Influence of a Rough Thin Layer on the Potential

Ionel Sorin Ciuperca, Ronan Perrussel, Clair Poignard

Influence of a Rough Thin Layer on the Potential

Sorin Ionel Ciuperca[1], Ronan Perrussel[2], and Clair Poignard[3]

[1]Université de Lyon, ICJ UMR CNRS 5208, F-69003 Lyon, France
[2]Laboratoire Ampère UMR CNRS 5005, Université de Lyon, École Centrale de Lyon, F-69134 Écully, France
[3]INRIA Bordeaux-Sud-Ouest, Team MC2, IMB UMR CNRS 5251, Université Bordeaux 1, F-33405 Talence, France

In this paper, we study the behavior of the steady-state potentials in a material composed by an interior medium surrounded by a rough thin layer and embedded in an ambient bounded medium. The roughness of the layer is supposed to be ε^{α}-periodic, ε being the small thickness of the layer and α is a parameter, that describes the roughness of the layer: if $\alpha < 1$ the layer is weakly rough, while if $\alpha > 1$, the layer is very rough. Depending on the roughness of the membrane, the approximate transmission conditions are very different. We present and validate numerically the rigorous approximate transmissions proved by Poignard (*Math. Meth. Appl. Sci.*, vol. 32, pp. 435–453, 2009) and Ciuperca *et al.* (Research reports INRIA RR-6812 and RR-6975). This paper extends previous works of Perrussel and Poignard (*IEEE Trans. Magn.*, vol. 44, pp. 1154–1157, 2008) in which the layer had a constant thickness.

***Index Terms*—Asymptotic model.**

I. Introduction

IN DOMAINS with rough thin layers, numerical difficulties appear due to the complex geometry of the rough layer when computing the steady-state potentials. We present here how these difficulties may be avoided by replacing the rough layer by appropriate transmission conditions. Particularly, we show that considering only the mean effect of the roughness is not sufficient to obtain the potential with a good accuracy when the layer is too rough.

For the sake of simplicity, we present here the two-dimensional case, however, the approximate conditions are still valid for the three-dimensional case. The results presented in this paper together with the result of Perrussel and Poignard [1] provide a survey of the different configurations that appear when dealing with thin layers for steady-state potential or diffusion problems.

A. Statement of the Problem

Let Ω be a smooth bounded domain of $\mathbb{R}^2$ with connected boundary $\partial\Omega$. For ε and α strictly positive, we split Ω into three subdomains: $\mathcal{O}^1$, $\mathcal{O}^m_\varepsilon$ and $\mathcal{O}^0_\varepsilon$. $\mathcal{O}^1$ is a smooth domain strictly embedded in Ω (see Fig. 1). We denote by Γ its connected boundary. The domain $\mathcal{O}^m_\varepsilon$ is a thin oscillating layer surrounding $\mathcal{O}_1$. We denote by $\Gamma^\alpha_\varepsilon$ the oscillating boundary of $\mathcal{O}^m_\varepsilon$: $\Gamma^\alpha_\varepsilon = \partial\mathcal{O}^m_\varepsilon \setminus \Gamma$. The parameter α is related to the ratio of period of the oscillating boundary Γ and the mean membrane thickness. It can be seen as a roughness parameter: for $\alpha = 0$, the membrane thickness is constant, for $\alpha = 1$ the period of the oscillations is similar to the membrane thickness, for $\alpha > 1$ the membrane is very rough.

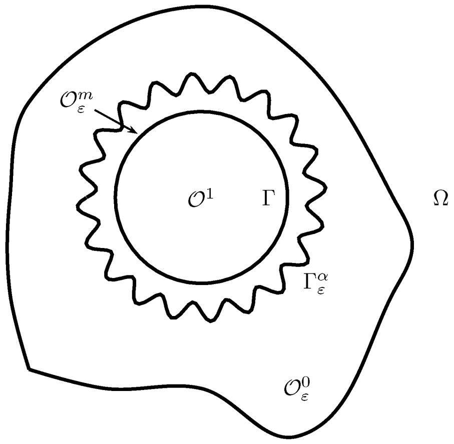

Fig. 1. Geometry of the problem.

