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THE DISTANCE $dist(\mathcal{B}, X)$ WHEN $\mathcal{B}$ IS A BOUNDARY OF $B(X^{**})$

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ABSTRACT. Let X be a real Banach space and $\mathcal{B}$ a boundary of the unit ball $B(X^{**})$ of the bidual X^{**} (which means that for each $x^* \in X^*$ there is $b \in \mathcal{B}$ such that $\langle b, x^* \rangle = \|x^*\|$). We show that $dist(\mathcal{B}, X) = dist(B(X^{**}), X)$ where $dist(A, X)$ denotes the sup of all $dist(a, X)$ with $a \in A$. Since $\overline{\text{co}}^{w^*}(\mathcal{B}) = B(X^{**})$ this is in contrast with the fact that in general strict inequality can occur between $dist(K, X)$ and $dist(\overline{\text{co}}^{w^*}(K), X)$ even for a w^*-compact $K \subset X^{**}$.

If K is a w^*-compact subset of a dual Banach space X^*, a subset $\mathcal{B}$ of K is said to be a *(James) boundary* of K if every $x \in X$ attains its maximum on K at an element of $\mathcal{B}$, that is to say if for every $x \in X$ there is $b \in \mathcal{B}$ such that $\langle b, x \rangle = \sup \langle K, x \rangle$. For instance, K itself and the set of extreme points of $\overline{\text{co}}^{w^*}(K)$ (which is contained in K) are boundaries of K, but there are boundaries that do not meet the set of extreme points (see [2] and references therein).

For two sets $A, C \subset X$ and $x \in X$, $dist(x, C)$ denotes the usual distance from x to C and $dist(A, C) = \sup\{dist(a, C) : a \in A\}$ denotes the (non-symmetric) greatest distance from A to C. It is easy to see that $dist(A, C) = dist(\overline{A}, C)$ and, if C is a convex subset, then $dist(A, C) = dist(\overline{\text{co}}(A), C)$. In this short note we prove

Theorem 1. *If X is a real Banach space and $\mathcal{B}$ a boundary of the unit ball $B(X^{**})$ then $dist(\mathcal{B}, X) = dist(B(X^{**}), X)$ (which equals 0 if X is reflexive and 1 if X is not reflexive).*

Before passing to the proof we shortly explain its context and fix the notation.

The topic of our theorem is part of the general question to which extent and in which sense $\overline{\text{co}}^{w^*}(\mathcal{B})$ can be recovered by a boundary $\mathcal{B}$ (cf. [2, 3, 4, 5, 8]). When K is a w^*-compact subset of a bidual Banach space X^{**} and $\mathcal{B} = K$, the distance $dist(K, X)$ is, in general, different from the distance $dist(\overline{\text{co}}^{w^*}(K), X)$. Actually, it is known (see [4]) that

$$dist(\overline{\text{co}}^{w^*}(K), X) \leq 5dist(K, X) \tag{0.1}$$

while [5, Prop. 10] exhibits a Banach space X and a w^*-compact subset $K \subset X^{**}$ such that $dist(\overline{\text{co}}^{w^*}(K), X) = 3dist(K, X) > 0$, which shows that the factor 5 above cannot be replaced by 1. It is not known whether (0.1)

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holds if K is replaced by an arbitrary boundary $\mathcal{B}$ of $\overline{\mathrm{co}}^{w^*}(\mathcal{B})$. In the special case $\overline{\mathrm{co}}^{w^*}(K) = B(X^{**})$, however, our theorem shows that (0.1) holds with 1 instead of 5 and for any boundary $\mathcal{B}$ of $B(X^{**})$. Note that in this case one has $dist(B(X^{**}), X) = 0$ if X is reflexive and $dist(B(X^{**}), X) = 1$ if X is not reflexive.

Our notation is standard (cf. [6, 1]). The underlying scalar field we consider are the reals. If $(X, \|\cdot\|)$ is a Banach space, let $B(X)$ and $S(X)$ be the closed unit ball and unit sphere of X, respectively, and X^* its topological dual. The weak*-topology of the dual Banach space X^* is denoted by w^* and the weak topology of X by w. $\mathrm{co}(A)$ denotes the convex hull of the set A, $\overline{\mathrm{co}}(A)$ is the $\|\cdot\|$-closure of $\mathrm{co}(A)$ and $\overline{\mathrm{co}}^{w^*}(A)$ the w^*-closure of $\mathrm{co}(A)$. An *ℓ_1-sequence* is a sequence which is equivalent to the canonical basis of $\ell_1 = \ell_1(\mathbb{N})$. A Banach space X is said to contain an *asymptotically*[1] *isometric copy of* ℓ_1 if there exists an ℓ_1-sequence (a_n) in $B(X)$ and a sequence $(\delta_n)_{n\in\mathbb{N}} \subset]0,1]$ converging to 1 satisfying the following inequality for every finite sequence $(\lambda_i)_{1\le i\le n}$ of $\mathbb{R}$:

$$\sum_{1\le i\le n} \delta_i|\lambda_i| \le \Big\| \sum_{1\le i\le n} \delta_i a_i \Big\|. \tag{0.2}$$

