Proving Fixed Points

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Abstract

We propose a method to characterize the fixed points described in Tarski's theorem for complete lattices. The method is deductive: the least and greatest fixed points are "proved" in some inference system defined from deduction rules. We also apply the method to two other fixed point theorems, a generalization of Tarski's theorem to chain-complete posets and Bourbaki-Witt's theorem. Finally, we compare the method with the traditional iterative method resorting to ordinals and the original impredicative method used by Tarski.

We are interested in fixed points of maps defined over partially ordered sets (abbreviated as posets). Consider Tarski's fixed point theorem. Asserting the existence of fixed points under certain conditions, it is proved according to one of these two methods. In the *impredicative method*, the fixed points are characterized by a property (expressing extremality) using a quantification over a domain that includes the fixed point itself. In the *iterative method*, the fixed points are iteratively computed by using a transfinite induction. The impredicative method corresponds to a logical specification that is essentially not constructive[1]. As for the iterative method, it seems to be more constructive, in that it corresponds to an iteration. However, first, it assumes the machinery of ordinals and second, it therefore requires a specific computation for limit ordinals, the next value being then computed from an infinite set of preceding values.

Is there a proof method that not only is not impredicative, but also does not resort to ordinals? In this paper, we positively answer by proposing an alternative method, where fixed points are (inductively) *proved* in inference systems: this is a deductive method. It corresponds to a proof construction, using forward chaining for deduction rules, as used in logic programming, for instance in Datalog, a query language for deductive databases.

The paper is organized as follows. After recalling Tarski's theorem, the first section deals with inference systems, and their interpretations, either inductive or coinductive. In the second section, we introduce the deductive method, and apply it to different fixed point theorems: Tarski's theorem, its generalization to chain-complete posets and Bourbaki-Witt's theorem. Finally, we compare the deductive method with the iterative and impredicative methods.

1 Induction and Coinduction for Inference Systems

Generally speaking, following Aczel's classical presentation [1], an *inference system* over a set $\mathscr{U}$ of judgments is a set of deduction rules. A *deduction rule* is an ordered pair (A, c), where $A \subseteq \mathscr{U}$ is the set of *premises* or *antecedents* and $c \in \mathscr{U}$ is the *conclusion*. A rule is usually written as follows:

$$\frac{A}{c}\,.$$

Its intuitive interpretation is that the judgment c can be deduced from the set of judgments A.

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[1] In a common sense: we do not specifically refer here to constructive mathematics.

Fixed Point Approach. The first way to assign a definitional meaning to an inference system is to consider the fixed points of the associated inference operator.

Indeed there exists a canonical Galois connection between the set of inference systems over $\mathscr{U}$ ordered by inclusion and the set of isotone (order-preserving) maps from $2^{\mathscr{U}}$ to $2^{\mathscr{U}}$ ordered point-wise. If Φ is an inference system over $\mathscr{U}$, then the associated operator $\varphi : 2^{\mathscr{U}} \to 2^{\mathscr{U}}$ is defined as follows:

$$\varphi(S) = \{c \in \mathscr{U} \mid \exists A \subseteq S.\ (A,c) \in \Phi\}.$$

The application of φ to S gives the conclusions that can be inferred in one step from S by using the inference rules: φ is thus called the *inference operator* associated to the system Φ. For instance, if S is the empty set, then $\varphi(S)$ is the set of the *axioms* of the inference system, in other words the conclusions of the rules without premises. Conversely, given an isotone operator $\varphi : 2^{\mathscr{U}} \to 2^{\mathscr{U}}$, the inference system containing all the rules (A,c), with $c \in \varphi(A)$, and only these rules, belongs to the inverse image of φ: this is the greatest element of the inverse image of φ.

Let Φ be an inference system and φ its associated inference operator. Since the powerset $2^{\mathscr{U}}$ is a complete lattice, by applying Tarski's fixed point theorem [10, p. 286], we obtain that the inference operator φ possesses both a *least fixed point* and a *greatest fixed point*.

We recall here Tarski's fixed point theorem, with two proofs following the impredicative and the iterative methods respectively.

