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SOME GENERALIZED EUCLIDEAN AND 2-STAGE EUCLIDEAN
NUMBER FIELDS THAT ARE NOT NORM-EUCLIDEAN

JEAN-PAUL CERRI

Abstract. We give examples of Generalized Euclidean but not norm-Euclidean number fields of degree strictly greater than 2. In the same way we give examples of 2-stage Euclidean but not norm-Euclidean number fields of degree strictly greater than 2. In both cases, no such examples were known.

1. Introduction

In 1985, Johnson, Queen and Sevilla [9] introduced a generalization of the classical notion of Euclidean number field.

Definition 1.1. A number field $K$ is said to be Generalized Euclidean or simply G.E. if for every $(\alpha, \beta) \in \mathbb{Z}_K \times \mathbb{Z}_K \setminus \{0\}$ such that the ideal $(\alpha, \beta)$ is principal, there exists $\Upsilon \in \mathbb{Z}_K$ such that
\[ |N_{K/Q}(\alpha - \Upsilon \beta)| < |N_{K/Q}(\beta)|. \]

If $(\alpha, \beta)$ is principal, we thus have at our disposal the Euclidian algorithm to compute a gcd of $\alpha$ and $\beta$ because it is easy to see that $(\beta, \alpha - \Upsilon \beta)$ is principal again, and so on. Note that if $K$ is norm-Euclidean then $K$ is G.E. and that if $K$ has class number 1, then $K$ is G.E. if and only if $K$ is norm-Euclidean. If we want to illustrate the difference between “G.E.” and “norm-Euclidean”, the interesting case is when $K$ is not principal, G.E. but not norm-Euclidean. The following result was established by Johnson, Queen and Sevilla in [9].

Theorem 1.1. The quadratic number field $\mathbb{Q}(\sqrt{d})$ is G.E. but not norm-Euclidean for $d = 10$ and $d = 65$. The quadratic number field $\mathbb{Q}(\sqrt{d})$ is not G.E. for $d = 15, 26, 30, 35, 39, 51, 78, 87, 102, 115, 195$ and 230.

Furthermore, Johnson, Queen and Sevilla conjectured that $K = \mathbb{Q}(\sqrt{d})$ (with $d > 1$ squarefree) is G.E. if and only if $K$ is norm-Euclidean or $d = 10$ or 65.

Another variation on norm-Euclidean number fields has been introduced by Cooke [7].

Definition 1.2. Let $m$ be a rational integer $\geq 1$. The number field $K$ is $m$-stage Euclidean if and only if for every $\alpha \in \mathbb{Z}_K$ and every $\beta \in \mathbb{Z}_K \setminus \{0\}$ there exists a positive rational integer $k \leq m$ and $k$ pairs $(q_i, r_i)$ $(1 \leq i \leq k)$ of elements of $\mathbb{Z}_K$
such that
\[\alpha = \beta q_1 + r_1,\]
\[\beta = r_1 q_2 + r_2,\]
\[\vdots\]
\[r_{k-2} = r_{k-1} q_k + r_k,\]
and \(|N_{K/Q}(r_k)| < |N_{K/Q}(\beta)|\).

When it is well defined, let us put
\[\lfloor q_1, q_2, \ldots, q_k \rfloor = q_1 + \frac{1}{q_2 + \frac{1}{q_3 + \cdots + \frac{1}{q_k}}},\]
where \(a_k\) and \(b_k\) are given by
\[a_1 = q_1, \quad b_1 = 1,\]
\[a_2 = a_1 q_2 + 1, \quad b_2 = q_2,\]
and recursively by
\[a_k = a_{k-1} q_k + a_{k-2}, \quad b_k = q_k b_{k-1} + b_{k-2}.\]

Since
\[\frac{\alpha}{\beta} = \frac{a_k}{b_k} + (-1)^k \frac{r_k}{b_k},\]
this definition is equivalent to the following.

**Definition 1.3.** The number field \(K\) is \(m\)-stage Euclidean if and only if for every \(\xi \in K\), there exists a positive rational integer \(k \leq m\), and \(k\) elements \(q_1, q_2, \ldots, q_k \in \mathbb{Z}_K\) such that
\[\left|N_{K/Q}(\xi - \lfloor q_1, q_2, \ldots, q_k \rfloor)\right| < \frac{1}{|N_{K/Q}(b_k)|}.\]

As in the previous case, norm-Euclidean implies \(m\)-stage Euclidean, but contrary to what happens with the G.E. condition, we have the following result [7].