Given a sequence (x_n) in a Banach space X we call $(\sum_{i\in F_n} \lambda_i f_i)$ a *block sequence of* (x_n) if (F_n) is a sequence of finite pairwise disjoint subsets of $\mathbb{N}$ and $(\lambda_i)_{i\in\mathbb{N}}$ is a sequence of scalars such that $\sum_{i\in F_n} |\lambda_i| = 1$ for all $n \in \mathbb{N}$. The dual unit ball of a Banach space X is said to be *w^*-block compact* if each sequence in $B(X^*)$ admits a w^*-convergent block sequence.

Proof of the theorem.
We will distinguish the two cases that X contains an asymptotically isometric copy of ℓ_1 and that it does not. This distinction comes naturally from Morillon's proof [7] of James' theorem.

First case: X contains an asymptotically isometric copy of ℓ_1.
We simply adapt the proof of [7, Th. 3]:

Proposition 2. *Let X be a Banach space containing an asymptotically isometric copy of ℓ_1 and let $\mathcal{B}$ be a boundary of $B(X^{**})$. Then there exists $b \in \mathcal{B}$ such that $dist(b, X) = dist(B(X^{**}), X)$ and so $dist(\mathcal{B}, X) = dist(B(X^{**}), X)$.*

Proof. Let $(a_i)_{i\in\mathbb{N}} \subset B(X)$ be an asymptotically isometric copy of ℓ_1. As in [7, Th. 3] let $g_n \in X^*$ be, for each $n \in \mathbb{N}$, a Hahn-Banach extension to X of the functional on $\overline{[(a_i)_{i\in\mathbb{N}}]}$ defined by $a_i \mapsto -\delta_i$ if $i < n$, and $a_i \mapsto \delta_i$ if $i \ge n$. Then $\|g_n\| \le 1$ by (0.2). Let $g \in X^*$ be a w^*-cluster point of $\{g_n : n \ge 1\}$ and set $h = (\sum_{k\in\mathbb{N}} 2^{-k} g_k) - g$. Clearly $\|h\| \le 2$ because $\|g\| \le 1$ but also $\|h\| \ge 2$ because by construction

$$\sup_{i\in\mathbb{N}} \langle h, a_i\rangle = \sup_{i\in\mathbb{N}} \delta_i\Big(\sum_{n\le i} 2^{-n} - \sum_{i<n} 2^{-n} + 1\Big) = 2.$$

[1] In the literature the notion "asymptotic l^p" can have a different meaning.

Since $\mathcal{B}$ is a boundary of $B(X^{**})$ there is $b \in \mathcal{B}$ such that $\langle b, h\rangle = 2$ and so $\langle b, g_n\rangle = 1$ for all $n \in \mathbb{N}$ and $\langle b, g\rangle = -1$. Let $x \in X$. If either $\langle g, x\rangle \geq 0$ or $\langle g, x\rangle \leq -2$ then $\|b - x\| \geq |\langle b - x, g\rangle| \geq 1$. Otherwise, $-2 < \langle g, x\rangle < 0$ and there exists some $m \in \mathbb{N}$ (depending on x) such that $-2 < \langle g_m, x\rangle < 0$. Hence $-3 < \langle x - b, g_m\rangle < -1$ and again $\|b - x\| \geq \langle b - x, g_m\rangle > 1$. This shows that $dist(b, X) = 1$ and completes the proof. $\square$

SECOND CASE: X DOES NOT CONTAIN AN ASYMPTOTICALLY ISOMETRIC COPY OF ℓ_1.

Lemma 3. *Let X be a non-reflexive Banach space without an asymptotically isometric copy of ℓ_1. Then*

(1) *X^* contains a w^*-null sequence that does not converge weakly;*
(2) *for each $\epsilon > 0$ there are an element $z_\epsilon \in S(X^{**})$ and a sequence (f_n) in $B(X^*)$ such that $f_n \xrightarrow{w^*} 0$ and $\langle z_\epsilon, f_n\rangle \geq 1 - \epsilon$ for all $n \in \mathbb{N}$.*

Proof. (1) Except for some routine arguments, we use essentially [7, Th. 2] for this first part of the lemma. Suppose that X has no asymptotically isometric copy of ℓ_1 and that in X^* each w^*-null sequence converges weakly. It follows from [7, Th. 2] that a Banach space without an asymptotically isometric copy of ℓ_1 has a w^*-block compact dual unit ball. Therefore, since a block sequence of an ℓ_1-sequence is again an ℓ_1-sequence and since an ℓ_1-sequence cannot converge weakly, the dual X^* does not contain copies of ℓ_1; moreover, via Eberlein-Šmulyan's theorem, the dual unit ball is weakly compact because by Rosenthal's ℓ_1-Theorem each sequence in it admits a weak Cauchy subsequence which w^*-converges and hence converges weakly. Hence X is reflexive which proves (1).