Theorem 1 (Tarski)**.** *Let $(\mathscr{E}, \leq)$ be a complete lattice. Let $\eta : \mathscr{E} \to \mathscr{E}$ be an isotone map over $\mathscr{E}$. Then η has a least fixed point* $\operatorname{lfp}\eta$ *and a greatest fixed point* $\operatorname{gfp}\eta$.

Impredicative method. Let $S = \{x \in \mathscr{E} \mid \eta(x) \leq x\}$ and x in S. Since η is isotone, we have $\eta^2(x) \leq \eta(x)$, hence $\eta(x) \in S$. Since $\mathscr{E}$ is a complete lattice, S has a greatest lower bound, $\wedge S$. We have $\wedge S \leq x$, hence $\eta(\wedge S) \leq \eta(x)$ by isotony. Therefore $\eta(\wedge S)$ is a lower bound of S, hence $\eta(\wedge S) \leq \wedge S$, and $\wedge S \in S$. Then $\eta(\wedge S) \in S$, hence $\wedge S \leq \eta(\wedge S)$. Finally, $\wedge S$ is a fixed point. Since all fixed point belongs to S, we can deduce the first equality, and by duality, the second one. □

This characterization of the least and greatest fixed points as the smallest η-closed[2] set and the greatest η-consistent[3] set respectively can be found in original Tarski's paper [10].

Iterative method. Let $(\Delta_\alpha(\eta))_\alpha$ and $(\nabla_\alpha(\eta))_\alpha$ be the sequences defined over ordinals as follows:

$$\Delta_\alpha(\eta) \quad = \quad \eta(\textstyle\bigvee_{\beta \mid \beta<\alpha} \Delta_\beta(\eta)) \quad \text{and} \quad \nabla_\alpha(\eta) \quad = \quad \eta(\textstyle\bigwedge_{\beta \mid \beta<\alpha} \nabla_\beta(\eta)).$$

By transfinite induction, we have for any ordinal α, $\Delta_\alpha(\eta) \leq \eta(\Delta_\alpha(\eta))$ and the sequence $(\Delta_\beta(\eta))_{\beta<\alpha}$ is increasing. Assume for contradiction that the sequence is strictly increasing. We get an injective map from the class of ordinals into a set, which is a contradiction, by Hartogs' lemma. Consider the smallest ordinal λ such that there exists $\beta > \lambda$ satisfying $\Delta_\lambda(\eta) = \Delta_\beta(\eta)$. By transfinite induction, we have that the sequence $(\Delta_\alpha(\eta))_\alpha$ is stationary from rank λ. Moreover, $\Delta_\lambda(\eta)$ is a fixed point of η. Since by transfinite induction we have that any fixed point is an upper bound of the sequence $(\Delta_\alpha(\eta))_\alpha$, we deduce the first equality. By duality, we deduce the second one. □

A slight variant of this proof can be found in Cousot's article [5], for instance.

[2] We say that x is η-closed if $\eta(x) \leq x$.

[3] We say that x is η-consistent if $x \leq \eta(x)$.

Deductive Method. In contrast with the fixed point approach, the deductive method starts from the *proofs* admissible in an inference system. These proofs are represented as trees, called *proof trees*. These are trees whose nodes are labeled with judgments in $\mathscr{U}$ and such that for all nodes n, the label c of n and the labels A of the sons of n correspond to an inference rule (A,c) in Φ. The conclusion of a proof is the label of its root node. A proof d is *well-founded* if it has no infinite branch; d is *ill-founded* otherwise. Note that an ill-founded proof is always infinite. A well-founded proof is finite if and only if it only uses rules with a finite set of premises.

In the deductive method, the *inductive interpretation* of the inference system Φ is the set $\Delta(\Phi)$ of the conclusions of the well-founded proofs, while the *coinductive interpretation* is the set $\nabla(\Phi)$ of the conclusions of all the proofs, ill-founded or well-founded. The following theorem shows that the interpretations defined using fixed points and using proofs coincide.

