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SOME GENERALIZED EUCLIDEAN AND 2-STAGE EUCLIDEAN NUMBER FIELDS THAT ARE NOT NORM-EUCLIDEAN

JEAN-PAUL CERRI

Abstract. We give examples of Generalized Euclidean but not norm-Euclidean number fields of degree strictly greater than 2. In the same way we give examples of 2-stage Euclidean but not norm-Euclidean number fields of degree strictly greater than 2. In both cases, no such examples were known.

1. Introduction

In 1985, Johnson, Queen and Sevilla [9] introduced a generalization of the classical notion of Euclidean number field.

Definition 1.1. A number field $K$ is said to be Generalized Euclidean or simply G.E. if for every $(\alpha, \beta) \in \mathbb{Z}_K \times \mathbb{Z}_K \setminus \{0\}$ such that the ideal $(\alpha, \beta)$ is principal, there exists $\Upsilon \in \mathbb{Z}_K$ such that $|N_{K/Q}(\alpha - \Upsilon \beta)| < |N_{K/Q}(\beta)|$.

If $(\alpha, \beta)$ is principal, we thus have at our disposal the Euclidian algorithm to compute a gcd of $\alpha$ and $\beta$ because it is easy to see that $(\beta, \alpha - \Upsilon \beta)$ is principal again, and so on. Note that if $K$ is norm-Euclidean then $K$ is G.E. and that if $K$ has class number 1, then $K$ is G.E. if and only if $K$ is norm-Euclidean. If we want to illustrate the difference between “G.E.” and “norm-Euclidean”, the interesting case is when $K$ is not principal, G.E. but not norm-Euclidean. The following result was established by Johnson, Queen and Sevilla in [9].

Theorem 1.1. The quadratic number field $\mathbb{Q}(\sqrt{d})$ is G.E. but not norm-Euclidean for $d = 10$ and $d = 65$. The quadratic number field $\mathbb{Q}(\sqrt{d})$ is not G.E. for $d = 15, 26, 30, 35, 39, 51, 78, 87, 102, 115, 195$ and $230$.

Furthermore, Johnson, Queen and Sevilla conjectured that $K = \mathbb{Q}(\sqrt{d})$ (with $d > 1$ squarefree) is G.E. if and only if $K$ is norm-Euclidean or $d = 10$ or 65.

Another variation on norm-Euclidean number fields has been introduced by Cooke [7].

Definition 1.2. Let $m$ be a rational integer $\geq 1$. The number field $K$ is $m$-stage Euclidean if and only if for every $\alpha \in \mathbb{Z}_K$ and every $\beta \in \mathbb{Z}_K \setminus \{0\}$ there exists a positive rational integer $k \leq m$ and $k$ pairs $(q_i, r_i)$ $(1 \leq i \leq k)$ of elements of $\mathbb{Z}_K$...
such that
\[
\alpha = \beta q_1 + r_1, \\
\beta = r_1 q_2 + r_2, \\
\vdots \\
r_{k-2} = r_{k-1} q_k + r_k,
\]
and \(|N_{K/Q}(r_k)| < |N_{K/Q}(\beta)|\).

When it is well defined, let us put
\[
[q_1, q_2, \ldots, q_k] = q_1 + \frac{1}{q_2 + \frac{1}{q_3 + \cdots + \frac{1}{q_k}}},
\]
where \(a_k\) and \(b_k\) are given by
\[
\begin{align*}
a_1 &= q_1, \quad b_1 = 1, \\
a_2 &= a_1 q_2 + 1, \quad b_2 = q_2,
\end{align*}
\]
and recursively by
\[
a_k = a_{k-1} q_k + a_{k-2}, \quad b_k = q_k b_{k-1} + b_{k-2}.
\]
Since
\[
\frac{\alpha}{\beta} = \frac{a_k}{b_k} + (-1)^{k+1} \frac{r_k}{b_k \beta'},
\]
this definition is equivalent to the following.

**Definition 1.3.** The number field \(K\) is \(m\)-stage Euclidean if and only if for every \(\xi \in K\), there exists a positive rational integer \(k \leq m\), and \(k\) elements \(q_1, q_2, \ldots, q_k \in \mathbb{Z}_K\) such that
\[
\left| N_{K/Q}(\xi - [q_1, q_2, \ldots, q_k]) \right| < \frac{1}{|N_{K/Q}(\beta)|}.
\]

As in the previous case, norm-Euclidean implies \(m\)-stage Euclidean, but contrary to what happens with the G.E. condition, we have the following result [7].