**Theorem 1.2.** A number field \(K\) with unit rank \(r \geq 1\) is principal if and only if \(K\) is \(m\)-stage Euclidean for some \(m\).

As a consequence, if we want to study the difference between \(m\)-stage Euclidean and norm-Euclidean, we have to look at number fields with class number 1 and find some example where \(K\) is principal, \(m\)-stage Euclidean but not norm-Euclidean. The following result was established by Cooke [7].

**Theorem 1.3.** For \(d = 14, 22, 23, 31, 38, 43, 46, 53, 61, 69, 89, 93, 97, \mathbb{Q}(\sqrt{d})\) is 2-stage euclidean but not norm-Euclidean.

Furthermore, Cooke and Weinberger [8] proved that, under GRH, every principal number field \(K\) with unit rank \(r \geq 1\) is 4-stage Euclidean, and even 2-stage Euclidean if \(K\) has at least one real place.
For both notions (G.E. and $m$-stage Euclidean), no examples of number fields of degree strictly greater than 2 were known. Our main results are the following.

**Theorem 1.4.** None of the totally real number fields enumerated in Table 1 are principal. They all are G.E. except for the second cubic number field of discriminant 3969, defined by \( x^3 - 21x - 35 \), which is neither principal nor G.E.

\[
\begin{array}{|c|c|c|c|}
\hline
n & D_K & P(x) & h & M(K) \\
\hline
3 & 1957 & x^3 - x^2 - 9x + 10 & 2 & 2 \\
3 & 2597 & x^3 - x^2 - 9x + 8 & 3 & 5/2 \\
3 & 2777 & x^3 - x^2 - 14x + 23 & 2 & 5/3 \\
3 & 3969 & x^3 - 21x - 28 & 3 & 4/3 \\
3 & 3969 & x^3 - 21x - 35 & 3 & 7/3 \\
3 & 3981 & x^3 - x^2 - 11x + 12 & 2 & 3/2 \\
3 & 4212 & x^3 - 12x - 10 & 3 & 7/2 \\
3 & 4312 & x^3 - x^2 - 16x + 8 & 3 & 11/4 \\
3 & 5684 & x^3 - 14x - 14 & 3 & 9/2 \\
4 & 21025 & x^4 - 17x^3 + 36 & 2 & 1 \\
4 & 32625 & x^4 - x^3 - 19x^2 + 4x + 76 & 2 & 1 \\
4 & 46400 & x^4 - 22x^3 + 116 & 2 & 5/4 \\
4 & 51200 & x^4 - 20x^3 + 50 & 2 & 7/2 \\
\hline
\end{array}
\]

**Table 1.** Here, \( n \) is the degree of the field \( K \), \( D_K \) its discriminant, \( P(x) \) its equation, \( h \) its class number and \( M(K) \) its Euclidean minimum.

**Theorem 1.5.** The totally real number fields of degree 3 and of discriminants < 15000 which are principal but not norm-Euclidean (82 cases) are 2-stage norm-Euclidean. The same is true for degree 4 and discriminants 18432, 34816, 35152 and for degree 5 and discriminant 390625. In all these cases, the number field is principal, not norm-Euclidean, but 2-stage norm-Euclidean.

Details on the number fields appearing in Theorem 1.5 are available from [6]. In Section 2, we recall other definitions and general results. In Section 3 and 4, we study the case of Generalized Euclidean number fields and the case of 2-stage Euclidean number fields, respectively.

### 2. The Algorithm, Generalities

Let \( K \) be a number field of degree \( n \). We have designed an algorithm which allows us to compute the Euclidean minimum of \( K \), in particular when \( K \) is totally real [5], but also in the general case [3]. According to theoretical results [4], this algorithm can also give the upper part of the Euclidean spectrum of \( K \) and this yields new examples of number fields with interesting properties.

