

# Link between Copula and Tomography

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## Abstract

An important problem in statistics is to determine a joint probability distribution from its marginals and an important problem in Computed Tomography (CT) is to reconstruct an image from its projections. In the bivariate case, the marginal probability density functions  $f_1(x)$  and  $f_2(y)$  are related to their joint distribution  $f(x, y)$  via horizontal and vertical line integrals. Interestingly, this is also the case of a very limited angle X ray CT problem where  $f(x, y)$  is an image representing the distribution of the material density and  $f_1(x)$ ,  $f_2(y)$  are the horizontal and vertical line integrals. The problem of determining  $f(x, y)$  from  $f_1(x)$  and  $f_2(y)$  is an ill-posed undetermined inverse problem. In statistics the notion of *copula* is exactly introduced to characterize all the possible solutions to the problem of reconstructing a bivariate density from its marginals. In this paper, we elaborate on the possible link between Copula and CT and try to see whether we can use the methods used in one domain into the other.

*Key words:* Copula, Tomography, Joint and marginal distributions, Image reconstruction, Additive and Multiplicative Backprojection, Maximum Entropy, Archimedian Copulas.

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## 1. Introduction

2 The word *copula* originates from the Latin meaning *link, chain, union*.  
3 In statistical literature, according to the seminal result in the copula's the-  
4 ory stated by Abe Sklar [1] in 1959, a copula is a function that connects  
5 a multivariate distribution function to its univariate marginal distributions.  
6 There is an increasing interest concerning copulas, widely used in Financial

7 Mathematics and in modelling of Environmental Data [2, 3]. Recently, in  
 8 Computational Biology, copulas were used for DNA analysis [4]. Copula  
 9 appears to be a powerful tool to model the structure of dependence [5, 6].  
 10 Copulas are useful for constructing joint distributions, particularly with non-  
 11 Gaussian random variables [7].

12 In 2D case, interpreting the joint probability density function  $f(x, y)$  as  
 13 an image and its marginal probability densities  $f_1(x)$  and  $f_2(y)$  as horizontal  
 14 and vertical line integrals:

$$f_1(x) = \int f(x, y)dy \quad \text{and} \quad f_2(y) = \int f(x, y)dx \quad (1)$$

15 we see that the problem of determining  $f(x, y)$  from  $f_1(x)$  and  $f_2(y)$  is an  
 16 ill-posed (inverse) problem [8–10]. It is a well known fact that while a dis-  
 17 tribution has a unique set of marginals, the converse is not necessarily true.  
 18 That is, many distributions may share a common subset of marginals. In  
 19 general, it is not possible to uniquely reconstruct a distribution from its  
 20 marginals. This is illustrated in Figure 1: Figure 1 (a) shows the forward  
 21 problem given by (1), whereas Figure 1 (b) illustrates the inverse problem.  
 22 As we will see later, all functions in the form of

$$f(x, y) = f_1(x) f_2(y) c(F_1(x), F_2(y)) \quad (2)$$

23 where  $F_1(x)$ ,  $F_2(y)$  are the marginal cumulative distributions functions (cdf's)  
 24 and  $c$  is any *copula* density function, are solutions of this problem. Interest-  
 25 ingly, this is very similar to the probability density function (pdf) reconstruc-  
 26 tion problem considered in [11], where a special *copula* was designed. The  
 27 approach in [11] could certainly be interpreted using the results presented  
 28 here.

29 In 1917, Johann Radon introduced the Radon transform (RT) [12, 13]  
 30 which was later used in CT [14]. Indeed, if we denote by  $f(x, y)$ , the spatial  
 31 distribution of the material density in a section of the body, a very simple  
 32 model that relates a line of the radiography image  $p(r, \theta)$  at direction  $\theta$  to  
 33  $f(x, y)$  is given by the Radon transform:

$$p(r, \theta) = \int_{L_{r,\theta}} f(x, y)dl = \int \int_{\mathbb{R}^2} f(x, y)\delta(r - x \cos \theta - y \sin \theta)dx dy, \quad (3)$$

34 where  $L_{r,\theta} = \{(x, y) : r = x \cos \theta + y \sin \theta\}$  and  $\delta$  is the Dirac's delta func-  
 35 tion. The experimental setup is presented in Figure 2.

36 If now we consider only the horizontal  $\theta = 0$  projection and the vertical  
 37  $\theta = \pi/2$  projection, we see easily the connexion between these two problems.