**Theorem 1.2.** A number field \(K\) with unit rank \(r \geq 1\) is principal if and only if \(K\) is \(m\)-stage Euclidean for some \(m\).

As a consequence, if we want to study the difference between \(m\)-stage Euclidean and norm-Euclidean, we have to look at number fields with class number 1 and find some example where \(K\) is principal, \(m\)-stage Euclidean but not norm-Euclidean. The following result was established by Cooke [7].

**Theorem 1.3.** For \(d = 14, 22, 23, 31, 38, 43, 46, 53, 61, 69, 89, 93, 97, \mathbb{Q}(\sqrt{d})\) is 2-stage euclidean but not norm-Euclidean.

Furthermore, Cooke and Weinberger [8] proved that, under GRH, every principal number field \(K\) with unit rank \(r \geq 1\) is 4-stage Euclidean, and even 2-stage Euclidean if \(K\) has at least one real place.
For both notions (G.E. and $m$-stage Euclidean), no examples of number fields of degree strictly greater than 2 were known. Our main results are the following.

**Theorem 1.4.** None of the totally real number fields enumerated in Table 1 are principal. They all are G.E. except for the second cubic number field of discriminant 3969, defined by $x^3 - 21x - 35$, which is neither principal nor G.E.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$D_K$</th>
<th>$P(x)$</th>
<th>$h$</th>
<th>$M(K)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>1957</td>
<td>$x^3 - x^2 - 9x + 10$</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>2597</td>
<td>$x^3 - x^2 - 9x + 8$</td>
<td>3</td>
<td>5/2</td>
</tr>
<tr>
<td>3</td>
<td>2777</td>
<td>$x^3 - x^2 - 14x + 23$</td>
<td>2</td>
<td>5/3</td>
</tr>
<tr>
<td>3</td>
<td>3969</td>
<td>$x^3 - 21x - 28$</td>
<td>3</td>
<td>4/3</td>
</tr>
<tr>
<td>3</td>
<td>3969</td>
<td>$x^3 - 21x - 35$</td>
<td>3</td>
<td>7/3</td>
</tr>
<tr>
<td>3</td>
<td>3981</td>
<td>$x^3 - x^2 - 11x + 12$</td>
<td>2</td>
<td>3/2</td>
</tr>
<tr>
<td>3</td>
<td>4212</td>
<td>$x^3 - 12x - 10$</td>
<td>3</td>
<td>7/2</td>
</tr>
<tr>
<td>3</td>
<td>4312</td>
<td>$x^3 - x^2 - 16x + 8$</td>
<td>3</td>
<td>11/4</td>
</tr>
<tr>
<td>3</td>
<td>5084</td>
<td>$x^3 - 14x - 14$</td>
<td>3</td>
<td>9/2</td>
</tr>
<tr>
<td>4</td>
<td>21025</td>
<td>$x^3 - 17x^2 + 36$</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>32625</td>
<td>$x^3 - x^2 - 19x^2 + 4x + 76$</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>46400</td>
<td>$x^3 - 22x^2 + 116$</td>
<td>2</td>
<td>5/4</td>
</tr>
<tr>
<td>4</td>
<td>51200</td>
<td>$x^3 - 20x^2 + 50$</td>
<td>2</td>
<td>7/2</td>
</tr>
</tbody>
</table>

Table 1. Here, $n$ is the degree of the field $K$, $D_K$ its discriminant, $P(x)$ its equation, $h$ its class number and $M(K)$ its Euclidean minimum.

**Theorem 1.5.** The totally real number fields of degree 3 and of discriminants $< 15000$ which are principal but not norm-Euclidean (82 cases) are 2-stage norm-Euclidean. The same is true for degree 4 and discriminants 18432, 34816, 35152 and for degree 5 and discriminant 390625. In all these cases, the number field is principal, not norm-Euclidean, but 2-stage norm-Euclidean.

Details on the number fields appearing in Theorem 1.5 are available from [6].

In Section 2, we recall other definitions and general results. In Section 3 and 4, we study the case of Generalized Euclidean number fields and the case of 2-stage Euclidean number fields, respectively.

### 2. The algorithm, generalities

Let $K$ be a number field of degree $n$. We have designed an algorithm which allows us to compute the Euclidean minimum of $K$, in particular when $K$ is totally real [5], but also in the general case [3]. According to theoretical results [4], this algorithm can also give the upper part of the Euclidean spectrum of $K$ and this yields new examples of number fields with interesting properties.