From now on, we suppose that \( K \) is totally real and that \( n > 2 \). We denote by \( \mathbb{Z}_K \) the ring of its integers and by \( N_{K/Q} \) its absolute norm. The Euclidean minimum

\[1\text{In }[2]\text{ and }[10]\text{ the Euclidean minimum of this number field is falsely announced to be 1.}\]
of an element $\xi \in K$ is
\[ m_K(\xi) = \inf_{\Upsilon \in \mathbb{Z}^K} \left| \frac{N_K}{Q}(\xi - \Upsilon) \right| \]
and the Euclidean minimum of $K$ is
\[ M(K) = \sup_{\xi \in K} m_K(\xi). \]

The set of values taken by $m_K$ is called the Euclidean spectrum of $K$. We know the following important result [4].

**Theorem 2.1.** The Euclidean spectrum of $K$ is the union of $\{0\}$ and of a strictly decreasing sequence of rationals $(r_i)_{i \geq 0}$ with limit 0. For each $k$, the set of $\xi \in K$ such that $m_K(\xi) = r_k$ is finite modulo $\mathbb{Z}_K$.

In fact, we have a stronger result, which can be formulated in terms of the inhomogeneous spectrum but we shall not need this in what follows.

**Corollary 2.2.** The set of $\xi \in K$ such that $m_K(\xi) \geq 1$ is finite modulo $\mathbb{Z}_K$.

Recall now that we have at our disposal an algorithm which can give us all the $\xi \in K$ with this property. Without going into details – these can be found in [5] – let us give nevertheless the theorem which justifies the algorithm and the main ideas that are behind it. Let us choose a constant $k > 0$ and a let us embed $K$ into $K \otimes_{\mathbb{Q}} \mathbb{R}$, which we can identify with $\mathbb{R}^n$, in which $\mathbb{Z}_K$ is a lattice. Under this identification an element $\xi$ of $K$ is viewed as $(\sigma_i(\xi))_{1 \leq i \leq n}$, where the $\sigma_i$ are the embeddings of $K$ into $\mathbb{R}$. The map $m_K$ extends to a map $m_{\mathbb{R}}$ from $\mathbb{R}^n$ to $\mathbb{R}^+$ in a natural way:
\[ m_{\mathbb{R}}(x) = \inf_{\Upsilon \in \mathbb{Z}^K} \left| \prod_{i=1}^n (x_i - \sigma_i(\Upsilon)) \right|. \]

Moreover, the product of two elements of $K$ is extended to the product coordinate by coordinate in $\mathbb{R}^n$. This new product of two elements $x, y \in \mathbb{R}^n$ will be denoted by $x \cdot y$. Let finally $\varepsilon$ be a non-torsion unit of $\mathbb{Z}_K$.

The main idea is to find in a fundamental domain $F$ associated to $\mathbb{Z}_K$ in $\mathbb{R}^n$, $s$ distinct bounded sets $T_i$ $(1 \leq i \leq s)$ with the property that for each such $T_i$ there exists an $X_i \in \mathbb{Z}_K$ and $s_i$ integers $n_{i,1}, \ldots, n_{i,s_i}$ $(s_i > 0)$ such that
\[ (\varepsilon \cdot T_i - X_i) \mathcal{H} \subset \bigcup_{1 \leq i \leq s_i} T_{n_{i,i}} \quad (i = 1, \ldots, s), \]
where
\[ \mathcal{H} = \{ x \in \mathbb{R}^n \text{ such that } m_{\mathbb{R}}(x) \leq k \}. \]

We consider the $T_i$ as the vertices of a directed graph $G$ and represent (1) by $s_i$ directed edges whose tail is $T_i$ and whose respective heads are the $T_{n_{i,l}}$ $(1 \leq l \leq s_i)$. To describe such an edge of $G$ we shall use the notation $T_i \rightarrow T_{n_{i,l}}(X_i)$. The set $\mathcal{C}$ of simple cycles of $G$ is nonempty and finite. Each element $c$ of $\mathcal{C}$ of length $j$ is in the form of the circular path, $T_0^c \rightarrow T_1^c(X'_0) \ldots \rightarrow T_{j-1}^c(X'_{j-2}) \rightarrow T_{j}^c(X'_{j-1})$, for some subset $\{T_1^c, \ldots, T_{j-1}^c\} \subseteq \{T_1, \ldots, T_s\}$, where $X'_j$ denotes the element $X \in \mathbb{Z}_K$ associated to $T_i^c$. This defines, in a unique way, $j$ elements of $K$, $\xi_0, \ldots, \xi_{j-1}$ by the formulae:
\[ \xi_r = \frac{\varepsilon^{j-r} X'_{r} + \varepsilon^{j-2} X'_{r+1} + \ldots + X'_{j-1+r}}{\varepsilon^j - 1}, \]
the indices being read modulo $j$. In this context, we say that $\xi_0, \ldots, \xi_{j-1}$ are associated to the cycle $c$

