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Abstract

We consider continuous-state and continuous-time control problem where the admissible trajectories of the system are constrained to remain on a network. Under suitable assumptions, we prove that the value function is continuous. We define a notion of viscosity solution of Hamilton-Jacobi equations on the network for which we prove a comparison principle. The value function is thus the unique viscosity solution of the Hamilton-Jacobi equation on the network.

Keywords Optimal control, graphs, networks, Hamilton-Jacobi equations, viscosity solutions

AMS 34H05, 49J15

1 Introduction

A network (or a graph) is a set of items, refered to as vertices or nodes, with connections between them refered to as edges. The main tools for the study of networks come from combinatorics and graph theory. But in the recent years there is an increasing interest in the investigation of dynamical systems and differential equation on networks, in particular in connection with problem of data transmission and traffic management (see for example Garavello-Piccoli [10], Engel et al [5]). In this perspective, the study of control problem on networks have interesting applications in various fields.

A typical optimal control problem is the *minimum time problem*, which consists in finding the shortest path between an initial position and a given target set. If the running cost is a fixed constant for each edge and the dynamics can go from one vertex to an adjacent one in each time step, the corresponding discrete-state discrete-time control problem can be studied via graph theory and matrix analysis. If instead the cost changes in a continuous way along the edges and the dynamics is continuous in time, the minimum time problem can be seen as a continuous-state continuous-time control problem where the admissible trajectories of the system are constrained to remain on the network. While state constraint control problems in closures of open sets are well studied ([15, 16], [3], [11]) there is to our knowledge much fewer literature on problems in closed sets. The results of Frankowska and Plaskacz [9, 8] do apply to some closed sets with empty interior, but not to networks with crosspoints (except in very particular cases).

The aim of this paper is therefore to study optimal control problems with dynamic constrained to a network and the related Hamilton-Jacobi-Bellman equation. Note that other types of optimal control problems could be considered as well, leading to other boundary conditions at the endpoints of the network. In most of the paper, we will consider for simplicity the toy model given by a star-shaped network, i.e. straight edges intersecting at the

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origin. Moreover we will often assume that the running cost is independent of the control. This simple model problem already contains many of the difficulties that we have to face in more general situations.

Since the dynamic is constrained to the network, the velocities tangent to the network vary from one edge to the other, hence the set of the admissible controls depends on the state of the system. If the set of admissible controls varies in a continuous way, the corresponding control problem can be studied via standard viscosity solution techniques (see Koike[12]). But for a network, the set of admissible controls drastically changes from a point in the interior of an edge, where only one direction is admissible (with positive and negative velocities), to a vertex where the admissible directions are given by all the edges connected to the node. Therefore, even if the data of the problem are regular, the corresponding Hamiltonian when restricted to the network has a discontinuous structure. Problem with discontinuous Hamiltonians have been recently studied by various authors (Tourin[20], Soravia[17], Deckelnick-Elliott[4], Bressan-Hong[2]), but the approaches and the results considered in these papers do not seem to be applicable because of the particular structure of the considered domain.

Assuming that the set of the admissible control laws - i.e. the control laws for which the corresponding trajectory remain on the graph - is not empty, the control problem is well posed and the corresponding value function satisfies a dynamic programming principle. We introduce a first set of assumptions which guarantees that the value function is a continuous function on the network (with respect to the intrinsic geodetic distance).

The next step is to introduce a definition of weak solution which ensures the uniqueness of the continuous solution via a comparison theorem. While in the interior of an edge we can test the equation with a smooth test function, the main difficulties arise at the vertices where the network does not have a regular differential structure. At a vertex, we consider a concept of derivative similar to the one of Dini's derivative, see for example[1], hence regular test functions are the ones which admit derivatives in each directions of the edges entering in the node. We give a definition of viscosity solution on the network using the previous class of test functions. It is worthwhile to observe that this definition reduces to the classical one of viscosity solution if the graph is composed of two parallel segments entering in a node, see [1].

With this definition, the intrinsic geodetic distance, fixed one argument, is a regular function w.r.t. the other argument and it can be used in the comparison theorem as a penalization term in the classical doubling argument of viscosity solution theory.

We conclude observing that this paper is a first attempt to study Hamilton-Jacobi-Bellman equations and viscosity solutions on a network. Several points remain open such as more general control problems, problem with boundary conditions, stochastic control problem, etc...

2 Setting of the problem

We consider a planar network with a finite number of edges and vertices. A network in $\mathbb{R}^2$ is a pair $(\mathcal{V}, \mathcal{E})$ where

i) $\mathcal{V}$ is a finite subset of $\mathbb{R}^2$ whose elements are said vertices

ii) $\mathcal{E}$ is a finite set of regular arcs of $\mathbb{R}^2$, said edges, whose extrema are elements of $\mathcal{V}$.

We say that two vertices are adjacent if they are connected by an edge. We say that a vertex belongs to $\partial\mathcal{V}$ (resp., $int(\mathcal{V})$) if there is only one (resp., more than one) edge connected to it. We assume that the edges cross each other transversally. We denote by $\overline{\mathcal{G}}$ the union of all the edges in $\mathcal{E}$ and all the vertices in $\mathcal{V}$. We denote by $\mathcal{G}$ the set $\overline{\mathcal{G}}\backslash\partial\mathcal{V}$.
Unless explicitely mentioned, we focus for simplicity on the model case of a star-shaped network with N straight edges, $N > 1$, i.e.

$$\mathcal{G} = \{O\} \cup \bigcup_{j=1}^{N} J_j \subset \mathbb{R}^2, \quad O = (0,0), \quad J_j = (0,1)e_j, \tag{2.1}$$

where $(e_j)_{j=1,\dots,N}$ is a set of unit vectors in $\mathbb{R}^2$ s.t. $e_j \neq e_k$ if $j \neq k$. Note that $e_j = -e_k$ is possible. Then, $\partial\mathcal{V} = \{e_j, j = 1, \dots, N\}$ and $\mathrm{int}(\mathcal{V}) = \{O\}$. Except in § 4.2, we assume that there is at least a pair (j,k), $j \neq k$ s.t. e_j is not aligned with e_k.

The general case will be dealt with in a forthcoming paper, where we will also consider structures made of several manifolds of different dimensions crossing each other transversally. For any $x \in \mathcal{G}$, we denote by $T_x(\mathcal{G}) \subset \mathbb{R}^2$ the set of the tangent directions to the network, i.e.

$$p \in T_x(\mathcal{G}) \iff \text{there exists } \xi \in \mathcal{C}^1([0,T];\mathbb{R}^2) \text{ such that } \xi(t) \in \mathcal{G},\ \xi(0) = x \text{ and } \dot{\xi}(0) = p. \tag{2.2}$$

It is easy to prove that $p \in T_x(\mathcal{G})$ if and only if there exist sequences $(t_n)_{n\in\mathbb{N}}$, $t_n > 0$ and $(x_n)_{n\in\mathbb{N}}$, $x_n \in \mathcal{G}$, such that $t_n \to 0^+$ and $(x_n - x)/t_n \to p$.
We now introduce the optimal control problem on $\mathcal{G}$. We start by making some assumptions on the structure of the problem.
Call B the unit ball of $\mathbb{R}^2$ centered at O. Take for A a compact set of $\mathbb{R}^2$ containing 0, and a continuous function $f : B \times A \to \mathbb{R}^2$ such that

$$|f(x,a) - f(y,a)| \le L|x-y|, \quad \forall x, y \in B,\ a \in A. \tag{2.3}$$

The assumption (2.3) implies that there exists $M > 0$ such that

$$|f(x,a)| \le M, \quad \forall x \in B,\ a \in A. \tag{2.4}$$

Additional assumptions will be made below. For $x \in \overline{\mathcal{G}}$, we consider the dynamical system

$$\begin{cases} \dot{y}(t;x,\alpha) = f(y(t;x,\alpha),\alpha(t)), & t > 0, \\ y(0) = x. \end{cases} \tag{2.5}$$

Remark 2.1. *We have chosen to parametrize the dynamics by a function f defined on $B \times A$, i.e. on a much larger set than $\mathcal{G} \times A$. We could have also defined f on $\mathcal{G} \times A$ only. This would have been equivalent since by Whitney extension theorem one can extend any Lipschitz function defined on $\mathcal{G}$ to a Lipschitz function defined on B. In fact, all the assumptions made below on f involve $f|_{\mathcal{G}\times A}$ only. Yet, it seemed to us that defining f on $B \times A$ led to simpler notations.*

Denoting by $\mathcal{A}$ the class of the control laws, i.e. the set of measurable functions from $[0,+\infty)$ to A, we introduce the subset $\mathcal{A}_x \subset \mathcal{A}$ of the admissible control laws, i.e. the control laws for which the dynamics (2.5) is constrained on the network $\mathcal{G}$:

$$\mathcal{A}_x = \{\alpha \in \mathcal{A} :\ y(t;x,\alpha) \in \mathcal{G}, \quad \forall t > 0\}. \tag{2.6}$$

Assumption 2.1.

$$\mathcal{A}_x \text{ is not empty for any } x \in \overline{\mathcal{G}}. \tag{2.7}$$

We will always consider $\alpha \in \mathcal{A}_x$ in (2.5).
We also define for $x \in \overline{\mathcal{G}}$,

$$A_x = \{a \in A \text{ s.t. } \exists \theta > 0 : y(t;x,a) \in \mathcal{G}, \forall t, 0 < t < \theta\}. \tag{2.8}$$

From (2.3), we see that for all $a \in A_x$, $f(x,a) \in T_x(\mathcal{G})$.

Assumption 2.2.

$$A_x = A \cap \mathbb{R}e_j, \qquad \text{if } x \in J_j,\ j = 1,\dots,N, \tag{2.9}$$

$$A_O = \cup_{j=1}^N \{a \in A \cap \mathbb{R}e_j : f(O,a) \in \mathbb{R}^+ e_j\}, \tag{2.10}$$

$$A_{e_j} \subset A \cap \mathbb{R}e_j, \quad \text{and} \quad \inf_{a \in A_{e_j}} f(e_j,a)\cdot e_j < 0, \qquad j = 1,\dots,N. \tag{2.11}$$

Remark 2.2. *The set $A \cap \mathbb{R}e_j$ is not empty since it contains* 0.

Remark 2.3. *The continuity of f and (2.9) imply that $f(O,0) = 0$. Indeed, assuming that e_1 and e_2 are not aligned, take $x_n \to O$, $x_n \in J_1$ and $y_n \to O$, $y_n \in J_2$, we know that $|e_1 \wedge (f(x_n,0) - f(O,0))| \to 0$ and that $|e_2 \wedge (f(y_n,0) - f(O,0))| \to 0$, where we denote by $\cdot \wedge \cdot$ the exterior product. This implies the claim.*

Remark 2.4. *Assumption 2.2 says that the set of admissible controls laws contains locally constant function (for t small) with nonzero value. The assumption in (2.11) at the vertices in $\partial\mathcal{V}$ tells us that there exist controls which make the trajectory enter $\mathcal{G}$; this assumption is classical in the context of state constrained problems.*

Assumption 2.3. *For all $x \in \mathcal{G}\backslash\{O\}$, there exists $\tau > 0$ such that for all $\alpha \in \mathcal{A}_x$, $\alpha(t) \in A_x$ for almost all $t \in [0, \tau]$.*

Assumption 2.3 says that for small times, an admissible control at x cannot take values outside A_x (except maybe on a negligeable set of times).