We denote by $\mathcal{O}^0$ the domain $\mathcal{O}^0 = \Omega \setminus \overline{\mathcal{O}^1}$. Define the two following piecewise-constant conductivities on the domain Ω, denoted by σ and $\tilde\sigma$, by

$$\sigma(z) = \begin{cases} \sigma_1, & \text{if } z \in \mathcal{O}^1 \\ \sigma_m, & \text{if } z \in \mathcal{O}^m_\varepsilon, \\ \sigma_0, & \text{elsewhere} \end{cases} \quad \tilde\sigma(z) = \begin{cases} \sigma_1, & \text{if } z \in \mathcal{O}^1 \\ \sigma_0, & \text{if } z \in \Omega \setminus \mathcal{O}^1 \end{cases}$$

where σ_1, σ_m, and σ_0 are given positive constants.[1]

Let u^ε and u^0 be defined by

$$\begin{cases} \nabla.(\sigma\nabla u^\varepsilon) = 0, & \text{in } \Omega, \\ u^\varepsilon|_{\partial\Omega} = g, \end{cases}, \quad \begin{cases} \nabla.(\tilde\sigma\nabla u^0) = 0, & \text{in } \Omega \\ u^0|_{\partial\Omega} = g \end{cases} \tag{1}$$

where g is a sufficiently smooth boundary data. For different values of α, we present how to define the potential u^1_α such that u^ε is approached by $u^\varepsilon = u^0 + \varepsilon u^1_\alpha + o(\varepsilon)$ for ε tending to zero.[2]

Manuscript received December 23, 2009; accepted February 18, 2010. Current version published July 21, 2010. Corresponding author: C. Poignard (e-mail: clair.poignard@inria.fr).

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/TMAG.2010.2044027

[1]The same following results hold if σ_0, σ_1, and σ_m are given complex numbers with imaginary parts (and, respectively, real parts) with the same sign.

[2]The notation $o(\varepsilon)$ means that $\|u^\varepsilon - (u^0 + \varepsilon u^1_\alpha)\|$ goes to zero faster than ε as ε goes to zero. We refer to [2]–[4] for a precise description of the involved norms and the accuracy of the convergence.

B. Heuristics of the Derivation of the Conditions

Suppose Γ is a smooth closed curve of $\mathbb{R}^2$ of length 1 and parametrize it by the curvilinear coordinate $\Gamma = \{\Psi(\theta), \theta \in [0,1]\}$. Let n be the (outward) normal to $\partial\mathcal{O}_1$. Γ_ε is described by

$$\Gamma_\varepsilon^\alpha = \left\{\Phi(\eta,\theta) =^{\text{def}} \Psi(\theta) + \varepsilon f(\theta/\varepsilon^\alpha) n(\theta),\ \theta \in [0,1]\right\}$$

where f is a smooth one-periodic and positive function and α is a positive parameter, that describes the roughness of the layer.

The heuristics of the derivation consists in performing a suitable change of variables in the domain $\mathcal{O}_\varepsilon^m$ in order to make appear the small parameter ε in the partial differential equations. Then, supposing that the potential u^ε can be written as a formal sum $u^0 + \varepsilon u_\alpha^1 + \cdots$ and identifying the terms with the same power in ε, we infer the approximated transmission conditions. We emphasize that for rough thin layers, these conditions can be very different, depending on the roughness of the layer. Actually, deriving the above function Φ with respect to θ makes appear the term $\varepsilon^{1-\alpha} n$, which is not negligible as soon as $\alpha \geq 1$, providing rather different terms in the identification process.