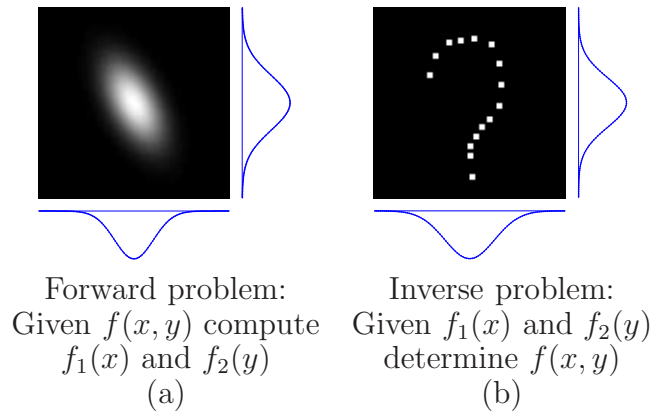


Figure 1: Forward and inverse problems

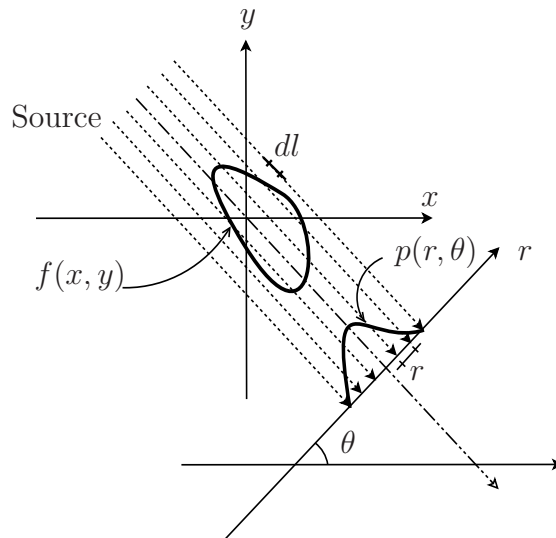


Figure 2: X ray Computed Tomography: 2D parallel geometry.

38 The main object of this paper is to explore in more details these relations,  
 39 and exploit the similarity between the two problems as a new approach to  
 40 image reconstruction in Computed Tomography.

41 The rest of this paper is organized as follows: In section 2, we present a  
 42 summary of the necessary definitions and properties of copulas and highlight  
 43 methods to generate a copula. In section 3, we present the main tomographic  
 44 image reconstruction methods based on the Radon inversion formula. In sec-

45 tion 4, we will be in the heart of the link and relations between the notions  
 46 of these two previous sections. Section 5 and 6 are devoted to details con-  
 47 cerning our method. Some preliminary results from our Copula-Tomography  
 48 Matlab package are shown.

## 49 2. Copula

50 In this section, we give a few definitions and properties of copulas that we  
 51 need in the rest of the paper. For more details about this section we refer to  
 52 [15]. First, we note by  $F(x, y)$  a bivariate cumulative distribution function  
 53 (cdf), by  $f(x, y)$  its bivariate probability density function (pdf), by  $F_1(x)$ ,  
 54  $F_2(y)$  its marginal cdf's and  $f_1(x)$ ,  $f_2(y)$  their corresponding pdf's with their  
 55 classical relations :

$$\begin{aligned} F(x, y) &= \int_{-\infty}^x \int_{-\infty}^y f(s, t) \, ds \, dt, & f(x, y) &= \frac{\partial^2 F(x, y)}{\partial x \partial y}, \\ F_1(x) &= \int_{-\infty}^x f_1(s) \, ds = F(x, \infty), & F_2(y) &= \int_{-\infty}^y f_2(t) \, dt = F(\infty, y), \\ f_1(x) &= \frac{dF_1(x)}{dx} = \int f(x, y) \, dy, & f_2(y) &= \frac{dF_2(y)}{dy} = \int f(x, y) \, dx. \end{aligned}$$

56 **Definition** Bivariate Copula: A bivariate copula, or shortly a copula is a  
 57 function  $C$  from  $[0, 1]^2$  to  $[0, 1]$  with the following properties:

- 58 •  $\forall u, v \in [0, 1], C(u, 0) = 0 = C(0, v)$ ,
- 59 •  $\forall u, v \in [0, 1], C(u, 1) = u$  and  $C(1, v) = v$  and
- 60 •  $C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0$   
 61 for all  $u_1, u_2, v_1, v_2 \in [0, 1]$  such that  $u_1 \leq u_2, v_1 \leq v_2$ .

62 **Theorem 2.1.** *Sklar's Theorem (for proof, see [16]): Let  $F$  be a two-dimensional*  
 63 *distribution function with marginal distributions functions  $F_1$  and  $F_2$ . Then*  
 64 *there **exists** a copula  $C$  such that:*

$$F(x, y) = C(F_1(x), F_2(y)). \quad (4)$$

65 **Conversely**, for any univariate distribution functions  $F_1$  and  $F_2$  and any  
 66 copula  $C$ , the function  $F$  is a two-dimensional distribution function with  
 67 marginals  $F_1$  and  $F_2$ , given by (4).

68 **Lemma 2.2.** *If the marginal functions are continuous, then the copula  $C$  is*  
 69 ***unique**, and is given by*

$$C(u, v) = F(F_1^{-1}(u), F_2^{-1}(v)). \quad (5)$$

70 **Definition** Copula Density: From (4) and differentiating (5) gives the den-  
 71 sity of a copula

$$c(u, v) = \frac{\partial^2 C}{\partial u \partial v} = \frac{f(F_1^{-1}(u), F_2^{-1}(v))}{f_1(F_1^{-1}(u)) f_2(F_2^{-1}(v))}, \quad (6)$$

72 and thus

$$f(x, y) = f_1(x) f_2(y) c(F_1(x), F_2(y)). \quad (7)$$

73 An usual simple example is the **product** or **independent** copula:

$$\Pi(u, v) = uv \longrightarrow c(u, v) = 1, \quad (u, v) \in [0, 1]^2. \quad (8)$$

