charged harmonic oscillator in uniform magnetic field

In this section, we propose a simple and elementary method for the calculation of Feynman propagator of two-dimensional charged harmonic oscillator in uniform magnetic field.

We have:

$$\begin{aligned}\mathrm{K}((\vec{\mathrm{r}},\mathrm{t}),(\vec{\mathrm{r}}',\mathrm{t}_0))&=\langle r|e^{-\frac{i}{\hbar}H(t-t_0)}|r'\rangle=\langle r|e^{-\frac{i}{\hbar}H(t-t_0)}I|r'\rangle\\&=\mathrm{e}^{-\mathrm{i}\omega(\mathrm{t}-\mathrm{t}_0)}\sum_n\overline{\Phi_{jm}(\vec{r})}e^{-i(2j\alpha-2m\beta)}\Phi jm(\vec{r}\,')\end{aligned} \tag{4.1}$$

With $\alpha=\omega\tau,\ \beta=\omega_c\tau$ and $\tau=(t-t_0)$

From the orthogonality of the basis $\varphi_{jm}(z)$ and (3.5) we deduce that:

$$\begin{aligned}\mathrm{K}((\vec{\mathrm{r}},\mathrm{t}),(\vec{\mathrm{r}}',\mathrm{t}_0))=\left(\frac{\mu\omega}{\pi\hbar}\right)\mathrm{e}^{-\mathrm{i}\omega(\mathrm{t}-\mathrm{t}_0)}\int\Big[\overline{G}(\bar{a}z_1,\bar{b}z_2,\vec{r}\,')\times\\G(az_1,bz_2,\vec{r})\Big]d\mu(z)\end{aligned} \tag{4.2}$$

With $a=e^{-i(\alpha-\beta)/2}$ and $b=e^{-i(\alpha+\beta)/2}$

By substituting the expressions under the integral by (3.5) we write:

$$\begin{aligned}\mathrm{K}((\vec{\mathrm{r}},\mathrm{t}),(\vec{\mathrm{r}}',\mathrm{t}_0))=\frac{\lambda^2}{\pi}\int\exp\Big[-\lambda^2\frac{\vec{r}_1^{\,2}+\vec{r}_2^{\,2}}{2}+\lambda[a\bar{z}_1(x_2-iy_2)+b\bar{z}_2(x_1-iy_1)]+\\\lambda[az_1(x_1+iy_1)+bz_2(x_2+iy_2)]-ab(z_1\bar{z}_2+\bar{z}_1z_2)\Big]d\mu(z_1)d\mu(z_2)\end{aligned} \tag{4.3}$$

This integral is invariant by changing $z_2\leftrightarrow\bar{z}_2$, and then we can use the well known formula:

$$\left(\frac{1}{\pi}\right)^n\int\prod_{i=1}^n dx_i dy_i\exp(-\bar{z}^tXz+A^tz+\bar{z}^t\overline{B})=(\det(X))^{-1}\exp(A^tX^{-1}\overline{B}) \tag{4.4}$$

We find that $X=\begin{pmatrix}1 & ab\\ ab & 1\end{pmatrix}$

And $$\det(X) = 1 - a^2 b^2 = 1 - e^{-i\alpha}$$

By an elementary calculation we find that:

$$A^t X^{-1} \overline{B} = \lambda^2 \left\{ -a^2 b^2 (\vec{r}_1^{\,2} + \vec{r}_2^{\,2}) + a^2 (x_1 + iy_1)(x_2 + iy_2) + b^2 (x_2 - iy_2)(x_1 - iy_1) \right\} / (1 - a^2 b^2) \tag{4.5}$$

Then the propagator may be written:

$$\mathrm{K}((\vec{\mathrm{r}}, \mathrm{t}), (\vec{\mathrm{r}}', \mathrm{t}_0)) = \left(\frac{\mu\omega}{\pi\hbar} \right) \frac{\mathrm{e}^{-\mathrm{i}\omega(\mathrm{t}-\mathrm{t}_0)}}{1 - e^{-i2\alpha}} \exp\left\{ \lambda^2 \left[-(\vec{r}_1^{\,2} + \vec{r}_2^{\,2})(1 + a^2 b^2)/2 + (a^2 + b^2)(x_1 x_2 + y_1 y_2) - i(a^2 - b^2)(x_1 y_2 - y_1 x_2) \right] / (1 - a^2 b^2) \right\} \tag{4.6}$$

It is easy to verify the following identities

$$\frac{e^{-2i\alpha}}{(1 - a^2 b^2)} = \frac{1}{2i \sin\alpha}, \qquad \frac{(1 + a^2 b^2)}{(1 - a^2 b^2)} = -i \frac{\cos\alpha}{2 \sin\alpha}\,. \tag{4.7}$$