Three kinds of roughness appear: the weakly rough layer —$\alpha < 1$— the very rough case —$\alpha > 1$— and the case $\alpha = 1$. In the following, it is convenient to denote by $\tilde{f}$ the mean value of f

$$\tilde{f} = \int_0^1 f(\theta) \mathrm{d}\theta.$$

II. Weakly Oscillating Thin Layer

In this section, we suppose that $\alpha < 1$, meaning that the thin layer is weakly rough. This is the simplest case of rough thin layer in the sense that the intuition, which consists in considering the mean effect of the roughness, is valid. More precisely, $u^\varepsilon = u^0 + \varepsilon \tilde{u}^1 + o(\varepsilon)$, where the first-order coefficient $\tilde{u}^1$ is the solution to the following problem[3]:

$$\begin{cases} \Delta \tilde{u}^1 = 0,\ \tilde{u}^1|_{\partial\Omega} = 0, & \text{in } \mathcal{O}^0 \cup \mathcal{O}^1, \\ [\tilde{\sigma}\partial\tilde{u}^1]_\Gamma = (\sigma_e - \sigma_m)\nabla_\Gamma.(\tilde{f}\nabla_\Gamma u^0|_\Gamma), & \\ [\tilde{u}^1]_\Gamma = \tilde{f}\left(\frac{1}{\sigma_0} - \frac{1}{\sigma_m}\right)\sigma_1 \partial_n u^0|_{\Gamma^-} & \end{cases} \quad (2)$$

where ∇_Γ, $\nabla_\Gamma.$, and ∂_n denote, respectively, the surface gradient, the tangential divergence and the normal derivatives along Γ. Therefore, according to [1], we infer that if $\alpha < 1$, the first order approximation amounts to replacing the weakly oscillating thin layer by a thin layer with a constant thickness equal to $\tilde{f}$.

III. The Case $\alpha = 1$

Suppose now that $\alpha = 1$. As theoretically shown in Ciuperca *et al.* [3], considering only the mean value of f is not sufficient to provide an accurate approximation of the potential. Actually, it is necessary to define appropriate boundary layer correctors.

[3]We denote by $[w]_\Gamma$ the jump of a function w on Γ.

A. Boundary Layer Corrector in the Infinite Strip

The key-point of the derivation of the equivalent transmission conditions consists in taking advantage of the periodicity of the roughness. This is performed by unfolding and upscaling the rough thin layer into the infinite strip $\mathbb{R} \times [0,1]$.

Define the closed curves $\mathcal{C}_1$ and $\mathcal{C}_0$, which are trigonometrically oriented by

$$\mathcal{C}_0 = \{0\} \times [0,1], \quad \mathcal{C}_1 = \{(f(y), y), \quad \forall y \in [0,1]\}.$$

The outward normals to $\mathcal{C}_0$ and $\mathcal{C}_1$ equal

$$n_{\mathcal{C}_0} = \begin{pmatrix}1\\0\end{pmatrix}, \quad n_{\mathcal{C}_1} = \frac{1}{\sqrt{1+(f'(y))^2}}\begin{pmatrix}1\\-f'(y)\end{pmatrix}. \quad (3)$$

According to [3], there exists a unique couple (A^0, a^0) where A^0 is a continuous vector field and a^0 is constant such that

$$A^0 \text{ is } 1\text{−periodic in } y, \quad \Delta A^0 = 0, \ \text{ in } \mathbb{R}\times[0,1] \quad (4a)$$

$$\sigma_0 \partial_n A^0|_{\mathcal{C}_1^+} - \sigma_m \partial_n A^0|_{\mathcal{C}_1^-} = (\sigma_m - \sigma_0) n_{\mathcal{C}_1} \quad (4b)$$

$$\sigma_m \partial_n A^0|_{\mathcal{C}_0^+} - \sigma_1 \partial_n A^0|_{\mathcal{C}_0^-} = -(\sigma_m - \sigma_0) n_{\mathcal{C}_0} \quad (4c)$$

$$A^0 \to_{x\to-\infty} 0, \quad A^0 - a^0 \to_{x\to+\infty} 0 \quad (4d)$$

where the convergences at infinity are exponential. We emphasize that a^0 is not imposed but is a floating potential, which has to be calculated numerically.