Assumption 2.4. *We assume that there exist positive constants $\underline{\zeta_j}$ and $\overline{\zeta_j}$, $j = 1, \dots, N$, s.t.*

$$\overline{co}(f(O, A \cap \mathbb{R}e_j)) = [-\underline{\zeta_j}, \overline{\zeta_j}]e_j, \text{ with } \quad \overline{\zeta_j} \geq \underline{\zeta_j} > 0, \ \forall j = 1, \dots, N. \tag{2.12}$$

Remark 2.5. *Assumption 2.4 will be mostly used for proving a comparison principle. It also implies the continuity of the value function, for which weaker assumptions can be made, see Remarks 2.8 and 2.9.*

Remark 2.6. *Assumption 2.4 implies controllability near O.*

Remark 2.7. *Note that if $e_j = -e_k$ then $\underline{\zeta_j} = \overline{\zeta_j} = \underline{\zeta_k} = \overline{\zeta_k}$. Indeed, from (2.12) and the continuity of f we get that $\overline{\zeta_j} = \underline{\zeta_k}$ and $\overline{\zeta_k} = \underline{\zeta_j}$. The last part of (2.12) implies $\overline{\zeta_j} \geq \underline{\zeta_j} = \overline{\zeta_k}$ and $\overline{\zeta_k} \geq \underline{\zeta_k} = \overline{\zeta_j}$, which yields the claim.*
In particular, if $\mathcal{G} = -\mathcal{G}$ and $A = -A$, then (2.12) and the continuity of f imply that for all j, $\underline{\zeta_j} = \overline{\zeta_j}$. For example, take the cross-shaped network as follows: $N = 4$, $J_1 = (0, e_1)$, $J_2 = (0, e_2)$, $J_3 = (0, -e_1)$, $J_4 = (0, -e_2)$. We have $\overline{\zeta_1} = \underline{\zeta_3} = \overline{\zeta_3} = \underline{\zeta_1}$ and $\overline{\zeta_2} = \underline{\zeta_4} = \overline{\zeta_4} = \underline{\zeta_2}$.

Example 2.1. *Take for A the unit ball of $\mathbb{R}^2$ and $f(x, a) = g(x)a$ where $g : B \to \mathbb{R}$ is a Lipschitz continuous positive function: we can see that all the assumptions above are satisfied. In particular, let us show that Assumption 2.3 holds in the present case: take $x \in \mathcal{G}\backslash\{O\}$, for example $x \in J_1$ and $\alpha \in \mathcal{A}_x$. With M as in (2.4), take $\tau_x = |x|/(2M)$. It is easy to see that $y(t; x, \alpha) \in J_1$ for $t \in [0, \tau_x]$. This implies that $\int_0^t e_1 \wedge f(y(s; x, \alpha), \alpha(s))ds = 0$ for $t \in [0, \tau_x]$, and therefore $e_1 \wedge f(y(t; x, \alpha), \alpha(t)) = g(y(t; x, \alpha))e_1 \wedge \alpha(t) = 0$ for almost all $t \in [0, \tau_x]$. Therefore, since g is positive, $\alpha(t) \in A \cap \mathbb{R}e_1 = A_x$ for almost all $t \in [0, \tau_x]$.*

Example 2.2. *Take N unit vectors $(e_j)_{j=1,\dots,N}$, with $e_j = (\cos\theta_j, \sin\theta_j)$, $\theta_j \in [0, 2\pi)$. Choose $\underline{\zeta_j}$, $\overline{\zeta_j}$ $2N$ positive numbers satisfying the last condition in (2.12), and such that $\underline{\zeta_j} = \underline{\zeta_k} = \overline{\zeta_j} = \overline{\zeta_k}$ if $e_j = -e_k$. Take for A the unit ball of $\mathbb{R}^2$; let $\zeta : \mathbb{R} \to \mathbb{R}_+$ be a 2π-periodic and continuous function such that $\zeta(\theta_j) = \overline{\zeta_j}$ and $\zeta(-\theta_j) = \underline{\zeta_j}$, $j = 1, \dots, N$; Choose $f(x, a) = g(x)\zeta(\theta)a$ where $a = |a|(\cos\theta, \sin\theta)$ and $g : B \to \mathbb{R}$ is a Lipschitz continuous positive function. We can see that all the assumptions above are satisfied.*

Example 2.3. *Choose N unit vectors $(e_j)_{j=1,\dots,N}$ and $2N$ positive numbers $\underline{\zeta_j}$, $\overline{\zeta_j}$ as in Example 2.2. Take $A = \cup_{j=1}^N Ke_j$, $K = \{-1, 0, 1\}$. Choose*

$$f(x, a) = g(x) \sum_{j=1}^N \left(-\underline{\zeta_j} 1_{a=-e_j} + \overline{\zeta_j} 1_{a=e_j} \right) e_j$$

where $g : B \to \mathbb{R}$ is a Lipschitz continuous positive function. we can see that all the assumptions above are satisfied.

Example 2.4. *Take the cross shaped network as in Remark 2.7 and $A = Ke_1 \times Ke_2$, $K = \{-1, 0, 1\}$, $f(x, a) = g(x)a$ where $g : B \to \mathbb{R}$ is a Lipschitz continuous positive function: we can see that all the assumptions above are satisfied.*

Finally, we consider a continuous functions $\ell : \overline{\mathcal{G}} \times A \to \mathbb{R}$ such that

$$|\ell(x, a)| \leq M, \quad \forall x \in \overline{\mathcal{G}}, \, a \in A, \tag{2.13}$$

$$|\ell(x, a) - \ell(y, a)| \leq L|x - y|, \quad \forall x, y \in \overline{\mathcal{G}}, \, a \in A. \tag{2.14}$$

For $\lambda > 0$, we consider the cost functional

$$J(x, \alpha) = \int_0^\infty \ell(y(t; x, \alpha), \alpha(t))e^{-\lambda t}dt. \tag{2.15}$$

The value function of the constrained control problem on the network is

$$v(x) = \inf_{\alpha \in \mathcal{A}_x} J(x, \alpha), \qquad x \in \overline{\mathcal{G}}. \tag{2.16}$$

Proposition 2.1. *Under the assumptions above, the value function is continuous on* $\overline{\mathcal{G}}$.

Proof. For the continuity at $x \in \partial\mathcal{V}$, we refer to [1], proof of Theorem 5.2, page 274. We are going to study the continuity of the value function at $x \in \mathcal{G}$.
From (2.12) and (2.3), there exists a positive number r_0 and a constant C such that for all $x, z \in B_O(r_0)$, there exists $\alpha_{x,z} \in \mathcal{A}_x$ and $\tau_{x,z} \le C|x - z|$ with $y(\tau_{x,z}; x, \alpha_{x,z}) = z$.
For all $\varepsilon > 0$ small enough, we define

$$T_\varepsilon = -\frac{1}{\lambda}\log(\varepsilon/M), \quad C_\varepsilon = L\int_0^{T_\varepsilon} e^{(L-\lambda)t}dt, \quad \rho_\varepsilon = r_0 e^{-LT_\varepsilon}/4. \tag{2.17}$$

Consider now $x \in \mathcal{G}$. We want to prove that $\limsup_{z\to x} v(z) \le v(x)$. The inequality $\liminf_{z\to x} v(z) \ge v(x)$ is obtained in a similar way.
For all $\varepsilon > 0$ there exists a control $\overline{\alpha} \in \mathcal{A}_x$ such that $J(x, \bar{\alpha}) < v(x) + \varepsilon$. We distinguish two cases: a) $x \in \overline{B_O(r_0/2)}$; b)$x \notin B_O(r_0/2)$.
a) If $x \in \overline{B_O(r_0/2)}$, then if $z \in B_O(r_0)$, we construct $\tilde{\alpha} \in \mathcal{A}_z$ as follows:

$$\begin{array}{rcll} \tilde{\alpha}(t) & = & \alpha_{z,x}(t) & \text{if } t < \tau_{z,x}, \\ \tilde{\alpha}(t) & = & \overline{\alpha}(t - \tau_{z,x}) & \text{if } t > \tau_{z,x}. \end{array}$$

Since $\tau_{z,x} \le C|z - x|$, it is easy to prove that $v(z) \le J(z, \tilde{\alpha}) \le v(x) + \varepsilon + C|x - z|$. Sending ε to 0, we obtain that $\limsup_{z\to x} v(z) \le v(x)$ for $x \in \overline{B_O(r_0/2)}$.
b) If $x \notin B_O(r_0/2)$, we can assume that $x \in J_1$ and that $|x| > r_0/2$. Then for z close enough to x, z belongs to J_1. We take $z \in J_1$ such that $|x - z| \le \rho_\varepsilon$.
Therefore, the control $\overline{\alpha}$ is also admissible for z at least for a finite duration, (the first time T when $y(t; x, \overline{\alpha})$ or $y(t; z, \overline{\alpha})$ hits O, if it exists).
b1) If $T > T_\varepsilon$ or T does not exist, then both $y(t; z, \overline{\alpha})$ and $y(t; x, \overline{\alpha})$ remain in $J_1 \cup \{e_1\}$ for $t < T_\varepsilon$. For any $\tilde{\alpha} \in \mathcal{A}_z$ s.t. $\tilde{\alpha}(t) = \overline{\alpha}(t)$ for $t < T_\varepsilon$, we have that $|J(z, \tilde{\alpha}) - J(x, \tilde{\alpha})| \le C_\varepsilon|x - z| + 2\varepsilon$, where C_ε is defined in (2.17). Thus $v(z) \le J(z, \tilde{\alpha}) \le v(x) + C_\varepsilon|x - z| + 3\varepsilon$.
b2) If $y(T; x, \overline{\alpha}) = O$, then we construct the control $\tilde{\alpha} \in \mathcal{A}_z$ as follows

$$\begin{array}{rcll} \tilde{\alpha}(t) & = & \overline{\alpha}(t) & \text{if } t < T, \\ \tilde{\alpha}(t) & = & \alpha_{y(T;x,\overline{\alpha}),O}(t - T) & \text{if } T < t < T + \tau_{y(T;z,\overline{\alpha}),O}, \\ \tilde{\alpha}(t) & = & \overline{\alpha}(t - \tau_{y(T;z,\overline{\alpha}),O}) & \text{if } t > T + \tau_{y(T;z,\overline{\alpha}),O}. \end{array}$$

Note that this is possible since $|x - z| \le \rho_\varepsilon$ which implies $|y(T; z, \overline{\alpha})| \le e^{LT_\varepsilon}|x - z| < r_0/4$. Here again, we get that

$$v(z) \le J(z, \tilde{\alpha}) \le v(x) + \tilde{C}_\varepsilon|x - z| + \varepsilon,$$

for another constant $\tilde{C}_\varepsilon$.
b3) If $y(T; z, \overline{\alpha}) = O$, then we construct the control $\tilde{\alpha} \in \mathcal{A}_z$ as follows

$$\begin{array}{rcll} \tilde{\alpha}(t) & = & \overline{\alpha}(t) & \text{if } t < T, \\ \tilde{\alpha}(t) & = & \alpha_{O,y(T;x,\overline{\alpha})}(t - T) & \text{if } T < t < T + \tau_{O,y(T;x,\overline{\alpha})}, \\ \tilde{\alpha}(t) & = & \overline{\alpha}(t - \tau_{O,y(T;x,\overline{\alpha})}) & \text{if } t > T + \tau_{O,y(T;x,\overline{\alpha})}. \end{array}$$

Note that this is possible since $|x - z| \le \rho_\varepsilon$ which implies $|y(T; x, \overline{\alpha})| \le e^{LT_\varepsilon}|x - z| < r_0/4$. Here again, we get that

$$v(z) \le J(z, \tilde{\alpha}) \le v(x) + \tilde{C}_\varepsilon|x - z| + \varepsilon.$$

□

Remark 2.8. *Note that the assumption* $\overline{\zeta}_j \ge \underline{\zeta}_j$, $\forall j$, *has not been used.*

Remark 2.9. *It can be shown that Proposition 2.1 holds if for some indices* j, $\overline{\zeta}_j = \underline{\zeta}_j = 0$.