74 **Property 2.3.** Any copula  $C(u, v)$ , satisfies the inequality

$$W(u, v) \leq C(u, v) \leq M(u, v), \quad (9)$$

75 where the **Fréchet-Hoeffding upper bound copula**  $M(u, v)$  (or *comono-*  
 76 *tonicity copula*) is :

$$M(u, v) = \min(u, v), \quad (10)$$

78 and the **Fréchet-Hoeffding lower bound**  $W(u, v)$  (or *countermonotonic-*  
 79 *ity copula*) is:

$$W(u, v) = \max\{u + v - 1, 0\}, \quad (u, v) \in [0, 1]^2. \quad (11)$$

80 **Generating Copulas by the Inversion Method:** A straightforward  
 81 method is based directly on Sklar's theorem. Given  $F(x, y)$  the joint cdf of  
 82 two random variables  $X, Y$  and  $F_1(x)$  and  $F_2(y)$  their marginal cdf's, all  
 83 assumed to be continuous. The corresponding copula can be constructed by  
 84 using the unique inverse transformations (Quantile transform)  $x = F_1^{-1}(u)$ ,  
 85  $y = F_2^{-1}(v)$ , and the equation (5) where  $u, v$  are uniform on  $[0, 1]$ .

### 86 2.1. Archimedean Copulas

87 The Archimedean copulas (see [15] page 109) form an important class of  
 88 copulas which generalise the usual copulas.

89 **Theorem 2.4.** Let  $\varphi$  be a continuous, strictly decreasing function from  $[0, 1]$   
 90 to  $[0, \infty]$  such that  $\varphi(1) = 0$ , and let  $\varphi^{[-1]}$  be the pseudo-inverse<sup>1</sup> of  $\varphi$ . Let  
 91  $C$  be the function from  $[0, 1]^2$  to  $[0, 1]$  given by

$$C(u, v) = \varphi^{[-1]}(\varphi(u) + \varphi(v)). \quad (12)$$

92 Then  $C$  is a copula if and only if  $\varphi$  is convex.

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<sup>1</sup> $\varphi^{[-1]}(t) = \begin{cases} \varphi^{-1}(t), & 0 \leq t \leq \varphi(0) \\ 0, & \varphi(0) \leq t \leq \infty. \end{cases}$

93 Archimedean copulas are in the form (12) and the function  $\varphi$  is called the  
 94 generator of the copula.  $\varphi$  is a strict generator if  $\varphi(0) = \infty$ , then  $\varphi^{[-1]} = \varphi^{-1}$   
 95 and

$$C(u, v) = \varphi^{-1}(\varphi(u) + \varphi(v)). \quad (13)$$

96 **Property 2.5.** Any Archimedean copula  $C$  satisfies the following algebraic  
 97 properties:

- 98 •  $C(u, v) = C(v, u)$  meaning that  $C$  is symmetric;
- 99 •  $C(C(u, v), w) = C(u, C(v, w))$ ;
- 100 • If  $a > 0$ , then  $a\varphi$  is again a generator of  $C$ .

101 There are many families of Archimedean copulas constructed from differ-  
 102 ent generators  $\varphi_\alpha$  with a suitable parameter  $\alpha$ .

103 For example  $\varphi_\alpha(t) = \frac{1}{\alpha}(t^{-\alpha} - 1)$  and  $\varphi_\alpha(t) = \ln(1 - \alpha \ln t)$  yield succes-  
 104 sively to *Clayton* copula  $C_\alpha(u, v) = [\max(u^{-\alpha} + v^{-\alpha} - 1, 0)]^{-1/\alpha}$  and *Gumbel-*  
 105 *Hougaard* copula  $C_\alpha(u, v) = uv \exp(-\alpha \ln u \ln v)$ .

### 106 3. Tomography

107 In 2D, the mathematical problem of tomography is to determine the bi-  
 108 variate function  $f(x, y)$  from its line integrals  $p(r, \theta)$  (see Eq.(3)). Radon  
 109 has shown that this problem has a unique solution if we know  $p(r, \theta)$  for all  
 110  $\theta \in [0, \pi]$  and all  $r \in \mathbb{R}$ , then  $f(x, y)$  can be computed by the inverse Radon  
 111 transform (for details, see [17] ) :

$$f(x, y) = \left(-\frac{1}{2\pi^2}\right) \int_0^\pi \int_{-\infty}^{+\infty} \frac{\frac{\partial p(r, \theta)}{\partial r}}{r - x \cos \theta - y \sin \theta} dr d\theta \quad (14)$$

112 However, if the number of projections is limited, then the problem is ill-posed  
 113 and the problem has an infinite number of solutions.