Theorem 2 (Inductive and coinductive interpretations)**.** *Let Φ be an inference system and φ the associated inference operator. Then:*

$$\operatorname{lfp}\varphi \;=\; \Delta(\Phi) \quad \textit{and} \quad \operatorname{gfp}\varphi \;=\; \nabla(\Phi).$$

Proof. It is easy to show that $\Delta(\Phi)$ and $\nabla(\Phi)$ satisfy $\varphi(\Delta(\Phi)) \subseteq \Delta(\Phi)$ and $\nabla(\Phi) \subseteq \varphi(\nabla(\Phi))$ respectively. By Tarski's theorem, we deduce $\operatorname{lfp}\varphi \subseteq \Delta(\Phi)$ and $\nabla(\Phi) \subseteq \operatorname{gfp}\varphi$.

Assume a set S satisfying $\varphi(S) \subseteq S$. An induction over well-founded proofs d shows that the conclusion of d is in S, hence $\Delta(\Phi) \subseteq S$. Since $\operatorname{lfp}\varphi$ satisfies the assumption, the inclusion $\Delta(\Phi) \subseteq \operatorname{lfp}\varphi$ follows.

Finally, assume a set S satisfying $S \subseteq \varphi(S)$. For any judgment j in S, there exists a rule (K_j, j) in Φ, where $K_j \subseteq S$. We define a system of guarded recursive equations, with variables $(x_j)_{j\in S}$.

$$x_j = \frac{(x_k)_{k\in K_j}}{j}$$

The solution σ of this system is such that for all $j \in S$, the derivation $\sigma(x_j)$ is valid in Φ and proves j. Therefore, $S \subseteq \nabla(\Phi)$. Since $\operatorname{gfp}\varphi$ satisfies the assumption, the inclusion $\operatorname{gfp}\varphi \subseteq \nabla(\Phi)$ follows. □

The equality for the inductive case is proved by Aczel [1]. The equality for the coinductive case is proved in the author's PhD dissertation [6, p. 77]. Starting from this theorem, Leroy and the author show in their article about coinductive operational semantics [8, Sect. 2] how the deductive method accounts for induction and coinduction in the proof assistant Coq: for a concrete example, see the proof of Lemma 5 [8, Sect. 3], where the same proposition is proved following the impredicative method, the deductive method and in Coq.

2 Generalization of the Deductive Method

Theorem 2 asserts that the least and greatest fixed points of an isotone map over a powerset, a particular complete lattice, can also be defined as the inductive and coinductive interpretations of an inference system. We first generalize to any complete lattice, and then to more powerful fixed point theorems.

Tarski's Theorem Revisited. Given an isotone map η over a complete lattice $(\mathscr{E}, \leq)$, how can we define an inference system Φ over $\mathscr{E}$, or equivalently an inference operator $\varphi : 2^{\mathscr{E}} \to 2^{\mathscr{E}}$, whose inductive and coinductive interpretations produce the least and greatest fixed points of η?

A first attempt, defining $\varphi(S)$ as the image of S with η, trivially fails, since the least fixed point would be the empty set. However, a well-known result [4, Th. I.5.3] allows a complete lattice to be embedded in its powerset. Indeed, let $\gamma : 2^{\mathscr{E}} \to \{\leq x \mid x \in \mathscr{E}\}$ be the closure operator from the powerset of $\mathscr{E}$ to the set $\{\leq x \mid x \in \mathscr{E}\}$ of principal order ideals defined as follows: $\gamma(S) =\leq (\vee S)$[4]. Let $\delta : \{\leq x \mid x \in \mathscr{E}\} \to 2^{\mathscr{E}}$ be its adjoint embedding: $\delta(\leq x) =\leq x$. Now, the isomorphism ι from $\mathscr{E}$ to $\{\leq x \mid x \in \mathscr{E}\}$ is defined as follows: $\iota(x) =\leq x$. Finally, thanks to the embedding via a closure operator, we can associate to the map η the operator $\varphi : 2^{\mathscr{E}} \to 2^{\mathscr{E}}$, equal to $\delta \circ \iota \circ \eta \circ \iota^{-1} \circ \gamma$: we have $\varphi(S) =\leq \eta(\vee S)$. It turns out that this operator has the intended properties with respect to fixed points. It suffices to use the associated inference system to get the following theorem.

