Hamilton-Jacobi equations on networks

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Abstract

We consider continuous-state and continuous-time control problems where the admissible trajectories of the system are constrained to remain on a network. Under suitable assumptions, we prove that the value function is continuous. We define a notion of constrained viscosity solution of Hamilton-Jacobi equations on the network and we study related comparison principles. Under suitable assumptions, we prove in particular that the value function is the unique constrained viscosity solution of the Hamilton-Jacobi equation on the network.

Keywords Optimal control, graphs, networks, Hamilton-Jacobi equations, viscosity solutions

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1 Introduction

A network (or a graph) is a set of items, referred to as vertices or nodes, with connections between them referred to as edges. The main tools for the study of networks come from combinatorics and graph theory. But in the recent years there is an increasing interest in the investigation of dynamical systems and differential equation on networks, in particular in connection with problem of data transmission and traffic management (see for example Garavello-Piccoli [10], Engel et al [5]). In this perspective, the study of control problems on networks has interesting applications in various fields. Note that partial differential operators on ramified spaces have also been investigated, see e.g. [16], [15].

A typical optimal control problem is the *minimum time problem*, which consists of finding the shortest path between an initial position and a given target set. If the running cost is a fixed constant for each edge and the dynamics can go from one vertex to an adjacent one at each time step, the corresponding discrete-state discrete-time control problem can be studied via graph theory and matrix analysis. If instead the cost changes in a continuous way along the edges and the dynamics is continuous in time, the minimum time problem can be seen as a continuous-state continuous-time control problem where the admissible trajectories of the system are constrained to remain on the network. While control problems with state constrained in closures of open sets are well studied ([17, 18], [3], [11]) there is to our knowledge much fewer literature on problems in closed sets with empty interior. The results of Frankowska and Plaskacz [9, 8] do apply to some closed sets with empty interior, but not to networks with crosspoints (except in very particular cases).

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The aim of this paper is therefore to study optimal control problems whose dynamics is constrained to a network and the related Hamilton-Jacobi-Bellman equation. Note that other types of optimal control problems could be considered as well, leading to other boundary conditions at the endpoints of the network. In most of the paper, we will consider for simplicity the toy model given by a star-shaped network, i.e. straight edges intersecting at the origin. This simple model problem already contains many of the difficulties that we have to face in more general situations. Moreover we will sometimes assume that the running cost is independent of the control.

Since the dynamics is constrained to the network, the velocities tangent to the network vary from one edge to another, hence the set of the admissible controls depends on the state of the system. If the set of admissible controls varies in a continuous way, the corresponding control problem can be studied via standard viscosity solution techniques (see Koike[12]). But for a network, the set of admissible controls drastically changes from a point in the interior of an edge, where only one direction is admissible (with possibly positive and negative velocities), to a vertex where the admissible directions are given by all the edges connected to it. Therefore, even if the data of the problem are regular, the corresponding Hamiltonian when restricted to the network has a discontinuous structure. Problem with discontinuous Hamiltonians have been recently studied by various authors (Tourin[22], Soravia[19], Deckelnick-Elliott[4], Bressan-Hong[2]), but the approaches and the results considered in these papers do not seem to be applicable because of the particular structure of the considered domain.

Assuming that the set of the admissible control laws - i.e. the control laws for which the corresponding trajectory remains on the graph - is not empty, the control problem is well posed and the corresponding value function satisfies a dynamic programming principle. We introduce a first set of assumptions which guarantees that the value function is continuous on the network (with respect to the intrinsic geodetic distance).

The next step is to introduce a definition of weak solution which ensures the uniqueness of the continuous solution via a comparison theorem. While in the interior of an edge we can test the equation with a smooth test function, the main difficulties arise at the vertices where the network does not have a regular differential structure. At a vertex, we consider a concept of derivative similar to that of Dini's derivative, see for example[1], hence regular test functions are the ones which admit derivatives in the directions of the edges entering in the node. We give a definition of viscosity solution on the network using the previous class of test functions. It is worthwhile to observe that this definition reduces to the classical one of viscosity solution if the graph is composed of two parallel segments entering in a node, see [1].

With this definition, the intrinsic geodetic distance, fixed one argument, is a regular function w.r.t. the other argument and it can be used in the comparison theorem as a penalty term in the classical doubling argument of viscosity solution theory.

We conclude observing that this paper is a first attempt to study Hamilton-Jacobi-Bellman equations and viscosity solutions on a network. Several points remain open such as more general control problems, problem with boundary conditions, stochastic control problems.

The paper is organized as follows: the control problem and the basic assumptions are set in Section 2. In Section 3, we define useful notions and prove preliminary results, before proposing a definition of viscosity solutions of an Hamilton-Jacobi equation on the network in Section 4; then, we prove that this notion is equivalent to the classical one if the network is made of two parallel segments sharing one endpoint. We also prove that the value function of the control problem is a viscosity solution. Comparison principles are studied in Section 5. Finally, in Section 6 we study a case when the value function may be discontinuous and we propose a notion of discontinuous viscosity solution.

2 Setting of the problem and basic assumptions

We consider a planar network with a finite number of edges and vertices. A network in $\mathbb{R}^2$ is a pair $(\mathcal{V},\mathcal{E})$ where

i) $\mathcal{V}$ is a finite subset of $\mathbb{R}^2$ whose elements are said vertices

ii) $\mathcal{E}$ is a finite set of regular arcs of $\mathbb{R}^2$, said edges, whose extrema are elements of $\mathcal{V}$.

We say that two vertices are adjacent if they are connected by an edge. We say that a vertex belongs to $\partial\mathcal{V}$ (resp., $int(\mathcal{V})$) if there is only one (resp., more than one) edge connected to it. We assume that the edges cross each other transversally. We denote by $\overline{\mathcal{G}}$ the union of all the edges in $\mathcal{E}$ and all the vertices in $\mathcal{V}$. We denote by $\mathcal{G}$ the set $\overline{\mathcal{G}}\backslash\partial\mathcal{V}$.
Except when explicitly mentioned, we focus for simplicity on the model case of a star-shaped network with N straight edges, $N > 1$, i.e.

$$\mathcal{G} = \{O\} \cup \bigcup_{j=1}^{N} J_j \subset \mathbb{R}^2, \quad O = (0,0), \quad J_j = (0,1)e_j, \tag{2.1}$$

where $(e_j)_{j=1,\dots,N}$ is a set of unit vectors in $\mathbb{R}^2$ s.t. $e_j \neq e_k$ if $j \neq k$. Note that $e_j = -e_k$ is possible. Then, $\partial\mathcal{V} = \{e_j, j = 1,\dots,N\}$ and $\mathrm{int}(\mathcal{V}) = \{O\}$. We will use the notation $\partial\mathcal{G} \equiv \partial\mathcal{V}$. Except in § 4.2, we assume that there is at least a pair (j,k), $j \neq k$ s.t. e_j is not aligned with e_k.
The general case will be dealt with in a forthcoming paper, where we will also consider structures made of several manifolds of different dimensions crossing each other transversally. Hereafter, the notation $\mathbb{R}_+$ stands for the interval $[0,+\infty)$.
For any $x \in \mathcal{G}$, we denote by $T_x(\mathcal{G}) \subset \mathbb{R}^2$ the set of the tangent directions to the network, i.e.

$$p \in T_x(\mathcal{G}) \iff \exists T > 0 \text{ and } \xi \in \mathcal{C}^1([0,T];\mathbb{R}^2) \text{ s.t. } \xi(t) \in \mathcal{G},\ \forall t \le T,\ \xi(0) = x \text{ and } \dot{\xi}(0) = p. \tag{2.2}$$

It is easy to prove that $p \in T_x(\mathcal{G})$ if and only if there exist sequences $(t_n)_{n\in\mathbb{N}}$, $t_n > 0$ and $(x_n)_{n\in\mathbb{N}}$, $x_n \in \mathcal{G}$, such that $t_n \to 0^+$ and $(x_n - x)/t_n \to p$.
We now introduce the optimal control problem on $\mathcal{G}$. We start by making some assumptions on the structure of the problem.
Call B the closed unit ball of $\mathbb{R}^2$ centered at O. Take for A a compact set of $\mathbb{R}^2$ and a continuous function $f : B \times A \to \mathbb{R}^2$ such that

$$|f(x,a) - f(y,a)| \le L|x - y|, \quad \forall x,y \in B,\ a \in A. \tag{2.3}$$

The assumption (2.3) implies that there exists $M > 0$ such that

$$|f(x,a)| \le M, \quad \forall x \in B,\ a \in A. \tag{2.4}$$

Additional assumptions will be made below. For $x \in \overline{\mathcal{G}}$, we consider the dynamical system

$$\begin{cases} \dot{y}(t;x,\alpha) = f(y(t;x,\alpha),\alpha(t)), & t > 0, \\ y(0) = x. \end{cases} \tag{2.5}$$

Remark 2.1. *We have chosen to parameterize the dynamics by a function f defined on $B \times A$, i.e. on a much larger set than $\mathcal{G} \times A$. We could also have defined f on $\mathcal{G} \times A$ only. This would have been equivalent since by Whitney extension theorem one can extend any Lipschitz function defined on $\mathcal{G}$ to a Lipschitz function defined on B. In fact, all the assumptions made below on f involve $f|_{\mathcal{G}\times A}$ only. Yet, it seemed to us that defining f on $B \times A$ led to simpler notations.*

Denoting by $\mathcal{A}$ the class of the control laws, i.e. the set of measurable functions from $[0,+\infty)$ to A, we introduce the subset $\mathcal{A}_x \subset \mathcal{A}$ of the admissible control laws, i.e. the control laws for which the dynamics (2.5) is constrained on the network $\mathcal{G}$:

$$\mathcal{A}_x = \{\alpha \in \mathcal{A} :\ y(t;x,\alpha) \in \mathcal{G}, \quad \forall t > 0\}. \tag{2.6}$$

Assumption 2.1.

$$\mathcal{A}_x \textit{ is not empty for any } x \in \overline{\mathcal{G}}. \tag{2.7}$$

We will always consider $\alpha \in \mathcal{A}_x$ in (2.5).
We also define for $x \in \overline{\mathcal{G}}$,

$$A_x = \{a \in A \text{ s.t. } \exists \theta > 0 : y(t; x, a) \in \mathcal{G}, \forall t, 0 < t < \theta\}. \tag{2.8}$$

From the continuity of f, we see that for all $a \in A_x$, $f(x,a) \in T_x(\mathcal{G})$.

Assumption 2.2. *We assume that there exist non empty subsets A^j of A, $j = 1, \dots, N$, such that*

$$A_x = A^j, \qquad \text{if } x \in J_j, \quad j = 1, \dots, N, \tag{2.9}$$

$$A_O = \cup_{j=1}^N \{a \in A^j : f(O, a) \in \mathbb{R}_+ e_j\}, \tag{2.10}$$

$$A_{e_j} = \{a \in A^j : \; f(e_j, a) \cdot e_j \le 0\} \ne \emptyset, \text{ and } \inf_{a \in A_{e_j}} f(e_j, a) \cdot e_j < 0, \quad j = 1, \dots, N. \tag{2.11}$$

and such that $A^j = A^k$ if $e_j = -e_k$.

Remark 2.2. *Assumption (2.9) says that the set of constant controls for which the trajectories leaving $x \in J_j$ stay in $\mathcal{G}$ for a positive time is nonempty and does not depend on $x \in J_j$. Assumption (2.10) characterizes the set of constant controls for which the trajectories leaving O stay in $\mathcal{G}$ for a positive time: a further assumption will be needed to state A_O is not empty. The assumption in (2.11) at the vertices in $\partial\mathcal{V}$ tells us that there exist controls which make the trajectory enter $\mathcal{G}$; this assumption is classical in the context of state constrained problems.*

Assumption 2.3. *For all $x \in \mathcal{G}\backslash\{O\}$, there exists $\tau > 0$ such that for all $\alpha \in \mathcal{A}_x$, $\alpha(t) \in A_x$ for almost all $t \in [0, \tau]$.*

Assumption 2.3 says that for small durations, an admissible control law at x cannot take values outside A_x (except maybe on a negligeable set of times).