114 To present briefly the main classical methods in CT, we start by decom-  
 115 posing the inverse RT in the following parts:

$$\text{Derivative } \mathcal{D}: \bar{p}_\theta(r) = \frac{\partial p(r, \theta)}{\partial r}, \quad (15)$$

116

$$\text{Hilbert Transform } \mathcal{H}: \tilde{\bar{p}}(r', \theta) = \frac{1}{\pi} \text{p.v.} \int_{-\infty}^{+\infty} \frac{\bar{p}(r, \theta)}{r - r'} dr \quad (16)$$

117 where p.v. is the Cauchy principal value.

$$\text{Backprojection } \mathcal{B}: f(x, y) = \frac{1}{2\pi} \int_0^\pi \tilde{\bar{p}}(r' = x \cos \theta + y \sin \theta, \theta) d\theta. \quad (17)$$

118 Then defining the one dimensional inverse Fourier transform  $\mathcal{F}_1^{-1}$  by

$$\text{Inverse Fourier } \mathcal{F}_1^{-1}: P(\Omega, \theta) = \int p(r, \theta) \exp [j\Omega r] dr.$$

119 Using the properties of the Fourier transform  $\mathcal{F}_1$  and the derivative  $\mathcal{D}$ , from  
120 (15) we have:

$$\bar{P}(\Omega, \theta) = \Omega P(\Omega, \theta),$$

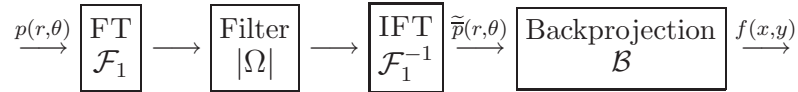
121 the relation between  $\mathcal{H}$  and  $\mathcal{F}_1$  yields :

$$\tilde{P}(\Omega, \theta) = \text{sgn}(\Omega) \bar{P}(\Omega, \theta) = \text{sgn}(\Omega) \Omega P(\Omega, \theta) = |\Omega| P(\Omega, \theta).$$

122 Finally the *filtered backprojection* which is currently the most used recon-  
123 struction method is performed by the following formula :

$$f(x, y) = \mathcal{B} \mathcal{H} \mathcal{D} p(r, \theta) = \mathcal{B} \mathcal{F}_1^{-1} |\Omega| \mathcal{F}_1 p(r, \theta) \quad (18)$$

124 that is



125 In X-ray CT, if we have a great number of projections uniformly dis-  
126 tributed over the angles interval  $[0, \pi]$ , the filtered backprojection (FBP) or  
127 even the simple backprojection (BP) image are good solutions to the inverse  
128 CT problem [18]. But, when we are restricted to only two projections, the  
129 FBP or BP images are not correct reconstruction [19–21].

#### 130 4. Link between Copula and Tomography

131 Now, let consider the particular case where we have only two projections  
132  $\theta = 0$  and  $\theta = \pi/2$ . Then

$$p_0(r) = \int \int f(x, y) \delta(r - x) dx dy = \int f(r, y) dy,$$

$$p_{\pi/2}(r) = \int \int f(x, y) \delta(r - y) dx dy = \int f(x, r) dx$$

133 and if we let  $f_1 = p_0$  and  $f_2 = p_{\pi/2}$  we can deduce the following new methods,  
134 inspired by the reconstruction approaches in CT, for the inverse problem that  
135 consists in determining the probability density  $f(x, y)$  from its marginals  
136  $f_1(x)$  and  $f_2(y)$ :

137 **Backprojection:**

$$f(x, y) = \frac{1}{2}(f_1(x) + f_2(y)). \quad (19)$$

138 **Filtered Backprojection:**

$$f(x, y) = \frac{1}{2} \left( \int \frac{\frac{\partial f_1}{\partial x}(x')}{x' - x} dx' + \int \frac{\frac{\partial f_2}{\partial y}(y')}{y' - y} dy' \right) \quad (20)$$

139 which can also be implemented in the Fourier domain as it follows

$$\begin{aligned} f(x, y) &= \frac{1}{2} \int e^{+j\xi x} |\xi| \left( \int e^{-j\xi x'} f_1(x') dx' \right) d\xi \\ &+ \frac{1}{2} \int e^{+j\nu y} |\nu| \left( \int e^{-j\nu y'} f_2(y') dy' \right) d\nu. \end{aligned}$$

## 140 5. How to use Copula in Tomography

141 The definition and the notion of copula give us the possibility to propose  
 142 new X ray CT methods. Let first consider the case of two projections. In this  
 143 case, immediately, we can propose a first use which corresponds to the case  
 144 of independent copula, as given in (8). We call this method *Multiplicative*  
 145 *Backprojection (MBP)*(see [22])

146 **MBP:**

$$f(x, y) = f_1(x) f_2(y) \quad (21)$$

147 If we compare the equation (19) to (21) instead of the classical BP which  
 148 is an additive operation or *Additive Backprojection*, the name MBP comes  
 149 naturally. In Figure 3 we give comparisons of BP and MBP. As we can see  
 150 on the image original 1, at least the image obtained by MBP is better than  
 151 the one obtained by BP and it satisfies exactly the marginals.

152 We may still do better if we choose another copula rather than the in-  
 153 dependent copula, by proposing the following method that we call *Copula*  
 154 *Backprojection (CopBP)*.

155 **CopBP:**

$$f(x, y) = f_1(x) f_2(y) c(F_1(x), F_2(y)) \quad (22)$$

156 where  $c(u, v)$  is a parametrized copula.