(2) Fix $\epsilon > 0$. By (1) of this lemma there exists a w^*-null sequence (x_n^*) in $B(X^*)$ that does not converge weakly. So, passing to a subsequence if necessary, there exist $z \in S(X^{**})$ and $\epsilon_0 > 0$ such that $\langle z, x_n^*\rangle \geq \epsilon_0 > 0$, $\forall n \geq 1$. For each $n \geq 1$ we define $a(n)$ as follows

$$a(n) := \inf\{\|x^*\| : x^* \in \mathrm{co}(\{x_k^* : k \geq n\})\}.$$

Obviously, $\epsilon_0 \leq a(n) \leq a(n+1) \leq 1$. Let

$$a := \lim_{n \geq 1} a(n) = \sup_{n \geq 1} a(n).$$

We choose $\eta > 0$ and $n_0 \in \mathbb{N}$ such that $\frac{a-\eta}{a+\eta} > 1 - \epsilon$ and $a - \eta < a(n_0)$. Now for each $k \in \mathbb{N}$ pick $v_k \in \mathrm{co}(\{x_n^* : n \geq n_0 + k\})$ so that $a(n_0 + k) \leq \|v_k\| \leq a(n_0 + k) + \eta$. Thus for each $k \geq 1$ we have

$$a - \eta < a(n_0) \leq a(n_0 + k) \leq \|v_k\| \leq a(n_0 + k) + \eta \leq a + \eta.$$

Let $f_k := v_k/(a + \eta)$, $\forall k \in \mathbb{N}$. Observe that $\|f_k\| \leq 1$. Then:

(a) By the definition of $a(n_0)$ and (f_k), every $u \in \mathrm{co}(\{f_k : k \geq 1\})$ satisfies $u \in \mathrm{co}(\{x_n^*/(a + \eta) : n \geq n_0\})$ and so:

$$1 \geq \|u\| \geq \frac{a(n_0)}{a + \eta} > \frac{a - \eta}{a + \eta} \geq 1 - \epsilon.$$

Thus, $\mathrm{co}(\{f_k : k \geq 1\}) \cap \frac{a-\eta}{a+\eta} B(X^*) = \emptyset$. By the Hahn-Banach separation theorem there exists $z_\epsilon \in S(X^{**})$ such that $\langle z_\epsilon, u\rangle \geq (a-\eta)/(a+\eta) \geq 1-\epsilon$ for every $u \in \mathrm{co}(\{f_k : k \geq 1\}$.

(b) Since f_k is a finite convex combination of elements of $\{x_n^*/M : n \geq n_0 + k\}$ and $x_n^* \xrightarrow{w^*} 0$, necessarily $f_k \xrightarrow{w^*} 0$. ☐

Remark. The second part of Lemma 3 gives an improvement (probably known as folklore) of the Josefson-Nissenzweig Theorem for non-Grothendieck spaces. (A Grothendieck space is a Banach space in whose dual w^*-convergent sequences converge weakly.)

Proposition 4. *Let X be a Banach space without an asymptotically isometric copy of ℓ_1 and let $\mathcal{B}$ be a boundary of $B(X^{**})$. Then $dist(\mathcal{B}, X) = dist(B(X^{**}), X)$.*

Proof. If X is reflexive the statement is trivially true. Assume that X is not reflexive. Let $\epsilon > 0$ and choose a w^*-null sequence $(f_n)_{n\geq 1} \subset B(X^*)$ according to Lemma 3. By Simons' equality [8, p. 69] we have

$$\sup\{\limsup_{n\to\infty}\langle b, f_n\rangle : b \in \mathcal{B}\} = \sup\{\limsup_{n\to\infty}\langle z, f_n\rangle : z \in B(X^{**})\}.$$

This expression is at least $1-\epsilon$ by Lemma 3. Hence there is $b \in \mathcal{B}$ such that $\limsup_{n\to\infty}\langle b, f_n\rangle > 1-2\epsilon$ and so there is a subsequence $(f_{n_k})_{k\geq 1}$ such that $\langle b, f_{n_k}\rangle > 1-2\epsilon$ for all k. Let v be a w^*-cluster point of $\{f_{n_k} : k \geq 1\}$ in X^{***}. Then $v \in X^\perp$ because $(f_{n_k})_{k\geq 1}$ is w^*-null in X^*. Furthermore $\|v\| \leq 1$ and $\langle v, b\rangle \geq 1-2\epsilon$. Now for any $x \in X$ we have that $\|b-x\| \geq \langle v, b-x\rangle = \langle v, b\rangle \geq 1-2\epsilon$ hence $dist(b, X) \geq 1-2\epsilon$. Since ϵ was arbitrary this proves the proposition. ☐

The theorem is clear from the two Propositions.

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