Assumption 2.4. *We assume that there exist constants $\underline{\zeta_j} > 0$ and $\overline{\zeta_j} > 0$, $j = 1, \dots, N$, s.t.*

$$\overline{co}(f(O, A^j)) = [-\underline{\zeta_j}, \overline{\zeta_j}]e_j, \quad \forall j = 1, \dots, N, \tag{2.12}$$

where $\overline{co}(F)$ stands for the closed convex hull of F.

Remark 2.3. *We will see that Assumption 2.4 implies the continuity of the value function, for which weaker assumptions can be made, see Remark 2.6. We will also see that a strenghtened version of Assumption 2.4 may be used for proving a comparison principle, see Assumption 5.1.*

Remark 2.4. *Assumption 2.4 implies controllability near O.*

Remark 2.5. *Note that if $e_j = -e_k$ then, from (2.12) and the continuity of f, $\underline{\zeta_j} = \overline{\zeta_k}$ and $\underline{\zeta_k} = \overline{\zeta_j}$.*

Example 2.1. *Take for A the unit ball of $\mathbb{R}^2$ and $f(x,a) = g(x)a$ where $g : B \to \mathbb{R}$ is a Lipschitz continuous positive function: we can see that all the assumptions above are satisfied. In particular, let us show that Assumption 2.3 holds in the present case: take $x \in \mathcal{G}\backslash\{O\}$, for example $x \in J_1$ and $\alpha \in \mathcal{A}_x$. With M as in (2.4), take $\tau_x = |x|/(2M)$. It is easy to see that $y(t; x, \alpha) \in J_1$ for $t \in [0, \tau_x]$. This implies that $\int_0^t e_1 \wedge f(y(s; x, \alpha), \alpha(s))ds = 0$ for $t \in [0, \tau_x]$, and therefore $e_1 \wedge f(y(t; x, \alpha), \alpha(t)) = g(y(t; x, \alpha))e_1 \wedge \alpha(t) = 0$ for almost all $t \in [0, \tau_x]$. Therefore, since g is positive, $\alpha(t) \in A^1 = A \cap \mathbb{R}e_1 = A_x$ for almost all $t \in [0, \tau_x]$.*

Example 2.2. *Take N unit vectors $(e_j)_{j=1,\dots,N}$, with $e_j = (\cos\theta_j, \sin\theta_j)$, $\theta_j \in [0, 2\pi)$. Choose $\underline{\zeta_j}$, $\overline{\zeta_j}$ $2N$ positive numbers such that $\underline{\zeta_j} = \overline{\zeta_k}$ and $\underline{\zeta_k} = \overline{\zeta_j}$ if $e_j = -e_k$. Take for A the unit ball of $\mathbb{R}^2$; let $\zeta : \mathbb{R} \to \mathbb{R}_+$ be a 2π-periodic and continuous function such that $\zeta(\theta_j) = \overline{\zeta_j}$ and $\zeta(-\theta_j) = \underline{\zeta_j}$, $j = 1, \dots, N$; Choose $f(x, a) = g(x)\zeta(\theta)a$ where $a = |a|(\cos\theta, \sin\theta)$ and $g : B \to \mathbb{R}$ is a Lipschitz continuous positive function. We can see that all the assumptions above are satisfied.*

Example 2.3. *Choose N unit vectors $(e_j)_{j=1,\dots,N}$ and $2N$ positive numbers $\underline{\zeta_j}$, $\overline{\zeta_j}$ as in Example 2.2. Take $A = \cup_{j=1}^N K e_j$, $K = \{-1, 1\}$. Choose*

$$f(x,a) = g(x) \sum_{j=1}^{N} \left(-\underline{\zeta_j} 1_{a=-e_j} + \overline{\zeta_j} 1_{a=e_j} \right) e_j$$

where $g : B \to \mathbb{R}$ is a Lipschitz continuous positive function. We can see that all the assumptions above are satisfied.

Example 2.4. *As a particular case of Example 2.3, one may take the cross shaped network $\mathcal{G} = \{O\} \cup \bigcup_{j=1}^4 J_j$, $J_1 = (0,1)e_1$, $J_2 = -(0,1)e_1$, $J_3 = (0,1)e_2$, $J_4 = -(0,1)e_2$, e_1 and e_2 being two orthogonal unit vectors. One may choose $A = Ke_1 \cup Ke_2$, $K = \{-1, 1\}$ and $f(x, a) = g(x)a$ where $g : B \to \mathbb{R}$ is a Lipschitz continuous positive function.*

Finally, we consider a continuous functions $\ell : B \times A \to \mathbb{R}$ such that

$$|\ell(x,a)| \le M, \quad \forall x \in B,\, a \in A, \tag{2.13}$$
$$|\ell(x,a) - \ell(y,a)| \le L|x-y|, \quad \forall x, y \in B,\, a \in A. \tag{2.14}$$

For $\lambda > 0$, we consider the cost functional

$$J(x,\alpha) = \int_0^\infty \ell(y(t;x,\alpha), \alpha(t)) e^{-\lambda t} dt. \tag{2.15}$$

The value function of the constrained control problem on the network is

$$v(x) = \inf_{\alpha \in \mathcal{A}_x} J(x,\alpha), \qquad x \in \overline{\mathcal{G}}. \tag{2.16}$$

Assumption 2.1 and the assumptions on ℓ are enough for the dynamic programming principle:

$$v(x) = \inf_{\alpha \in \mathcal{A}_x} \left\{ \int_0^t \ell(y(s;x,\alpha), \alpha(s)) e^{-\lambda s} ds + e^{-\lambda t} v(y(t;x,\alpha)) \right\}. \tag{2.17}$$

The proof is standard along the arguments in Propositions III.2.5 or IV.5.5 in [1].

Proposition 2.1. *Under the assumptions above, the value function is continuous on $\overline{\mathcal{G}}$.*

Proof. For the continuity at $x \in \partial\mathcal{V}$, we refer to [1], proof of Theorem 5.2, page 274. We are going to study the continuity of the value function at $x \in \mathcal{G}$.
Consider now $x \in \mathcal{G}$. We want to prove that $\limsup_{z\to x} v(z) \le v(x)$. The inequality $\liminf_{z\to x} v(z) \ge v(x)$ is obtained in a similar way.
For any $\varepsilon > 0$, there exists a control $\overline{\alpha} \in \mathcal{A}_x$ such that $J(x, \bar{\alpha}) < v(x) + \varepsilon$.
The following observation will be useful: from the controllability assumption (2.12) and from (2.3), there exist a positive number r_0 and a constant C such that for all $z_1, z_2 \in B_O(r_0) \cap \mathcal{G}$, there exists $\alpha_{z_1,z_2} \in \mathcal{A}_{z_1}$ and $\tau_{z_1,z_2} \le C|z_1 - z_2|$ with $y(\tau_{z_1,z_2}; z_1, \alpha_{z_1,z_2}) = z_2$.
We distinguish two cases: a) $x \in \overline{B_O(r_0/2)}$; b) $x \notin \overline{B_O(r_0/2)}$.
a) If $x \in \overline{B_O(r_0/2)}$, then if $z \in B_O(r_0)$, we construct $\tilde{\alpha} \in \mathcal{A}_z$ as follows:

$$\begin{array}{rcll} \tilde{\alpha}(t) & = & \alpha_{z,x}(t) & \text{if } t < \tau_{z,x}, \\ \tilde{\alpha}(t) & = & \overline{\alpha}(t - \tau_{z,x}) & \text{if } t > \tau_{z,x}. \end{array}$$

Since $\tau_{z,x} \le C|z-x|$, it is easy to prove that $v(z) \le J(z,\tilde{\alpha}) \le v(x)+\varepsilon+CM|x-z|$. Sending ε to 0, we obtain that $\limsup_{z\to x} v(z) \le v(x)$ for $x \in \overline{B_O(r_0/2)}$.
b) If $x \notin \overline{B_O(r_0/2)}$, we can assume that $x \in J_1$. Then for z close enough to x, z belongs to J_1. We take $z \in J_1$ such that $|x-z| \le \rho_\varepsilon$.
Therefore, the control $\overline{\alpha}$ is also admissible for z at least for a finite duration, (the first time T when $y(t;x,\overline{\alpha})$ or $y(t;z,\overline{\alpha})$ hits O or e_1, if it exists). For brevity, we will only discuss the case when $y(T;x,\overline{\alpha}) = O$ or $y(T;z,\overline{\alpha}) = O$, if T exists. The other cases $y(T;x,\overline{\alpha}) = e_1$ or $y(T;z,\overline{\alpha}) = e_1$ can be dealt with in a similar way by using the controllability assumption (2.11), see [1] for example.
We define

$$T_\varepsilon = -\frac{1}{\lambda}\log(\varepsilon\lambda/(2M)), \quad C_\varepsilon = L\int_0^{T_\varepsilon} e^{(L-\lambda)t}dt, \quad \rho_\varepsilon = r_0 e^{-LT_\varepsilon}/4. \tag{2.18}$$

b1) If $T > T_\varepsilon$ or T does not exist, then both $y(t;z,\overline{\alpha})$ and $y(t;x,\overline{\alpha})$ remain in $J_1 \cup \{e_1\}$ for $t < T_\varepsilon$. For any $\tilde{\alpha} \in \mathcal{A}_z$ s.t. $\tilde{\alpha}(t) = \overline{\alpha}(t)$ for $t < T_\varepsilon$, we have that $|J(z,\tilde{\alpha}) - J(x,\tilde{\alpha})| \le C_\varepsilon|x-z| + \varepsilon$, where C_ε is defined in (2.18). Thus $v(z) \le J(z,\tilde{\alpha}) \le v(x) + C_\varepsilon|x-z| + 3\varepsilon$.
b2) If $y(T;x,\overline{\alpha}) = O$, then we construct the control $\tilde{\alpha} \in \mathcal{A}_z$ as follows

$$\begin{array}{rcll}
\tilde{\alpha}(t) &=& \overline{\alpha}(t) & \text{if } t < T,\\
\tilde{\alpha}(t) &=& \alpha_{y(T;z,\overline{\alpha}),O}(t-T) & \text{if } T < t < T + \tau_{y(T;z,\overline{\alpha}),O},\\
\tilde{\alpha}(t) &=& \overline{\alpha}(t - \tau_{y(T;z,\overline{\alpha}),O}) & \text{if } t > T + \tau_{y(T;z,\overline{\alpha}),O}.
\end{array}$$

Note that this is possible since $|x-z| \le \rho_\varepsilon$ which implies $|y(T;z,\overline{\alpha})| \le e^{LT_\varepsilon}|x-z| \le r_0/4$.
Here again, we get that

$$v(z) \le J(z,\tilde{\alpha}) \le v(x) + \tilde{C}_\varepsilon|x-z| + \varepsilon,$$

for another constant $\tilde{C}_\varepsilon$.
b3) If $y(T;z,\overline{\alpha}) = O$, then we construct the control $\tilde{\alpha} \in \mathcal{A}_z$ as follows

$$\begin{array}{rcll}
\tilde{\alpha}(t) &=& \overline{\alpha}(t) & \text{if } t < T,\\
\tilde{\alpha}(t) &=& \alpha_{O,y(T;x,\overline{\alpha})}(t-T) & \text{if } T < t < T + \tau_{O,y(T;x,\overline{\alpha})},\\
\tilde{\alpha}(t) &=& \overline{\alpha}(t - \tau_{O,y(T;x,\overline{\alpha})}) & \text{if } t > T + \tau_{O,y(T;x,\overline{\alpha})}.
\end{array}$$

Note that this is possible since $|x-z| \le \rho_\varepsilon$ which implies $|y(T;x,\overline{\alpha})| \le e^{LT_\varepsilon}|x-z| \le r_0/4$.
Here again, we get that

$$v(z) \le J(z,\tilde{\alpha}) \le v(x) + \tilde{C}_\varepsilon|x-z| + \varepsilon.$$

□

Remark 2.6. *It can be shown that Proposition 2.1 holds if for some indices j, $\overline{\zeta}_j = \underline{\zeta}_j = 0$.*

We now give an example in which the value function is discontinuous: let (e_1,e_2) be an orthonormal basis of $\mathbb{R}^2$, $\mathcal{G} = (0,1)e_1 \cup \{O\} \cup (0,1)e_2$, $A = \{0, e_1, e_2\}$, $f(x,a) = a(1-2|x|)$. Take $\ell(x,a) = 1$ if $x_2 = 0$ and $\ell(x,a) = 1 - |x|$ if $x_1 = 0$. Assumption 2.4 is not satisfied. It is easy to compute the value function v at $x = (x_1,x_2)$: we have

$$\begin{aligned}
v(x_1,0) &= \frac{1}{\lambda}, \quad 0 < x_1 \le 1,\\
v(0,x_2) &= \frac{1}{2\lambda} + \frac{1-2x_2}{4+2\lambda}, \quad 0 \le x_2 < \frac{1}{2},\\
v(0,x_2) &= \frac{1-x_2}{\lambda}, \quad \frac{1}{2} \le x_2 \le 1.
\end{aligned} \tag{2.19}$$

The value function is discontinuous at O.