157 Here the main question is how to choose an appropriate copula for the  
 158 particular application. This problem can be thought as a way to introduce  
 159 some prior information, just enough to choose an appropriate family of cop-  
 160 ula. For example if we know that the joint density has only one mode, and



161 can be approximated by a bivariate Gaussian,  $\Phi^{-1}$  denoting the inverse of the  
 162 standard Gaussian cdf, then we can use a Gaussian copula whose expression  
 163 is given by

$$C_\rho(u, v) = \frac{A}{2\pi} \int_{-\infty}^{\Phi^{-1}(u)} \int_{-\infty}^{\Phi^{-1}(v)} \exp \left\{ \frac{-(s^2 - 2\rho st + t^2)}{2(1 - \rho^2)} \right\} ds dt$$

164 where  $A = (1 - \rho^2)^{-1/2}$  and  $\rho = 0$  correspond to copulas  $\Pi(u, v)$  in Eq.(8)  
 165 and where  $\rho = -1, +1$  give respectively the copulas  $W(u, v)$  and  $M(u, v)$  in  
 166 Equations (11) and (10). The corresponding Gaussian copula density is :

$$c_\rho(u, v) = A \exp \left\{ \frac{-A^2}{2} ((\rho\Phi^{-1}(u))^2 - 2\rho\Phi^{-1}(u)\Phi^{-1}(v) + (\rho\Phi^{-1}(v))^2) \right\}.$$

167 Finally, the function  $f(x, y)$  we are looking for, can be written as :

$$f(x, y) = Af_1(x)f_2(y) \exp \left\{ -\frac{(\rho^2x^2 - 2\rho xy + \rho^2y^2)}{2(1 - \rho^2)} \right\} \quad (23)$$

168 where  $\Phi^{-1}(u) = x$  and  $\Phi^{-1}(v) = y$ .

169 Figure 3 presents CopBP reconstructions obtained using this Gaussian  
 170 copula. We see the interest of such an approach compared to standard BP.

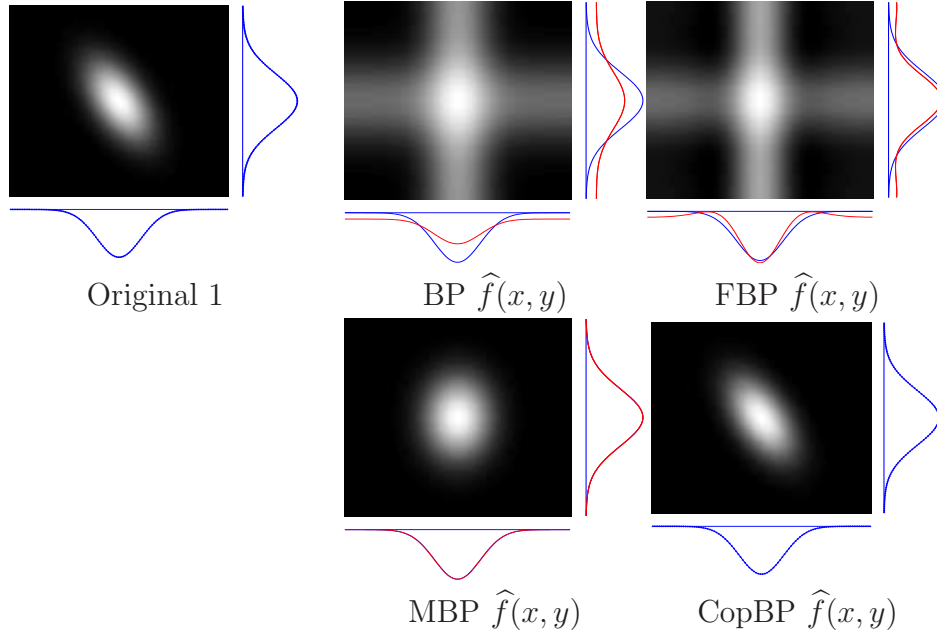
171 The particular reconstruction (23) is parametrized by the correlation co-  
 172 efficient  $\rho$  which is an hyperparameter of the reconstruction process. With  
 173 a value  $\rho = 0$ , that is with no correlations, the CopBP method reduces to  
 174 the multiplicative MBP method. The specification of  $\rho$  corresponds to the  
 175 encoding of some prior information in the reconstruction procedure which  
 176 helps to improve the quality of the reconstruction. For example, from physi-  
 177 cal or physiological knowledge, or from the experimental setting, the general  
 178 orientation of the underlying object is known. Another situation is the case  
 179 where a mean template for the object is available, for example as a result of  
 180 previous experiments.

181 The hyperparameter  $\rho$  may also be estimated from additional data. For  
 182 instance, using some additional measurements, e.g. a third (may be partial)  
 183 projection, it is easy to select the best value of  $\rho$  which minimizes the distance  
 184 between the actual projection and the one computed according to the model.

185 The general incorporation of prior information or additional data, with  
 186 the automatic determination of the hyperparameters is a work in progress  
 187 which is out of the scope of this Letter. What we want to emphasize through  
 188 this simple example is the interest of the CopBP approach for including a  
 189 such simple prior as the main orientation of the object, that leads to an  
 190 noticeable improvement of the reconstruction. This suggests that copula-  
 191 based approaches have a potential in the field of image reconstruction from  
 192 projections.