We now give an example in which the value function is discontinuous: let (e_1, e_2) be an orthonormal basis of $\mathbb{R}^2$, $\mathcal{G} = (0, 1)e_1 \cup \{O\} \cup (0, 1)e_2$, $A = \{0, e_1, e_2\}$, $f(x, a) = a(1 - 2|x|)$.

Take $\ell(x,a)=1$ if $x_2=0$ and $\ell(x,a)=1-|x|$ if $x_1=0$. Assumption 2.4 is not satisfied. It is easy to compute the value function u: we have

$$\begin{aligned} u(x_1,0) &= \frac{1}{\lambda}, \quad 0<x_1\le 1,\\ u(0,x_2) &= \frac{1}{2\lambda}+\frac{1-2x_2}{4+2\lambda}, \quad 0\le x_2<\frac{1}{2},\\ u(0,x_2) &= \frac{1-x_2}{\lambda}, \quad \frac{1}{2}\le x_2\le 1. \end{aligned} \tag{2.18}$$

The value function is discontinuous at O.

3 Preliminary notions

Hereafter, we make all the assumptions stated in § 2: except in § 4.2 and 5, all the theorems below will be stated without repeating the assumptions.

3.1 Test functions

We introduce the class of the admissible test functions for the PDE on the network

Definition 3.1. *We say that a function $\varphi:\overline{\mathcal{G}}\to\mathbb{R}$ is an admissible test function and we write $\varphi\in\mathcal{R}(\mathcal{G})$ if*

- *φ is continuous in $\overline{\mathcal{G}}$ and $\mathcal{C}^1$ in $\overline{\mathcal{G}}\setminus\{O\}$*
- *for any j, $j=1,\dots,N$, $\varphi|_{\overline{J_j}}\in\mathcal{C}^1(\overline{J_j})$.*

Therefore, for any $\zeta\in\mathbb{R}^2$ such that there exists a continuous function $z:[0,1]\to\mathcal{G}$ and a sequence $(t_n)_{n\in\mathbb{N}}$, $0<t_n\le 1$ with $t_n\to 0$ and

$$\lim_{n\to\infty}\frac{z(t_n)}{t_n}=\zeta,$$

the limit $\lim_{n\to\infty}\frac{\varphi(z(t_n))-\varphi(O)}{t_n}$ exists and does not depend on z and $(t_n)_{n\in\mathbb{N}}$. We define

$$D\varphi(O,\zeta)=\lim_{n\to\infty}\frac{\varphi(z(t_n))-\varphi(O)}{t_n}. \tag{3.1}$$

If $x\in\mathcal{G}\backslash\{O\}$ and $\zeta\in T_x(\mathcal{G})$, we agree to write $D\varphi(x,\zeta)=D\varphi(x)\cdot\zeta$.

Property 3.1. *Let us observe that $D\varphi(O,\rho\zeta)=\rho D\varphi(O,\zeta)$ for any $\rho>0$. Indeed, denoting by $\tau_n=t_n/\rho$, $\lim_{n\to\infty}z(t_n)/\tau_n=\rho\zeta$. Hence,*

$$\rho D\varphi(x,\zeta)=\lim_{n\to\infty}\frac{\varphi(z(t_n))-\varphi(O)}{\tau_n}=D\varphi(O,\rho\zeta).$$

As shown below, this property is not true if $\rho<0$.

If $\varphi\in\mathcal{C}^1(\mathbb{R}^2)$, then $\varphi_{|\mathcal{G}}\in\mathcal{R}(\mathcal{G})$ and $D\varphi(O,\zeta)=D\varphi(O)\cdot\zeta$ for any $\zeta\in\mathbb{R}^+e_j$, $j=1,\dots,N$. If $e_j=-e_k$ for some $j\neq k\in\{1,\dots,N\}$, $D\varphi(O,e_j)=-D\varphi(O,-e_j)$.
If φ is continuous and $\varphi|_{\bar{\mathcal{G}}\cap\mathbb{R}e_j}$ is $\mathcal{C}^1$ for $j=1,\dots,N$, then $\varphi\in\mathcal{R}(\mathcal{G})$ but the converse may not true if two edges are aligned: for example, if $e_j=-e_k$ for some $j\neq k\in\{1,\dots,N\}$, the function $x\mapsto b|x|$ belongs to $\mathcal{R}(\mathcal{G})$ and $D\varphi(O,e_j)=D\varphi(O,-e_j)=b$.

In the general case of curved edges, the following property holds for test functions: for any $\zeta\in\mathbb{R}^2$ such that there exists sequences $(t_n)_{n\in\mathbb{N}}$ and $(\zeta_n)_{n\in\mathbb{N}}$, $t_n\ge 0$, $\zeta_n\in\mathbb{R}^2$ with $t_n\zeta_n\in\mathcal{G}$ and $\zeta_n\to\zeta$, $t_n\to 0^+$, $\lim_{n\to\infty}\frac{\varphi(t_n\zeta_n)-\varphi(O)}{t_n}$ exists and it is independent of (ζ_n) and (t_n).

Property 3.2. *If $\varphi=g\circ\psi$ with $g\in\mathcal{C}^1$ and $\psi\in\mathcal{R}(\mathcal{G})$, then $\varphi\in\mathcal{R}(\mathcal{G})$ and*

$$D\varphi(O,\zeta)=g'(\psi(O))D\psi(O,\zeta). \tag{3.2}$$

3.2 A set of relaxed vector fields

Definition 3.2. *For $x \in \overline{\mathcal{G}}$, we introduce the set*

$$\tilde{f}(x) = \left\{ \eta \in T_x(\mathcal{G}) : \begin{array}{l} \exists (\alpha_n)_{n\in\mathbb{N}}, \alpha_n \in \mathcal{A}_x, \\ \exists (t_n)_{n\in\mathbb{N}} \end{array} \text{ s.t. } \begin{array}{l} t_n \to 0^+ \text{ and} \\ \lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n} f(y(t;x,\alpha_n),\alpha_n(t))dt = \eta \end{array} \right\}.$$

Note that the assumptions (2.9)- (2.11) and the continuity of f imply that

$$\overline{\mathrm{co}}\,(f(O,a) : a \in A \cap \mathbb{R}e_j) \subset \mathbb{R}e_j.$$

Indeed, take $x_n \to O$, $x_n \in J_j$, we know that $F(x_n) = \overline{\mathrm{co}}\,(f(x_n,a) : a \in A_{x_n}) \subset T_{x_n}(\mathcal{G})$. The claim is obtained by having x_n tend to O.

Proposition 3.1.

$$\tilde{f}(x) = \quad F(x) \quad \equiv \overline{\mathrm{co}}\,(f(x,a) : a \in A_x), \qquad \text{if } x \in \mathcal{G}\backslash\{O\}, \tag{3.3}$$

$$\tilde{f}(O) = \quad F(O) \quad \equiv \bigcup_{j=1}^{N} \left(\overline{\mathrm{co}}\,(f(O,a) : a \in A \cap \mathbb{R}e_j) \cap \mathbb{R}^+ e_j\right) = \bigcup_{j=1}^{N} [0, \overline{\zeta}_j] e_j, \tag{3.4}$$

$$\tilde{f}(e_j) = \quad F(e_j) \quad \equiv \overline{\mathrm{co}}\,(f(e_j,a) : a \in A \cap \mathbb{R}e_j) \cap \mathbb{R}^- e_j, \qquad j = 1,\dots,N. \tag{3.5}$$

Proof. Take first $x \in \mathcal{G}\backslash\{O\}$.
We can asssume that $x \in J_1$. The inclusion $F(x) \subset \tilde{f}(x)$ is obtained as follows: take $\zeta = \sum_{j=1}^{J} \mu_j f(x,a_j)$ with $a_j \in A_x$ and $\sum_j \mu_j = 1$, $0 \le \mu_j$. For t_n small enough, it is possible to construct a control $\alpha_n \in \mathcal{A}_x$ such that $\alpha_n(t) = a_j$ for $(\sum_{k<j}\mu_k)t_n < t \le (\sum_{k\le j}\mu_k)t_n$: we have $\frac{1}{t_n}\int_0^{t_n} f(y(t;x,\alpha_n),\alpha_n(t))dt = \frac{1}{t_n}\int_0^{t_n} f(x,\alpha_n(t))dt + o(1) = \sum_j \mu_j f(x,a_j) + o(1)$, so $\lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n} f(y(t;x,\alpha_n),\alpha_n(t))dt = \zeta$. Finally, for $\zeta \in F(x)$, we approximate ζ by $(\zeta_m)_{m\in\mathbb{N}}$, where ζ_m is a convex combination of $f(x,a)$, $a \in A_x$, and we conclude by a diagonal process.
For the opposite inclusion, since $x \in \mathcal{G}\backslash\{O\}$, we know from Assumption 2.3 that there exists $\tau > 0$, such that for all $\alpha \in \mathcal{A}_x$, $\alpha(t) \in A_x$ for $0 \le t < \tau$. Therefore, $\frac{1}{s}\int_0^s f(x,\alpha(t))dt \in F(x)$ for s small enough. This and the Lipschitz continuity of f w.r.t. its first argument imply that $\tilde{f}(x) \subset F(x)$. We have proved (3.3).
We now consider $x = O$. We first discuss the inclusion $F(O) \subset \tilde{f}(O)$: we take $\zeta = \sum_{j=1}^{J} \mu_j f(O,a_j)$ with $a_j \in A \cap \mathbb{R}e_1$ and we assume that $\zeta \in \mathbb{R}^+ e_1$. Up to a permutation of the indices, it is possible to assume that there exists J', $1 < J' \le J$ such that $f(O,a_j) \in \mathbb{R}^+ e_1$ for $j \le J'$ and that $f(O,a_j) \in \mathbb{R}^- e_1$ for $j > J'$. Then by a similar argument as above, $\zeta \in \tilde{f}(O)$. By a diagonal process, this implies that

$$\overline{\mathrm{co}}\,(f(O,a) : a \in A \cap \mathbb{R}e_1) \cap \mathbb{R}^+ e_1 \subset \tilde{f}(O).$$