B. Approximate Transmission Conditions

Our transmission conditions are then obtained with the help of the constant vector D and defined by

$$(\sigma_0 - \sigma_m)\left[\tilde{f} n_{\mathcal{C}_0} + \int_0^1 A^0(f(y), y)\, dy\right] + (\sigma_m - \sigma_1)\int_0^1 A^0(0,y) dy - \sigma_0 a^0$$

$$D = (\sigma_m - \sigma_0)\left[\int_0^1 A^0(f(y), y) f'(y) dy - \tilde{f}\begin{pmatrix}0\\1\end{pmatrix}\right].$$

The potential u_1^1 is then defined by:

$$\begin{cases} \Delta u_1^1 = 0, \quad in\ \mathcal{O}^0 \cup \mathcal{O}^1, \quad u_1^1\big|_{\partial\Omega} = 0, \\ \left[\tilde{\sigma}\partial_n u_1^1\right]_\Gamma = -\kappa D_1 \cdot \begin{pmatrix} \partial_n u^0|_{\Gamma+} \\ \nabla_\Gamma u^0|_{\Gamma+} \end{pmatrix} + D_2 \cdot \nabla_\Gamma \cdot \begin{pmatrix} \partial_n u^0|_{\Gamma+} \\ \nabla_\Gamma u^0|_{\Gamma+} \end{pmatrix} \\ \left[u_1^1\right]_\Gamma = a^0 \cdot \begin{pmatrix} \partial_n u^0|_{\Gamma+} \\ \nabla_\Gamma u^0|_{\Gamma+} \end{pmatrix} \end{cases}$$

where κ is the curvature of Γ.

Observe that our conditions are different than if we would only consider the mean effect of the roughness described by (2).

IV. Very Rough Thin Layer

We consider in this section the case of a very rough thin layer: the parameter α is strictly greater than 1.

For any $s \in \mathbb{R}$, we define the one-dimensional set $Q(s)$ by

$$\forall s \in \mathbb{R}, \quad Q(s) = \{\tau \in [0,1],\ s \leq f(\tau)\}$$

and we denote by q the Lebesgue-measure of Q

$$\forall s \in \mathbb{R}, \quad q(s) = \int_0^1 \chi_{Q(s)}(t)dt$$

$\chi_{Q(s)}$ is the characteristic function[4] of the set $Q(s)$. The approximate transmission conditions for the very rough thin layer require the two following numbers:

$$r_1 = \int_{f_{\min}}^{f_{\max}} \frac{q^2(s)}{\sigma_m(\Psi(t))\, q(s) + \sigma_0(\Psi(t))\,[1-q(s)]} ds$$

$$r_2 = \int_{f_{\min}}^{f_{\max}} \frac{q(s)\,[1-q(s)]}{\sigma_0(\Psi(t))\, q(s) + \sigma_m(\Psi(t))\,[1-q(s)]} ds$$

where $f_{\min}$ and $f_{\max}$ are the extremal values of the function f. According to [4], u^1_α satisfies the following problem:

$$\Delta u^1_\alpha = 0 \quad \text{in} \quad \mathcal{O}^0 \cup \mathcal{O}^1, \quad \text{with} \quad u^1_\alpha\big|_{\partial\Omega} = 0 \qquad (5)$$

and the following transmission conditions on Γ:

$$\left[\tilde{\sigma}\partial_n u^1_\alpha\right]_\Gamma = \nabla_\Gamma \cdot \left[(\sigma_0 - \sigma_m)\left(\tilde{f} + (\sigma_0 - \sigma_m) r_2\right) \times \nabla_\Gamma u^0|_\Gamma\right] \qquad (6)$$

$$\left[u^1_\alpha\right]_\Gamma = \sigma_m \left[(\sigma_m - \sigma_0) r_1 - \tilde{f}\right] \times \left(\frac{1}{\sigma_0} - \frac{1}{\sigma_m}\right) \sigma_1 \partial_n u^0|_{\Gamma^-}. \qquad (7)$$