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Example 1: One Gaussian



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Example 2: Four Gaussians

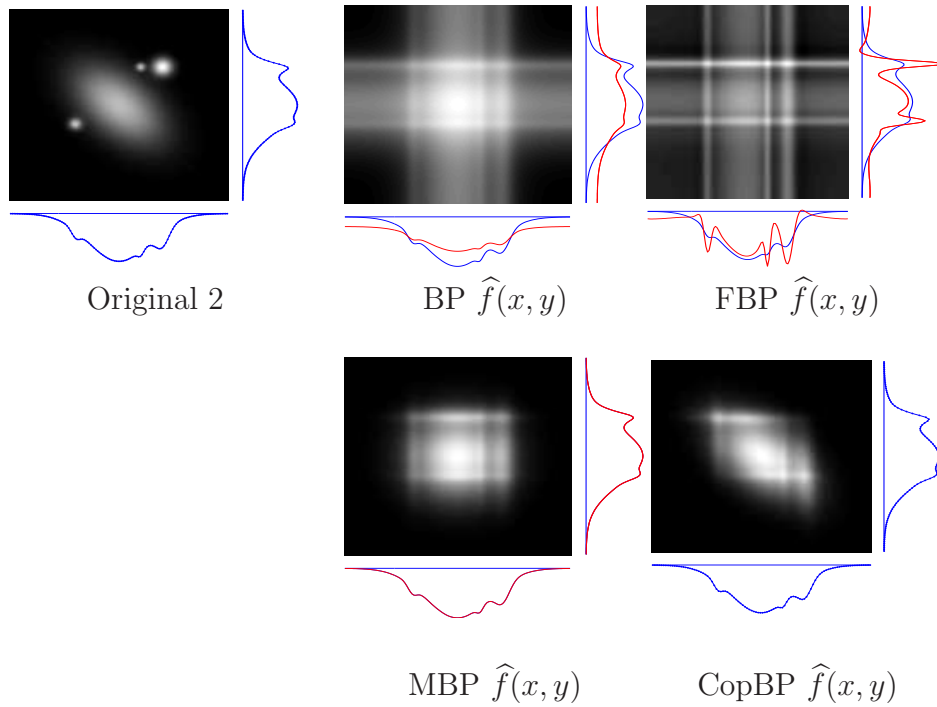


Figure 3: Comparison between BP, FBP, MBP and CopBP on two synthetic examples. This shows the improvement obtained with MBP and CopBP methods compared to standard Back Projection (BP) or Filtered Back Projections (FBP). It is noted that marginals of the BP and FBP reconstructions differ from the original data while marginals of MBP/CopBP perfectly agree with initial data.

193 **6. Maximum Entropy Copulas**

194 The selection of a particular copula is a difficult task. We propose here  
 195 to look at this ill-posed inverse problem using the maximum entropy (ME)  
 196 method. The principle of ME was first expounded by E.T. Jaynes in two  
 197 seminal papers in 1957 ([23, 24]). It is the way to assign a probability dis-  
 198 tribution to a quantity on which we have partial information. The classical  
 199 ME problem is to assign a probability law to a quantity on which we only  
 200 know a few moments. Here, the problem is a bit different, because the par-  
 201 tial information we have is not in terms of moments but in the form of the  
 202 following constraints:

$$\begin{cases} C_1 : \int f(x, y) dy = f_1(x), & \forall x \\ C_2 : \int f(x, y) dx = f_2(y), & \forall y \\ C_3 : \int \int f(x, y) dx dy = 1. \end{cases} \quad (24)$$

203 Hence, the goal is to find the most general copula, in the ME sense, com-  
 204 patible with available information, that is, with the marginals/projections at  
 205 hands.

206 *6.1. Problem's formulation*

207 Among all possible  $f(x, y)$  satisfying the constraints (24) choose the one  
 208 which optimizes a criterion  $J(f)$ , i.e :

$$\hat{f} := \text{maximize } \{J(f)\} \text{ subject to (24).}$$

209 Since the constraints are linear, if we choose a criterion which is a concave  
 210 function, then there is a unique solution to the problem. Many entropies  
 211 functional can serve as an objective function, e.g. [25–30] :

- 212 1.  $J_1(f) = - \int \int |f(x, y)|^2 dx dy$ , (-Energy or  $L_2$ -norm)
- 213 2.  $J_2(f) = - \int \int f(x, y) \ln f(x, y) dx dy$ , (Shannon Entropy)
- 214 3.  $J_3(f) = \int \int \ln f(x, y) dx dy$ , (Burg Entropy)
- 215 4.  $J_4(f) = \frac{1}{1 - \alpha} \left( 1 - \int \int f^\alpha(x, y) dx dy \right)$ , (Tsallis Entropy)
- 216 5.  $J_5(f) = \frac{1}{1 - \alpha} \ln \int \int f^\alpha(x, y) dx dy$ , (Rényi Entropy).