From now on, we suppose that $K$ is totally real and that $n > 2$. We denote by $\mathbb{Z}_K$ the ring of its integers and by $N_{K/Q}$ its absolute norm. The *Euclidean minimum*

\[\text{In [2] and [10] the Euclidean minimum of this number field is falsely announced to be 1.}\]
of an element $\xi \in K$ is

$$m_K(\xi) = \inf_{\Upsilon \in \mathbb{Z}} |N_{K/\mathbb{Q}}(\xi - \Upsilon)|$$

and the Euclidean minimum of $K$ is

$$M(K) = \sup_{\xi \in K} m_K(\xi).$$

The set of values taken by $m_K$ is called the Euclidean spectrum of $K$. We know the following important result [4].

**Theorem 2.1.** The Euclidean spectrum of $K$ is the union of $\{0\}$ and of a strictly decreasing sequence of rationals $(r_i)_{i \geq 0}$ with limit $0$. For each $k$, the set of $\xi \in K$ such that $m_K(\xi) = r_k$ is finite modulo $\mathbb{Z}_K$.

In fact, we have a stronger result, which can be formulated in terms of the inhomogeneous spectrum but we shall not need this in what follows.

**Corollary 2.2.** The set of $\xi \in K$ such that $m_K(\xi) \geq 1$ is finite modulo $\mathbb{Z}_K$.

Recall now that we have at our disposal an algorithm which can give us all the $\xi \in K$ with this property. Without going into details – these can be found in [5] – let us give nevertheless the theorem which justifies the algorithm and the main ideas that are behind it. Let us choose a constant $k > 0$ and a let us embed $K$ into $K \otimes_{\mathbb{Q}} \mathbb{R}$, which we can identify with $\mathbb{R}^n$, in which $\mathbb{Z}_K$ is a lattice. Under this identification an element $\xi$ of $K$ is viewed as $(\sigma_i(\xi))_{1 \leq i \leq n}$, where the $\sigma_i$ are the embeddings of $K$ into $\mathbb{R}$. The map $m_K$ extends to a map $m_{\mathbb{F}}$ from $\mathbb{R}^n$ to $\mathbb{R}^+$ in a natural way:

$$m_{\mathbb{F}}(x) = \inf_{\Upsilon \in \mathbb{Z}} \left| \prod_{i=1}^n (x_i - \sigma_i(\Upsilon)) \right|.$$

Moreover, the product of two elements of $K$ is extended to the product coordinate by coordinate in $\mathbb{R}^n$. This new product of two elements $x, y \in \mathbb{R}^n$ will be denoted by $x \cdot y$. Let finally $\varepsilon$ be a non-torsion unit of $\mathbb{Z}_K$.

The main idea is to find in a fundamental domain $\mathcal{F}$ associated to $\mathbb{Z}_K$ in $\mathbb{R}^n$, $s$ distinct bounded sets $\mathcal{T}_i$ $(1 \leq i \leq s)$ with the property that for each such $\mathcal{T}_i$ there exists an $X_i \in \mathbb{Z}_K$ and $s_i$ integers $n_{i,1}, \ldots, n_{i,s_i}$ $(s_i > 0)$ such that

$$(\varepsilon \cdot \mathcal{T}_i - X_i) \setminus \mathcal{H} \subset \bigcup_{1 \leq i \leq s_i} \mathcal{T}_{n_{i,i}} (i = 1, \ldots, s),$$

where

$$\mathcal{H} = \{ x \in \mathbb{R}^n \text{ such that } m_{\mathbb{F}}(x) \leq k \}.$$

We consider the $\mathcal{T}_i$ as the vertices of a directed graph $G$ and represent (1) by $s_i$ directed edges whose tail is $\mathcal{T}_i$ and whose respective heads are the $\mathcal{T}_{n_{i,i}}$ $(1 \leq l \leq s_i)$. To describe such an edge of $G$ we shall use the notation $\mathcal{T}_i \to \mathcal{T}_{n_{i,i}}(X_i)$. The set $\mathcal{C}$ of simple cycles of $G$ is nonempty and finite. Each element $c$ of $\mathcal{C}$ of length $j$ is in the form of the circular path, $\mathcal{T}_{0} \to \mathcal{T}_1(X_0') \to \mathcal{T}_2(X_1') \to \cdots \to \mathcal{T}_{j-1}(X_{j-2}') \to \mathcal{T}_j(X_{j-1}')$, for some subset $\{\mathcal{T}_1, \ldots, \mathcal{T}_{j-1}\} \subseteq \{\mathcal{T}_1, \ldots, \mathcal{T}_s\}$, where $X_j'$ denotes the element $X \in \mathbb{Z}_K$ associated to $\mathcal{T}_i$. This defines, in a unique way, $j$ elements of $K$, $\xi_0, \ldots, \xi_j$, by the formulae:

$$\xi_l = \frac{\varepsilon^{l-1} X_l' + \varepsilon^{l-2} X_{l+1}' + \cdots + X_{j-1+l}'}{\varepsilon^j - 1},$$

where

$$\varepsilon^{l-1} X_l' + \varepsilon^{l-2} X_{l+1}' + \cdots + X_{j-1+l}' = \varepsilon^{l-1} X_l - X_{j-1+l}.$$
the indices being read modulo $j$. In this context, we say that $\xi_0, \ldots, \xi_{j-1}$ are associated to the cycle $c$.

We denote by $E$ the finite set of all elements of $K$ associated to the elements of $C$. The $\xi_i$ associated to a cycle $c$ are in the same orbit modulo $Z_K$ under the action of $Z_K^*$ (in fact $\xi_{r+1} = \varepsilon \cdot \xi_r - X'_r$) and satisfy

$$m_K(\xi_0) = \ldots = m_K(\xi_{j-1}) =: m(c),$$

which is a rational number. Finally, define

$$m(G) = \max_{c \in C} m(c) = \max_{\xi \in E} m_K(\xi).$$

Let us say that $G$ is convenient if every infinite path of $G$ is ultimately periodic. The essential result is the following.

**Theorem 2.3.** Assume that $G$ is convenient and that there exists $T \in \{ T_1, \ldots, T_s \}$ and $x \in$ such that $m_T(x) > k$. Then

i) $m_T(x) \leq m(G)$.

ii) If $x \in K$, there exists $\xi \in E$ such that $x \equiv \xi \mod Z_K$.

In this situation we know all the potential $\xi \in K$ such that $m_K(\xi) > k$, and since computing $m_K(\xi)$ is possible (again see [5] for more details), we know in fact all the $\xi \in K$ such that $m_K(\xi) > k$. To identify the elements $\xi \in K$ such that $m_K(\xi) \geq 1$, it is sufficient to run the algorithm with $k = 0.999$, for instance.

3. Generalized Euclidean number fields

3.1. Generalities. From the definition of G.E. number fields and the definition of the map $m_K$, we have the following result.

**Proposition 3.1.** The field $K$ is G.E. if and only if for every $(\alpha, \beta) \in Z_K \times Z_K \setminus \{0\}$ such that $m_K(\alpha/\beta) \geq 1$, the ideal $(\alpha, \beta)$ is not principal.

**Remark 1.** Suppose that we have at our disposal the finite set $S$ of all $\xi \in K$ (modulo $Z_K$) such that $m_K(\xi) \geq 1$, and that for each such $\xi$ we have a representative $u/v$ where $(u, v) \in Z_K \times Z_K \setminus \{0\}$. Let $(\alpha, \beta) \in Z_K \times Z_K \setminus \{0\}$ such that $m_K(\alpha/\beta) \geq 1$. Then there exists $\xi \equiv u/v \mod S$ such that $\alpha/\beta = u/v + \gamma$ with $\gamma \in Z_K$. Since

$$(\alpha, \beta) = (\beta u/v + \gamma \beta, \beta) = (\beta u/v, \beta) = \beta/(u, v),$$

it is sufficient, for proving that $K$ is G.E., to check that for every $\xi \equiv u/v \in S$, $(u, v)$ is not principal.

3.2. A first example. The purpose of this subsection is to study in detail a particular case. Other results, obtained in another way, will be given in the next subsection. Let $K$ be the normal quartic field generated by any one of the roots of $P(X) = X^4 - 20X^2 + 50$.

The field $K$ is totally real, its discriminant is 51200, its class number is 2, and a $Z$-basis of $Z_K$ is $(e_1, e_2, e_3, e_4)$ with

$$e_1 = 1, \quad e_2 = \sqrt{2}, \quad e_3 = \sqrt{10 + 5\sqrt{2}}, \quad e_4 = \sqrt{10 - 5\sqrt{2}}.$$

Our algorithm shows that

$$M(K) = \frac{7}{2},$$
and that there is a unique $\xi \in K$ (modulo $\mathbb{Z}_K$) such that $m_K(\xi) \geq 1$. More precisely

$$\xi \equiv \frac{1}{2}(e_3 + e_4).$$

According to Remark 1, if we want to establish that $K$ is G.E., we have just to prove that the ideal $(2, e_3 + e_4)$ is not principal.