3 Preliminary notions for weak solutions

Hereafter, we make all the assumptions stated in § 2, except in § 4.2 and § 6. All the theorems below will be stated without repeating the assumptions.

3.1 Test functions

We introduce the class of the admissible test functions for the differential equation on the network

Definition 3.1. *We say that a function $\varphi : \overline{\mathcal{G}} \to \mathbb{R}$ is an admissible test function and we write $\varphi \in \mathcal{R}(\mathcal{G})$ if*

- *φ is continuous in $\overline{\mathcal{G}}$ and $\mathcal{C}^1$ in $\overline{\mathcal{G}} \setminus \{O\}$*
- *for any j, $j = 1, \dots, N$, $\varphi|_{\overline{J_j}} \in \mathcal{C}^1(\overline{J_j})$.*

Therefore, for any $\zeta \in \mathbb{R}^2$ such that there exists a continuous function $z : [0,1] \to \mathcal{G}$ and a sequence $(t_n)_{n\in\mathbb{N}}$, $0 < t_n \le 1$ with $t_n \to 0$ and

$$\lim_{n\to\infty} \frac{z(t_n)}{t_n} = \zeta,$$

the limit $\lim_{n\to\infty} \frac{\varphi(z(t_n))-\varphi(O)}{t_n}$ exists and does not depend on z and $(t_n)_{n\in\mathbb{N}}$. We define

$$D\varphi(O,\zeta) = \lim_{n\to\infty} \frac{\varphi(z(t_n)) - \varphi(O)}{t_n}. \tag{3.1}$$

If $x \in \mathcal{G}\backslash\{O\}$ and $\zeta \in T_x(\mathcal{G})$, we agree to write $D\varphi(x,\zeta) = D\varphi(x)\cdot\zeta$.

Property 3.1. *Let us observe that $D\varphi(O,\rho\zeta) = \rho D\varphi(O,\zeta)$ for any $\rho > 0$. Indeed, denoting by $\tau_n = t_n/\rho$, $\lim_{n\to\infty} z(t_n)/\tau_n = \rho\zeta$. Hence,*

$$\rho D\varphi(O,\zeta) = \lim_{n\to\infty} \frac{\varphi(z(t_n)) - \varphi(O)}{\tau_n} = D\varphi(O,\rho\zeta).$$

As shown below, this property is not true if $\rho < 0$.

If $\varphi \in \mathcal{C}^1(\mathbb{R}^2)$, then $\varphi_{|\mathcal{G}} \in \mathcal{R}(\mathcal{G})$ and $D\varphi(O,\zeta) = D\varphi(O)\cdot\zeta$ for any $\zeta \in \mathbb{R}_+ e_j$, $j = 1, \dots, N$. If $e_j = -e_k$ for some $j \neq k \in \{1, \dots, N\}$, $D\varphi(O, e_j) = -D\varphi(O, -e_j)$.
If φ is continuous and $\varphi|_{\bar{\mathcal{G}}\cap\mathbb{R}e_j}$ is $\mathcal{C}^1$ for $j = 1, \dots, N$, then $\varphi \in \mathcal{R}(\mathcal{G})$ but the converse may not be true if two edges are aligned: for example, if $e_j = -e_k$ for some $j \neq k \in \{1, \dots, N\}$, the function $x \mapsto b|x|$ belongs to $\mathcal{R}(\mathcal{G})$ and $D\varphi(O, e_j) = D\varphi(O, -e_j) = b$.

Property 3.2. *If $\varphi = g \circ \psi$ with $g \in \mathcal{C}^1$ and $\psi \in \mathcal{R}(\mathcal{G})$, then $\varphi \in \mathcal{R}(\mathcal{G})$ and*

$$D\varphi(O,\zeta) = g'(\psi(O)) D\psi(O,\zeta). \tag{3.2}$$

3.2 A set of relaxed vector fields

Let us use the notation

$$m_O = \min_{a \in \cup_{1\le k\le N} A^k} \ell(O,a). \tag{3.3}$$

We will sometimes make a further assumption:

Assumption 3.1. *The function $\ell : \overline{\mathcal{G}} \times A \to \mathbb{R}$ satisfies: for all $j = 1, \dots, N$,*

$$(0, m_O) \in \overline{\text{co}}\left((f(O,a), \ell(O,a)) : a \in A^j\right). \tag{3.4}$$

Note that from Assumption 2.4, for all $j = 1, \dots, N$, $0 \in \overline{\text{co}}\left((f(O,a) : a \in A^j\right)$.

Example 3.1. *From Assumption 2.4, Assumption 3.1 is always satisfied if $\ell(O,a)$ does no depend on a.*

Example 3.2. *In the examples 2.1- 2.4, we can take $\ell(x,a)=q(x)|a|^\nu+p(x)$, where $\nu\geq 0$ and q and p are Lipschitz function defined on $\overline{\mathcal{G}}$ with $q(O)\geq 0$.*

Definition 3.2. *For $x\in\overline{\mathcal{G}}$, we introduce the sets*

$$\tilde{f}(x)=\left\{\eta\in T_x(\mathcal{G}):\begin{array}{l}\exists(\alpha_n)_{n\in\mathbb{N}},\alpha_n\in\mathcal{A}_x,\\ \exists(t_n)_{n\in\mathbb{N}}\end{array}\text{ s.t. }\begin{array}{l}t_n\to 0^+\text{ and}\\ \displaystyle\lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}f(y(t;x,\alpha_n),\alpha_n(t))dt=\eta\end{array}\right\}$$

and

$$\widetilde{f\ell}(x)=\left\{(\eta,\mu)\in T_x(\mathcal{G})\times\mathbb{R}:\begin{array}{l}\exists(\alpha_n)_{n\in\mathbb{N}},\alpha_n\in\mathcal{A}_x,\\ \exists(t_n)_{n\in\mathbb{N}}\end{array}\text{ s.t.}\left|\begin{array}{l}t_n\to 0^+,\\ \displaystyle\lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}f(y(t;x,\alpha_n),\alpha_n(t))dt=\eta,\\ \displaystyle\lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}\ell(y(t;x,\alpha_n),\alpha_n(t))dt=\mu\end{array}\right.\right\}.$$

Proposition 3.1. *a) Under all the assumptions made in § 2,*

$$\widetilde{f\ell}(x)=\quad \mathrm{FL}(x)\quad\equiv\overline{\mathrm{co}}\left((f(x,a),\ell(x,a)):a\in A_x\right),\qquad \text{if } x\in\mathcal{G}\backslash\{O\},\tag{3.5}$$

$$\widetilde{f\ell}(O)\supset\quad\mathrm{FL}(O)\quad\equiv\bigcup_{j=1}^N\Big(\overline{\mathrm{co}}\left((f(O,a),\ell(O,a)):a\in A^j\right)\cap(\mathbb{R}^+e_j\times\mathbb{R})\Big),\tag{3.6}$$

$$\widetilde{f\ell}(e_j)=\quad\mathrm{FL}(e_j)\quad\equiv\overline{\mathrm{co}}\left((f(e_j,a),\ell(e_j,a)):a\in A^j\right)\cap(\mathbb{R}^-e_j\times\mathbb{R}).\tag{3.7}$$

b) Under all the assumptions made in § 2 and Assumption 3.1,

1. *For all $\zeta\in\tilde{f}(O)\cap\mathbb{R}_+e_j$, there exists $\xi\in\mathbb{R}$ such that $(\zeta,\xi)\in\mathrm{FL}(O)\cap(\mathbb{R}^+e_j\times\mathbb{R})$.*
2. *For all $\zeta\in\tilde{f}(O)$,*

$$\min\left\{\mu:(\zeta,\mu)\in\mathrm{FL}(O)\right\}=\min\left\{\mu:(\zeta,\mu)\in\widetilde{f\ell}(O)\right\}.\tag{3.8}$$

Proof. Take first $x\in\mathcal{G}\backslash\{O\}$.
We can assume that $x\in J_1$. The inclusion $\mathrm{FL}(x)\subset\widetilde{f\ell}(x)$ is obtained as follows: take $\zeta=\sum_{j=1}^J\mu_jf(x,a_j)$, $\xi=\sum_{j=1}^J\mu_j\ell(x,a_j)$ with $a_j\in A_x$ and $\sum_j\mu_j=1$, $0\leq\mu_j$. For t_n small enough, it is possible to construct a control $\alpha_n\in\mathcal{A}_x$ such that $\alpha_n(t)=a_j$ for $(\sum_{k<j}\mu_k)t_n<t\leq(\sum_{k\leq j}\mu_k)t_n$: we have $\frac{1}{t_n}\int_0^{t_n}f(y(t;x,\alpha_n),\alpha_n(t))dt=\frac{1}{t_n}\int_0^{t_n}f(x,\alpha_n(t))dt+o(1)=\sum_j\mu_jf(x,a_j)+o(1)$, so

$$\lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}f(y(t;x,\alpha_n),\alpha_n(t))dt=\zeta.$$

Similarly,

$$\lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}\ell(y(t;x,\alpha_n),\alpha_n(t))dt=\xi.$$

Finally, for $(\zeta,\xi)\in\mathrm{FL}(x)$, we approximate (ζ,ξ) by $(\zeta_m,\xi_m)_{m\in\mathbb{N}}$, where (ζ_m,ξ_m) is a convex combination of $(f(x,a),\ell(x,a))$, $a\in A_x$, and we conclude by a diagonal process.
For the opposite inclusion, since $x\in\mathcal{G}\backslash\{O\}$, we know from Assumption 2.3 that there exists $\tau>0$, such that for all $\alpha\in\mathcal{A}_x$, $\alpha(t)\in A_x$ for $0\leq t<\tau$. Therefore,

$$\left(\frac{1}{s}\int_0^s f(x,\alpha(t))dt,\frac{1}{s}\int_0^s\ell(x,\alpha(t))dt\right)\in\mathrm{FL}(x)$$

for s small enough. This and the Lipschitz continuity of f and ℓ w.r.t. their first argument imply that $\widetilde{f\ell}(x)\subset\mathrm{FL}(x)$. We have proved (3.5).

We now consider $x = O$. We first discuss the inclusion $\mathrm{FL}(O) \subset \widetilde{f\ell}(O)$: we take $\zeta = \sum_{j=1}^{J} \mu_j f(O, a_j)$, $\xi = \sum_{j=1}^{J} \mu_j \ell(O, a_j)$ with $a_j \in A^1$ and we assume that $\zeta \in \mathbb{R}_+ e_1$. Up to a permutation of the indices, it is possible to assume that there exists J', $1 < J' \leq J$ such that $f(O, a_j) \in \mathbb{R}_+ e_1$ for $j \leq J'$ and that $f(O, a_j) \in \mathbb{R}_- e_1$ for $j > J'$. Then by a similar argument as above, $(\zeta, \xi) \in \widetilde{f\ell}(O)$. By a diagonal process, this implies that

$$\overline{\mathrm{co}}\left((f(O,a), \ell(O,a)) : a \in A^1\right) \cap (\mathbb{R}_+ e_1 \times \mathbb{R}) \subset \widetilde{f\ell}(O).$$

Similarly $\overline{\mathrm{co}}\left((f(O,a), \ell(O,a)) : a \in A^j\right) \cap (\mathbb{R}_+ e_j \times \mathbb{R}) \subset \widetilde{f\ell}(O)$, so we have proved (3.6). The proof of (3.7) is similar.

To prove points b 1) and b 2), we consider $\zeta \in \tilde{f}(O)$ and make out two cases:

• $\zeta = 0$: from Assumption 2.4, $\mathrm{FL}(O) \cap (\{0\} \times \mathbb{R}) \neq \emptyset$.

From Assumption 3.1, $\min\{\xi : (0, \xi) \in \mathrm{FL}(O)\} = m_O$.