217 Our main contribution here is to find the generic expression for the solution of  
 218 these criteria. The main tool is the classical Lagrange multipliers technique  
 219 which consists in defining the Lagrangian functional

$$\begin{aligned} \mathcal{L}_g(f, \lambda_0, \lambda_1, \lambda_2) = & J(f) + \lambda_0 \left( 1 - \int \int f(x, y) dx dy \right) \\ & + \int \lambda_1(x) \left( f_1(x) - \int f(x, y) dy \right) dx \\ & + \int \lambda_2(y) \left( f_2(y) - \int f(x, y) dx \right) dy, \end{aligned}$$

and find its stationnary point which is defined as the solution of the following system of equations:

$$\begin{cases} \frac{\partial \mathcal{L}_g(f, \lambda_0, \lambda_1, \lambda_2)}{\partial f} = 0, \\ \frac{\partial \mathcal{L}_g(f, \lambda_0, \lambda_1, \lambda_2)}{\partial \lambda_i} = 0. \end{cases}$$

220 Here, we give the final expression, assuming that the integrals converge:

- 221 1.  $\hat{f}(x, y) = -\frac{1}{2}(\lambda_1(x) + \lambda_2(y) + \lambda_0)$ , (-Energy )
- 222 2.  $\hat{f}(x, y) = \exp(-\lambda_1(x) - \lambda_2(y) - \lambda_0)$ , (Shannon entropy)
- 223 3.  $\hat{f}(x, y) = \frac{1}{\lambda_1(x) + \lambda_2(y) + \lambda_0}$ , (Burg entropy)
- 224 4.  $\hat{f}(x, y) = \frac{1 - \alpha}{\alpha} (\lambda_1(x) + \lambda_2(y) + \lambda_0)^{\frac{1}{\alpha-1}}$ , (Tsallis and Renyi entropies).

225 Where  $\lambda_1(x)$ ,  $\lambda_2(y)$  and  $\lambda_0$  are obtained by replacing these expressions in  
 226 the constraints (24) and solving the resulting system of equations. When  
 227 solving the Lagrangian functional equation which is concave in  $f$ , we assume  
 228 that there exists a feasible  $f > 0$  with finite entropy. The results for Tsallis  
 229 and Renyi entropies leads to the same family of distribution depending on  
 230  $\alpha$  due to the monotonicity property of the logarithm function. For the two  
 231 criteria -Energy and Shannon entropy, we can find analytical solutions for  
 232  $\lambda_1(x)$ ,  $\lambda_2(y)$  and  $\lambda_0$ . For -Energy, we obtain:

$$233 \quad \lambda_1(x) = -2f_1(x) + \int \lambda_1(x) dx + 2, \quad \lambda_2(y) = -2f_2(y) + \int \lambda_2(y) dy + 2$$

234 and  $\lambda_0 = -2 - \int \lambda_1(x) dx - \int \lambda_2(y) dy$ , which finally gives:

$$\hat{f}(x, y) = f_1(x) + f_2(y) - 1. \quad (25)$$

235 This is nothing else but the standard Backprojection mechanism (up to scale  
 236 factor and a constant). Hence, the Backprojection method can be easily  
 237 interpreted as a minimum norm solution. For the Shannon entropy, we get:

$$238 \quad \lambda_1(x) = -\ln \left( f_1(x) \int \lambda_1(x) dx \right), \quad \lambda_2(y) = -\ln \left( f_2(y) \int \lambda_2(y) dy \right) \text{ and}$$

$$239 \quad \lambda_0 = \ln \left( \int \lambda_1(x) dx \int \lambda_2(y) dy \right) \text{ which yields}$$

$$\hat{f}(x, y) = f_1(x)f_2(y). \quad (26)$$

240 This is now the MBP we obtained as associate to an independent copula.  
 241 Unfortunately, in the cases of Burg, Tsallis and Renyi entropies, it is not  
 242 possible to find analytical expressions for  $\lambda_0$ ,  $\lambda_1$ , and  $\lambda_2$  as functions of  $f_1$   
 243 and  $f_2$ . Consequently a numerical approach is required, see for example [31].

244 Using equation (22) one can write all entropies in terms of copulas. For  
 245 example, if we denote the Shannon entropy by  $H(x, y)$  and the copula entropy  
 246 by  $H_c(u, v)$ , then :

$$H(x, y) = H(x) + H(y) + H_c(u, v).$$

247 The previous relation shows that the Shannon entropy of the bivariate dis-  
 248 tribution is the sum of the entropies provided by each marginal density and  
 249 the copula entropy. In Appendix, we provide the proof of this result in the  
 250 multivariate case, which is, to the best of our knowledge, original. This  
 251 result shall be of interest for multidimensional tomography, especially 3D  
 252 tomography. Therefore, maximizing the joint entropy, given the marginals,  
 253 is equivalent to maximize the entropy of the copula  $H_c(u, v)$ . Since we only  
 254 have here a domain constraint -the copula is defined on  $[0, 1]^2$ -, the Shannon  
 255 Maximum entropy copula is uniform,  $c(u, v) = 1$ , and we obtain the MBP  
 256 reconstruction (26). Now, if we look for a Shannon maximum entropy cop-  
 257 ula with an additional correlation constraint-that is we fix the correlation of  
 258 the underlying normalized random variables-,then we end with a Gaussian  
 259 copula, which in turn, lead us to the CopBP method with a Gaussian copula  
 260 (22). Along these lines, it seems possible to characterize the different families  
 261 of copula as maximum entropy solutions, possibly incorporating more prior  
 262 information. More generally, it will also be interesting to characterize the  
 263 copulas corresponding to the Burg/Rényi ME solutions.