**Theorem 3.2.** The field $K$ is not norm-Euclidean but it is G.E.

**Proof.** First of all, we note that $e_3 + e_4 = e_2 \cdot e_3$ so that we are reduced to proving that the ideal $(e_2, e_3)$ is not principal. Suppose on the contrary that it is principal so that we have

$$e_2 \mathbb{Z}_K + e_3 \mathbb{Z}_K = \nu \mathbb{Z}_K,$$

with $\nu \in \mathbb{Z}_K$. Since $N_{K/\mathbb{Q}}(e_2) = 4$ and $N_{K/\mathbb{Q}}(e_3) = 50$, we have

$$N_{K/\mathbb{Q}}(\nu) \mid 2 = \gcd(4, 50),$$

so that we have two possibilities: either $\nu \in \mathbb{Z}_K^*$ or $N_{K/\mathbb{Q}}(\nu) = \pm 2$.

**First case:** $\nu$ is a unit and we have in fact $e_2 \mathbb{Z}_K + e_3 \mathbb{Z}_K = \mathbb{Z}_K$.

In this case, there exist $u, v \in \mathbb{Z}_K$ such that

$$1 = e_2 \cdot u + e_3 \cdot v. \tag{2}$$

Let us write

$$\begin{align*}
u &= a + be_2 + ce_3 + de_4 \\
v &= a' + b'e_2 + c'e_3 + d'e_4,
\end{align*} \tag{3}$$

where $a, b, c, d, a', b', c', d' \in \mathbb{Z}$.

Since $e_2 \cdot e_3 = e_3 + e_4$, $e_2 \cdot e_4 = e_3 - e_4$ and $e_3 \cdot e_4 = 5e_2$, if we substitute (3) into (2) we obtain, by identification of the coefficients in our $\mathbb{Z}$-basis, that $2b + 10c' = 1$, which is clearly impossible.

**Second case:** $\nu$ has norm $\pm 2$.

Let us prove that this is impossible. If

$$\nu = a + be_2 + ce_3 + de_4$$

where $a, b, c, d \in \mathbb{Z}$, an easy computation leads to

$$\pm 2 = N_{K/\mathbb{Q}}(\nu)$$

$$= a^4 + 4b^4 + 50c^4 + 50d^4 - 4a^2b^2 - 20a^2c^2 - 20a^2d^2 - 40b^2c^2$$

$$- 40b^2d^2 + 100c^2d^2 + 40abc^2 - 40abd^2 + 200cd^3 - 200dc^3 + 80abcd.$$

This implies that

$$\pm 2 \equiv (a^2 - 2b^2)^2 \pmod{5},$$

which is impossible as neither of $\pm 2$ are quadratic residues $\pmod{5}$. \qed
3.3. Dedekind-Hasse criterion. In this subsection, we study the link between G.E. and a Euclidean-type map that we shall deduce from the Dedekind-Hasse criterion. This will lead us to define an easy test which allows to find new examples, without requiring detailed calculations as above. First of all, recall the Dedekind-Hasse criterion (see for instance [11]).

\[ \text{Theorem 3.3.} \quad \text{A number field } K \text{ has class number } 1 \text{ if and only if for every } \alpha, \beta \in \mathbb{Z}_K \setminus \{0\} \text{ such that } \beta \nmid \alpha, \text{ there exist } \gamma, \delta \in \mathbb{Z}_K \text{ such that} \]
\[ 0 < |N_{K/Q}(\alpha \gamma - \beta \delta)| < |N_{K/Q}(\beta)|. \]

This leads to the following natural definition.

\[ \text{Definition 3.1.} \quad \text{For every } \xi \in K \setminus \mathbb{Z}_K \text{ we shall denote by } h_K(\xi) \text{ the real number defined by} \]
\[ h_K(\xi) = \inf \{ m_K(\Upsilon \xi); \Upsilon \in \mathbb{Z}_K \text{ and } \Upsilon \xi \notin \mathbb{Z}_K \}. \]

This map has the following elementary properties, which we give here without proof.