Similarly $\overline{\mathrm{co}}\,(f(O,a) : a \in A \cap \mathbb{R}e_j) \cap \mathbb{R}^+ e_j \subset \tilde{f}(O)$, so we have proved that $F(O) \subset \tilde{f}(O)$.
For the opposite inclusion, consider sequences $\alpha_n \in \mathcal{A}_O$ and $t_n > 0$ such that $t_n \to 0^+$ and $\lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n} f(y(t;O,\alpha_n),\alpha_n(t))dt$ exists. We have two cases,
a) if $\lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n} f(y(t;O,\alpha_n),\alpha_n(t))dt = 0$ then it belongs to $F(O)$.
b) $\lim_{n\to\infty} \frac{1}{t_n}\int_0^{t_n} f(y(t;O,\alpha_n),\alpha_n(t))dt = \eta \ne 0$. Since $\alpha_n \in \mathcal{A}_O$, we know that $0 \ne \eta \in \cup_{j=1}^N \mathbb{R}^+ e_j$. Assume for example that $0 \ne \eta \in \mathbb{R}^+ e_1$. This implies that there exists s_n, $0 \le s_n < t_n$ such that $y(s_n;O,\alpha_n) = O$ and $y(t;O,\alpha_n) \in J_1$ for all t, $s_n < t \le t_n$. From Assumption 2.3, this implies that $\alpha_n(t) \in A \cap \mathbb{R}e_1$ for all t, $s_n < t < t_n$. Hence,

$$\frac{1}{t_n - s_n}\int_{s_n}^{t_n} f(O,\alpha_n(t))dt \in \overline{\mathrm{co}}\,(f(O,a) : a \in A \cap \mathbb{R}e_1) \cap \mathbb{R}^+ e_1.$$

Therefore, since $0 \in \overline{\mathrm{co}}\,(f(O,a) : a \in A \cap \mathbb{R}e_1)$, we get that

$$\frac{1}{t_n}\int_{s_n}^{t_n} f(O,\alpha_n(t))dt \in \overline{\mathrm{co}}\,(f(O,a) : a \in A \cap \mathbb{R}e_1) \cap \mathbb{R}^+ e_1.$$

This implies that

$$\frac{1}{t_n}\int_{s_n}^{t_n} f(y(t;O,\alpha_n),\alpha_n(t))dt + o(1) \in \overline{\mathrm{co}}\,(f(O,a) : a \in A \cap \mathbb{R}e_1) \cap \mathbb{R}^+ e_1.$$

But $\int_{s_n}^{t_n} f(y(t;O,\alpha_n),\alpha_n(t))dt = \int_0^{t_n} f(y(t;O,\alpha_n),\alpha_n(t))dt$. Hence

$$\frac{1}{t_n}\int_0^{t_n} f(y(t;O,\alpha_n),\alpha_n(t))dt + o(1) \in \overline{\mathrm{co}}\,(f(O,a) : a \in A \cap \mathbb{R}e_1) \cap \mathbb{R}^+ e_1,$$

and by passing to the limit $\eta \in \overline{\mathrm{co}}\,(f(O,a) : a \in A \cap \mathbb{R}e_1) \cap \mathbb{R}^+ e_1$.
The proof of (3.5) is similar. □

4 Viscosity solutions: the case when the running cost does not depend on a

4.1 Definition

We assume that $\ell(x,a) = \ell(x)$, for all $x \in \overline{\mathcal{G}}$ and $a \in A$. We now introduce the definition of viscosity solution for the equation

$$\lambda u(x) + \sup_{\zeta \in \tilde{f}(x)} \{-Du(x,\zeta)\} - \ell(x) = 0, \qquad x \in \mathcal{G} \tag{4.1}$$

with state constraint boundary conditions.

Definition 4.1.
- *A bounded and upper semicontinuous function $u : \overline{\mathcal{G}} \to \mathbb{R}$ is a subsolution of* (4.1) *in $\mathcal{G}$ if for any $x \in \mathcal{G}$, any $\varphi \in \mathcal{R}(\mathcal{G})$ s.t. $u - \varphi$ has a local maximum point at x, then*
$$\lambda u(x) + \sup_{\zeta \in \tilde{f}(x)} \{-D\varphi(x,\zeta)\} - \ell(x) \le 0;$$
- *A bounded and lower semicontinuous function $u : \overline{\mathcal{G}} \to \mathbb{R}$ is a supersolution of* (4.1) *if for any $x \in \overline{\mathcal{G}}$, any $\varphi \in \mathcal{R}(\mathcal{G})$ s.t. $u - \varphi$ has a local minimum point at x, then*
$$\lambda u(x) + \sup_{\zeta \in \tilde{f}(x)} \{-D\varphi(x,\zeta)\} - \ell(x) \ge 0;$$
- *A continuous function $u : \overline{\mathcal{G}} \to \mathbb{R}$ is a viscosity solution of* (4.1) *with state constraint boundary condition if it is a viscosity subsolution of* (4.1) *in $\mathcal{G}$ and supersolution of* (4.1) *in $\overline{\mathcal{G}}$.*

Remark 4.1. *At $x \in \mathcal{G}\backslash\{O\}$, the notion of sub, respectively super-solution in the definition 4.1 is equivalent to the standard definition of viscosity sub, respectively super-solution of the equation*

$$\lambda u(x) + \sup_{a \in A_x} \{-f(x,a)\cdot Du\} - \ell(x) = 0.$$

This is true because from Definition 3.1,

$$\sup_{\zeta \in \tilde{f}(x)} \{-D\varphi(x,\zeta)\} = \sup_{\zeta \in \tilde{f}(x)} \{-D\varphi(x)\cdot\zeta\} = \max_{\zeta \in F(x)} \{-D\varphi(x)\cdot\zeta\},$$

and because the maximum above is equal to $\sup_{a\in A_x}\{-D\varphi(x)\cdot f(x,a)\}$ (supremum of a linear functional). Similarly, at $x \in \partial\mathcal{V}$, the notion of supersolution in $\overline{\mathcal{G}}$ is equivalent to the standard definition.

4.2 Link with the classical definition of viscosity solutions

Let us compare our definition with the classical notion of viscosity solution in the particular network $\mathcal{G} = J_1 \cup \{0\} \cup J_2 = (-1,1) \subset \mathbb{R}$ where $J_1 = (-1,0)$ and $J_2 = (0,1)$. We denote by I the interval $[-1,1]$. We make the assumption that A is some compact subset of $\mathbb{R}$ containing 0, and that

$$|f(x,a) - f(y,a)| \le L|x-y|, \quad \text{for all } x,y \in I. \tag{4.2}$$

Note that Assumption 2.2 implies that for $x \in \mathcal{G}$, $A_x = A$ and that $\tilde{f}(0) = \overline{co}(f(0,a), a \in A)$. It is useful to recall the notion of viscosity solutions in the sense of Dini, or minimax viscosity solutions, in the special context considered here:

Definition 4.2. *Let u be a continuous function defined on I. The lower Dini derivative at $x \in \mathcal{G}$ in the direction $q = \eta e_1$, $\eta \in \mathbb{R}$, is*

$$\partial^- u(x, q) = \liminf_{t \to 0^+} \frac{u(x + tq) - u(x)}{t}.$$

The upper Dini derivative at $x \in \mathcal{G}$ in the direction q is

$$\partial^+ u(x, q) = \limsup_{t \to 0^+} \frac{u(x + tq) - u(x)}{t}.$$

Remark 4.2. *Similarly, at $x = 1$, it is possible to define Dini lower and upper derivatives in the direction $q = \eta e_1$ with $\eta < 0$. At $x = -1$, it is possible to define Dini lower and upper derivatives in the direction $q = \eta e_1$ with $\eta > 0$.*

Definition 4.3. *Let $u \in \mathcal{C}(I)$. The function u is a Dini subsolution of (4.1) in $\mathcal{G}$ if*

$$\lambda u(x) + \sup_{\zeta \in \tilde{f}(x)} \left\{ -\partial^+ u(x, \zeta) \right\} - \ell(x) \leq 0, \quad \forall x \in \mathcal{G}. \tag{4.3}$$

The function u is a Dini supersolution of (4.1) in I if

$$\lambda u(x) + \sup_{\zeta \in \tilde{f}(x)} \left\{ -\partial^- u(x, \zeta) \right\} - \ell(x) \geq 0, \quad \forall x \in I. \tag{4.4}$$

The function u is a constrained Dini solution of (4.1) if it is a Dini subsolution of (4.1) in $\mathcal{G}$ and a Dini supersolution of (4.1) in I.

Lemma 4.1. *If $u \in \mathcal{C}(I)$ is a Dini subsolution of (4.1) in $\mathcal{G}$, then it is a subsolution of (4.1) in $\mathcal{G}$ in the sense given by Definition 4.1. If $u \in \mathcal{C}(I)$ is a Dini supersolution of (4.1) in I, then it is a supersolution of (4.1) in I in the sense given by Definition 4.1.*

Proof. Assume that u is a Dini subsolution of (4.1) in $\mathcal{G}$. Let us focus on $x = 0$. Let $\varphi \in \mathcal{R}(\mathcal{G})$ be such that $u(0) = \varphi(0)$ and $u \leq \varphi$ in $B_\delta(0)$. This implies that $D\varphi(0, \zeta) \geq \partial^+ u(0, \zeta)$, forall $\zeta \in \mathbb{R}e_1$. Therefore,

$$\lambda u(0) + \sup_{\zeta \in \tilde{f}(0)} \{-D\varphi(0, \zeta)\} - \ell(0) \leq \lambda u(0) + \sup_{\zeta \in \tilde{f}(0)} \{-\partial^+ u(0, \zeta)\} - \ell(0) \leq 0.$$

A similar argument can be used at $x \neq 0$. We have proved that u is subsolution of (4.1) in $\mathcal{G}$ in the sense given by Definition 4.1.
The second assertion of Lemma 4.1 is proved similarly. □

Lemma 4.2. *If u is a constrained viscosity solution of (4.1) on I in the sense of Definition 4.1, then u is a standard constrained viscosity solution of*

$$\lambda u(x) + \sup_{\zeta \in A} \{-Du(x) \cdot \zeta\} - \ell(x) = 0 \tag{4.5}$$

on the interval I.

