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ADAPTIVE TUNING OF ELASTO-PLASTIC DAMPER

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ABSTRACT. Hysteretic dampers are frequency independent, and thereby potentially effective for several structural vibration modes, provided that the inherent amplitude dependence can be controlled. An adaptive tuning procedure is proposed, aiming at elimination of the amplitude dependence by adjusting the damper parameter(s) with respect to the magnitude of the damper motion. The procedure is demonstrated in terms of the bilinear elasto-plastic damper model, and optimality corresponds to maximum modal damping. A parametric solution for the damping ratio is obtained by a two-component system reduction technique, and maximization leads to an amplitude dependent expression for the optimal yield level. The amplitude is predicted from the most recent extremum of the damper response, and simultaneously used to adjust the yield level. Numerical examples demonstrate that the adaptive tuning procedure succeeds in controlling the amplitude dependence, resulting in equal damping for the first vibration modes.

1. INTRODUCTION

Flexible structures may be prone to excessive resonant response often due to fairly wide-banded excitation by e.g. waves, wind, traffic loading or earthquakes. An effective means to reduce excessive dynamic response comes in the form of external dampers, typically categorized as being of either passive [1], active [2], or semi-active [3] type. In general efficiency depends on a suitable placement and tuning of the damper with respect to the critical vibration mode(s) of the structure. If the dynamic response is dominated by resonance the characteristic frequency of the response is typically known and the response magnitude is inversely proportional to the modal damping ratio. Properly tuned and placed passive dampers are usually adequate in mitigating dynamic response dominated by resonance. However, semi-active dampers are furthermore able to change characteristics real time, adapting to changes in for example loading, environmental or structural conditions. In recent years semi-active dampers with variable mechanical or material properties have proven themselves useful in practice [3]. Some of the most promising types of semi-active devices are dampers with variable friction force [4], variable stiffness [5, 6] or variable oil-based yielding effects [7, 8]. In particular the so-called Magneto-Rheological (MR) dampers have received considerable attention in connection with structural damping and control of for example stay-cables [9–11]. Common for many of the semi-active dampers available is that they exhibit a significant hysteretic behavior, with clear rate-independent characteristics and a highly non-linear relation between energy dissipation (damping) and restoring forces (stiffness). Therefore, efficiency of these dampers relies on a proper interplay between viscosity and stiffness, where in particular stiffness has a deteriorating effect.

The analysis of non-linear damper models is typically based on an equivalent linear model obtained by harmonic averaging. Damping is evaluated in terms of the dissipated energy per period corresponding to the area inside a representative hysteresis loop. In the case of sinusoidal motion the energy per period dissipated by a linear viscous damper is proportional to frequency and motion amplitude squared. This implies that the properties of linear viscous dampers are frequency dependent, which only makes them optimal with respect to a single vibration mode. Conversely, energy dissipation by hysteretic dampers is frequency independent. This means that hysteretic dampers in principle may be optimal with respect to several vibration modes, provided that the inherent amplitude dependence is controlled. This suggests the introduction of an adaptive tuning

procedure where the parameters of the hysteretic damper are adjusted real-time in such a way that the influence of amplitude is cancelled. Different strategies for amplitude control have been proposed for the Reid sping model [6], the elastic-ideal plastic damper [7] and the ideal friction damper [12]. The aim of the present paper is therefore to formulate an adaptive tuning procedure where the amplitude dependence of a hysteretic damper is controlled to maintain optimal characteristics.

In the present paper the adaptive tuning procedure is illustrated in terms of the bilinear elasto-plastic damper model. The elasto-plastic damper is a fairly simple model that contains the most important characteristics of a fairly large family of hysteretic dampers, including for instance the ideal (Coulomb) friction damper or the MR damper. Furthermore, it has been analyzed extensively in the literature, typically based on an equivalent linear model where the parameters are found by means of averaging. For the bilinear elasto-plastic model equivalent parameters have been determined in [13–15] for harmonic response analysis or in [16–18] for stochastic damper response. Similar solutions based on harmonic averaging have also been obtained for the double bilinear model [19], the so-called flag-shaped model [20] and for more general hysteretic models [21–24]. The equivalent linear model for the elasto-plastic damper is briefly presented in Section 2.

For damping of flexible structures the interaction between damper and structure is important, and the tuning of the external damper is crucial for effective energy dissipation. Too little effect of the damper leaves the structure virtually undamped, while too much simply locks the structure at the location of the damper. For flexible structures, where dynamic response is governed mainly by resonance, the effect of the damper on the individual vibration modes is conveniently described by the modal damping ratio. An expression for the damping ratio is obtained via an accurate representation of the damped vibration modes. In the present paper (Section 3) the shape of the damped vibration mode is described by a two-component representation, where the two components are the modes of the structure without the damper and with the damper fully locked, respectively. This approach has been introduced for discrete structures with external viscous dampers in [25]. This representation implies a reduced set of equations of motion, which in the frequency domain leads to an expression for the natural frequency of the damped vibration mode, and thereby to a parametric expression for the corresponding damping ratio, [25, 26]. Solutions of similar format have been obtained for continuous systems, such as cables [27] and beams [28].

By substitution of the linear equivalent model for the elasto-plastic damper into the solution for the modal damping ratio the optimal tuning of the elasto-plastic damper is obtained by maximizing the damping ratio with respect to the governing parameters of the elasto-plastic model. This leads to an optimal relation between the yield force level of the elasto-plastic damper and the motion amplitude in the case of sinusoidal response. This relation constitutes the basis of the adaptive tuning procedure, as discussed in Section 4. The main aspects of the procedure are 1) the interpretation of amplitude in the case of irregular response and 2) the adaptive tuning of the elasto-plastic damper by optimal and sequential adjustments of the yield force level.