On the other hand, for all sequences $t_n \to 0^+$ and $\alpha_n \in \mathcal{A}_O$,

$$\liminf_{n \to \infty} \frac{1}{t_n} \int_0^{t_n} \ell(y(t; O, \alpha_n), \alpha_n(t)) dt \geq m_O.$$

Therefore,

$$\min\{\xi : (0, \xi) \in \mathrm{FL}(O)\} \leq \min\left\{\xi : (0, \xi) \in \widetilde{f\ell}(O)\right\},$$

and this inequality is in fact an identity, because $\mathrm{FL}(O) \subset \widetilde{f\ell}(O)$.

• $\zeta \neq 0$: we can suppose that $0 \neq \zeta \in \mathbb{R}_+ e_1$. There exist sequences $\alpha_n \in \mathcal{A}_O$ and $t_n > 0$ such that $t_n \to 0^+$, $\lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} f(y(t; O, \alpha_n), \alpha_n(t)) dt = \zeta$. Up to an extraction, we may assume that $\lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} \ell(y(t; O, \alpha_n), \alpha_n(t)) dt = \mu$.

Since $0 \neq \zeta \in \mathbb{R}_+ e_1$, there exists s_n, $0 \leq s_n < t_n$ such that $y(s_n; O, \alpha_n) = O$ and $y(t; O, \alpha_n) \in J_1$ for all t, $s_n < t \leq t_n$. From Assumption 2.3, this implies that $\alpha_n(t) \in A^1$ for all t, $s_n < t < t_n$. Hence,

$$\begin{aligned}&\left(\frac{1}{t_n - s_n} \int_{s_n}^{t_n} f(O, \alpha_n(t)) dt, \frac{1}{t_n - s_n} \int_{s_n}^{t_n} \ell(O, \alpha_n(t)) dt\right)\\ &\in \overline{\mathrm{co}}\left((f(O,a), \ell(O,a)) : a \in A^1\right) \cap (\mathbb{R}_+ e_1 \times \mathbb{R}).\end{aligned}$$

Therefore, since $(0, m_O) \in \overline{\mathrm{co}}\left((f(O,a), \ell(O,a)) : a \in A^1\right)$ from Assumption 3.1, we get that

$$\begin{aligned}&\left(\frac{1}{t_n} \int_{s_n}^{t_n} f(O, \alpha_n(t)) dt, \frac{1}{t_n} \int_{s_n}^{t_n} \ell(O, \alpha_n(t)) dt + \frac{s_n}{t_n} m_O\right)\\ &\in \overline{\mathrm{co}}\left((f(O,a), \ell(O,a)) : a \in A^1\right) \cap (\mathbb{R}_+ e_1 \times \mathbb{R}).\end{aligned}$$

Up to the extraction of a subsequence, we may say that $\frac{1}{t_n} \int_{s_n}^{t_n} \ell(O, \alpha_n(t)) dt + \frac{s_n}{t_n} m_O$ converges to a real number ξ. Moreover, from the Lipschitz continuity of f,

$$\zeta = \lim_{n\to\infty} \frac{1}{t_n} \int_{s_n}^{t_n} f(y(t; O, \alpha_n), \alpha_n(t)) dt = \lim_{n\to\infty} \frac{1}{t_n} \int_{s_n}^{t_n} f(O, \alpha_n(t)) dt,$$

and we see that $(\zeta, \xi) \in \mathrm{FL}(O) \cap (\mathbb{R}_+ e_1 \times \mathbb{R})$, which proves point b 1).

We also see that

$$\xi \leq \lim \frac{1}{t_n} \int_0^{t_n} \ell(O, \alpha_n(t)) dt = \mu,$$

where the last identity comes from the Lipschitz continuity of ℓ. We have proved point b 2), since $\xi \leq \mu$ is true for all μ such that $(\zeta, \mu) \in \widetilde{f\ell}(O)$. $\square$

4 Viscosity solutions

Hereafter, unless explicitly mentioned, we make all the assumptions of § 2.

4.1 Definition of viscosity solutions

We now introduce the definition of a constrained viscosity solution of

$$\lambda u(x) + \sup_{(\zeta,\xi)\in\widetilde{f\ell}(x)} \{-Du(x,\zeta) - \xi\} = 0, \tag{4.1}$$

in $\overline{\mathcal{G}}$.

Definition 4.1.
- *A bounded and upper semi-continuous function $u : \overline{\mathcal{G}} \to \mathbb{R}$ is a subsolution of* (4.1) *in $\mathcal{G}$ if for any $x \in \mathcal{G}$, any $\varphi \in \mathcal{R}(\mathcal{G})$ s.t. $u - \varphi$ has a local maximum point at x, then*
$$\lambda u(x) + \sup_{(\zeta,\xi)\in\widetilde{f\ell}(x)} \{-D\varphi(x,\zeta) - \xi\} \le 0; \tag{4.2}$$
- *A bounded and lower semi-continuous function $u : \overline{\mathcal{G}} \to \mathbb{R}$ is a supersolution of* (4.1) *if for any $x \in \overline{\mathcal{G}}$, any $\varphi \in \mathcal{R}(\mathcal{G})$ s.t. $u - \varphi$ has a local minimum point at x, then*
$$\lambda u(x) + \sup_{(\zeta,\xi)\in\widetilde{f\ell}(x)} \{-D\varphi(x,\zeta) - \xi\} \ge 0; \tag{4.3}$$
- *A continuous function $u : \overline{\mathcal{G}} \to \mathbb{R}$ is a constrained viscosity solution of* (4.1) *in $\overline{\mathcal{G}}$ if it is a viscosity subsolution of* (4.1) *in $\mathcal{G}$ and supersolution of* (4.1) *in $\overline{\mathcal{G}}$.*

Remark 4.1. *At $x \in \mathcal{G}\backslash\{O\}$, the notion of sub, respectively super-solution in Definition 4.1 is equivalent to the standard definition of viscosity sub, respectively super-solution of the equation*

$$\lambda u(x) + \sup_{a\in A_x} \{-f(x,a)\cdot Du - \ell(x,a)\} = 0.$$

This is true because

1. *any test function in $\mathcal{R}(\mathcal{G})$ is $\mathcal{C}^1$ in a neighborhood of x from Definition 3.2,*
2. *$\widetilde{f\ell}(x) = \mathrm{FL}(x)$ since $x \neq O$,*

so

$$\sup_{(\zeta,\xi)\in\widetilde{f\ell}(x)} \{-D\varphi(x,\zeta) - \xi\} = \max_{(\zeta,\xi)\in\mathrm{FL}(x)} \{-D\varphi(x)\cdot\zeta - \xi\},$$

and because the maximum above is equal to $\sup_{a\in A_x}\{-D\varphi(x)\cdot f(x,a) - \ell(x,a)\}$. Similarly, at $x \in \partial\mathcal{V}$, the notion of supersolution in $\overline{\mathcal{G}}$ is equivalent to the standard definition.

Remark 4.2. *Assume that Assumption 3.1 is satisfied. In view of Proposition 3.1 (in particular point b. for $x = O$), (4.2) is equivalent to*

$$\lambda u(x) + \sup_{(\zeta,\xi)\in\mathrm{FL}(x)} \{-D\varphi(x,\zeta) - \xi\} \le 0, \tag{4.4}$$

and (4.3) is equivalent to

$$\lambda u(x) + \sup_{(\zeta,\xi)\in\mathrm{FL}(x)} \{-D\varphi(x,\zeta) - \xi\} \ge 0. \tag{4.5}$$

4.2 Link with the classical definition of viscosity solutions

Let us compare our definition with the classical notion of viscosity solution in the particular network $\mathcal{G} = J_1 \cup \{0\} \cup J_2 = (-1,1) \subset \mathbb{R}$ where $J_1 = (-1,0)$ and $J_2 = (0,1)$. Here $N = 2$. We denote by I the interval $[-1,1]$. We assume that A is some compact subset of $\mathbb{R}$. Note that from Assumption 2.2, we may say that for $x \in \mathcal{G}$, $A_x = A$ and that $\widetilde{f\ell}(0) = \overline{co}((f(0,a),\ell(O,a)), a \in A) = \mathrm{FL}(0)$. We aim at comparing the solutions in the sense of Definition 4.1 with the classical notion of constrained viscosity solution of the Hamilton-Jacobi equation

$$\lambda u + H(x, DU) = 0 \tag{4.6}$$

in $\overline{\mathcal{G}}$, with

$$H(x,p) = \sup_{a\in A}\{-f(x,a)\cdot p - \ell(x,a)\} = \max_{(\zeta,\xi)\in \mathrm{FL}(x)}\{-\zeta\cdot p - \xi\}. \tag{4.7}$$

It is useful to recall the notion of viscosity solutions in the sense of Dini, or minimax viscosity solutions, in the special context considered here:

Definition 4.2. *Let u be a continuous function defined on I. The lower Dini derivative at $x \in \mathcal{G}$ in the direction $q = \eta e_1$, $\eta \in \mathbb{R}$, is*

$$\partial^- u(x,q) = \liminf_{t\to 0^+} \frac{u(x+tq)-u(x)}{t}.$$

The upper Dini derivative at $x \in \mathcal{G}$ in the direction q is

$$\partial^+ u(x,q) = \limsup_{t\to 0^+} \frac{u(x+tq)-u(x)}{t}.$$

Remark 4.3. *Similarly, at $x = 1$, it is possible to define Dini lower and upper derivatives in the direction $q = \eta e_1$ with $\eta < 0$. At $x = -1$, it is possible to define Dini lower and upper derivatives in the direction $q = \eta e_1$ with $\eta > 0$.*

Definition 4.3. *Let $u \in \mathcal{C}(I)$. The function u is a Dini subsolution of (4.6) in $\mathcal{G}$ if*

$$\lambda u(x) + \sup_{(\zeta,\xi)\in \mathrm{FL}(x)} \left\{-\partial^+ u(x,\zeta) - \xi\right\} \le 0, \quad \forall x \in \mathcal{G}. \tag{4.8}$$

The function u is a Dini supersolution of (4.6) in I if

$$\lambda u(x) + \sup_{(\zeta,\xi)\in \mathrm{FL}(x)} \left\{-\partial^- u(x,\zeta) - \xi\right\} \ge 0, \quad \forall x \in I. \tag{4.9}$$

The function u is a constrained Dini solution of (4.6) if it is a Dini subsolution of (4.6) in $\mathcal{G}$ and a Dini supersolution of (4.6) in I.