We denote by $\mathcal{E}$ the finite set of all elements of $K$ associated to the elements of $\mathcal{C}$. The $\xi_i$ associated to a cycle $c$ are in the same orbit modulo $Z_K$ under the action of $Z_K^*$ (in fact $\xi_{i+1} = \varepsilon_i \cdot \xi_i - X_i^\prime$) and satisfy
$$m\mathcal{F}(\xi_0) = \ldots = m\mathcal{F}(\xi_{j-1}) =: m(c),$$
which is a rational number. Finally, define
$$m(G) = \max_{c \in \mathcal{C}} m(c) = \max_{\xi \in \mathcal{E}} m\mathcal{F}(\xi).$$
Let us say that $G$ is convenient if every infinite path of $G$ is ultimately periodic. The essential result is the following.

**Theorem 2.3.** Assume that $G$ is convenient and that there exists $T \in \{T_1, \ldots, T_s\}$ and $x \in T$ such that $m\mathcal{F}(x) > k$. Then

i) $m\mathcal{F}(x) \leq m(G)$.

ii) If $x \in K$, there exists $\xi \in \mathcal{E}$ such that $x \equiv \xi \mod Z_K$.

In this situation we know all the potential $\xi \in K$ such that $m_K(\xi) > k$, and since computing $m_K(\xi)$ is possible (again see [5] for more details), we know in fact all the $\xi \in K$ such that $m_K(\xi) > k$. To identify the elements $\xi \in K$ such that $m_K(\xi) \geq 1$, it is sufficient to run the algorithm with $k = 0.999$, for instance.

### 3. Generalized Euclidean number fields

#### 3.1. Generalities

From the definition of G.E. number fields and the definition of the map $m_K$, we have the following result.

**Proposition 3.1.** The field $K$ is G.E. if and only if for every $(\alpha, \beta) \in Z_K \times Z_K \setminus \{0\}$ such that $m_K(\alpha/\beta) \geq 1$, the ideal $(\alpha, \beta)$ is not principal.

**Remark 1.** Suppose that we have at our disposal the finite set $S$ of all $\xi \in K$ (modulo $Z_K$) such that $m_K(\xi) \geq 1$, and that for each such $\xi$ we have a representative $u/v$ where $(u, v) \in Z_K \times Z_K \setminus \{0\}$. Let $(\alpha, \beta) \in Z_K \times Z_K \setminus \{0\}$ such that $m_K(\alpha/\beta) \geq 1$. Then there exists $\xi \equiv u/v$ in $S$ such that $\alpha/\beta = u/v + \gamma$ with $\gamma \in Z_K$. Since
$$(\alpha, \beta) = (\beta u/v + \gamma \beta, \beta) = (\beta u/v, \beta) = \beta uu/v,$$
it is sufficient, for proving that $K$ is G.E., to check that for every $\xi \equiv u/v \in S$, $(u, v)$ is not principal.

#### 3.2. A first example

The purpose of this subsection is to study in detail a particular case. Other results, obtained in another way, will be given in the next subsection. Let $K$ be the normal quartic field generated by any one of the roots of $P(X) = X^4 - 20X^2 + 50$.

The field $K$ is totally real, its discriminant is 51200, its class number is 2, and a $\mathbb{Z}$-basis of $Z_K$ is $(e_1, e_2, e_3, e_4)$ with
$$e_1 = 1, \quad e_2 = \sqrt{2}, \quad e_3 = \sqrt{10 + 5\sqrt{2}}, \quad e_4 = \sqrt{10 - 5\sqrt{2}}.$$ 
Our algorithm shows that
$$M(K) = \frac{7}{2},$$
and that there is a unique \( \xi \in K \) (modulo \( \mathbb{Z}_K \)) such that \( m_K(\xi) \geq 1 \). More precisely

\[
\xi \equiv \frac{1}{2}(e_3 + e_4).
\]

According to Remark 1, if we want to establish that \( K \) is G.E., we have just to prove that the ideal \( (2, e_3 + e_4) \) is not principal.