V. Numerical Simulations

In order to verify the convergence rate stated in the previous sections, we consider a problem where the geometry and the boundary conditions are ε^α-periodic, for three different α: 0, 1, and 2. The computational domain Ω is delimited by the circles of radius 2 and of radius 0.2 centered in 0, while $\mathcal{O}^1$ is the intersection of Ω with the concentric disk of radius 1. The rough layer is then described by $f(y) = 1 + \varepsilon^{1/\alpha}/2\sin(y)$. One period of the domain is shown Fig. 2(a), for $\alpha = 1$ and Fig. 2(a) for $\alpha = 2$. The Dirichlet boundary data is identically 1 on the outer circle and 0 on the inner circle.

The mesh generator *Gmsh* [5] and the finite element library *Getfem++* [6] enables us to compute the five potentials u^ε, u^0, $\tilde{u}^1$, u^1_1, and u^1_2. The rough thin layer is supposed slightly insulating. The conductivities σ_0, σ_1 and σ_m, respectively, equal to 3, 1, and 0.1. We denote by $\tilde{u}^1$ the solution to Problem (2).

[4]The characteristic function of a set S is the function $\chi_S(t)$ equal to 1 if $t \in S$ and 0 if $t \notin S$.

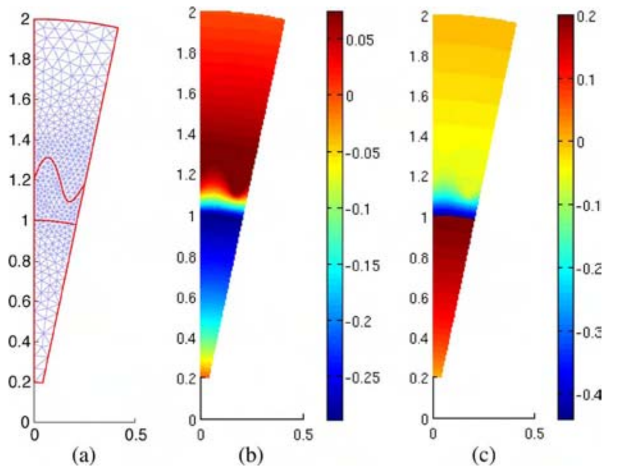

Fig. 2. Representation of one period of the domain and the corresponding errors with approximate solutions u^0 and $u^0 + \varepsilon u^1$. $\varepsilon = 2\pi/30$. Do not consider the error in the rough layer because a proper reconstruction of the solution in it is not currently implemented. (a) One period. (b) Error order 0. (c) Error order 1.

TABLE I
Coefficients Issued From the Solution to Problem (4). Three Significant Digits are Kept

a^0_1	a^0_2	D_1	D_2
19.3	0	-0.0499	-3.87

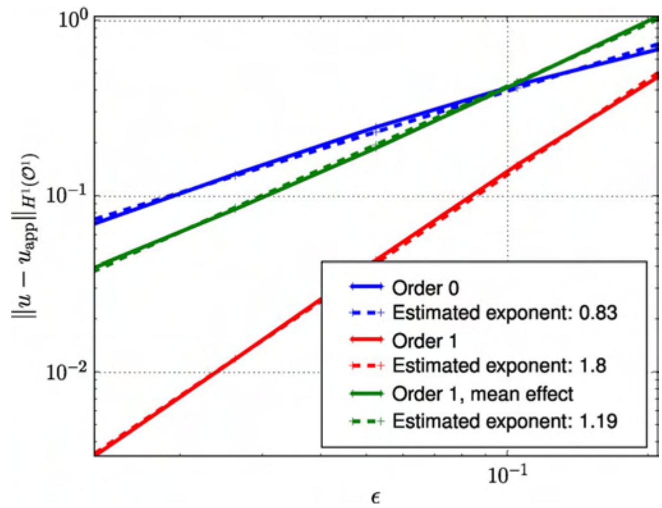

Fig. 3. H^1-Error in the cytoplasm versus ϵ for three approximate solutions.