Lemma 4.1. *If $u \in \mathcal{C}(I)$ is a Dini subsolution of (4.6) in $\mathcal{G}$, then it is a subsolution of (4.1) in $\mathcal{G}$ in the sense given by Definition 4.1. If $u \in \mathcal{C}(I)$ is a Dini supersolution of (4.6) in I, then it is a supersolution of (4.1) in I in the sense given by Definition 4.1.*

Proof. Assume that u is a Dini subsolution of (4.6) in $\mathcal{G}$. Let us focus on $x = 0$. Let $\varphi \in \mathcal{R}(\mathcal{G})$ be such that $u(0) = \varphi(0)$ and $u \le \varphi$ in $B_\delta(0)$. This implies that $D\varphi(0,\zeta) \ge \partial^+ u(0,\zeta)$, for all $\zeta \in \mathbb{R}e_1$. Therefore,

$$\lambda u(0) + \sup_{(\zeta,\xi)\in \mathrm{FL}(x)} \{-D\varphi(0,\zeta) - \xi\} \le \lambda u(0) + \sup_{(\zeta,\xi)\in \mathrm{FL}(x)} \{-\partial^+ u(0,\zeta) - \xi\} \le 0.$$

A similar argument can be used at $x \neq 0$. We have proved that u is subsolution of (4.6) in $\mathcal{G}$ in the sense given by Definition 4.1.
The second assertion of Lemma 4.1 is proved similarly. □

Lemma 4.2. *If u is a constrained viscosity solution of (4.1) on I in the sense of Definition 4.1, then u is a standard constrained viscosity solution of (4.6) on the interval I.*

Proof. In view of Remark 4.1, it is enough to test u at the origin 0.
Let $\varphi \in \mathcal{C}^1(I)$ be supertangent w.r.t. u at 0. We know that for any control $a \in A$, $(f(0,a), \ell(0,a)) \in \mathrm{FL}(0)$ and that $D\varphi(0)\cdot f(0,a) = D\varphi(0, f(0,a))$. Therefore

$$\sup_{a\in A}\{-D\varphi(0)\cdot f(0,a) - \ell(0,a)\} = \sup_{a\in A}\{-D\varphi(0,f(0,a)) - \ell(0,a)\} = \sup_{(\zeta,\xi)\in \mathrm{FL}(0)}\{-D\varphi(0,\zeta) - \xi\}.$$

Combining this and Definition 4.1, we get that

$$\lambda u(0) + \sup_{a\in A}\{-D\varphi(0)\cdot f(0,a) - \ell(0,a)\} \le 0.$$

Similarly, if $\varphi \in \mathcal{C}^1(I)$ is subtangent w.r.t. u at 0, then

$$\lambda u(0) + \sup_{a\in A}\{-D\varphi(0)\cdot f(0,a) - \ell(0,a)\} = \lambda u(0) + \sup_{(\zeta,\xi)\in \mathrm{FL}(0)} \{-D\varphi(0,\zeta) - \xi\} \geq 0.$$

$\square$

Proposition 4.1. *A function $u \in \mathcal{C}(I)$ is a constrained viscosity solution of (4.1) in the sense of Definition 4.1 if and only if it is a standard constrained viscosity solution of (4.6) on I.*

Proof. From Lemma 4.2, we know that a constrained viscosity solution of (4.1) in the sense of Definition 4.1 is a standard constrained viscosity solution of (4.6).
We have to prove the converse implication. In view of Remark 4.1, the two notions may differ only at the point $x_0 = 0$, so in particular, we need not consider the endpoints ± 1 of I.
From [1], Theorem 2.40 page 128, a function $u \in \mathcal{C}(I)$ is a viscosity subsolution (supersolution) of (4.6) in $\mathcal{G}$ if and only if it is a Dini subsolution (supersolution).
From this result and Lemma 4.1, we see that if $u \in \mathcal{C}(I)$ is a viscosity subsolution (supersolution) of (4.6) in $\mathcal{G}$, then it is a subsolution (supersolution) in the sense given by Definition 4.1.
$\square$

Remark 4.4. *The equivalence between viscosity and Dini solutions was first proved by P-L. Lions and P. Souganidis in [13, 14] for Lipschitz continuous functions. The use of Dini derivative for Hamilton-Jacobi equations goes back to Subbotin [20, 21] for Lipschitz functions, see the works of H. Frankowska [6, 7] for generalized versions.*

4.3 Existence

Theorem 4.1. *The value function v is a constrained viscosity solution of* (4.1) *in* $\overline{\mathcal{G}}$.

Proof. We recall that v satisfies the dynamic programming principle (2.17).

The value function v is a subsolution: it is enough to check that v is a subsolution at $x = O$. Let $\varphi \in \mathcal{R}(\mathcal{G})$ be such that $v - \varphi$ has a maximum point at O, i.e.

$$v(O) - v(z) \geq \varphi(O) - \varphi(z) \qquad \forall z \in B(O,r)\cap \mathcal{G}.$$

For $(\zeta,\xi) \in \widetilde{f\ell}(O)$, there exists $\alpha_n \in \mathcal{A}_O$ and $t_n \to 0^+$ such that

$$\begin{aligned}
\zeta &= \lim_{n\to\infty} \frac{y(t_n; O, \alpha_n)}{t_n} = \lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n} f(y(t;O,\alpha_n),\alpha_n(t))dt,\\
\xi &= \lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n} \ell(y(t;O,\alpha_n),\alpha_n(t))dt.
\end{aligned}$$

Take $T > 0$ such that $y(t) = y(t; O, \alpha) \in B(O,r)\cap\mathcal{G}$ for any $t \leq T$ and all $\alpha \in \mathcal{A}_O$. From (2.17)

$$\begin{aligned}
&\varphi(O) - \varphi(y(t;O,\alpha_n))\\
\leq& v(O) - v(y(t;O,\alpha_n)) \leq \int_0^t \ell(y(s;O,\alpha_n),\alpha_n(s))e^{-\lambda s}ds + v(y(t;O,\alpha_n))(e^{-\lambda t}-1).
\end{aligned}$$

By (3.1),

$$-D\varphi(O,\zeta) = \lim_{n\to\infty} \frac{\varphi(O) - \varphi(t_n\zeta)}{t_n}.$$

Since $t_n\zeta = y(t_n; O, \alpha_n) + o(t_n)$ and φ is Lipschitz continuous, we deduce that

$$-D\varphi(O,\zeta) = \lim_{n\to\infty} \frac{\varphi(O) - \varphi(y(t_n;O,\alpha_n))}{t_n}.$$

On the other hand,

$$\lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} \ell(y(s;O,\alpha_n),\alpha_n(s))e^{-\lambda s}ds = \xi.$$

Therefore

$$-D\varphi(O,\zeta) - \xi \le \lim_{n\to\infty} \frac{1}{t_n} \big(v(y(t_n;O,\alpha_n))(e^{-\lambda t_n}-1)\big) = -\lambda v(O).$$

Since the latter holds for any $(\zeta,\xi) \in \widetilde{f\ell}(O)$, we conclude that v is a subsolution at $x = O$.

The value function v is a supersolution Let $\varphi \in \mathcal{R}(\mathcal{G})$ be such that $v - \varphi$ has a minimum point at O, i.e.

$$v(O) - v(z) \le \varphi(O) - \varphi(z) \qquad \forall z \in B(O,r) \cap \mathcal{G}.$$

We can always assume that $\varphi(O) = v(O)$ and $v(z) \ge \varphi(z)$, $\forall z \in B(O,r) \cap \mathcal{G}$. From (2.17), for $\varepsilon > 0$ and $t > 0$, there exists $\alpha \in \mathcal{A}_O$ (depending on ε and t) such that

$$\begin{aligned} v(O) + t\varepsilon &\ge \int_0^t \ell(y(s;O,\alpha),\alpha(s))e^{-\lambda s}ds + e^{-\lambda t}v(y(t;O,\alpha)) \\ &\ge \int_0^t \ell(y(s;O,\alpha),\alpha(s))ds + e^{-\lambda t}v(y(t;0,\alpha)) + o(t), \end{aligned}$$

from the the continuity of ℓ.
For t sufficiently small, we get

$$\varphi(O) - \varphi(y(t;O,\alpha_n)) - \int_0^t \ell(y(s;O,\alpha),\alpha(s))ds + (1-e^{-\lambda t})\varphi(y(t;O,\alpha_n)) \ge -t\varepsilon + o(t).$$

There exist sequences $t_n \to 0$ and $\alpha_n \in \mathcal{A}_O$, ζ and ξ such that $\zeta = \lim_{n\to\infty} \frac{y(t_n,O,\alpha_n)}{t_n}$ and $\xi = \lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n} \ell(y(s;O,\alpha_n),\alpha_n(s))ds$ hence $(\zeta,\xi) \in \widetilde{f\ell}(O) \subset T_O(\mathcal{G}) \times \mathbb{R}$.
We clearly have

$$\frac{\varphi(O) - \varphi(y(t_n;O,\alpha_n))}{t_n} - \frac{1}{t_n}\int_0^{t_n} \ell(y(s;O,\alpha_n),\alpha_n(s))ds + \frac{(1-e^{-\lambda t_n})}{t_n}\varphi(y(t_n;0,\alpha_n)) \ge -\varepsilon + o(1).$$

But, as above, $\lim_{n\to\infty} \frac{\varphi(O)-\varphi(y(t_n;O,\alpha_n))}{t_n} = -D\varphi(O,\zeta)$. Therefore,

$$\lambda v(O) + \sup_{(\eta,\mu)\in\widetilde{f\ell}(O)} \{-D\varphi(O,\eta) - \mu\} \ge \lambda v(O) - D\varphi(O,\zeta) - \xi \ge -\varepsilon.$$

From the arbitrariness of ε, we get that

$$\lambda v(O) + \sup_{(\eta,\mu)\in\widetilde{f\ell}(O)} \{-D\varphi(O,\eta) - \mu\} \ge 0.$$

We conclude that v is a supersolution at $x = O$. □

5 Comparison principle

5.1 The case when the running cost does not depend on a

Here we assume that the running cost does not depend on a, so Assumption 3.1 is automatically satisfied.
The arguments in the proof of Proposition 3.1 yield the following:

$$\tilde f(x) = \quad F(x) \quad \equiv \overline{\mathrm{co}}\,(f(x,a) : a \in A_x), \qquad \text{if } x \in \mathcal{G}\backslash\{O\}, \tag{5.1}$$

$$\tilde f(O) = \quad F(O) \quad \equiv \bigcup_{j=1}^N \big(\overline{\mathrm{co}}\,\big(f(O,a) : a \in A^j\big) \cap \mathbb{R}^+ e_j\big) = \bigcup_{j=1}^N [0,\overline{\zeta}_j]e_j, \tag{5.2}$$

$$\tilde f(e_j) = \quad F(e_j) \quad \equiv \overline{\mathrm{co}}\,\big(f(e_j,a) : a \in A^j\big) \cap \mathbb{R}^- e_j, \qquad j = 1,\dots,N. \tag{5.3}$$

It is also easy to check that (4.2) is equivalent to

$$\lambda u(x) + \sup_{\zeta \in F(x)} \{-D\varphi(x,\zeta)\} - \ell(x) \le 0, \tag{5.4}$$

and that (4.3) is equivalent to

$$\lambda u(x) + \sup_{\zeta \in F(x)} \{-D\varphi(x,\zeta)\} - \ell(x) \ge 0. \tag{5.5}$$

We define the geodetic distance on $\overline{\mathcal{G}}$ by

$$d(x,y) = \begin{cases} |x-y| & \text{if } x,y \in \overline{J_j}, j = 1,\dots,N, \\ |x|+|y| & \text{if } x \in \overline{J_i}, y \in \overline{J_j}, i \neq j, \end{cases}$$

and the modified geodetic distance $\widetilde{d}(x,y)$:

$$\begin{array}{rcll} \widetilde{d}(x,y) & = & |x-y|/\overline{\zeta_i}, & \text{if } x,y \in \overline{\mathcal{G}} \cap \mathbb{R}e_i, \\ \widetilde{d}(x,y) & = & |x|/\overline{\zeta_i} + |y|/\overline{\zeta_j}, & \text{if } x \in \overline{J_i} \text{ and } y \in \overline{J_j}. \end{array} \tag{5.6}$$

For the comparison theorem, we need an easy lemma:

Lemma 5.1. *For any $y \in \mathcal{G}$, the functions $x \mapsto d(x,y)$ and $x \mapsto \widetilde{d}(x,y)$ are admissible test functions. For any $x \in \mathcal{G}$, the functions $y \mapsto d(x,y)$ and $y \mapsto \widetilde{d}(x,y)$ are admissible test functions.*

For the comparison theorem, we also need a further assumption:

Assumption 5.1. *The positive constants $\underline{\zeta}_j$ and $\overline{\zeta_j}$ in (2.12) are such that*

$$\overline{\zeta_j} \ge \underline{\zeta}_j > 0, \quad \forall j = 1,\dots,N. \tag{5.7}$$

Remark 5.1. *Note that if $e_j = -e_k$ then Assumption 5.1 implies that $\underline{\zeta_j} = \overline{\zeta_j} = \underline{\zeta_k} = \overline{\zeta_k}$. Indeed, from Remark 2.5, we know that $\overline{\zeta_j} = \underline{\zeta_k}$ and $\overline{\zeta_k} = \underline{\zeta_j}$. Then (5.7) implies $\overline{\zeta_j} \ge \underline{\zeta_j} = \overline{\zeta_k}$ and $\overline{\zeta_k} \ge \underline{\zeta_k} = \overline{\zeta_j}$, which yields the claim.*
In particular, if $\mathcal{G} = -\mathcal{G}$ and $A = -A$, then (2.12), (5.7) and the continuity of f imply that for all j, $\underline{\zeta_j} = \overline{\zeta_j}$. For example, take the cross-shaped network as follows: $N = 4$, $J_1 = (0, e_1)$, $J_2 = (0, e_2)$, $J_3 = (0, -e_1)$, $J_4 = (0, -e_2)$. We have $\overline{\zeta_1} = \underline{\zeta_3} = \overline{\zeta_3} = \underline{\zeta_1}$ and $\overline{\zeta_2} = \underline{\zeta_4} = \overline{\zeta_4} = \underline{\zeta_2}$.