**Theorem 3.2.** *The field \( K \) is not norm-Euclidean but it is G.E.*

**Proof.** First of all, we note that \( e_3 + e_4 = e_2 \cdot e_3 \) so that we are reduced to proving that the ideal \( (e_2, e_3) \) is not principal. Suppose on the contrary that it is principal so that we have

\[
e_2 \mathbb{Z}_K + e_3 \mathbb{Z}_K = \nu \mathbb{Z}_K,
\]

with \( \nu \in \mathbb{Z}_K \). Since \( N_{K/\mathbb{Q}}(e_2) = 4 \) and \( N_{K/\mathbb{Q}}(e_3) = 50 \), we have

\[
N_{K/\mathbb{Q}}(\nu) | 2 = \gcd(4, 50),
\]

so that we have two possibilities: either \( \nu \in \mathbb{Z}_K^* \) or \( N_{K/\mathbb{Q}}(\nu) = \pm 2 \).

*First case:* \( \nu \) is a unit and we have in fact \( e_2 \mathbb{Z}_K + e_3 \mathbb{Z}_K = \mathbb{Z}_K \).

In this case, there exist \( u, v \in \mathbb{Z}_K \) such that

\[
1 = e_2 \cdot u + e_3 \cdot v.
\]

Let us write

\[
\begin{cases}
    u = a + b e_2 + c e_3 + d e_4 \\
    v = a' + b' e_2 + c' e_3 + d' e_4,
\end{cases}
\]

where \( a, b, c, d, a', b', c', d' \in \mathbb{Z} \).

Since \( e_2 \cdot e_3 = e_3 + e_4, e_2 \cdot e_4 = e_3 - e_4 \) and \( e_3 \cdot e_4 = 5e_2 \), if we substitute (3) into (2) we obtain, by identification of the coefficients in our \( \mathbb{Z} \)-basis, that \( 2b + 10c' = 1 \), which is clearly impossible.

*Second case:* \( \nu \) has norm \( \pm 2 \).

Let us prove that this is impossible. If

\[
\nu = a + be_2 + ce_3 + de_4
\]

where \( a, b, c, d \in \mathbb{Z} \), an easy computation leads to

\[
\pm 2 = N_{K/\mathbb{Q}}(\nu) = a^4 + 4b^4 + 50c^4 + 50d^4 - 4a^2b^2 - 20a^2c^2 - 20a^2d^2 - 40b^2c^2 - 40b^2d^2 - 40c^2d^2 - 400abd^2 + 200cd^3 - 200dc^3 + 80abcd.
\]

This implies that

\[
\pm 2 \equiv (a^2 - 2b^2)^2 \pmod{5},
\]

which is impossible as neither of \( \pm 2 \) are quadratic residues (mod5). \( \square \)
3.3. Dedekind-Hasse criterion. In this subsection, we study the link between G.E. and a Euclidean-type map that we shall deduce from the Dedekind-Hasse criterion. This will lead us to define an easy test which allows to find new examples, without requiring detailed calculations as above. First of all, recall the Dedekind-Hasse criterion (see for instance [11]).

**Theorem 3.3.** A number field $K$ has class number 1 if and only if for every $\alpha, \beta \in \mathbb{Z}_K \backslash \{0\}$ such that $\beta \nmid \alpha$, there exist $\gamma, \delta \in \mathbb{Z}_K$ such that

\[0 < |N_{K/Q}(\alpha \gamma - \beta \delta)| < |N_{K/Q}(\beta)|.\]

This leads to the following natural definition.

**Definition 3.1.** For every $\xi \in K \backslash \mathbb{Z}_K$ we shall denote by $h_K(\xi)$ the real number defined by

\[h_K(\xi) = \inf \{m_K(\Upsilon \xi); \Upsilon \in \mathbb{Z}_K \text{ and } \Upsilon \xi \notin \mathbb{Z}_K\}.\]

This map has the following elementary properties, which we give here without proof.