Theorem 3 (Tarski revisited). *Let $(\mathscr{E}, \leq)$ be a complete lattice. Let $\eta : \mathscr{E} \to \mathscr{E}$ be an isotone map over $\mathscr{E}$. Consider the inference system Φ over $\mathscr{E}$ containing all the rules, and only these rules, of the following form:*

$$\frac{S}{s} \quad \big(S \subseteq \mathscr{E}, s \leq \eta(\vee S)\big).$$

Then:

$$\leq (\mathrm{lfp}\,\eta) \quad = \quad \Delta(\Phi) \quad \textit{and} \quad \leq (\mathrm{gfp}\,\eta) \quad = \quad \nabla(\Phi).$$

Proof. It is easy to show that the inference operator φ associated to Φ is equal to $\delta \circ \iota \circ \eta \circ \iota^{-1} \circ \gamma$, with the preceding notation. As $\leq (\mathrm{lfp}\,\eta) = \mathrm{lfp}\,\varphi$ and $\leq (\mathrm{gfp}\,\eta) = \mathrm{gfp}\,\varphi$, we can conclude by applying Theorem 2. □

Two Other Fixed Point Theorems. The deductive characterization of fixed points is still pertinent when we consider two other well-known fixed point theorems, where the assumptions for the poset and the map are weakened.

A poset is *chain-complete* if any chain, including the empty chain, has a least upper bound[5]. Tarski's theorem can be extended to chain-complete posets, as shown for instance by Markowsky [9, Th. 9]. We again resort to the preceding inference system to characterize the least fixed point, with two restrictions: we only consider chains as premises and greatest judgments as conclusions.

Theorem 4 (Fixed point theorem for chain-complete poset). *Let $(\mathscr{E}, \leq)$ be a chain-complete poset and $\eta : \mathscr{E} \to \mathscr{E}$ an isotone map. Let Φ be the inference system over $\mathscr{E}$ containing all the rules, and only these rules, of the following form:*

$$\frac{C}{\eta(\vee C)} \quad \big(C \subseteq \mathscr{E}, C \textit{ chain}\big).$$

Then η has a least fixed point $\mathrm{lfp}\,\eta$ satisfying:

$$\leq (\mathrm{lfp}\,\eta) \quad = \quad \leq \Delta(\Phi).$$

Proof. We first show that $\Delta(\Phi)$ is a chain, precisely that for any x_1 and any x_2 in $\Delta(\Phi)$, we have $x_1 \leq x_2$ or $x_2 \leq x_1$. We proceed by induction over well-founded proofs. Consider a well-founded proof ended with the rule $(C_1, \eta(\vee C_1))$. Assume as inductive hypothesis that for any $y_1 \in C_1$ and any $y_2 \in \Delta(\Phi)$, we have $y_1 \leq y_2$ or $y_2 \leq y_1$. Let x_2 be in $\Delta(\Phi)$. There exists a chain C_2 included in $\Delta(\Phi)$ such that $x_2 = \eta(\vee C_2)$. There are two cases.

First, assume $\exists y_1 \in C_1 \,.\, \forall y_2 \in C_2 \,.\, y_2 \leq y_1$. We therefore have, for some y_1 in C_1 and for all y_2 in C_2: $y_2 \leq y_1$, $y_2 \leq \vee C_1$ and $\vee C_2 \leq \vee C_1$. We deduce by monotony $x_2 \leq \eta(\vee C_1)$.

[4] Here, and in the following, given $x \in \mathscr{E}$, we denote by $\leq x$ the principal order ideal $\{y \in \mathscr{E} \mid y \leq x\}$. Likewise, if $X \subseteq \mathscr{E}$, we denote by $\leq X$ the union $\bigcup_{x \in X} (\leq x)$.

[5] A chain-complete poset has therefore a bottom element, $\vee \emptyset$.

Second, assume $\forall y_1 \in C_1 \,.\, \exists y_2 \in C_2 \,.\, \neg(y_2 \leq y_1)$. We therefore have, for all y_1 in C_1 and some dependent point y_2 in C_2: $y_1 \leq y_2$ (inductive hypothesis), $y_1 \leq \vee C_2$ and $\vee C_1 \leq \vee C_2$. We deduce by monotony $\eta(\vee C_1) \leq x_2$.