Theorem 5.1 (Comparison principle)**.** *We assume that $\ell(x,a)$ does not depend on a. Under all the assumptions made in § 2 and Assumption 5.1, if u and v are respectively a subsolution of* (4.1) *in $\mathcal{G}$ and a supersolution of* (4.1) *in $\mathcal{G}$ such that*

$$u \le v \qquad \text{on } \partial\mathcal{G}, \tag{5.8}$$

then $u \le v$ in $\overline{\mathcal{G}}$.

Proof. We use the standard argument consisting of doubling the variables, see [1] page 292. Note that $u - v$ is bounded and upper semi-continuous on $\overline{\mathcal{G}}$.
We assume by contradiction that there exist $x_0 \in \overline{\mathcal{G}}$, $\chi > 0$ such that

$$u(x_0) - v(x_0) = \max_{\overline{\mathcal{G}}}(u - v) = \chi, \tag{5.9}$$

and we consider

$$\Phi_\varepsilon(x,y) = u(x) - v(y) - \frac{\widetilde{d}^2(x,y)}{2\varepsilon}, \qquad x,y \in \mathcal{G}.$$

Let $(x_\varepsilon, y_\varepsilon)$ be a maximum point of Φ_ε; we have

$$\chi = \Phi_\varepsilon(x_0,x_0) \le \Phi_\varepsilon(x_\varepsilon, y_\varepsilon). \tag{5.10}$$

From $\Phi_\varepsilon(x_\varepsilon, x_\varepsilon) \le \Phi_\varepsilon(x_\varepsilon, y_\varepsilon)$, we get $\frac{\widetilde{d}^2(x_\varepsilon,y_\varepsilon)}{2\varepsilon} \le v(x_\varepsilon) - v(y_\varepsilon)$ and since v is bounded,

$$\widetilde{d}(x_\varepsilon, y_\varepsilon) \le C\sqrt{\varepsilon}. \tag{5.11}$$

Hence $x_\varepsilon, y_\varepsilon$ converge for $\varepsilon \to 0$ to a point $\overline{x}$ and, by (5.8), $\overline{x} \in \mathcal{G}$. Therefore we can assume that for ε sufficiently small, $x_\varepsilon, y_\varepsilon \in \mathcal{G}$ and, by standard arguments, we can prove that

$$\lim_{\varepsilon \to 0} \frac{\widetilde{d}^2(x_\varepsilon, y_\varepsilon)}{2\varepsilon} = 0.$$

Moreover, $x \mapsto u(x) - (v(y_\varepsilon) + \frac{\widetilde{d}^2(x,y_\varepsilon)}{2\varepsilon})$ has a maximum point at x_ε and by Lemma 5.1,

$$\lambda u(x_\varepsilon) + \sup_{\zeta \in \tilde{f}(x_\varepsilon)} \left\{ D\left(x \mapsto \frac{\widetilde{d}^2(x, y_\varepsilon)}{2\varepsilon} \right)(x_\varepsilon, \zeta) \right\} - \ell(x_\varepsilon) \le 0. \tag{5.12}$$

Similarly, $y \mapsto v(y) - (u(x_\varepsilon) - \frac{\widetilde{d}^2(x_\varepsilon,y)}{2\varepsilon})$ has a minimum at y_ε and by Lemma 5.1,

$$\lambda v(y_\varepsilon) + \sup_{\zeta \in \tilde{f}(y_\varepsilon)} \left\{ D\left(y \mapsto -\frac{\widetilde{d}^2(x_\varepsilon, y)}{2\varepsilon} \right)(y_\varepsilon, \zeta) \right\} - \ell(y_\varepsilon) \ge 0. \tag{5.13}$$

If $x_\varepsilon = y_\varepsilon$, subtracting (5.13) from (5.12) we get

$$\lambda(u(x_\varepsilon) - v(x_\varepsilon)) \le 0,$$

and letting $\varepsilon \to 0$, we obtain the contradiction $\chi \le 0$. Hence we can assume $x_\varepsilon \ne y_\varepsilon$.

1^{st} **case: $x_\varepsilon \ne O$, $y_\varepsilon \ne O$:** From (5.12) and (5.13), taking into account Remark 4.1, we get

$$\begin{aligned}\lambda(u(x_\varepsilon) - v(y_\varepsilon)) \le &- \sup_{a \in A_{x_\varepsilon}} \left\{ D\left(x \mapsto \frac{\widetilde{d}^2(x, y_\varepsilon)}{2\varepsilon} \right)(x_\varepsilon, f(x_\varepsilon, a)) \right\} \\ &+ \sup_{a \in A_{y_\varepsilon}} \left\{ D\left(y \mapsto -\frac{\widetilde{d}^2(x_\varepsilon, y)}{2\varepsilon} \right)(y_\varepsilon, f(y_\varepsilon, a)) \right\} + \ell(x_\varepsilon) - \ell(y_\varepsilon).\end{aligned} \tag{5.14}$$

- If x_ε, y_ε are on the same edge, for example, $x_\varepsilon \in \bar{J}_1$ and $y_\varepsilon \in \bar{J}_1$, then $\widetilde{d}^2(x_\varepsilon, y_\varepsilon) = |x_\varepsilon - y_\varepsilon|^2 / \overline{\zeta_1}^2$, hence by (5.14), (2.3), (2.9) and (2.14),

$$\begin{aligned}&\lambda(u(x_\varepsilon) - v(y_\varepsilon)) \\ \le& \frac{\widetilde{d}(x_\varepsilon, y_\varepsilon)}{\overline{\zeta}_1 \varepsilon} \left(- \sup_{a \in A_{x_\varepsilon}} \left\{ \frac{x_\varepsilon - y_\varepsilon}{|x_\varepsilon - y_\varepsilon|} \cdot f(x_\varepsilon, a) \right\} + \sup_{a \in A_{y_\varepsilon}} \left\{ \frac{x_\varepsilon - y_\varepsilon}{|x_\varepsilon - y_\varepsilon|} \cdot f(y_\varepsilon, a) \right\} \right) \\ &+ \ell(x_\varepsilon) - \ell(y_\varepsilon) \\ \le& L \frac{\widetilde{d}^2(x_\varepsilon, y_\varepsilon)}{\varepsilon} + L|x_\varepsilon - y_\varepsilon|,\end{aligned} \tag{5.15}$$

  (note that $(x_\varepsilon - y_\varepsilon)/|x_\varepsilon - y_\varepsilon| \in T_{x_\varepsilon}(\mathcal{G}) = T_{y_\varepsilon}(\mathcal{G})$), which yields the desired contradiction by having ε tend to 0.
- If x_ε, y_ε are not on the same edge, for example $x_\varepsilon \in \overline{J_1} \backslash \{O\}$ and $y_\varepsilon \in \overline{J_2} \backslash \{O\}$ then $\widetilde{d}^2(x_\varepsilon, y_\varepsilon) = (|x_\varepsilon| / \overline{\zeta_1} + |y_\varepsilon| / \overline{\zeta_2})^2$, hence by (5.14)

$$\begin{aligned}\lambda(u(x_\varepsilon) - v(y_\varepsilon)) \le \frac{\widetilde{d}(x_\varepsilon, y_\varepsilon)}{\varepsilon} &\left(-\frac{1}{\overline{\zeta}_1} \sup_{a \in A_{x_\varepsilon}} \left\{ \frac{x_\varepsilon}{|x_\varepsilon|} \cdot f(x_\varepsilon, a) \right\} + \frac{1}{\overline{\zeta}_2} \sup_{a \in A_{y_\varepsilon}} \left\{ -\frac{y_\varepsilon}{|y_\varepsilon|} \cdot f(y_\varepsilon, a) \right\} \right) \\ &+ \ell(x_\varepsilon) - \ell(y_\varepsilon),\end{aligned} \tag{5.16}$$

(note that $x_\varepsilon/|x_\varepsilon| \in T_{x_\varepsilon}(\mathcal{G})$ and $y_\varepsilon/|y_\varepsilon| \in T_{y_\varepsilon}(\mathcal{G})$). From (2.3), we get

$$\begin{aligned}\lambda(u(x_\varepsilon)-v(y_\varepsilon)) \le \frac{\widetilde{d}(x_\varepsilon,y_\varepsilon)}{\varepsilon}\left(-\frac{1}{\overline{\zeta_1}}\sup_{a\in A_{x_\varepsilon}}\left\{\frac{x_\varepsilon}{|x_\varepsilon|}\cdot f(O,a)\right\}+\frac{1}{\overline{\zeta_2}}\sup_{a\in A_{y_\varepsilon}}\left\{-\frac{y_\varepsilon}{|y_\varepsilon|}\cdot f(O,a)\right\}\right)\\ +\ell(x_\varepsilon)-\ell(y_\varepsilon)+L\frac{\widetilde{d}^2(x_\varepsilon,y_\varepsilon)}{\varepsilon}.\end{aligned} \tag{5.17}$$

From (2.9) and (2.12),

$$-\frac{1}{\overline{\zeta_1}}\sup_{a\in A_{x_\varepsilon}}\left\{\frac{x_\varepsilon}{|x_\varepsilon|}\cdot f(O,a)\right\}+\frac{1}{\overline{\zeta_2}}\sup_{a\in A_{y_\varepsilon}}\left\{-\frac{y_\varepsilon}{|y_\varepsilon|}\cdot f(O,a)\right\}=-1+\underline{\zeta_2}/\overline{\zeta_2}\le 0,$$

and we obtain the desired contradiction from (5.17) and (2.14).

2^{nd} **case: either $x_\varepsilon = O$ and $y_\varepsilon \neq O$ or $x_\varepsilon \neq O$ and $y_\varepsilon = O$:** Assume $x_\varepsilon = O$ and $y_\varepsilon \neq O$ for example $y_\varepsilon \in J_2\backslash\{O\}$ (we proceed similarly in the other cases). Take $\zeta \in \tilde{f}(O)$ where $\tilde{f}(O)$ is given by (5.2). We know that $\overline{\mathrm{co}}\left(f(O,a) : a \in A^j\right)$ is contained in $\mathbb{R}e_j$; therefore, $\delta(\zeta) \equiv D\{x \mapsto \widetilde{d}(x,y_\varepsilon)\}(O,\zeta) = -\frac{y_\varepsilon}{\overline{\zeta_2}|y_\varepsilon|}\cdot\zeta$ if ζ is aligned with e_2 or $\delta(\zeta) = |\zeta|/\overline{\zeta_j}$ if $\zeta \in \tilde{f}(O)\cap\mathbb{R}e_j$ is not aligned with e_2.
From (5.12) and (5.13), we get

$$\begin{aligned}\lambda(u(O)-v(y_\varepsilon)) \le \frac{\widetilde{d}(O,y_\varepsilon)}{\varepsilon}\left(-\sup_{\zeta\in\tilde{f}(O)}\{\delta(\zeta)\}+\sup_{a\in A_{y_\varepsilon}}\left\{-\frac{y_\varepsilon}{\overline{\zeta_2}|y_\varepsilon|}\cdot f(y_\varepsilon,a)\right\}\right)\\ +\ell(O)-\ell(y_\varepsilon).\end{aligned} \tag{5.18}$$

From (2.3), we get that

$$\begin{aligned}\lambda(u(O)-v(y_\varepsilon)) \le \frac{\widetilde{d}(O,y_\varepsilon)}{\varepsilon}\left(-\sup_{\zeta\in\tilde{f}(O)}\{\delta(\zeta)\}+\sup_{a\in A_{y_\varepsilon}}\left\{-\frac{y_\varepsilon}{\overline{\zeta_2}|y_\varepsilon|}\cdot f(O,a)\right\}\right)\\ +\ell(O)-\ell(y_\varepsilon)+L\frac{\widetilde{d}^2(O,y_\varepsilon)}{\varepsilon}.\end{aligned} \tag{5.19}$$