**Proposition 3.4.** For every $\xi \in K \backslash \mathbb{Z}_K$ we have

1. $0 < h_K(\xi) \leq m_K(\xi)$;
2. For every $\alpha \in \mathbb{Z}_K$, $h_K(\xi + \alpha) = h_K(\xi)$;
3. For every $\varepsilon \in \mathbb{Z}_K^*$, $h_K(\varepsilon \xi) = h_K(\xi)$.

We can now reformulate Dedekind-Hasse criterion as follows.

**Theorem 3.5.** A number field $K$ has class number 1 if and only if for every $\xi \in K \backslash \mathbb{Z}_K$ we have $h_K(\xi) < 1$.

**Proof.** The norm being multiplicative, (4) can be reformulated: for every $\xi \in K \backslash \mathbb{Z}_K$ there exist $\gamma, \delta \in \mathbb{Z}_K$ such that

\[0 < |N_{K/Q}(\gamma \xi - \delta)| < 1,
\]

which leads to $m_K(\gamma \xi) < 1$. Since (5) cannot be true if $\gamma \xi \in \mathbb{Z}_K$, we have $h_K(\xi) < 1$. Conversely, since $|N_{K/Q}(\gamma \xi - \delta)| = 0$ implies $\gamma \xi \in \mathbb{Z}_K$ which is excluded in the definition of $h_K$, we see that if $h_K(\xi) < 1$ then (5) is true. \(\square\)

Now consider a number field $K$ and put

\[S = \{\xi \in K; m_K(\xi) \geq 1\}.
\]

Suppose that $K$ is not norm-euclidean so that $S \neq \emptyset$. We have the following result.

**Theorem 3.6.** One of the following three possibilities holds:

1. For every $\xi \in S$, $h_K(\xi) < 1$. Then $K$ has class number 1 and is not G.E.
2. For every $\xi \in S$, $h_K(\xi) \geq 1$. Then $K$ is G.E. (and not principal).
3. There exist $\xi, \mu \in S$ such that $h_K(\xi) < 1$ and $h_K(\mu) \geq 1$. Then $K$ is not principal. If in addition, there exists $\xi = \alpha/\beta \in S$ (with $\alpha, \beta \in \mathbb{Z}_K$) with $h_K(\xi) < 1$ and such that $(\alpha, \beta)$ is principal, then $K$ is not G.E. Otherwise it is G.E.

**Proof.** Clearly we have the three cases.

**Case 1.** The result is a consequence of Theorem 3.5 and of the fact that when the field is principal norm-Euclidean and G.E. are synonymous.
Case 2. Theorem 3.5 indicates that $K$ is not principal. By Proposition 3.1 it is sufficient to prove that for every $\xi = \alpha/\beta \in S$ where $\alpha, \beta \in \mathbb{Z}_K$, the ideal $(\alpha, \beta)$ is not principal. Otherwise, we have $(\alpha, \beta) = \nu \mathbb{Z}_K$ with $\nu \in \mathbb{Z}_K$. By hypothesis $h_K(\xi) \geq 1$ so that for every $X, Y \in \mathbb{Z}_K$ with $X\xi \not\in \mathbb{Z}_K$ we have

$$|N_{K/\mathbb{Q}}(X\alpha - Y\beta)| \geq |N_{K/\mathbb{Q}}(\beta)|.$$

Now $\nu$ can be written $\nu = X\alpha - Y\beta$ with $X, Y \in \mathbb{Z}_K$ and $X\xi \not\in \mathbb{Z}_K$. Otherwise $\nu \in \beta \mathbb{Z}_K$ so that $\beta \mid \nu$. But this implies that $\nu$ and $\beta$ are associates and we have $(\alpha, \beta) = \beta \mathbb{Z}_K$ which implies $\beta \mid \alpha$ and $\xi \in \mathbb{Z}_K$, which is impossible. We deduce from this that $|N_{K/\mathbb{Q}}(\nu)| \geq |N_{K/\mathbb{Q}}(\beta)|$. Since $N_{K/\mathbb{Q}}(\nu) \mid N_{K/\mathbb{Q}}(\beta)$ we have $|N_{K/\mathbb{Q}}(\nu)| = |N_{K/\mathbb{Q}}(\beta)|$, and since $\nu \mid \beta$, $\nu$ and $\beta$ are associates, which is impossible by the previous argument.