\[ \text{Proposition 3.4.} \quad \text{For every } \xi \in K \setminus \mathbb{Z}_K \text{ we have} \]
\[ (1) \quad 0 < h_K(\xi) \leq m_K(\xi); \]
\[ (2) \quad \text{For every } \alpha \in \mathbb{Z}_K, \ h_K(\xi + \alpha) = h_K(\xi); \]
\[ (3) \quad \text{For every } \varepsilon \in \mathbb{Z}_K^*, \ h_K(\varepsilon \xi) = h_K(\xi). \]

We can now reformulate Dedekind-Hasse criterion as follows.

\[ \text{Theorem 3.5.} \quad \text{A number field } K \text{ has class number } 1 \text{ if and only if for every } \xi \in K \setminus \mathbb{Z}_K \text{ we have} \]
\[ h_K(\xi) < 1. \]

\[ \text{Proof.} \quad \text{The norm being multiplicative, } (4) \text{ can be reformulated: for every } \xi \in K \setminus \mathbb{Z}_K \text{ there exist } \gamma, \delta \in \mathbb{Z}_K \text{ such that} \]
\[ 0 < |N_{K/Q}(\gamma \xi - \delta)| < 1, \]
which leads to \( m_K(\gamma \xi) < 1 \). Since (5) cannot be true if \( \gamma \xi \in \mathbb{Z}_K \), we have \( h_K(\xi) < 1 \).

Conversely, since \( |N_{K/Q}(\gamma \xi - \delta)| = 0 \) implies \( \gamma \xi \in \mathbb{Z}_K \) which is excluded in the definition of \( h_K \), we see that if \( h_K(\xi) < 1 \) then (5) is true. \( \square \)

Now consider a number field \( K \) and put
\[ S = \{ \xi \in K; \ m_K(\xi) \geq 1 \}. \]
Suppose that \( K \) is not norm-euclidean so that \( S \neq \emptyset \). We have the following result.

\[ \text{Theorem 3.6.} \quad \text{One of the following three possibilities holds:} \]
\[ (1) \quad \text{For every } \xi \in S, \ h_K(\xi) < 1. \text{ Then } K \text{ has class number } 1 \text{ and is not G.E.} \]
\[ (2) \quad \text{For every } \xi \in S, \ h_K(\xi) \geq 1. \text{ Then } K \text{ is G.E. (and not principal).} \]
\[ (3) \quad \text{There exist } \xi, \mu \in S \text{ such that } h_K(\xi) < 1 \text{ and } h_K(\mu) \geq 1. \text{ Then } K \text{ is not principal. If in addition, there exists } \xi = \alpha/\beta \in S \text{ (with } \alpha, \beta \in \mathbb{Z}_K) \text{ with } h_K(\xi) < 1 \text{ and such that } (\alpha, \beta) \text{ is principal, then } K \text{ is not G.E. Otherwise it is G.E.} \]

\[ \text{Proof.} \quad \text{Clearly we have the three cases.} \]

\[ \text{Case 1.} \quad \text{The result is a consequence of Theorem 3.5 and of the fact that when the field is principal norm-Euclidean and G.E. are synonymous.} \]
Case 2. Theorem 3.5 indicates that \( K \) is not principal. By Proposition 3.1 it is sufficient to prove that for every \( \xi = \alpha/\beta \in S \) where \( \alpha, \beta \in \mathbb{Z}_K \), the ideal \((\alpha, \beta)\) is not principal. Otherwise, we have \((\alpha, \beta) = \nu \mathbb{Z}_K\) with \( \nu \in \mathbb{Z}_K \). By hypothesis \( h_K(\xi) \geq 1 \) so that for every \( X, Y \in \mathbb{Z}_K \) with \( X\xi \notin \mathbb{Z}_K \) we have

\[
|N_{K/Q}(X\alpha - Y\beta)| \geq |N_{K/Q}(\beta)|.
\]

Now \( \nu \) can be written \( \nu = X\alpha - Y\beta \) with \( X, Y \in \mathbb{Z}_K \) and \( X\xi \notin \mathbb{Z}_K \). Otherwise this implies that \( \nu \) and \( \beta \) are associates and we have \((\alpha, \beta) = \beta \mathbb{Z}_K\) which implies \( \beta \mid \alpha \) and \( \xi \in \mathbb{Z}_K \), which is impossible. We deduce from this that \( |N_{K/Q}(\nu)| \geq |N_{K/Q}(\beta)| \). Since \( N_{K/Q}(\nu) \mid N_{K/Q}(\beta) \) we have \( |N_{K/Q}(\nu)| = |N_{K/Q}(\beta)| \), and since \( \nu \mid \beta \), \( \nu \) and \( \beta \) are associates, which is impossible by the previous argument.