A. The Case $\alpha = 1$

The computed coefficients[5] issued from Problem (4) are given in Table I.

The numerical convergence rates for the H^1-norm in $\mathcal{O}^1$ of the three following errors $u^\varepsilon - u^0$, $u^\varepsilon - u^0 - \varepsilon u^1$ and $u^\varepsilon - u^0 - \varepsilon\tilde{u}^1$ as ε goes to zero are given in Fig. 3. As predicted by the theory, the rates are close to 1 for the order 0 and for the order 1 with the mean effect, whereas it is close to 2 for the "real" order 1 equal to $u^\varepsilon - u^0 - \varepsilon u^1$.

[5]The convergences at the infinity in Problem (4) are exponential, hence, we just have to compute problem (4) for $|x| \le M$, with M large enough to obtain a^0 with a good accuracy.

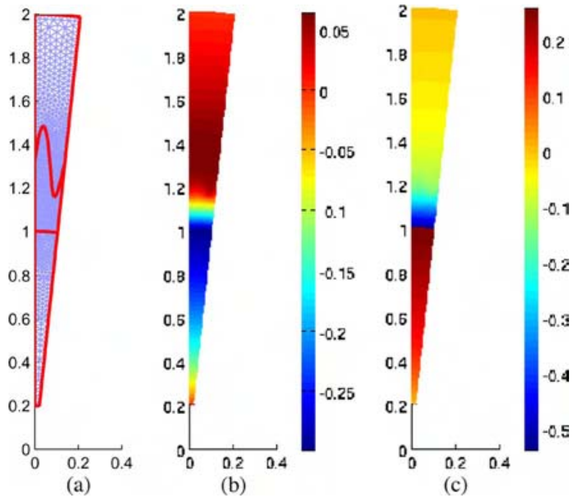

Fig. 4. Representation of one period of the domain and the corresponding errors with approximate solutions u and $u + \varepsilon u_2^1$. $\varepsilon = 2\pi/60$. Do not consider the error inside the rough layer because a proper reconstruction of the solution in it is not currently implemented. (a) One period. (b) Error order 0. (c) Error order 1.

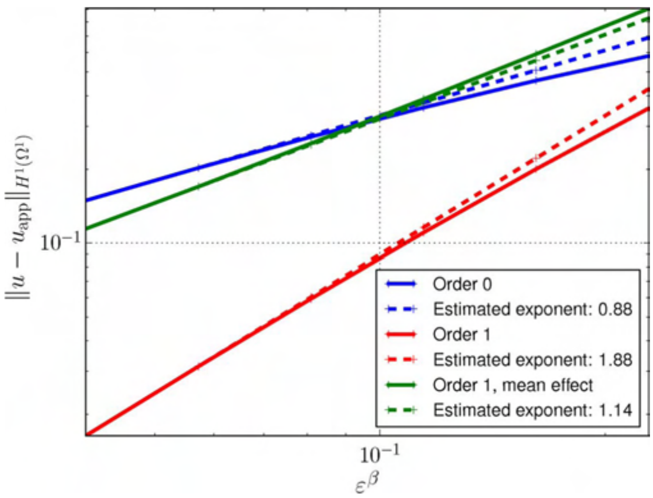

Fig. 5. H^1-error versus ε for three approximate solutions. We choose $\alpha = 2$.

B. The Very Rough Thin Layer With $\alpha = 2$

Suppose now that $\alpha = 2$. One period of the domain is shown Fig. 4(a). The computed coefficients for quantifying the roughness effect are $r_1 = 5.87$ and $r_2 = 0.413$ (three significant digits are kept).

The numerical convergence rates for both the H^1- and the L^2-norms in Ω^1 of the three following errors $u_\varepsilon - u^0$, $u_\varepsilon - u^0 - \varepsilon u_\alpha^1$ and $u_\varepsilon - u - \varepsilon\tilde{u}^1$ as ε goes to zero are given in Fig. 5 for $\alpha = 2$. Observe that the numerical convergence rates are similar to the rates shown for $\alpha = 1$. More precisely they are close to 1 for $u_\varepsilon - u^0$ and for $u_\varepsilon - (u + \varepsilon\tilde{u}^1)$, whereas the convergence rate is close to 2 for $u_\varepsilon - (u^0 + \varepsilon u_2^1)$.