Case 3. Theorem 3.5 indicates that $K$ is not principal. The second assertion is a consequence of Proposition 3.1. Indeed, as previously, if $h_K(\xi) \geq 1$ and $\xi = \alpha/\beta$ then $(\alpha, \beta)$ is not principal and this case is not an obstruction for $K$ to be G.E.

Finally, the only possibilities for contradicting G.E. come from the $\xi = \alpha/\beta \in S$ such that $h_K(\xi) < 1$ and $(\alpha, \beta)$ is principal.

\begin{corollary}
Suppose that $K$ is not norm-\textit{Euclidean} and that, with the above notation, $S$ modulo $\mathbb{Z}_K$ is composed of a single orbit under the (multiplicative) action of $\mathbb{Z}_K$ modulo $\mathbb{Z}_K$, i.e. that if $\xi, \mu \in S$ there exists an $\varepsilon \in \mathbb{Z}_K$ and an $\alpha \in \mathbb{Z}_K$ such that $\mu = \varepsilon \xi + \alpha$. Then either $K$ is principal and not G.E. or $K$ is not principal but is G.E.
\end{corollary}

\begin{proof}
If $K$ is principal, we are in case 1. Otherwise, since all the elements of $S$, which are in the same orbit, have the same image by $h_K$ (Proposition 3.4), we cannot be in case 3 of Theorem 3.6. Finally, we are in case 2 and $K$ is G.E.
\end{proof}

\begin{remark}
To simplify notation and vocabulary, we shall often, by abuse of language, speak indifferently of $\xi \in K$ or $\xi \in K \mod \mathbb{Z}_K$. For instance we shall speak of orbits in $S$ under the action of $\mathbb{Z}_K$; in this context $S$ and these orbits should be understood modulo $\mathbb{Z}_K$.
\end{remark}

\begin{corollary}
The totally real number fields of degree 3 and discriminants 1957, 2777, 3981 are G.E. The totally real number fields of degree 4 and discriminants 46400 and 51200 are G.E.
\end{corollary}

\begin{proof}
In fact, in all these cases, our algorithm establish that we are under the previous hypotheses. For discriminant 1957, we have $M(K) = 2$ and one orbit with one element in $S$. For discriminant 2777, we have $M(K) = 5/3$ and one orbit with 2 elements in $S$. For discriminant 3981, we have $M(K) = 3/2$ and one orbit with one element in $S$. For discriminant 46400, we have $M(K) = 5/4$ and one orbit with 3 elements in $S$. For discriminant 51200, we have $M(K) = 7/2$ and one orbit with one element in $S$.
\end{proof}

And now, if there are several orbits in $S$, and we want to use Theorem 3.6, we have to see whether, for one element $\xi$ by orbit, and for every orbit, we have $h_K(\xi) \geq 1$, in which case necessarily $K$ is G.E. The problem is now: how can we compute $h_K(\xi)$? Our algorithm gives us every such $\xi$ by its coordinates in a $\mathbb{Z}$-basis of $\mathbb{Z}_K$. These coordinates are of the form $(a_1/d, a_2/d, \ldots, a_n/d)$ where $a_i \in \mathbb{Z}$ for every $i$ and $d \in \mathbb{Z}_{>0}$. Furthermore we can compute $m_K(\mu)$ for every $\mu \in K$. Hence,
it is easy to see that, to compute $h_K(\xi)$, it is sufficient to compute $m_K(\Upsilon \xi)$ for every $\Upsilon$ with coordinates in $\{0, 1, \ldots, d-1\}$ for our basis, such that $\Upsilon \xi \notin \mathbb{Z}_K$. This is easy to check. By definition, the value of $h_K(\xi)$ will be the minimum of these $m_K(\Upsilon \xi)$. Of course if for every $\xi$ and every such $\Upsilon$ we have $\Upsilon \xi \in S \mod \mathbb{Z}_K$, then $K$ is G.E. Using this last approach we have established the following result.

**Theorem 3.9.** The following totally real number fields of degree $n$ are G.E. but not norm-Euclidean:

- when $n = 3$, the fields with discriminants $2597, 4212, 4312, 5684$;
- when $n = 4$, the fields with discriminants $21025, 32625$.