Case 3. Theorem 3.5 indicates that \( K \) is not principal. The second assertion is a consequence of Proposition 3.1. Indeed, as previously, if \( h_K(\xi) \geq 1 \) and \( \xi = \alpha/\beta \) then \((\alpha, \beta)\) is not principal and this case is not an obstruction for \( K \) to be G.E. Finally, the only possibilities for contradicting G.E. come from the \( \xi = \alpha/\beta \in S \) such that \( h_K(\xi) < 1 \) and \((\alpha, \beta)\) is principal. \( \square \)

**Corollary 3.7.** Suppose that \( K \) is not norm-Euclidean and that, with the above notation, \( S \) modulo \( \mathbb{Z}_K \) is composed of a single orbit under the (multiplicative) action of \( \mathbb{Z}_K^* \) modulo \( \mathbb{Z}_K \), i.e. that if \( \xi, \mu \in S \) there exists an \( \varepsilon \in \mathbb{Z}_K^* \) and an \( \alpha \in \mathbb{Z}_K \) such that \( \mu = \varepsilon \xi + \alpha \). Then either \( K \) is principal and not G.E. or \( K \) is not principal but is G.E.

**Proof.** If \( K \) is principal, we are in case 1. Otherwise, since all the elements of \( S \), which are in the same orbit, have the same image by \( h_K \) (Proposition 3.4), we cannot be in case 3 of Theorem 3.6. Finally, we are in case 2 and \( K \) is G.E. \( \square \)

**Remark 2.** To simplify notation and vocabulary, we shall often, by abuse of language, speak indifferently of \( \xi \in K \) or \( \xi \in K \mod \mathbb{Z}_K \). For instance we shall speak of orbits in \( S \) under the action of \( \mathbb{Z}_K^* \); in this context \( S \) and these orbits should be understood modulo \( \mathbb{Z}_K \).

**Corollary 3.8.** The totally real number fields of degree 3 and discriminants 1957, 2777, 3981 are G.E. The totally real number fields of degree 4 and discriminants 46400 and 51200 are G.E.

**Proof.** In fact, in all these cases, our algorithm establish that we are under the previous hypotheses. For discriminant 1957, we have \( M(K) = 2 \) and one orbit with one element in \( S \). For discriminant 2777, we have \( M(K) = 5/3 \) and one orbit with 2 elements in \( S \). For discriminant 3981, we have \( M(K) = 3/2 \) and one orbit with one element in \( S \). For discriminant 46400, we have \( M(K) = 5/4 \) and one orbit with 3 elements in \( S \). For discriminant 51200, we have \( M(K) = 7/2 \) and one orbit with one element in \( S \). \( \square \)

And now, if there are several orbits in \( S \), and we want to use Theorem 3.6, we have to see whether, for one element \( \xi \) by orbit, and for every orbit, we have \( h_K(\xi) \geq 1 \), in which case necessarily \( K \) is G.E. The problem is now: how can we compute \( h_K(\xi) \)? Our algorithm gives us every such \( \xi \) by its coordinates in a \( \mathbb{Z} \)-basis of \( \mathbb{Z}_K \). These coordinates are of the form \((a_1/d, a_2/d, \ldots, a_n/d)\) where \( a_i \in \mathbb{Z} \) for every \( i \) and \( d \in \mathbb{Z}_{>0} \). Furthermore we can compute \( m_K(\mu) \) for every \( \mu \in K \). Hence,
it is easy to see that, to compute \( h_K(\xi) \), it is sufficient to compute \( m_K(\Upsilon \xi) \) for every \( \Upsilon \) with coordinates in \( \{0, 1, \ldots, d-1\} \) for our basis, such that \( \Upsilon \xi \notin \mathbb{Z}_K \). This is easy to check. By definition, the value of \( h_K(\xi) \) will be the minimum of these \( m_K(\Upsilon \xi) \). Of course if for every \( \xi \) and every such \( \Upsilon \) we have \( \Upsilon \xi \in S \mod \mathbb{Z}_K \), then \( K \) is G.E. Using this last approach we have established the following result.

**Theorem 3.9.** The following totally real number fields of degree \( n \) are G.E. but not norm-Euclidean:

- when \( n = 3 \), the fields with discriminants 2597, 4212, 4312, 5684;
- when \( n = 4 \), the fields with discriminants 21025, 32625.