Proof. In view of Remark 4.1, it is enough to compare the two notions at the point $x_0 = 0$. Let $\varphi \in \mathcal{C}^1(I)$ be supertangent w.r.t. u at 0. For any control $a \in A$, there exists $t_a > 0$ such that a is admissible up to time t_a. Given a control $\alpha \in \mathcal{A}_{y(t_a;0,a)}$, define

$$\bar{\alpha}(t) = \begin{cases} a, & t \leq t_a; \\ \alpha, & t > t_a. \end{cases}$$

We have that $\lim_{t_n \to 0} y(t_n; 0, \bar{\alpha})/t_n = f(0, a)$. Hence, since φ is smooth,

$$D\varphi(0) \cdot f(0, a) = \lim_n \frac{\varphi(y(t_n; 0, \bar{\alpha})) - \varphi(0)}{t_n} = D\varphi(0, f(0, a)).$$

By Definition 4.1,

$$\lambda u(0) + \sup_{a \in A} \{-D\varphi(0) \cdot f(0, a)\} - \ell(0) \leq \lambda u(0) + \sup_{\zeta \in \tilde{f}(0)} \{-D\varphi(0, \zeta)\} - \ell(0) \leq 0.$$

Now, let $\varphi \in \mathcal{C}^1(I)$ be subtangent w.r.t. u at 0 and let $\zeta \in \tilde{f}(0)$. Since $\tilde{f}(0) = \overline{\text{co}}(f(0,a), a \in A)$,

$$\lambda u(0) + \sup_{a \in A}\{-D\varphi(0) \cdot f(0,a)\} - \ell(0) = \lambda u(0) + \sup_{\zeta \in \tilde{f}(0)}\{-D\varphi(0,\zeta)\} - \ell(0) \geq 0.$$

□

Proposition 4.1. *A function $u \in \mathcal{C}(I)$ is a constrained viscosity solution of (4.1) in the sense of Definition 4.1 if and only if it is a standard constrained viscosity solution of (4.5) on I.*

Proof. From Lemma 4.2, we know that a constrained viscosity solution of (4.1) in the sense of Definition 4.1 is a standard constrained viscosity solution of (4.5).
We have to prove the converse implication. In view of Remark 4.1, the two notions may differ only at the point $x_0 = 0$, so we need not consider the endpoints ± 1 of I.
From [1], Theorem 2.40 page 128, a function $u \in \mathcal{C}(I)$ is a viscosity subsolution (supersolution) of (4.5) in $\mathcal{G}$ if and only if it is a Dini subsolution (supersolution).
From this result and Lemma 4.1, we see that if $u \in \mathcal{C}(I)$ is a viscosity subsolution (supersolution) of (4.5) in $\mathcal{G}$, then it is a subsolution (supersolution) in the sense given by Definition 4.1.
□

Remark 4.3. *The equivalence between viscosity and Dini solutions was first proved by P-L. Lions and P. Souganidis in [13, 14] for Lipschitz continuous functions. The use of Dini derivative for Hamilton Jacobi equations goes back to Subbotin [18, 19] for Lipschitz functions, see the works of H. Frankowska [6, 7] for generalized versions.*

4.3 Existence and uniqueness

The proof of the following result is standard and it is based on the dynamic programming principle.

Theorem 4.1. *The value function v is a viscosity solution of* (4.1) *with state constraint boundary conditions.*

Proof. We first claim that v satisfies the following dynamic programming principle

$$v(x) = \inf_{\alpha \in \mathcal{A}_x} \left\{ \int_0^t \ell(y(s;x,\alpha))e^{-\lambda s} ds + e^{-\lambda t} v(y(t;x,\alpha)) \right\}. \tag{4.6}$$

The proof is standard along the argument in Propositions III.2.5 or IV.5.5 in [1].

The value function v is a subsolution: it is enough to check that v is a subsolution at $x = O$. Let $\varphi \in \mathcal{R}(\mathcal{G})$ be such that $v - \varphi$ has a maximum point at O, i.e.

$$v(O) - v(z) \geq \varphi(O) - \varphi(z) \qquad \forall z \in B(O,r) \cap \mathcal{G}.$$

Let $d \in \tilde{f}(O)$, then there exists $\alpha_n \in \mathcal{A}_O$ and $t_n \to 0^+$ such that

$$d = \lim_{n\to\infty} \frac{y(t_n; O, \alpha_n)}{t_n} = \lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} f(y(t; O, \alpha_n), \alpha_n(t)) dt.$$

Let $T > 0$ such that $y(t) = y(t; O, \alpha) \in B(O,r) \cap \mathcal{G}$ for any $t \leq T$ and all $\alpha \in \mathcal{A}_O$. From (4.6)

$$\begin{aligned} &\varphi(O) - \varphi(y(t; O, \alpha_n)) \\ \leq & v(O) - v(y(t; O, \alpha_n)) \leq \int_0^t \ell(y(s; O, \alpha_n)) e^{-\lambda s} ds + v(y(t; O, \alpha_n))(e^{-\lambda t} - 1), \end{aligned}$$

and therefore by (3.1),

$$\begin{aligned} -D\varphi(O,d) &= \lim_{n\to\infty} \frac{\varphi(O) - \varphi(y(t_n; O, \alpha_n))}{t_n} \\ &\leq \lim_{n\to\infty} \frac{1}{t_n} \left(\int_0^{t_n} \ell(y(s; O, \alpha_n)) e^{-\lambda s} ds + v(y(t_n; O, \alpha_n))(e^{-\lambda t_n} - 1) \right) \\ &= \ell(O) - \lambda v(O). \end{aligned}$$

Since the previous inequality holds for any $d \in \tilde{f}(O)$, we conclude that v is a subsolution at $x = O$.

The value function v is a supersolution Let $\varphi \in \mathcal{R}(\mathcal{G})$ be such that $v - \varphi$ has a minimum point at O, i.e.

$$v(O) - v(z) \le \varphi(O) - \varphi(z) \qquad \forall z \in B(O,r) \cap \mathcal{G}.$$

For $\varepsilon > 0$, let $\alpha \in \mathcal{A}_O$ be an ε-optimal control. For brevity, we use the notation $y(t) = y(t; O, \alpha)$. Hence by (4.6) and the continuity of ℓ and f we have

$$\begin{aligned} v(O) + t\varepsilon &\ge \int_0^t \ell(y(s))e^{-\lambda s} ds + e^{-\lambda t} v(y(t)) \\ &\ge \int_0^t \ell(O)e^{-\lambda s} ds + e^{-\lambda t} v(y(t)) + o(t). \end{aligned}$$

For t sufficiently small, we get

$$\varphi(O) - \varphi(y(t)) - \int_0^t \ell(O)e^{-\lambda s} ds + (1 - e^{-\lambda t}) v(y(t)) \ge -t\varepsilon + o(t).$$

Since $\alpha \in \mathcal{A}_O$, there exists a $t_n \to 0$ and ζ such that $\zeta = \lim_{n\to\infty} \frac{y(t_n)}{t_n}$, hence $\zeta \in \tilde{f}(O) \subset T_O(\mathcal{G})$. From

$$\frac{\varphi(O) - \varphi(y(t_n))}{t_n} - \frac{1}{t_n}\int_0^{t_n} \ell(O)e^{-\lambda s} ds + \frac{(1 - e^{-\lambda t_n})}{t_n} v(y(t_n)) \ge -\varepsilon + o(1),$$

and the arbitrariness of ε, we get for $t_n \to 0^+$

$$\lambda v(O) + \sup_{d \in \tilde{f}(O)} \{-D\varphi(O,d)\} - \ell(O) \ge \lambda v(O) - D\varphi(O,\zeta) - \ell(O) \ge 0.$$

We conclude that v is a supersolution at $x = O$. $\square$

We define the geodetic distance on $\mathcal{G}$ by

$$d(x,y) = \begin{cases} |x - y| & \text{if } x, y \in J_j \cup \{O\}, j = 1, \dots, N, \\ |x| + |y| & \text{if } x \in J_i, y \in J_j, i \ne j, \end{cases}$$

and the modified geodetic distance $\widetilde{d}(x,y)$:

$$\begin{array}{rcll} \widetilde{d}(x,y) &=& |x - y| / \overline{\zeta_i}, & \text{if } x, y \in \overline{\mathcal{G}} \cap \mathbb{R}e_i, \\ \widetilde{d}(x,y) &=& |x| / \overline{\zeta_i} + |y| / \overline{\zeta_j}, & \text{if } x \in \overline{J_i} \text{ and } y \in \overline{J_j}. \end{array} \tag{4.7}$$

For the comparison theorem we need an easy preliminary lemma

Lemma 4.3. *For any $y \in \mathcal{G}$, the functions $x \mapsto d(x,y)$ and $x \mapsto \widetilde{d}(x,y)$ are admissible test functions. For any $x \in \mathcal{G}$, the functions $y \mapsto d(x,y)$ and $y \mapsto \widetilde{d}(x,y)$ are admissible test functions.*

Theorem 4.2 (Comparison principle)**.** *If u and v are respectively a subsolution of* (4.1) *in $\mathcal{G}$ and a supersolution of* (4.1) *in $\mathcal{G}$ such that*

$$u \le v \qquad \text{on } \partial\mathcal{G}, \tag{4.8}$$

then $u \le v$ in $\overline{\mathcal{G}}$.

Proof. We use the standard argument consisting of doubling the variables, see [1] page 292. Note that $u - v$ is bounded and upper semi-continuous on $\overline{\mathcal{G}}$. We assume by contradiction that there exist $x_0 \in \overline{\mathcal{G}}$, $\chi > 0$ such that

$$u(x_0) - v(x_0) = \max_{\overline{\mathcal{G}}}(u - v) = \chi, \tag{4.9}$$

and we consider

$$\Phi_\varepsilon(x,y) = u(x) - v(y) - \frac{\widetilde{d}^2(x,y)}{2\varepsilon}, \qquad x, y \in \mathcal{G}.$$

Let $(x_\varepsilon, y_\varepsilon)$ be a maximum point of Φ_ε; we have

$$\chi = \Phi_\varepsilon(x_0, x_0) \le \Phi_\varepsilon(x_\varepsilon, y_\varepsilon). \tag{4.10}$$

From $\Phi_\varepsilon(x_\varepsilon, x_\varepsilon) \le \Phi_\varepsilon(x_\varepsilon, y_\varepsilon)$, we get $\frac{\widetilde{d}^2(x_\varepsilon, y_\varepsilon)}{2\varepsilon} \le v(x_\varepsilon) - v(y_\varepsilon)$ and since v is bounded,

$$\widetilde{d}(x_\varepsilon, y_\varepsilon) \le C\sqrt{\varepsilon}. \tag{4.11}$$

Hence $x_\varepsilon, y_\varepsilon$ converge for $\varepsilon \to 0$ to a point $\overline{x}$ and, by (4.8), $\overline{x} \in \mathcal{G}$. Therefore we can assume that for ε sufficiently small, $x_\varepsilon, y_\varepsilon \in \mathcal{G}$ and, by standard arguments, we can prove that

$$\lim_{\varepsilon \to 0} \frac{\widetilde{d}^2(x_\varepsilon, y_\varepsilon)}{2\varepsilon} = 0.$$

Moreover, $x \mapsto u(x) - (v(y_\varepsilon) + \frac{\widetilde{d}^2(x, y_\varepsilon)}{2\varepsilon})$ has a maximum point at x_ε and by Lemma 4.3,

$$\lambda u(x_\varepsilon) + \sup_{\zeta \in \tilde{f}(x_\varepsilon)} \left\{ D\left(x \mapsto \frac{\widetilde{d}^2(x, y_\varepsilon)}{2\varepsilon}\right)(x_\varepsilon, \zeta) \right\} - \ell(x_\varepsilon) \le 0. \tag{4.12}$$

Similarly, $y \mapsto v(y) - (u(x_\varepsilon) - \frac{\widetilde{d}^2(x_\varepsilon, y)}{2\varepsilon})$ has a minimum at y_ε and by Lemma 4.3,

$$\lambda v(y_\varepsilon) + \sup_{\zeta \in \tilde{f}(y_\varepsilon)} \left\{ D\left(y \mapsto -\frac{\widetilde{d}^2(x_\varepsilon, y)}{2\varepsilon}\right)(y_\varepsilon, \zeta) \right\} - \ell(y_\varepsilon) \ge 0. \tag{4.13}$$

If $x_\varepsilon = y_\varepsilon$, subtracting (4.13) from (4.12) we get

$$\lambda(u(x_\varepsilon) - v(x_\varepsilon)) \le 0,$$

and letting $\varepsilon \to 0$, we obtain the contradiction $\chi \le 0$. Hence we can assume $x_\varepsilon \neq y_\varepsilon$.