Thus, from (5.2), we get that

$$\begin{aligned}-\sup_{\zeta\in\tilde{f}(O)}\{\delta(\zeta)\}+\sup_{a\in A_{y_\varepsilon}}\left\{-\frac{y_\varepsilon}{\overline{\zeta_2}|y_\varepsilon|}\cdot f(O,a)\right\} &= -\max_{j=1,\dots,N}\ \max_{\zeta\in[0,\overline{\zeta}_j]e_j}\delta(\zeta)+\sup_{a\in A^2}\left\{-\frac{e_2\cdot f(O,a)}{\overline{\zeta_2}}\right\}\\ &= -\max_{j=1,\dots,N}\ \max_{\zeta\in[0,\overline{\zeta}_j]e_j}\delta(\zeta)+\underline{\zeta}_2/\overline{\zeta_2}\\ &= -1+\underline{\zeta}_2/\overline{\zeta_2}\le 0,\end{aligned}$$

which, with (2.14), yields the desired contradiction. □

Theorem 5.2. *We assume that $\ell(x,a)$ does not depend on a. Under all the assumptions made in § 2 and Assumption 5.1, if u and v are respectively a subsolution of* (4.1) *in $\mathcal{G}$ and a supersolution of* (4.1) *in $\overline{\mathcal{G}}$ then $u \le v$ in $\overline{\mathcal{G}}$.*

Proof. The proof resembles that of Theorem 5.1, with more technicalities near $\partial\mathcal{V}$, see [1], page 278. We skip it for brevity. □

5.2 A case when the running cost depends on a

We consider a particular case when the running cost depends on a; a more general setting will be studied in a forthcoming paper.
Here we further assume that A is the unit ball of $\mathbb{R}^2$, that

$$A^j = [-1,1]e_j, \quad j = 1,\dots,N; \tag{5.20}$$

and that

$$f(O,a) = \sum_{j=1}^{N} \frac{\mu_j}{c_j} 1_{a\in\mathbb{R}e_j} a, \qquad \text{for } a \in A, \tag{5.21}$$

with

- $\mu_j > 0$, $j = 1,\dots,N$,
- $\mu_j = \mu_k$ and $c_j = c_k = 2$ if $e_j = -e_k$, $j \neq k \in \{1,\dots,N\}$,
- $c_j = 1$ if $e_j \neq -e_k$ $\forall k \neq j$.

We easily obtain that

$$A_O = \cup_{j=1}^{N}[0,1]e_j, \quad \text{and} \quad \tilde{f}(O) = \cup_{j=1}^{N}\mu_j[0,1]e_j = f(O,A_O), \tag{5.22}$$

and that Assumption 2.4 is satisfied with $\overline{\zeta_j} = \underline{\zeta_j} = \mu_j$, $j = 1,\dots,N$. We also assume that

$$\ell(O,a) = \mathcal{L}(a\cdot e_j), \qquad \text{for } a \in A^j, \tag{5.23}$$

where $\mathcal{L}$ is a smooth and convex function defined on $[-1,1]$ such that $0 = \mathcal{L}(0) \le \mathcal{L}(t)$ for all $t \in [-1,1]$. Note that $\mathcal{L}$ must be even if there exists j,k such that $e_k = -e_j$. Assumption 3.1 is clearly satisfied.
We see that for all function $\varphi \in \mathcal{R}(\mathcal{G})$,

$$\max_{(\zeta,\xi)\in \mathrm{FL}(O)} \{-D\varphi(O,\zeta) - \xi\} = \max_{j=1,\dots,N} \max_{(\zeta,\xi)\in \mathrm{FL}_j(O)} \{-D\varphi(O,\zeta) - \xi\}$$

where

$$\mathrm{FL}_j(O) = \overline{\mathrm{co}}\Big((\mu_j t e_j, \mathcal{L}(t)) : t\in[-1,1] \Big) \cap (\mathbb{R}_+ e_j \cap \mathbb{R})$$

But since $\mathrm{FL}_j(O) \subset \mathbb{R}_+ e_j \cap \mathbb{R}$, the function $(\zeta,\xi) \mapsto -D\varphi(O,\zeta) - \xi$ is linear on $\mathrm{FL}_j(O)$: indeed, it coincides with the linear function $-D\varphi_{|\overline{J_j}}(O)\cdot\zeta - \xi$. Thus,

$$\max_{(\zeta,\xi)\in \mathrm{FL}(O)} \{-D\varphi(O,\zeta) - \xi\} = \max_{j=1,\dots,N} \max_{(\zeta,\xi)\in \mathrm{FL}_j(O)} \{-D\varphi_{|\overline{J_j}}(O)\cdot\zeta - \xi\}$$

and, from the convexity of $\mathcal{L}$, we have that

$$\begin{aligned}\max_{(\zeta,\xi)\in \mathrm{FL}(O)} \{-D\varphi(O,\zeta) - \xi\} &= \max_{j=1,\dots,N} \max_{t\in[0,1]} \{-D\varphi_{|\overline{J_j}}(O)\cdot \mu_j t e_j - \mathcal{L}(t)\} \\ &= \max_{a\in A_O} \{-D\varphi(O, f(O,a)) - \ell(O,a)\}.\end{aligned}$$

Therefore, (4.2) is equivalent to

$$\lambda u(x) + \sup_{a\in A_x} \{-D\varphi(x, f(x,a)) - \ell(x,a)\} \le 0,$$

and (4.3) is equivalent to

$$\lambda u(x) + \sup_{a\in A_x} \{-D\varphi(x, f(x,a)) - \ell(x,a)\} \ge 0.$$

The following Assumption plays a role similar to Assumption 5.1 in § 5.1:

Assumption 5.2. *We assume that the Legendre transform of $\mathcal{L}$ defined by*

$$\mathcal{L}^*(\delta) = \max_{\alpha\in[-1,1]} \{\delta\alpha - \mathcal{L}(\alpha)\} \tag{5.24}$$

satisfies

$$\mathcal{L}^*(\delta) \geq \mathcal{L}^*(-\delta), \qquad \forall \delta \geq 0, \tag{5.25}$$

and that

$$\text{if } \delta \geq 0, \text{ then the maximum in (5.24) is reached in } [0,1]. \tag{5.26}$$

Example 5.1. *The function $\mathcal{L}(t) = |t|^\nu$, $\nu > 1$ satisfies Assumption 5.2.*

Theorem 5.3 (Comparison principle)**.** *With the assumptions made above, if u and v are respectively a subsolution of (4.1) in $\mathcal{G}$ and a supersolution of (4.1) in $\mathcal{G}$ such that (5.8) holds, then $u \leq v$ in $\overline{\mathcal{G}}$.*

Proof. We assume by contradiction that there exist $x_0 \in \overline{\mathcal{G}}$, $\chi > 0$ such that $u(x_0) - v(x_0) = \max_{\overline{\mathcal{G}}}(u - v) = \chi$, and we consider

$$\Phi_\varepsilon(x,y) = u(x) - v(y) - \frac{\widetilde{d}^2(x,y)}{2\varepsilon}, \qquad x, y \in \mathcal{G},$$

where $\widetilde{d}$ is defined by (5.6) with $\overline{\zeta_j} = \mu_j$. Let $(x_\varepsilon, y_\varepsilon)$ be a maximum point of Φ_ε; we have $\chi = \Phi_\varepsilon(x_0, x_0) \leq \Phi_\varepsilon(x_\varepsilon, y_\varepsilon)$. From $\Phi_\varepsilon(x_\varepsilon, x_\varepsilon) \leq \Phi_\varepsilon(x_\varepsilon, y_\varepsilon)$, we get $\frac{\widetilde{d}^2(x_\varepsilon,y_\varepsilon)}{2\varepsilon} \leq v(x_\varepsilon) - v(y_\varepsilon)$ and since v is bounded, $\widetilde{d}(x_\varepsilon, y_\varepsilon) \leq C\sqrt{\varepsilon}$. Hence $x_\varepsilon, y_\varepsilon$ converge for $\varepsilon \to 0$ to a point $\overline{x}$ and, by (5.8), $\overline{x} \in \mathcal{G}$. Therefore we can assume that for ε sufficiently small, $x_\varepsilon, y_\varepsilon \in \mathcal{G}$ and, by standard arguments, we can prove that $\lim_{\varepsilon\to 0} \frac{\widetilde{d}^2(x_\varepsilon,y_\varepsilon)}{2\varepsilon} = 0$. Moreover, $x \mapsto u(x) - (v(y_\varepsilon) + \frac{\widetilde{d}^2(x,y_\varepsilon)}{2\varepsilon})$ has a maximum point at x_ε and by Lemma 5.1,

$$\lambda u(x_\varepsilon) + \sup_{a\in A_{x_\varepsilon}} \left\{ D\left(x \mapsto \frac{\widetilde{d}^2(x,y_\varepsilon)}{2\varepsilon}\right)(x_\varepsilon, f(x_\varepsilon,a)) - \ell(x_\varepsilon,a) \right\} \leq 0. \tag{5.27}$$

Similarly, $y \mapsto v(y) - (u(x_\varepsilon) - \frac{\widetilde{d}^2(x_\varepsilon,y)}{2\varepsilon})$ has a minimum at y_ε and by Lemma 5.1,

$$\lambda v(y_\varepsilon) + \sup_{a\in A_{y_\varepsilon}} \left\{ D\left(y \mapsto -\frac{\widetilde{d}^2(x_\varepsilon,y)}{2\varepsilon}\right)(y_\varepsilon, f(y_\varepsilon,a)) - \ell(y_\varepsilon,a) \right\} \geq 0. \tag{5.28}$$

If $x_\varepsilon = y_\varepsilon$, subtracting (5.28) from (5.27) we get

$$\lambda(u(x_\varepsilon) - v(x_\varepsilon)) \leq 0,$$

and letting $\varepsilon \to 0$, we obtain the contradiction $\chi \leq 0$. Hence we can assume $x_\varepsilon \neq y_\varepsilon$.

1^{st} case: $x_\varepsilon \neq O$, $y_\varepsilon \neq O$: From (5.27) and (5.28), we get

$$\begin{aligned}\lambda(u(x_\varepsilon) - v(y_\varepsilon)) \leq &- \sup_{a\in A_{x_\varepsilon}} \left\{ D\left(x \mapsto \frac{\widetilde{d}^2(x,y_\varepsilon)}{2\varepsilon}\right)(x_\varepsilon, f(x_\varepsilon,a)) - \ell(x_\varepsilon,a) \right\} \\ &+ \sup_{a\in A_{y_\varepsilon}} \left\{ D\left(y \mapsto -\frac{\widetilde{d}^2(x_\varepsilon,y)}{2\varepsilon}\right)(y_\varepsilon, f(y_\varepsilon,a)) - \ell(y_\varepsilon,a) \right\}.\end{aligned} \tag{5.29}$$

- If x_ε, y_ε are on the same edge, for example, $x_\varepsilon \in \bar{J}_1$ and $y_\varepsilon \in \bar{J}_1$, then $\widetilde{d}^2(x_\varepsilon, y_\varepsilon) = |x_\varepsilon - y_\varepsilon|^2/\mu_1^2$, hence by (5.29), (2.3), (2.9) and (2.14),

$$\begin{aligned}
&\lambda(u(x_\varepsilon) - v(y_\varepsilon)) \\
\le &\left(\begin{array}{l} - \sup_{a \in A_{x_\varepsilon}} \left\{ \dfrac{\widetilde{d}(x_\varepsilon, y_\varepsilon)}{\varepsilon \mu_1} \dfrac{x_\varepsilon - y_\varepsilon}{|x_\varepsilon - y_\varepsilon|} \cdot f(x_\varepsilon, a) - \ell(x_\varepsilon, a) \right\} \\ + \sup_{a \in A_{y_\varepsilon}} \left\{ \dfrac{\widetilde{d}(x_\varepsilon, y_\varepsilon)}{\varepsilon \mu_1} \dfrac{x_\varepsilon - y_\varepsilon}{|x_\varepsilon - y_\varepsilon|} \cdot f(y_\varepsilon, a) - \ell(y_\varepsilon, a) \right\} \end{array} \right) \\
\le & L \frac{\widetilde{d}^2(x_\varepsilon, y_\varepsilon)}{\varepsilon} + L|x_\varepsilon - y_\varepsilon|,
\end{aligned}$$

  (note that $(x_\varepsilon - y_\varepsilon)/|x_\varepsilon - y_\varepsilon| \in T_{x_\varepsilon}(\mathcal{G}) = T_{y_\varepsilon}(\mathcal{G})$), which yields the desired contradiction by having ε tend to 0.