**Proof.** We just give a typical example. For \( n = 3 \) and discriminant 2597, we have two orbits in \( S \), the first one \( O_1 \) with 2 elements \((\pm (e_1 + 2e_2 + 2e_3))/3 \mod \mathbb{Z}_K\) where \((e_i)\) is the \( \mathbb{Z} \)-basis of \( \mathbb{Z}_K \) returned by PARI [1]) and te second one \( O_2 \) with 1 element \((e_1 + e_2 + e_3)/2 \mod \mathbb{Z}_K\). Then we can easily check that \( \mathbb{Z}_K \cdot O_1 = O_1 \cup \{0\} \) and that \( \mathbb{Z}_K \cdot O_2 = O_2 \cup \{0\} \). The same thing happens in other cases with sometimes more complicated equalities but always with \( \mathbb{Z}_K \cdot O \subseteq S \cup \{0\} \).  

**Remark 3.** If we want to treat all the non principal number fields of degree \( 3 \) and discriminant \( < 6000 \), it remains to study the two number fields with discriminant 3969. In these cases, our previous method does not work because we have some \( \xi = \alpha/\beta \in S \) such that \( h_K(\xi) < 1 \). The first one, \( K_1 \), is defined by \( x^3 - 21x - 28 \). For this field, \( S \) is composed of five orbits \( O_i \), \( 1 \leq i \leq 5 \). For 4 of them, say for \( 1 \leq i \leq 4 \), we have \( \mathbb{Z}_K \cdot O_i \subseteq S \cup \{0\} \) but for the last one \( O_5 \) this is not true. Take an element \( \alpha/\beta \) of \( O_5 \); here we can take \( \alpha = 3e_1 + 2e_2 + 2e_3 \) and \( \beta = 6 \) where \((e_1, e_2, e_3)\) is the \( \mathbb{Z} \)-basis returned by PARI [1]. We can then prove directly as in Section 3.2 that the ideal \( (\alpha, \beta) \) is not principal. We conclude that \( K_1 \) is G.E.

For the second field, \( K_2 \), defined by \( x^3 - 21x - 35 \) the situation is different. Here \( S \) is composed of seven orbits \( O_i \), \( 1 \leq i \leq 7 \) and four of them, say \( O_i \), \( 1 \leq i \leq 4 \), are such that \( \mathbb{Z}_K \cdot O_i \subseteq S \cup \{0\} \). Now if we look at the three others, we find that two of them contain an \( \alpha/\beta \) for which \( (\alpha, \beta) \) is principal. For completeness these \((\alpha, \beta)\) are \((7e_1 + 12e_2 + 4e_3, 21)\) and \((7e_1 + 5e_2 + 11e_3, 21)\) with the usual notation. Consequently \( K_2 \) is not G.E. All the computations, which are long and complicated - in particular for \( K_2 \) - have been done by hand and checked using PARI [1]. We do not give them here for lack of space and because they are not especially enlightening.

Finally, we put all these results together to give us Theorem 1.4.

### 4. The 2-stage Euclidean number fields

Let us begin with an example. Let \( K \) be the totally real cubic number field with discriminant 3988. Using our algorithm we see that the upper part of the Euclidean spectrum of \( K \) has five elements, more precisely

\[
\text{sp}(K) \cap [1, \infty) = \{19/8, 11/8, 5/4, 19/16, 133/128\}.
\]

The set \( S \) is composed of five orbits, respectively the orbits of \( ae_1 + be_2 + ce_3 \) with \((a, b, c) = (0, 1/2, 1/2), (1/2, 1/2, 0), (1/2, 1/2, 1/2), (0, 3/4, 1/2) \) and \((0, 3/8, 1/2)\), where \((e_1, e_2, e_3)\) is the \( \mathbb{Z} \)-basis of \( \mathbb{Z}_K \) returned by PARI [1]. These orbits have
respectively 1, 1, 1, 2 and 4 elements. For one element $\xi$ by orbit, we try to find $q_1, q_2 \in \mathbb{Z}_K$ such that

$$
\left| N_{K/Q}(\xi - q_1 - \frac{1}{q_2}) \right| < \frac{1}{|N_{K/Q}(q_2)|},
$$

by testing “small” $q_1 \in \mathbb{Z}_K$ and “small” $q_2 \in \mathbb{Z}_K \setminus \{0\}$. In each case this is possible, so that for every $\xi \in S$, (6) is true. Finally this implies that $K$ is 2-stage norm-Euclidean. Using exactly the same approach we have established the results of Theorem 1.5.

Remark 4. Obviously these fields, which are principal and not norm-Euclidean, are not G.E.

References