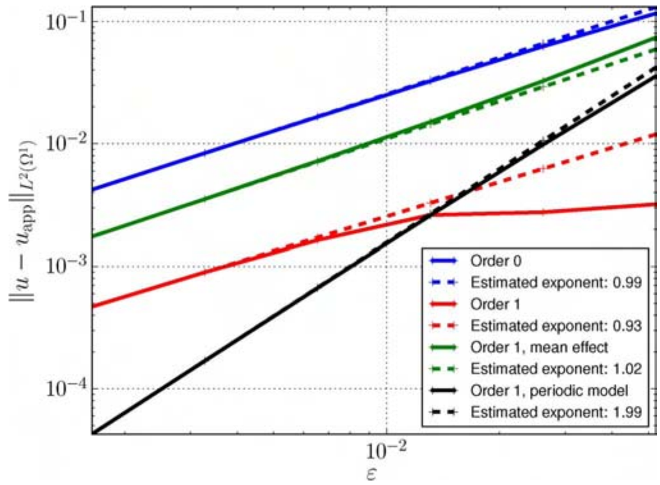

Fig. 6. L^2-error in the cytoplasm versus ε for four approximate solutions.

VI. Conclusion

In this paper, we have presented approximate transmission conditions that tackle the numerical difficulties due to the computation of the steady-state potential in a rough thin layer. The rough layer is rigorously replaced by appropriate conditions that avoid to mesh it.

Depending on the roughness of the layer, three different conditions have to be used. These three transmission conditions describe all the configurations of rough thin layers: from the weakly oscillating thin layer to the very rough membrane.

The main feature of the paper consists in the fact that considering the mean effect of the roughness is not sufficient when the layer is too rough.

To conclude, Fig. 6 demonstrates that when $\alpha = 1$ the convergence rate of the approximate transmission conditions for very rough thin layers decreases dramatically. Therefore to compute the steady-state potential with a good accuracy, it is necessary to choose appropriately one of the above transmission conditions.

References

[1] C. Poignard, P. Dular, R. Perrussel, L. Krähenbühl, L. Nicolas, and M. Schatzman, "Approximate conditions replacing thin layer," *IEEE Trans. Mag.* vol. 44, no. 6, pp. 1154–1157, Jun. 2008. [Online]. Available: http://hal.archives-ouvertes.fr/hal-00165049/fr

[2] C. Poignard, "Approximate transmission conditions through a weakly oscillating thin layer," *Math. Meth. App. Sci.* vol. 32, pp. 435–453, 2009. [Online]. Available: http://www.math.u-bordeaux1.fr/~pclair/membWosci-18pp.html

[3] I. Ciuperca, M. Jai, and C. Poignard, "Approximate transmission conditions through a rough thin layer. The case of the periodic roughness," to appear in *Eur. J. Appl. Math.* [Online]. Available: http://hal.inria.fr/inria-00356124/fr/ [Online]. Available: http://www.math.u-bordeaux1.fr/~pclair/periodicroughness-20pp.pdf, Res. Rep. INRIA RR-6812

[4] I. Ciuperca, R. Perrussel, and C. Poignard, Two-scale analysis for very rough thin layers. an explicit characterization of the polarization tensor INRA, Res. Rep. INRIA RR-6975. [Online]. Available: http://hal.inria.fr/inria-00401835/fr/ [Online]. Available: http://hal.inria.fr/inria-00384198/fr/

[5] C. Geuzaine and J. F. Remacle, GMSH mesh generator. [Online]. Available: http://geuz.org/gmsh

[6] Y. Renard and J. Pommier, Getfem Finite Element Library. [Online]. Available: http://home.gna.org/getfem