Since $\Delta(\Phi)$ is a chain, $\big(\Delta(\Phi), \eta(\vee\Delta(\Phi))\big)$ is a rule. We have $\eta(\vee\Delta(\Phi)) \in \Delta(\Phi)$, hence $\eta(\vee\Delta(\Phi)) \leq \vee\Delta(\Phi)$ and then $\eta^2(\vee\Delta(\Phi)) \leq \eta(\vee\Delta(\Phi))$ by monotony. Assume x is η-closed: $\eta(x) \leq x$. It is easy to show by induction over well-founded proofs that x is an upper bound of $\Delta(\Phi)$. We deduce that first $\vee\Delta(\Phi) \leq \eta(\vee\Delta(\Phi))$ and second $\vee\Delta(\Phi)$ is the least fixed point of η. Finally $\leq (\mathrm{lfp}\,\eta) = \leq \Delta(\Phi)$ since $\vee\Delta(\Phi) \in \Delta(\Phi)$. □

Actually, as shown by Markowsky [9, Th. 9], the preceding theorem can be considered as a corollary of Bourbaki-Witt's theorem [3]. We also give a proof of this theorem by using the same inference system as in Theorem 4.

Theorem 5 (Bourbaki-Witt)**.** *Let $\mathscr{E}$ be a chain-complete poset and $\eta : \mathscr{E} \to \mathscr{E}$ an expansive*[6] *map. Let Φ be the inference system over $\mathscr{E}$ containing all the rules, and only these rules, of the following form:*

$$\frac{C}{\eta(\vee C)} \quad \big(C \subseteq \mathscr{E}, C \text{ chain}\big).$$

Then η has a fixed point $\mathrm{fp}\,\eta$ *satisfying:*

$$\leq (\mathrm{fp}\,\eta) \quad = \quad \leq \Delta(\Phi).$$

Proof. We show that $\Delta(\Phi)$ is a chain, which allows to conclude. Indeed, if $\Delta(\Phi)$ is a chain, the inference rule $\big(\Delta(\Phi), \eta(\vee\Delta(\Phi))\big)$ belongs to Φ. From $\eta(\vee\Delta(\Phi)) \in \Delta(\Phi)$, we deduce $\eta(\vee\Delta(\Phi)) \leq \vee\Delta(\Phi)$. Since η is expansive, we deduce that $\vee\Delta(\Phi)$ is a fixpoint of η. Moreover, $\vee\Delta(\Phi)$ belongs to $\Delta(\Phi)$. Therefore $\leq \Delta(\Phi) = \leq (\vee\Delta(\Phi))$.

To show that $\Delta(\Phi)$ is a chain, we introduce the notion of useful proofs. A well-founded proof, say ended by the rule $(C, \eta(\vee C))$, is said *useful* if for any well-founded proof, say ended by the rule $(D, \eta(\vee D))$, we have:

$$(\vee D < \vee C) \Rightarrow (\eta(\vee D) \leq \vee C).$$

Two crucial properties about usefulness can be asserted. First, for any useful proof ended by $(C, \eta(\vee C))$ and any well-founded proof ended by $(D, \eta(\vee D))$, we have either $\eta(\vee C) = \eta(\vee D)$, $\eta(\vee C) \leq \vee D$ or $\eta(\vee D) \leq \vee C$. Second, all well-founded proofs are useful.

Thanks to these properties, it is easy to conclude that $\Delta(\Phi)$ is a chain. Indeed, let x_1 and x_2 be in $\Delta(\Phi)$. There exists two well-founded proofs, respectively ended by $(C_1, \eta(\vee C_1))$ and $(C_2, \eta(\vee C_2))$, such that $x_1 = \eta(\vee C_1)$ and $x_2 = \eta(\vee C_2)$. Since the former proof is useful, we have by applying the first property either $x_1 = x_2$, $x_1 \leq \vee C_2$ or $x_2 \leq \vee C_1$. Since η is expansive, we deduce $x_1 \leq x_2$ or $x_2 \leq x_1$.

Let us prove the two properties about usefulness.