264 Some simulations are reported Figure 3. The aim of these simulations  
 265 from our Copula-Tomography package (which can be downloaded from [32])  
 266 is to illustrate the link between copula in tomography in the case of only two

267 projections. The original 1 image simulated is a Gaussian and the original  
 268 2 image is formed by four Gaussians. We performed BP, FBP, MBP and  
 269 CopBP on these images. We observe that for the MBP and the CopBP, the  
 270 two projections on the reconstructed images match those from the original  
 271 images which is not the case for the BP and the FBP.

## 272 7. Conclusion

273 The main contribution of this paper is to highlight a link between the  
 274 notion of *copulas* in statistics and X-ray CT for small number of projections.  
 275 This link brings up possible new approaches for image reconstruction in CT.  
 276 We first presented the bivariate copulas and the image reconstruction prob-  
 277 lem in CT. We highlight the connexion between the two problems that consist  
 278 in i) determining a joint bivariate pdf from its two marginals and ii) the CT  
 279 image reconstruction from only two horizontal and vertical projections. We  
 280 emphasize that in both cases, we have the same inverse problem for the de-  
 281 termination of a bivariate function (an image) from the line integrals. We  
 282 have indicated the potential of copula-based reconstruction methods, intro-  
 283 ducing the MBP (Multiplicative Back Projection) and CopBP (Copula Back  
 284 Projection) methods. Current work addresses the characterization of family  
 285 of copulas as well as the estimation of copulas parameters in the reconstruc-  
 286 tion process. We also intend to improve the results by accounting for more  
 287 projections in the method, while keeping the copula approach.

## 288 Appendix A. Relation with Shannon entropy in high dimension

289 From the  $n$ -dimensional version of Sklar's theorem [1, 16], we have

$$F(x_1, \dots, x_n) = C(F_1(x_1), \dots, F_n(x_n)). \quad (\text{A.1})$$

290 Now taking the partial derivative in Eq.(A.1), since  $u_i = F_i(x_i)$  it follows  
 291 that the probability density function can be expressed by

$$f(x_1, \dots, x_n) = c(u_1, \dots, u_n) \prod_{i=1}^n f_i(x_i). \quad (\text{A.2})$$

Notice also that the differentials  $du_i = dF_i(x_i) = f_i(x_i) dx_i$ ,

and  $\mathbf{dx} = \prod_{i=1}^n dx_i$ . Hence  $\mathbf{du} = \prod_{i=1}^n f_i(x_i) dx_i$ , and we remark that

$$\int_{I^{n-1}} c(\mathbf{u}) \prod_{\substack{j=1 \\ j \neq i}}^n du_j = \int_{\mathbb{R}^{n-1}} \frac{f(x_1, \dots, x_n)}{f_i(x_i)} \prod_{\substack{j=1 \\ j \neq i}}^n dx_j = \frac{f_i(x_i)}{f_i(x_i)} = 1.$$

292

From the Shannon entropy and using the expression of  $f(\mathbf{x})$  in Eq.(A.2):

*Proof.*

$$\begin{aligned}
H(\mathbf{x}) &= - \int_{\mathbb{R}^n} \left( c(\mathbf{u}) \prod_{i=1}^n f_i(x_i) \right) \ln \left( c(\mathbf{u}) \prod_{i=1}^n f_i(x_i) \right) d\mathbf{x} \\
&= - \int_{\mathbb{R}^n} \left( c(\mathbf{u}) \prod_{i=1}^n f_i(x_i) \right) \left( \sum_{i=1}^n \ln f_i(x_i) \right) \prod_{i=1}^n dx_i - \int_{\mathbb{R}^n} c(\mathbf{u}) \ln c(\mathbf{u}) \prod_{i=1}^n f_i(x_i) dx_i \\
&= - \sum_{i=1}^n \int_{\mathbb{R}^n} \left( c(\mathbf{u}) \prod_{\substack{j=1 \\ j \neq i}}^n f_j(x_j) dx_j \right) f_i(x_i) \ln f_i(x_i) dx_i - \int_{I^n} c(\mathbf{u}) \ln c(\mathbf{u}) d\mathbf{u} \\
&= - \sum_{i=1}^n \left( \int_{I^{n-1}} c(\mathbf{u}) \prod_{\substack{j=1 \\ j \neq i}}^n du_j \right) \left( \int_{\mathbb{R}} f_i(x_i) \ln f_i(x_i) dx_i \right) + H_c(\mathbf{u}) \\
&= - \sum_{i=1}^n \int_{\mathbb{R}} f_i(x_i) \ln f_i(x_i) dx_i + H_c(\mathbf{u}) \\
&= \sum_{i=1}^n H(x_i) + H_c(\mathbf{u}). \tag{A.3}
\end{aligned}$$

293

□

294

Eq.(A.3) shows that the entropy  $H(\mathbf{x}) = - \int_{\mathbb{R}^n} f(\mathbf{x}) \ln f(\mathbf{x}) d\mathbf{x}$  of the  
295 joint multivariate distribution is the sum of the entropies provide by each  
296 marginal density  $H(x_i)$  and the copula entropy  $H_c(\mathbf{u})$ .

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