1^{st} **case:** $x_\varepsilon \neq O$**,** $y_\varepsilon \neq O$**:** From (4.12) and (4.13), taking into account Remark 4.1, we get

$$\begin{aligned} \lambda(u(x_\varepsilon) - v(y_\varepsilon)) \le &- \sup_{a \in A_{x_\varepsilon}} \left\{ D\left(x \mapsto \frac{\widetilde{d}^2(x, y_\varepsilon)}{2\varepsilon}\right)(x_\varepsilon, f(x_\varepsilon, a)) \right\} \\ &+ \sup_{a \in A_{y_\varepsilon}} \left\{ D\left(y \mapsto -\frac{\widetilde{d}^2(x_\varepsilon, y)}{2\varepsilon}\right)(y_\varepsilon, f(y_\varepsilon, a)) \right\} + \ell(x_\varepsilon) - \ell(y_\varepsilon). \end{aligned} \tag{4.14}$$

- If x_ε, y_ε are on the same edge, for example, $x_\varepsilon \in \bar{J}_1$ and $y_\varepsilon \in \bar{J}_1$, then $\widetilde{d}^2(x_\varepsilon, y_\varepsilon) = |x_\varepsilon - y_\varepsilon|^2/\overline{\zeta_1}^2$, hence by (4.14), (2.3), (2.9) and (2.14),

$$\begin{aligned} &\lambda(u(x_\varepsilon) - v(y_\varepsilon)) \\ \le &\frac{\widetilde{d}(x_\varepsilon, y_\varepsilon)}{\overline{\zeta}_1 \varepsilon} \left(- \sup_{a \in A_{x_\varepsilon}} \left\{ \frac{x_\varepsilon - y_\varepsilon}{|x_\varepsilon - y_\varepsilon|} \cdot f(x_\varepsilon, a) \right\} + \sup_{a \in A_{y_\varepsilon}} \left\{ \frac{x_\varepsilon - y_\varepsilon}{|x_\varepsilon - y_\varepsilon|} \cdot f(y_\varepsilon, a) \right\} \right) \\ &+ \ell(x_\varepsilon) - \ell(y_\varepsilon) \\ \le &L \frac{\widetilde{d}^2(x_\varepsilon, y_\varepsilon)}{\varepsilon} + L|x_\varepsilon - y_\varepsilon|, \end{aligned} \tag{4.15}$$

  (note that $(x_\varepsilon - y_\varepsilon)/|x_\varepsilon - y_\varepsilon| \in T_{x_\varepsilon}(\mathcal{G}) = T_{y_\varepsilon}(\mathcal{G})$), which yields the desired contradiction by having ε tend to 0.

- If x_ε, y_ε are not on the same edge, for example $x_\varepsilon \in \overline{J_1} \backslash \{O\}$ and $y_\varepsilon \in \overline{J_2} \backslash \{O\}$ then $\widetilde{d}^2(x_\varepsilon, y_\varepsilon) = (|x_\varepsilon|/\overline{\zeta_1} + |y_\varepsilon|/\overline{\zeta_2})^2$, hence by (4.14)

$$\begin{aligned} \lambda(u(x_\varepsilon) - v(y_\varepsilon)) \le &\frac{\widetilde{d}(x_\varepsilon, y_\varepsilon)}{\varepsilon} \left(-\frac{1}{\overline{\zeta}_1} \sup_{a \in A_{x_\varepsilon}} \left\{ \frac{x_\varepsilon}{|x_\varepsilon|} \cdot f(x_\varepsilon, a) \right\} + \frac{1}{\overline{\zeta}_2} \sup_{a \in A_{y_\varepsilon}} \left\{ -\frac{y_\varepsilon}{|y_\varepsilon|} \cdot f(y_\varepsilon, a) \right\} \right) \\ &+ \ell(x_\varepsilon) - \ell(y_\varepsilon), \end{aligned} \tag{4.16}$$

(note that $x_\varepsilon/|x_\varepsilon| \in T_{x_\varepsilon}(\mathcal{G})$ and $y_\varepsilon/|y_\varepsilon| \in T_{y_\varepsilon}(\mathcal{G})$). From (2.3), we get

$$\begin{aligned}\lambda(u(x_\varepsilon) - v(y_\varepsilon)) \le \frac{\widetilde{d}(x_\varepsilon, y_\varepsilon)}{\varepsilon} \left(-\frac{1}{\overline{\zeta_1}} \sup_{a \in A_{x_\varepsilon}} \left\{ \frac{x_\varepsilon}{|x_\varepsilon|} \cdot f(O,a) \right\} + \frac{1}{\overline{\zeta_2}} \sup_{a \in A_{y_\varepsilon}} \left\{ -\frac{y_\varepsilon}{|y_\varepsilon|} \cdot f(O,a) \right\} \right) \\ + \ell(x_\varepsilon) - \ell(y_\varepsilon) + L \frac{\widetilde{d}^2(x_\varepsilon, y_\varepsilon)}{\varepsilon}.\end{aligned} \tag{4.17}$$

From (2.9) and (2.12),

$$-\frac{1}{\overline{\zeta_1}} \sup_{a \in A_{x_\varepsilon}} \left\{ \frac{x_\varepsilon}{|x_\varepsilon|} \cdot f(O,a) \right\} + \frac{1}{\overline{\zeta_2}} \sup_{a \in A_{y_\varepsilon}} \left\{ -\frac{y_\varepsilon}{|y_\varepsilon|} \cdot f(O,a) \right\} = -1 + \underline{\zeta_2}/\overline{\zeta_2} \le 0,$$

and we obtain the desired contradiction from (4.17) and (2.14).

2^{nd} **case: either $x_\varepsilon = O$ and $y_\varepsilon \neq O$ or $x_\varepsilon \neq O$ and $y_\varepsilon = O$:** Assume $x_\varepsilon = O$ and $y_\varepsilon \neq O$ for example $y_\varepsilon \in \overline{J_2} \backslash \{O\}$ (we proceed similarly in the other cases). Take $\zeta \in \tilde{f}(O)$ where $\tilde{f}(O)$ is given by (3.4). We know that $\overline{\mathrm{co}}\,(f(O,a) : a \in A \cap \mathbb{R}e_j)$ is contained in $\mathbb{R}e_j$; therefore, $\delta(\zeta) \equiv D\{x \mapsto \widetilde{d}(x, y_\varepsilon)\}(O,\zeta) = -\frac{y_\varepsilon}{\overline{\zeta_2}|y_\varepsilon|} \cdot \zeta$ if ζ is aligned with e_2 or $\delta(\zeta) = |\zeta|/\overline{\zeta_j}$ if $\zeta \in \tilde{f}(O) \cap \mathbb{R}e_j$ is not aligned with e_2.
From (4.12) and (4.13), we get

$$\begin{aligned}\lambda(u(O) - v(y_\varepsilon)) \le \frac{\widetilde{d}(O, y_\varepsilon)}{\varepsilon} \left(-\sup_{\zeta \in \tilde{f}(O)} \{\delta(\zeta)\} + \sup_{a \in A_{y_\varepsilon}} \left\{ -\frac{y_\varepsilon}{\overline{\zeta_2}|y_\varepsilon|} \cdot f(y_\varepsilon, a) \right\} \right) \\ + \ell(O) - \ell(y_\varepsilon).\end{aligned} \tag{4.18}$$

From (2.3), we get that

$$\begin{aligned}\lambda(u(O) - v(y_\varepsilon)) \le \frac{\widetilde{d}(O, y_\varepsilon)}{\varepsilon} \left(-\sup_{\zeta \in \tilde{f}(O)} \{\delta(\zeta)\} + \sup_{a \in A_{y_\varepsilon}} \left\{ -\frac{y_\varepsilon}{\overline{\zeta_2}|y_\varepsilon|} \cdot f(O, a) \right\} \right) \\ + \ell(O) - \ell(y_\varepsilon) + L \frac{\widetilde{d}^2(O, y_\varepsilon)}{\varepsilon}.\end{aligned} \tag{4.19}$$

Thus, from (3.4), we get that

$$\begin{aligned} -\sup_{\zeta \in \tilde{f}(O)} \{\delta(\zeta)\} + \sup_{a \in A_{y_\varepsilon}} \left\{ -\frac{y_\varepsilon}{\overline{\zeta_2}|y_\varepsilon|} \cdot f(O,a) \right\} &= -\max_{j=1,\dots,N} \max_{\zeta \in [0,\overline{\zeta}_j]e_j} \delta(\zeta) + \sup_{a \in A \cap \mathbb{R}e_2} \left\{ -\frac{e_2 \cdot f(O,a)}{\overline{\zeta_2}} \right\} \\ &= -\max_{j=1,\dots,N} \max_{\zeta \in [0,\overline{\zeta}_j]e_j} \delta(\zeta) + \underline{\zeta}_2/\overline{\zeta_2} \\ &= -1 + \underline{\zeta}_2/\overline{\zeta_2} \le 0, \end{aligned}$$

which, with (2.14), yields the desired contradiction. □

Theorem 4.3. *If u and v are respectively a subsolution of* (4.1) *in $\mathcal{G}$ and a supersolution of* (4.1) *in $\overline{\mathcal{G}}$ then $u \le v$ in $\overline{\mathcal{G}}$.*

Proof. The proof resembles that of Theorem 4.2, with more technicalities near $\partial\mathcal{V}$, see [1], page 278. We skip it for brevity. □

Let us go back to the example above in which the value function is discontinuous: let (e_1, e_2) be an orthogonal basis of $\mathbb{R}^2$, $\mathcal{G} = (0,1)e_1 \cup \{O\} \cup (0,1)e_2$, $A = \{0, e_1, e_2\}$, $f(x,a) = (1-2|x|)a$, hence Assumption 2.4 is not satisfied. Take $\ell(x,a) = 1$ if $x_2 = 0$ and $\ell(x,a) = 1-|x|$ if $x_1 = 0$. The value function u is given by (2.18). We see that u is only lower semi-continuous at the origin and continuous in $\mathcal{G}\backslash\{O\}$.
Moreover it is easy to see that u is a supersolution and its upper semi-continuous envelope u^* is a subsolution. From this we conclude that in this case, the comparison theorem fails since otherwise we should have $u \ge u^*$ in $\mathcal{G}$ and therefore u would be continuous in $\mathcal{G}$.