**Proof.** We just give a typical example. For $n = 3$ and discriminant $2597$, we have two orbits in $S$, the first one $O_1$ with 2 elements $(\pm(e_1 + 2e_2 + 2e_3)/3$ modulo $\mathbb{Z}_K$ where $(e_i)$ is the $\mathbb{Z}$-basis of $\mathbb{Z}_K$ returned by PARI [1]) and the second one $O_2$ with 3 elements $((e_1 + e_2 + e_3)/2$ modulo $\mathbb{Z}_K$). Then we can easily check that $\mathbb{Z}_K \cdot O_1 = O_1 \cup \{0\}$ and that $\mathbb{Z}_K \cdot O_2 = O_2 \cup \{0\}$. The same thing happens in other cases with sometimes more complicated equalities but always with $\mathbb{Z}_K \cdot O \subseteq S \cup \{0\}$. \qed

**Remark 3.** If we want to treat all the non principal number fields of degree 3 and discriminant < 6000, it remains to study the two number fields with discriminant 3969. In these cases, our previous method does not work because we have some $\xi = \alpha/\beta \in S$ such that $h_K(\xi) < 1$. The first one, $K_1$, is defined by $x^3 - 21x - 28$. For this field, $S$ is composed of five orbits $O_i$, $1 \leq i \leq 5$. For 4 of them, say for $1 \leq i < 4$, we have $\mathbb{Z}_K \cdot O_i \subseteq S \cup \{0\}$ but for the last one $O_5$ this is not true. Take an element $\alpha/\beta$ of $O_5$: here we can take $\alpha = 3e_1 + 2e_2 + 2e_3$ and $\beta = 6$ where $(e_1, e_2, e_3)$ is the $\mathbb{Z}$-basis returned by PARI [1]. We can then prove directly as in Section 3.2 that the ideal $(\alpha, \beta)$ is not principal. We conclude that $K_1$ is G.E.

For the second field, $K_2$, defined by $x^3 - 21x - 35$ the situation is different. Here $S$ is composed of seven orbits $O_i$, $1 \leq i \leq 7$ and four of them, say $O_4$, with $1 \leq i \leq 4$, are such that $\mathbb{Z}_K \cdot O_4 \subseteq S \cup \{0\}$. Now if we look at the three others, we find that two of them contain an $\alpha/\beta$ for which $(\alpha, \beta)$ is principal. For completeness these $\alpha/\beta$ are $(7e_1 + 12e_2 + 5e_3, 21)$ and $(7e_1 + 5e_2 + 11e_3, 21)$ with the usual notation. Consequently $K_2$ is not G.E. All the computations, which are long and complicated - in particular for $K_2$ - have been done by hand and checked using PARI [1]. We do not give them here for lack of space and because they are not especially enlightening.

Finally, we put all these results together to give us Theorem 1.4.

4. The 2-stage Euclidean number fields

Let us begin with an example. Let $K$ be the totally real cubic number field with discriminant 3988. Using our algorithm we see that the upper part of the Euclidean spectrum of $K$ has five elements, more precisely

$$sp(K) \cap [1, \infty) = \{19/8, 11/8, 5/4, 19/16, 133/128\}.$$  

The set $S$ is composed of five orbits, respectively the orbits of $ae_1 + be_2 + ce_3$ with $(a, b, c) = (0, 1/2, 1/2), (1/2, 1/2, 0), (1/2, 1/2, 1/2), (0, 3/4, 1/2) and (0, 3/8, 1/2)$, where $(e_1, e_2, e_3)$ is the $\mathbb{Z}$-basis of $\mathbb{Z}_K$ returned by PARI [1]. These orbits have
respectively 1, 1, 1, 2 and 4 elements. For one element $\xi$ by orbit, we try to find $q_1, q_2 \in \mathbb{Z}_K$ such that

$$\left| N_{K/Q}(\xi - q_1 - \frac{1}{q_2}) \right| < \frac{1}{|N_{K/Q}(q_2)|},$$

by testing “small” $q_1 \in \mathbb{Z}_K$ and “small” $q_2 \in \mathbb{Z}_K \setminus \{0\}$. In each case this is possible, so that for every $\xi \in S$, (6) is true. Finally this implies that $K$ is 2-stage norm-Euclidean. Using exactly the same approach we have established the results of Theorem 1.5.

**Remark 4.** Obviously these fields, which are principal and not norm-Euclidean, are not G.E.

**References**