Suppose that a useful proof ends by the rule $(C, \eta(\vee C))$. We prove by induction that for any well-founded proof, say ended by $(D, \eta(\vee D))$, we have either $\eta(\vee C) = \eta(\vee D)$, $\eta(\vee C) \leq \vee D$ or $\eta(\vee D) \leq \vee C$. Let us consider a proof concluded by the rule $\big(D, \eta(\vee D)\big)$ and where the sub-proof of each premise y in D is ended by the rule $\big(D_y, y\big)$. Assume the inductive hypothesis: for any y in D, we have either $\eta(\vee C) = y$, $\eta(\vee C) \leq \vee D_y$ or $y \leq \vee C$. There are three cases.
First, suppose that there exists y in D such that $\eta(\vee C) = y$. We deduce $\eta(\vee C) \leq \vee D$.
Otherwise, second, suppose there exists y in D such that $\eta(\vee C) \leq \vee D_y$. Since η is expansive, we have $\eta(\vee C) \leq y$, hence $\eta(\vee C) \leq \vee D$.

[6] A map $\eta : \mathscr{E} \to \mathscr{E}$ is *expansive* (also inflationary, progressive) if any point is η-consistent: $\forall x \in \mathscr{E} \,.\, x \leq \eta(x)$.

Otherwise, third, for all y in D, we have $y \leq \vee C$. We deduce $\vee D \leq \vee C$. If $\vee D = \vee C$, we deduce $\eta(\vee C) = \eta(\vee D)$. Otherwise, we have $\vee D < \vee C$. Since the proof ended by $(C, \eta(\vee C))$ is useful, we deduce $\eta(\vee D) \leq \vee C$.

Let us prove that all well-founded proofs are useful. We proceed by induction over well-founded proofs. Let us consider a proof concluded by the rule $\big(C, \eta(\vee C)\big)$ and where the sub-proof of each premise x in C is concluded by the rule $\big(C_x, x\big)$ Assume the inductive hypothesis: for all x in C, the sub-proof with conclusion x is useful. Let us consider a well-founded proof ended by $(D, \eta(\vee D))$, and suppose $\vee D < \vee C$. We have to show $\eta(\vee D) \leq \vee C$. The inductive hypothesis and the first property give for any x in C either $x = \eta(\vee D)$, $x \leq \vee D$ or $\eta(\vee D) \leq \vee C_x$. We first notice that there exists x in C such that either $x = \eta(\vee D)$ or $\eta(\vee D) \leq \vee C_x$. Otherwise, we would have for any x in C, $x \leq \vee D$, hence $\vee C \leq \vee D$, contradiction. There are two cases.
First, suppose there exists x in C such that $x = \eta(\vee D)$. We deduce $\eta(\vee D) \leq \vee C$. Otherwise, second, there exists x in C such that $\eta(\vee D) \leq \vee C_x$. Since η is expansive, we deduce by transitivity $\eta(\vee D) \leq x$, hence $\eta(\vee D) \leq \vee C$. □

The notion of useful proofs comes from the notion of *extreme points*, which is standard in the proof of the Bourbaki-Witt's theorem following the impredicative method, as given for instance in Lang's book [7, pp. 881–884].

Comparison of the Methods. Tarski's theorem and its two extensions to chain-complete posets are usually proved with the iterative method or the impredicative method, two methods that are not clearly connected. We now suggest that the deductive method allows these two methods to be connected.

First, to each inference system Φ used in Theorems 3, 4 and 5, we can associate an inference operator φ over $2^{\mathscr{E}}$. Thus, the inductive interpretation $\Delta(\Phi)$ of Φ can also be defined as the smallest φ-closed set, as expressed in Theorems 1 and 2. It turns out that the definition of a φ-closed set is very akin to the definition of an admissible set, as found in the standard proof of Bourbaki-Witt's theorem following the impredicative method [7, pp. 881–884].

Second, the deductive method can be considered as an abstraction of the iterative method: it abstracts away from the iterative process involved in proof construction. The following proposition precisely describes their relationship in the case of Tarski's theorem. The *height* of a well-founded proof tree d is defined as follows: it is the least ordinal greater than the height of each immediate proof sub-tree of d. Given an inference system Φ over $\mathscr{E}$ and an ordinal α, we say that $x \in \mathscr{E}$ has complexity α if there is a well-founded proof of x with height less or equal to α. We denote by $\Delta_\alpha(\Phi)$ the set of all x with complexity α.