5 Viscosity solutions: a case when the running cost depends on a

We consider a particular case when the running cost depends on a; a more general setting will be studied in a forthcoming paper.
Here we further assume that A is the unit ball of $\mathbb{R}^2$ and that for all $a \in A \cap \cup_{j=1}^N \mathbb{R}e_j$

$$f(O,a) = \sum_{j=1}^N \frac{\mu_j}{c_j} 1_{a\in\mathbb{R}e_j} a,$$

with

- $\mu_j > 0$, $j = 1, \dots, N$,
- $\mu_j = \mu_k$ and $c_j = c_k = 2$ if $e_j = -e_k$, $j \neq k \in \{1, \dots, N\}$,
- $c_j = 1$ if $e_j \neq -e_k$ $\forall k \neq j$.

We easily obtain that

$$\tilde{f}(O) = \cup_{j=1}^N \mu_j [0,1] e_j = f(O, A_O), \tag{5.1}$$

and that Assumption 2.4 is satisfied with $\overline{\zeta_j} = \underline{\zeta_j} = \mu_j$, $j = 1, \dots, N$. We also assume that

$$\ell(O,a) = \sum_{j=1}^N \ell_j(a \cdot e_j) 1_{a\in\mathbb{R}e_j}, \qquad \text{for } a \in A, \tag{5.2}$$

where ℓ_j are convex functions defined on $[-1,1]$ and vanishing at 0. Therefore,

$$\ell(O,a) = \sum_{j=1}^N \ell_j(1_{a\in\mathbb{R}e_j} a \cdot e_j), \qquad \text{for } a \in A. \tag{5.3}$$

We now introduce the definition of viscosity solution for the equation

$$\lambda u(x) + \sup_{a\in A_x} \{-Du(x, f(x,a)) - \ell(x,a)\} = 0 \tag{5.4}$$

with state constraint boundary conditions.

Definition 5.1.
- *An upper semicontinuous function $u : \overline{\mathcal{G}} \to \mathbb{R}$ is a subsolution of (5.4) in $\mathcal{G}$ if for any $x \in \mathcal{G}$, any $\varphi \in \mathcal{R}(\mathcal{G})$ s.t. $u - \varphi$ has a local maximum point at x, then*

$$\lambda u(x) + \sup_{a\in A_x} \{-D\varphi(x, f(x,a)) - \ell(x,a)\} \le 0;$$

- *A lower semicontinuous function $u : \overline{\mathcal{G}} \to \mathbb{R}$ is a supersolution of (5.4) if for any $x \in \overline{\mathcal{G}}$, any $\varphi \in \mathcal{R}(\mathcal{G})$ s.t. $u - \varphi$ has a local minimum point at x, then*

$$\lambda u(x) + \sup_{a\in A_x} \{-D\varphi(x, f(x,a)) - \ell(x,a)\} \ge 0;$$

- *A continuous function $u : \overline{\mathcal{G}} \to \mathbb{R}$ is a viscosity solution of (5.4) with state constraint boundary conditions if it is a viscosity subsolution of (5.4) in $\mathcal{G}$ and supersolution of (5.4) in $\overline{\mathcal{G}}$.*

Theorem 5.1. *With the assumptions made at the beginning of § 5, the value function v is a viscosity solution of (5.4) with state constraint boundary conditions.*

Proof. From Remark 4.1, it is enough to test the definition at the origin.

The value function v is a subsolution Let $\varphi \in \mathcal{R}(\mathcal{G})$ be such that $v - \varphi$ has a maximum point at $x_0 = O$, i.e.

$$v(O) - v(z) \geq \varphi(O) - \varphi(z) \qquad \forall z \in B(O,r) \cap \mathcal{G}.$$

Take $a \in A_O$, for example $a = a_1 e_1$. Then there exists $\alpha \in \mathcal{A}_O$ and $t_a > 0$ such that $\alpha(s) = a$ for all $s \in [0, t_a]$. Let $0 < T < t_a$ be such that $y(t) = y(t; O, \alpha) \in B(O,r) \cap \mathcal{G}$ for any $t \leq T$. From (4.6)

$$\varphi(O) - \varphi(y(t)) \leq v(O) - v(y(t)) \leq \int_0^t \ell(y(s), \alpha(s)) e^{-\lambda s} ds + v(y(t))(e^{-\lambda t} - 1).$$

Moreover, since $\alpha(s) = a$ in $[0,T]$, and f is continuous, for all $t_n \to 0^+$,

$$\lim_{n\to\infty} \frac{y(t_n; O, a)}{t_n} = \lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} f(y(t; O, a), a) dt = f(O, a).$$

From (3.1), we get

$$\begin{aligned} &- D\varphi(O, f(O,a)) = \lim_{n\to\infty} \frac{\varphi(O) - \varphi(y(t_n))}{t_n} \\ &\leq \lim_{n\to\infty} \frac{1}{t_n} \left(\int_0^{t_n} \ell(y(s), a) e^{-\lambda s} ds + v(y(t_n))(e^{-\lambda t_n} - 1) \right) = \ell(O, a) - \lambda v(x). \end{aligned}$$

Since the latter inequality holds for any $a \in A_0$, we conclude that v satisfies

$$\lambda v(O) + \sup_{a \in A_O} [-D\varphi(O, f(O,a)) - \ell(O,a)] \leq 0.$$

Hence v is a subsolution.

The value function v is a supersolution Let $\varphi \in \mathcal{R}(\mathcal{G})$ be such that $v - \varphi$ has a minimum point at $x_0 = O$, i.e.

$$v(O) - v(z) \leq \varphi(O) - \varphi(z) \qquad \forall z \in B(O,r) \cap \mathcal{G}.$$

For $\varepsilon > 0$, let $\alpha \in \mathcal{A}_0$ be an ε-optimal control. Hence by (4.6) and the continuity of ℓ and f we have

$$\begin{aligned} v(O) + t\varepsilon &\geq \int_0^t \ell(y(s; O, \alpha), \alpha(s)) e^{-\lambda s} ds + e^{-\lambda t} v(y(t; O, \alpha)) \\ &\geq \int_0^t \ell(O, \alpha(s)) e^{-\lambda s} ds + e^{-\lambda t} v(y(t; O, \alpha)) + o(t). \end{aligned}$$

For t sufficiently small, we get

$$\varphi(O) - \varphi(y(t)) - \int_0^t \ell(O, \alpha(s)) e^{-\lambda s} ds + (1 - e^{-\lambda t}) v(y(t; O, \alpha)) \geq -t\varepsilon + o(t).$$

Since $\alpha \in \mathcal{A}_0$, there exists a $t_n \to 0^+$ such that

$$\lim_{n\to\infty} \frac{y(t_n; O, \alpha)}{t_n} = \lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} f(y(s; O, \alpha), \alpha(s)) ds = f(O, \overline{a}).$$

where $\overline{a} \in A_O$ (the existence of $\overline{a}$ comes from (5.1)). On the other hand,

$$\frac{\varphi(O) - \varphi(y(t_n))}{t_n} - \frac{1}{t_n} \int_0^{t_n} \ell(O, \alpha(s)) e^{-\lambda s} ds + \frac{(1 - e^{-\lambda t_n})}{t_n} v(y(t_n)) \geq -\varepsilon + o(1).$$

Moreover, by (5.2), the Jensen's inequality and the continuity of ℓ_i

$$\begin{aligned} &\liminf_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} \ell(O, \alpha(s)) e^{-\lambda s} ds = \liminf_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} \ell(O, \alpha(s)) ds \\ &= \liminf_{n\to\infty} \sum_{j=1}^N \frac{1}{t_n} \int_0^{t_n} \ell_j (1_{\alpha(s) \in \mathbb{R}e_j} \alpha(s) \cdot e_j) ds \\ &\geq \sum_{j=1}^N \ell_j \left(\lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} 1_{\alpha(s) \in \mathbb{R}e_j} \alpha(s) \cdot e_j ds \right), \end{aligned}$$

where $\lim_{n\to\infty}\frac{1}{t_n}\int_0^{t_n}1_{\alpha(s)\in\mathbb{R}e_j}\alpha(s)\cdot e_j ds = 1_{\overline{a}\in\mathbb{R}e_j}\overline{a}\cdot e_j$, from the fact that $\alpha\in\mathcal{A}_O$, (2.3) and the special structure of $f(O,\cdot)$. Hence, letting $n\to\infty$ and using the arbitrariness of ε, we get the existence of $\overline{a}\in A_O$ such that

$$D\varphi(O,f(O,\overline{a}))+\ell(O,\overline{a})-\lambda v(O)\le 0,$$

which yields

$$\lambda v(O)+\sup_{a\in A_O}\{-D\varphi(O,f(O,a))-\ell(O,a)\}\ge 0.$$

□

We conclude by stating a comparison principle in a simple case. More general results will be given in a forthcoming paper. We suppose that ℓ satisfies (5.2) and that there exists a convex and regular function $\mathcal{L}:[-1,1]\to\mathbb{R}$ such that $\mathcal{L}(0)=0$ with

- $\ell_j=\mathcal{L}$ if $e_k\neq -e_j$, $\forall k\neq j$,
- $\ell_j(x)+\ell_k(-x)=\mathcal{L}(x)$ $\forall x\in[-1,1]$ if $e_k=-e_j$.