And

$$\frac{(a^2 + b^2)}{(1 - a^2 b^2)} = -i \frac{\cos\beta}{\sin\alpha}, \qquad \frac{(a^2 - b^2)}{(1 - a^2 b^2)} = \frac{\sin\beta}{\sin\alpha} \tag{4.8}$$

Substitute these relations in (4.5) we obtain the exact expression of Feynman Propagator of a charged harmonic oscillator in constant magnetic field:

$$\mathrm{K}((\vec{\mathrm{r}}, \mathrm{t}), (\vec{\mathrm{r}}', \mathrm{t}_0)) = \left(\frac{\mu\omega}{2i\pi\hbar} \right) \frac{1}{\sin(\omega\tau)} \exp\left\{ \frac{i\mu\omega}{\hbar} \left[\frac{\cos(\omega\tau)}{2\sin(\omega\tau)} (\vec{r}_1^{\,2} + \vec{r}_2^{\,2}) + - \frac{\cos(\omega_c \tau)}{\sin(\omega\tau)} (x_1 x_2 + y_1 y_2) - \frac{\sin(\omega_c \tau)}{\sin(\omega\tau)} (x_1 y_2 - y_1 x_2) \right] \right\} \tag{4.9}$$

I leave the reader to compare between our method and Schwinger's method, Reference [1] part B.

It is important to emphasize that the expression (4.1) may be obtained by the application of the transformation from the coordinates representations to the harmonic oscillator basis and the formula (3.7).

Conclusion

The Bargmann Fock space can be found from the expansion of the generating function of harmonic oscillator and well known to undergraduates. The introduction of this space in the education facilities and minimizes the time of calculation and more, it is very useful in the theory of angular momentum and other field [10]. In our papers, and since 2005, we have been trying to use this space and the generating function to solve many interesting applications that allow students to have a very broad view on many fields of physics and to give them a useful background for further advanced studies.

Finally I hope that people learn to help each other against this difficult life and that is the most important thing that we have learned from Sciences.

5. References

[1] A. Aragão, H. Boschi-Filho, C. Farina, and F. A. Barone "Non-Relativistic Propagators via Schwinger's Method" Brazilian Journal of Physics, vol. 37, no. 4, December, 2007.

[2] F.A.Barone, H. C. Boschi-Filho, and Farina,' Three methods for calculating the Feynman propagator' Am. J. Phys., Vol. 71, No. 5, May 2003, 490.

[3] Surarit Pepore and Bodinchat Sukbot , Chinese jour.of phys. vol.47 n. 6 dec. 2009.

[4] F. Benamira and L. Guechi arXiv: [math-ph] /0112003 v1, feb. 2008.

[5] H. Ahmedov, I. Duru, and A. Gumrukcuoglu, arXiv: [math-ph] /0511095, nov. 2005.

[6] B. R. Holstein, "The harmonic oscillator propagator" Amer. J. Phys., 67 (1998).

[7] M. Moshinsky, E. Sadurn´ı and A. del Campo, SIGMA 3 (2007), 110.

[8] C. Grosche and F.Steiner," Classification of solvable Feynman path integrals" arXiv: [math-ph] /9302053.

[9] M. B. Bouloudenine and T. Boudjedaa, Int. J. Theor. Phys. 47, 1156–72 (2008).

[10] M. Hage-Hassan, "On Fock-Bargmann space, Dirac delta function, Feynman propagator, angular momentum and SU(3) multiplicity free" arXiv: [math-ph] /0909.0129, (2010).

[11] A.K. Ghatak and S. Lokanathan, "Quantum Mechanics" Macmillan India LTD (1997).

[12] Y. Peleg, R. Pnini, E. Zaarur, "Quantum Mechanics", " Schaum's series (1998).

[13] L. I. Schiff, "Quantum Mechanics", Third Edition. New York: McGraw-Hill, 1968.

[14] E. Merzbacher, "Quantum Mechanics", Third Edition. New York: Wiley, 1998.

[15] A. Messiah," Mécanique quantique 1-2" E. Dunod, Paris (1965).

[16] L. D. Landau and E. M. Lifshitz. "Quantum Mechanics: Non-relativistic Theory" Pergamon Press, Oxford, 1977.

[17] J. J. Sakurai, "Modern Quantum Mechanics" (Addison Wesley 1994).

[18] M. Hage-Hassan, "Angular momentum and the polar basis of harmonic oscillator" arXiv: [math-ph] /0903.0708.

[19] V. Bargmann, "On the representations of the Rotation group" Rev. Mod. Phys.34, 4, (1962).

[20] J. Schwinger, In "Quantum Theory of angular momentum" Ed. L.C. Biedenharn and H. Van Dam, Acad. Press, NY, (1965).