- If x_ε, y_ε are not on the same edge, for example $x_\varepsilon \in \overline{J_1} \backslash \{O\}$ and $y_\varepsilon \in \overline{J_2} \backslash \{O\}$ then $\widetilde{d}^2(x_\varepsilon, y_\varepsilon) = (|x_\varepsilon|/\mu_1 + |y_\varepsilon|/\mu_2)^2$, hence by (5.29)

$$\lambda(u(x_\varepsilon) - v(y_\varepsilon)) \le \left(\begin{array}{l} - \sup_{a \in A_{x_\varepsilon}} \left\{ \dfrac{\widetilde{d}(x_\varepsilon, y_\varepsilon)}{\varepsilon \mu_1} \dfrac{x_\varepsilon}{|x_\varepsilon|} \cdot f(x_\varepsilon, a) - \ell(x_\varepsilon, a) \right\} \\ + \sup_{a \in A_{y_\varepsilon}} \left\{ -\dfrac{\widetilde{d}(x_\varepsilon, y_\varepsilon)}{\varepsilon \mu_2} \dfrac{y_\varepsilon}{|y_\varepsilon|} \cdot f(y_\varepsilon, a) - \ell(y_\varepsilon, a) \right\} \end{array} \right),$$

  (note that $x_\varepsilon/|x_\varepsilon| \in T_{x_\varepsilon}(\mathcal{G})$ and $y_\varepsilon/|y_\varepsilon| \in T_{y_\varepsilon}(\mathcal{G})$. From (2.3), we get

$$\begin{aligned}
\lambda(u(x_\varepsilon) - v(y_\varepsilon)) &\le \left(\begin{array}{l} - \sup_{a \in A_{x_\varepsilon}} \left\{ \dfrac{\widetilde{d}(x_\varepsilon, y_\varepsilon)}{\varepsilon \mu_1} \dfrac{x_\varepsilon}{|x_\varepsilon|} \cdot f(O, a) - \ell(O, a) \right\} \\ + \sup_{a \in A_{y_\varepsilon}} \left\{ -\dfrac{\widetilde{d}(x_\varepsilon, y_\varepsilon)}{\varepsilon \mu_2} \dfrac{y_\varepsilon}{|y_\varepsilon|} \cdot f(O, a) - \ell(O, a) \right\} \end{array} \right) + L \frac{\widetilde{d}^2(x_\varepsilon, y_\varepsilon)}{\varepsilon} \\
&= \left(\begin{array}{l} - \sup_{a \in [-1,1]e_1} \left\{ \dfrac{\widetilde{d}(x_\varepsilon, y_\varepsilon)}{\varepsilon} e_1 \cdot a - \ell(O, a) \right\} \\ + \sup_{a \in [-1,1]e_2} \left\{ -\dfrac{\widetilde{d}(x_\varepsilon, y_\varepsilon)}{\varepsilon} e_2 \cdot a - \ell(O, a) \right\} \end{array} \right) + L \frac{\widetilde{d}^2(x_\varepsilon, y_\varepsilon)}{\varepsilon} \\
&= -\mathcal{L}^* \left(\frac{\widetilde{d}(x_\varepsilon, y_\varepsilon)}{\varepsilon} \right) + \mathcal{L}^* \left(-\frac{\widetilde{d}(x_\varepsilon, y_\varepsilon)}{\varepsilon} \right) + L \frac{\widetilde{d}^2(x_\varepsilon, y_\varepsilon)}{\varepsilon},
\end{aligned}$$

  and we obtain the desired contradiction from (5.25).

2^{nd} case: either $x_\varepsilon = O$ and $y_\varepsilon \neq O$ or $x_\varepsilon \neq O$ and $y_\varepsilon = O$: Assume $x_\varepsilon = O$ and $y_\varepsilon \neq O$ for example $y_\varepsilon \in \overline{J_2} \backslash \{O\}$ (we proceed similarly in the other cases). For any $a \in A_O$, $\delta(a) \equiv D\{x \mapsto \widetilde{d}(x, y_\varepsilon)\}(O, f(O, a)) = -e_2 \cdot a$ if a is aligned with e_2 or $\delta(a) = |a|$ if $a \in A_O$ is not aligned with e_2.
From (5.27) and (5.28), we get

$$\lambda(u(O) - v(y_\varepsilon)) \le - \sup_{a \in A_O} \left\{ \frac{\widetilde{d}(O, y_\varepsilon)}{\varepsilon} \delta(a) - \ell(O, a) \right\} + \mathcal{L}^* \left(-\frac{\widetilde{d}(O, y_\varepsilon)}{\varepsilon} \right) + L \frac{\widetilde{d}^2(O, y_\varepsilon)}{\varepsilon},$$

and the desired contradiction follows, because

$$\sup_{a \in A_O} \left\{ \frac{\widetilde{d}(O, y_\varepsilon)}{\varepsilon} \delta(a) - \ell(O, a) \right\} = \mathcal{L}^* \left(\frac{\widetilde{d}(O, y_\varepsilon)}{\varepsilon} \right)$$

from (5.26). $\square$

Similarly, we have the following theorem:

Theorem 5.4. *With the same assumptions as in Theorem 5.3, if u and v are respectively a subsolution of (4.1) in $\mathcal{G}$ and a supersolution of (4.1) in $\overline{\mathcal{G}}$ then $u \le v$ in $\overline{\mathcal{G}}$.*

6 A case when the value function may be discontinuous

In this section, we keep all the assumptions made in § 2 except Assumption 2.4, which we replace by

Assumption 6.1. *We assume that for all $j \in \{1, \dots, N\}$,*

$$\max_{a\in A^j} \big(f(O,a)\cdot e_j\big)_+ > 0. \tag{6.1}$$

Remark 6.1. *With the other assumptions, Assumption 6.1 says that for any $j \in \{1, \dots, N\}$, one can choose a trajectory departing from O and visiting J_j.*

With this new set of assumptions, the value function v is continuous at $\overline{\mathcal{G}}\backslash\{O\}$, but the example given at the end of §2 shows that the continuity of v at O is not guaranteed.

Proposition 6.1. *Under the assumptions above, the value function v is LSC at O. If furthermore $\max_{a\in A^j}\big(f(O,a)\cdot e_j\big)_- > 0$, then $v_{|\overline{J_j}}$ is continuous.*

Proof. The proof is similar to that of Proposition 2.1. □

Hereafter we denote by v^* the upper semicontinuous envelope of v: $v^*(x) = \limsup_{y\to x} v(y)$. We now consider a generalized notion of constrained viscosity solution for the equation (4.1) in $\overline{\mathcal{G}}$, adapted to the present case:

Definition 6.1.
- *A bounded function $u : \overline{\mathcal{G}} \to \mathbb{R}$, continuous in $\mathcal{G}\backslash\{O\}$ is a subsolution of (4.1) in $\mathcal{G}$ if for any $\varphi \in \mathcal{R}(\mathcal{G})$ s.t. $u^* - \varphi$ has a local maximum point at O, then*

  $$\lambda u^*(O) + \min_j \lim_{\substack{z\to O\\ z\in J_j}} \sup_{\substack{a\in A^j\\ f(O,a)\in\mathbb{R}_+e_j}} \{-D\varphi(z)\cdot f(O,a) - \ell(O,a)\} \le 0, \tag{6.2}$$

  and if for any $\varphi \in \mathcal{R}(\mathcal{G})$ s.t. $u - \varphi$ has a local maximum point at $x \ne O$, then (5.4) holds.
- *A bounded and lower semicontinuous function $u : \overline{\mathcal{G}} \to \mathbb{R}$, continuous in $\mathcal{G}\backslash\{O\}$, is a constrained viscosity solution of (4.1) in $\overline{\mathcal{G}}$ if it is a viscosity subsolution of (4.1) in $\mathcal{G}$ in the latter sense and a supersolution of (4.1) in $\overline{\mathcal{G}}$ in the sense of Definition 4.1.*

Note that (6.2) has a meaning thank to Assumption 6.1.

Theorem 6.1. *The value function v is a constrained viscosity solution of (4.1) in $\overline{\mathcal{G}}$ in the sense of Definition 6.1.*

Proof.

The value function v is a subsolution: it is enough to check that v is a subsolution at $x = O$. We take $\varphi \in \mathcal{R}(\mathcal{G})$, such that $v^*(O) = \varphi(O)$ and $v^*(x) \le \varphi(x)$ for all $x \in \mathcal{G}$ near O. We assume by contradiction that

$$\lambda \varphi(O) + \min_j \lim_{\substack{z\to O\\ z\in J_j}} \sup_{\substack{a\in A^j\\ f(O,a)\in\mathbb{R}_+e_j}} \{-D\varphi(z)\cdot f(O,a) - \ell(O,a)\} > 0.$$

Then, from the continuity of φ and ℓ, there exists some $\varepsilon > 0$ such that $v^* \le \varphi$ in $B(0,\varepsilon)\cap\mathcal{G}$ and, for any $j = 1,\dots,N$,

$$\lambda\varphi(z) + \sup_{\substack{a \in A^j \\ f(O,a)\in\mathbb{R}_+e_j}} \{-D\varphi(z)\cdot f(O,a) - \ell(z,a)\} > \varepsilon, \qquad \forall z \in B(O,\varepsilon)\cap J_j.$$

This implies that there exists $\bar{a}_j \in A^j$ such that $f(O,\bar{a}_j) \in \mathbb{R}_+e_j$ and

$$\lambda\varphi(O) - (D\varphi)_{|\overline{J_j}}(O)\cdot f(O,\bar{a}_j) - \ell(O,\bar{a}_j) > \frac{3}{4}\varepsilon,$$

by using the continuity in $\overline{J_j}$ of all the involved functions. This yields that there exists $\eta(\varepsilon)$, $0 < \eta(\varepsilon) < \varepsilon$ such that

$$\lambda\varphi(z) - D\varphi(z)\cdot f(z,\bar{a}_j) - \ell(z,\bar{a}_j) > \frac{\varepsilon}{2}, \qquad \forall z \in B(O,\eta(\varepsilon))\cap J_j.$$

The fact that $f(O,\bar{a}_j) \in \mathbb{R}_+e_j$ and the Lipschitz continuity of f with respect to its first argument imply that

$$\big(f(x,\bar{a}_j)\cdot e_j\big)_- \le C|x|, \quad \forall x \in J_j, \tag{6.3}$$

so for all $z \in J_j$, $y(t;z,\bar{a}_j) \ne O$ for all $t \ge 0$ small enough. Thus, there exists $t > 0$ such that $y(s;x,\bar{a}_j) \in B(O,\eta(\varepsilon))\cap J_j$, for all $x \in B(O,\eta(\varepsilon)/2)\cap J_j$ and $s \le t$. From (2.17), we have that for any $z \in B(O,\eta(\varepsilon)/2)\cap J_j$,

$$\begin{aligned}
v(z) &\le \int_0^t \ell(y(s;z,\bar{a}_j),\bar{a}_j)e^{-\lambda s}ds + v(y(t;z,\bar{a}_j))e^{-\lambda t}\\
&\le \int_0^t \ell(y(s;z,\bar{a}_j),\bar{a}_j)e^{-\lambda s}ds + \varphi(y(t;z,\bar{a}_j))e^{-\lambda t}\\
&= \int_0^t \Big(\ell(y(s;z,\bar{a}_j),\bar{a}_j) + D\varphi(y(s;z,\bar{a}_j))\cdot f(y(s;z,\bar{a}_j),\bar{a}_j) - \lambda\varphi(y(s;z,\bar{a}_j))\Big)e^{-\lambda s}ds + \varphi(z).
\end{aligned}$$

Thus, for any $z \in B(O,\eta(\varepsilon)/2)\cap J_j$,

$$\begin{aligned}
v(z)-\varphi(z) &\le \int_0^t \Big(\ell(y(s;z,\bar{a}_j),\bar{a}_j) + D\varphi(y(s;z,\bar{a}_j))\cdot f(y(s;z,\bar{a}_j),\bar{a}_j) - \lambda\varphi(y(s;z,\bar{a}_j))\Big)e^{-\lambda s}ds\\
&\le -\frac{\varepsilon}{2\lambda}(1-e^{-\lambda t}),
\end{aligned}$$

which yields the desired contradiction, because $1 - e^{-\lambda t} > 0$.

The value function v is a supersolution: the proof is exactly the same as in the proof of Theorem 4.1. □

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