Therefore, for all $a\in A\cap\mathbb{R}e_j$, $\ell(O,a)=\mathcal{L}(a\cdot e_j)$.
Note that $\mathcal{L}$ must be even if there exists j,k such that $e_k=-e_j$.
We also assume that the Legendre transform of $\mathcal{L}$ defined by

$$\mathcal{L}^*(\delta)=\max_{\alpha\in[-1,1]}\{\delta\alpha-\mathcal{L}(\alpha)\} \tag{5.5}$$

satisfies

$$\mathcal{L}^*(\delta)\ge\mathcal{L}^*(-\delta),\qquad \forall\delta\ge 0, \tag{5.6}$$

and that

$$\text{if } \delta\ge 0\text{, then the maximum in (5.5) is reached in } [0,1]. \tag{5.7}$$

Example 5.1. *Assume for simplicity that $e_j\neq -e_k$ if $j\neq k$ and that for any $j=1,\dots,N$, $\ell_j(t)=\alpha t^2+\beta t$ with $\alpha>0$ and $\beta<0$. Then it can be checked that all the assumptions above hold.*

Theorem 5.2 (Comparison principle)**.** *With the assumptions made at the beginning of § 5 and ℓ satisfying the set of assumptions stated immediately above, if u and v are respectively a subsolution of (5.4) in $\mathcal{G}$ and a supersolution of (5.4) in $\mathcal{G}$ such that (4.8) holds, then $u\le v$ in $\overline{\mathcal{G}}$.*

Proof. We assume by contradiction that there exist $x_0\in\overline{\mathcal{G}}$, $\chi>0$ such that $u(x_0)-v(x_0)=\max_{\overline{\mathcal{G}}}(u-v)=\chi$, and we consider

$$\Phi_\varepsilon(x,y)=u(x)-v(y)-\frac{\widetilde{d}^2(x,y)}{2\varepsilon},\qquad x,y\in\mathcal{G},$$

where $\widetilde{d}$ is defined by (4.7) with $\overline{\zeta_j}=\mu_j$. Let $(x_\varepsilon,y_\varepsilon)$ be a maximum point of Φ_ε; we have $\chi=\Phi_\varepsilon(x_0,x_0)\le\Phi_\varepsilon(x_\varepsilon,y_\varepsilon)$. From $\Phi_\varepsilon(x_\varepsilon,x_\varepsilon)\le\Phi_\varepsilon(x_\varepsilon,y_\varepsilon)$, we get $\frac{\widetilde{d}^2(x_\varepsilon,y_\varepsilon)}{2\varepsilon}\le v(x_\varepsilon)-v(y_\varepsilon)$ and since v is bounded, $\widetilde{d}(x_\varepsilon,y_\varepsilon)\le C\sqrt{\varepsilon}$. Hence $x_\varepsilon,y_\varepsilon$ converge for $\varepsilon\to 0$ to a point $\overline{x}$ and, by (4.8), $\overline{x}\in\mathcal{G}$. Therefore we can assume that for ε sufficiently small, $x_\varepsilon,y_\varepsilon\in\mathcal{G}$ and, by standard arguments, we can prove that $\lim_{\varepsilon\to 0}\frac{\widetilde{d}^2(x_\varepsilon,y_\varepsilon)}{2\varepsilon}=0$. Moreover, $x\mapsto u(x)-(v(y_\varepsilon)+\frac{\widetilde{d}^2(x,y_\varepsilon)}{2\varepsilon})$ has a maximum point at x_ε and by Lemma 4.3,

$$\lambda u(x_\varepsilon)+\sup_{a\in A_{x_\varepsilon}}\left\{D\left(x\mapsto\frac{\widetilde{d}^2(x,y_\varepsilon)}{2\varepsilon}\right)(x_\varepsilon,f(x_\varepsilon,a))-\ell(x_\varepsilon,a)\right\}\le 0. \tag{5.8}$$

Similarly, $y\mapsto v(y)-(u(x_\varepsilon)-\frac{\widetilde{d}^2(x_\varepsilon,y)}{2\varepsilon})$ has a minimum at y_ε and by Lemma 4.3,

$$\lambda v(y_\varepsilon)+\sup_{a\in A_{y_\varepsilon}}\left\{D\left(y\mapsto-\frac{\widetilde{d}^2(x_\varepsilon,y)}{2\varepsilon}\right)(y_\varepsilon,f(y_\varepsilon,a))-\ell(y_\varepsilon,a)\right\}\ge 0. \tag{5.9}$$

If $x_\varepsilon=y_\varepsilon$, subtracting (5.9) from (5.8) we get

$$\lambda(u(x_\varepsilon)-v(x_\varepsilon))\le 0,$$

and letting $\varepsilon\to 0$, we obtain the contradiction $\chi\le 0$. Hence we can assume $x_\varepsilon\neq y_\varepsilon$.

1^{st} **case:** $x_\varepsilon \neq O$**,** $y_\varepsilon \neq O$**:** From (5.8) and (5.9), we get

$$\begin{aligned}\lambda(u(x_\varepsilon)-v(y_\varepsilon)) \le & -\sup_{a\in A_{x_\varepsilon}}\left\{D\left(x\mapsto \frac{\widetilde{d}^2(x,y_\varepsilon)}{2\varepsilon}\right)(x_\varepsilon, f(x_\varepsilon,a))-\ell(x_\varepsilon,a)\right\}\\ & +\sup_{a\in A_{y_\varepsilon}}\left\{D\left(y\mapsto -\frac{\widetilde{d}^2(x_\varepsilon,y)}{2\varepsilon}\right)(y_\varepsilon, f(y_\varepsilon,a))-\ell(y_\varepsilon,a)\right\}.\end{aligned} \tag{5.10}$$

- If x_ε, y_ε are on the same edge, for example, $x_\varepsilon \in \bar{J}_1$ and $y_\varepsilon \in \bar{J}_1$, then $\widetilde{d}^2(x_\varepsilon,y_\varepsilon) = |x_\varepsilon - y_\varepsilon|^2/\mu_1^2$, hence by (5.10), (2.3), (2.9) and (2.14),

$$\begin{aligned}&\lambda(u(x_\varepsilon)-v(y_\varepsilon))\\ \le & \left(\begin{array}{l} -\sup\limits_{a\in A_{x_\varepsilon}}\left\{\dfrac{\widetilde{d}(x_\varepsilon,y_\varepsilon)}{\varepsilon\mu_1}\dfrac{x_\varepsilon-y_\varepsilon}{|x_\varepsilon-y_\varepsilon|}\cdot f(x_\varepsilon,a)-\ell(x_\varepsilon,a)\right\}\\ +\sup\limits_{a\in A_{y_\varepsilon}}\left\{\dfrac{\widetilde{d}(x_\varepsilon,y_\varepsilon)}{\varepsilon\mu_1}\dfrac{x_\varepsilon-y_\varepsilon}{|x_\varepsilon-y_\varepsilon|}\cdot f(y_\varepsilon,a)-\ell(y_\varepsilon,a)\right\}\end{array}\right)\\ \le & L\frac{\widetilde{d}^2(x_\varepsilon,y_\varepsilon)}{\varepsilon}+L|x_\varepsilon-y_\varepsilon|,\end{aligned}$$

  (note that $(x_\varepsilon - y_\varepsilon)/|x_\varepsilon - y_\varepsilon| \in T_{x_\varepsilon}(\mathcal{G}) = T_{y_\varepsilon}(\mathcal{G})$), which yields the desired contradiction by having ε tend to 0.

- If x_ε, y_ε are not on the same edge, for example $x_\varepsilon \in \overline{J_1}\backslash\{O\}$ and $y_\varepsilon \in \overline{J_2}\backslash\{O\}$ then $\widetilde{d}^2(x_\varepsilon,y_\varepsilon) = (|x_\varepsilon|/\mu_1 + |y_\varepsilon|/\mu_2)^2$, hence by (5.10)

$$\lambda(u(x_\varepsilon)-v(y_\varepsilon)) \le \left(\begin{array}{l} -\sup\limits_{a\in A_{x_\varepsilon}}\left\{\dfrac{\widetilde{d}(x_\varepsilon,y_\varepsilon)}{\varepsilon\mu_1}\dfrac{x_\varepsilon}{|x_\varepsilon|}\cdot f(x_\varepsilon,a)-\ell(x_\varepsilon,a)\right\}\\ +\sup\limits_{a\in A_{y_\varepsilon}}\left\{-\dfrac{\widetilde{d}(x_\varepsilon,y_\varepsilon)}{\varepsilon\mu_2}\dfrac{y_\varepsilon}{|y_\varepsilon|}\cdot f(y_\varepsilon,a)-\ell(y_\varepsilon,a)\right\}\end{array}\right),$$

  (note that $x_\varepsilon/|x_\varepsilon| \in T_{x_\varepsilon}(\mathcal{G})$ and $y_\varepsilon/|y_\varepsilon| \in T_{y_\varepsilon}(\mathcal{G})$. From (2.3), we get

$$\begin{aligned}\lambda(u(x_\varepsilon)-v(y_\varepsilon)) &\le \left(\begin{array}{l} -\sup\limits_{a\in A_{x_\varepsilon}}\left\{\dfrac{\widetilde{d}(x_\varepsilon,y_\varepsilon)}{\varepsilon\mu_1}\dfrac{x_\varepsilon}{|x_\varepsilon|}\cdot f(O,a)-\ell(O,a)\right\}\\ +\sup\limits_{a\in A_{y_\varepsilon}}\left\{-\dfrac{\widetilde{d}(x_\varepsilon,y_\varepsilon)}{\varepsilon\mu_2}\dfrac{y_\varepsilon}{|y_\varepsilon|}\cdot f(O,a)-\ell(O,a)\right\}\end{array}\right)+L\frac{\widetilde{d}^2(x_\varepsilon,y_\varepsilon)}{\varepsilon}\\ &= \left(\begin{array}{l} -\sup\limits_{a\in[-1,1]e_1}\left\{\dfrac{\widetilde{d}(x_\varepsilon,y_\varepsilon)}{\varepsilon}e_1\cdot a-\ell(O,a)\right\}\\ +\sup\limits_{a\in[-1,1]e_2}\left\{-\dfrac{\widetilde{d}(x_\varepsilon,y_\varepsilon)}{\varepsilon}e_2\cdot a-\ell(O,a)\right\}\end{array}\right)+L\frac{\widetilde{d}^2(x_\varepsilon,y_\varepsilon)}{\varepsilon}\\ &= -\mathcal{L}^*\left(\frac{\widetilde{d}(x_\varepsilon,y_\varepsilon)}{\varepsilon}\right)+\mathcal{L}^*\left(-\frac{\widetilde{d}(x_\varepsilon,y_\varepsilon)}{\varepsilon}\right)+L\frac{\widetilde{d}^2(x_\varepsilon,y_\varepsilon)}{\varepsilon},\end{aligned}$$

  and we obtain the desired contradiction from (5.6).

2^{nd} **case: either** $x_\varepsilon = O$ **and** $y_\varepsilon \neq O$ **or** $x_\varepsilon \neq O$ **and** $y_\varepsilon = O$**:** Assume $x_\varepsilon = O$ and $y_\varepsilon \neq O$ for example $y_\varepsilon \in \overline{J_2}\backslash\{O\}$ (we proceed similarly in the other cases). For any $a \in A_O$, $\delta(a) \equiv D\{x \mapsto \widetilde{d}(x,y_\varepsilon)\}(O, f(O,a)) = -e_2\cdot a$ if a is aligned with e_2 or $\delta(a) = |a|$ if $a \in A_O$ is not aligned with e_2.
From (5.8) and (5.9), we get

$$\lambda(u(O)-v(y_\varepsilon)) \le -\sup_{a\in A_O}\left\{\frac{\widetilde{d}(O,y_\varepsilon)}{\varepsilon}\delta(a)-\ell(O,a)\right\}+\mathcal{L}^*\left(-\frac{\widetilde{d}(O,y_\varepsilon)}{\varepsilon}\right)+L\frac{\widetilde{d}^2(O,y_\varepsilon)}{\varepsilon},$$

and the desired contradiction follows, because

$$\sup_{a\in A_O}\left\{\frac{\widetilde{d}(O,y_\varepsilon)}{\varepsilon}\delta(a)-\ell(O,a)\right\}=\mathcal{L}^*\left(\frac{\widetilde{d}(O,y_\varepsilon)}{\varepsilon}\right)$$

from (5.7). $\square$

Similarly, we have the following theorem:

Theorem 5.3. *With the same assumptions as in Theorem 5.2, if u and v are respectively a subsolution of (5.4) in $\mathcal{G}$ and a supersolution of (5.4) in $\overline{\mathcal{G}}$ then $u \le v$ in $\overline{\mathcal{G}}$.*

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