Proposition 1. *Let $(\mathscr{E}, \leq)$ be a complete lattice and $\eta : \mathscr{E} \to \mathscr{E}$ an isotone map over $\mathscr{E}$. Consider the transfinite sequence $(\Delta_\alpha(\eta))_\alpha$ defined as follows:*

$$\Delta_\alpha(\eta) \quad = \quad \eta(\textstyle\bigvee_{\beta|\beta<\alpha} \Delta_\beta(\eta)).$$

Consider the inference system Φ containing all the rules, and only these rules, of the following form:

$$\frac{S}{s} \quad \big(S \subseteq \mathscr{E}, s \leq \eta(\vee S)\big).$$

Then, for any ordinal α:

$$\leq \Delta_\alpha(\eta) = \Delta_\alpha(\Phi).$$

Proof. We proceed by transfinite induction for both inclusions.

Assume $x \leq \Delta_\alpha(\eta)$. Assume as inductive hypothesis that for any $\beta < \alpha$, we have $\leq \Delta_\beta(\eta) \subseteq \Delta_\beta(\Phi)$. Since $\big(\{\Delta_\beta(\eta) \mid \beta < \alpha\}, x\big)$ is a rule, we deduce that x belongs to $\Delta_\alpha(\Phi)$.

Assume $x \in \Delta_\alpha(\Phi)$. Assume as inductive hypothesis that for any $\beta < \alpha$, we have $\Delta_\beta(\Phi) \subseteq \leq \Delta_\beta(\eta)$. There exists a well-founded proof of x with height less or equal to α. This proof ends with a rule (S,x), with $x \leq \eta(\vee S)$. Each s in S belongs for some $\beta < \alpha$ to $\Delta_\beta(\Phi)$, hence by the inductive hypothesis to $\leq \Delta_\beta(\eta)$. We deduce $\vee S \leq \bigvee_{\beta|\beta<\alpha} \Delta_\beta(\eta)$, then by monotony $\eta(\vee S) \leq \Delta_\alpha(\eta)$ and finally $x \leq \Delta_\alpha(\eta)$. $\square$

An analogous proposition holds for the two other fixed point theorems.

3 Conclusion

We have presented a method to characterize the fixed points described in fixed point theorems for complete lattices and chain-complete posets. The method is deductive: the fixed points are "proved" in some inference system. The techniques used in the proofs of these theorems is consistent with the method: they are based on induction over well-founded proofs. Finally, we have sketched a comparison between the deductive method and the two traditional methods, impredicative and iterative. In brief, given an inference system, the impredicative method essentially corresponds to the characterization à la Tarski using the inference operator associated to the inference system, whereas the iterative method corresponds to an iterative construction of the well-founded proofs in the inference system. The connection that we have suggested deserves a further exploration, which we reserve to future work.

We have already experienced the deductive method in a work about coinductive operational semantics [8]. It turns out that the method is well-suited to the proof assistant Coq, which uses a calculus of inductive and coinductive constructions, an extension of type theory. The fixed points were expressed through inference systems and defined over powerset lattices. A motivation of the present work was to extend the method to fixed points defined in Coq over complete lattices or chain-complete posets. The iterative method resorts to ordinals, which are rarely used in Coq. Indeed, either they require to encode set theory, which is expensive, or they are implemented as constructive ordinals, which is restrictive. As for the impredicative method, it does not really fit with the calculus of inductive and coinductive constructions. Thus, the deductive method is a good candidate. A major problem to be solved is the use of classical logic. First, since the fixed points defined may have a non-terminating behavior, for instance when the fixed point is a function possibly divergent, adding classical logic axioms to the constructive logic of Coq is needed. In the same way, in the preceding proofs of the fixed point theorems, we have used the law of excluded middle. Second, as explained in the recent proposal of Bertot and Komendantsky [2], the addition of classical logic axioms should allow not only to reason about a fixed point with a terminating or non-terminating behavior, but also to extract a program computing the fixed point from the proof that the fixed point satisfies its specification, following the Curry-Howard correspondence.

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