The performance of the elasto-plastic damping is finally illustrated by numerical examples in Section 5, where the characteristics of the elasto-plastic damper are compared to those of the equivalent linear damper, whereas efficiency is evaluated by comparison with the corresponding optimal viscous damper.

2. PROPERTIES OF ELASTO-PLASTIC DAMPER

2.1. Non-linear damper model

The properties of the elasto-plastic damper model are conveniently described by the generic bilinear hysteresis loop shown in Fig. 1, which is composed of two elastic parts with stiffness k_1 and two plastic parts with stiffness $k_2 < k_1$. The relation between the elastic and the plastic part is governed by x_1 , representing the maximum elastic deformation at initial loading of the damper. The symmetric opening of the hysteresis loop around the line $k_2 X$ is defined in terms of the

FIGURE 1. Hysteresis loop of elasto-plastic model.

intersections of the hysteresis loop with the vertical axis. This defines an equivalent yield force

$$F_y = (k_1 - k_2)x_1 \quad (1)$$

whereby the yield limits are given as $k_2X \pm F_y$. It is noted that the equivalent yield force F_y is a convenient parameter for characterizing the hysteresis loops, but is not equal to the yield force at initial loading. In the numerical analysis the elasto-plastic force is computed on incremental form, as described in Appendix A. The approach is similar to the return algorithms used in numerical analysis of plasticity. However, in the present case the return to the yield limit is explicit due to its simple linear form.

The properties of the elasto-plastic damper are described in a non-dimensional form by normalizing displacements by x_1 and forces by k_1x_1 . This leads to the following non-dimensional parameters:

$$f = \frac{F}{k_1x_1} \quad , \quad x = \frac{X}{x_1} \quad , \quad a = \frac{A}{x_1} \quad , \quad \mu = \frac{F_y}{k_1x_1} = \frac{k_1 - k_2}{k_1} \quad (2)$$

where the normalized yield force μ appears as a convenient stiffness ratio that defines the yield limit at initial loading as μF_y . A closeup of the virgin part of the non-dimensional hysteresis loop is shown in Fig. 2. The maximum elastic deformation is scaled to unity, which means that the slope of the elastic part is a and the slope of the plastic part is $(1 - \mu)a$. Plastic deformation requires that $a > 1$ and $1/a$ therefore spans the interval from 0 to 1.

During harmonic motion the dissipated energy per period by the damper equals the area inside the hysteresis loop. For the elasto-plastic damper in Fig. 1 this area is

$$\Delta E = 4(k_1 - k_2)x_1(A - x_1) \quad (3)$$

Besides energy dissipation the elasto-plastic damper also introduces stiffness via an interpolation between the two limiting stiffness components k_1 and k_2 . Energy dissipation and stiffness effects are in the following represented by an equivalent linear model.

FIGURE 2. Characteristics of non-dimensional hysteresis loop.

2.2. Equivalent linear damper model

The analysis is based on a linear equivalent model with non-dimensional viscosity β and stiffness κ ,

$$f(t) = \frac{\beta}{\omega} \dot{x}(t) + \kappa x(t) \quad (4)$$

When assuming harmonic motion, with constant amplitude a and frequency ω , the equivalent linear parameters may be evaluated by harmonic averaging following e.g. Kryloff and Bogoliuboff, see [29],

$$\beta = \frac{1}{\pi a} \int_0^{2\pi} f \sin(\omega t) d(\omega t) \quad , \quad \kappa = \frac{1}{\pi a} \int_0^{2\pi} f \cos(\omega t) d(\omega t) \quad (5)$$

Harmonic averaging has been used in [13,14,21,23,24] to obtain equivalent linear parameters for the dynamic analysis of systems with hysteresis. For the elasto-plastic model the solution in [13] is formulated in terms of a parameter θ related to the amplitude via

$$\cos \theta = 1 - \frac{2}{a} \quad (6)$$

Evaluation of the integrals in (5) leads to the following expressions for the equivalent linear viscous and stiffness parameters, see [13],

$$\beta = \begin{cases} 0 & , \quad a \leq 1 \\ \frac{\mu}{2\pi} [1 - \cos 2\theta] & , \quad a > 1 \end{cases} \quad (7)$$

$$\kappa = \begin{cases} 1 & , \quad a \leq 1 \\ \frac{\mu}{2\pi} [2\theta + 2\pi \frac{(1-\mu)}{\mu} - \sin 2\theta] & , \quad a > 1 \end{cases} \quad (8)$$

It follows immediately by elimination of θ by means of (6) that β is related to the dissipated energy by $k_1 \beta = \Delta E / (\pi a^2 x_1^2)$. This shows that β/ω is similar to the equivalent linear viscous parameter obtained by (3) in terms of energy balance.

Figure 3 shows 3D surface plots of β and κ as function of μ and $1/a$. As demonstrated in the following section the optimal tuning requires a trade-off between minimum stiffness and optimal viscosity. It seen in Fig. 3 that minimum stiffness occurs at $\mu = 1$ and $1/a = 0$ while maximum viscosity is obtained for $\mu = 1$ and $1/a = 0.5$.

FIGURE 3. Surface plot of damping and stiffness parameters.

3. TUNING OF ELASTO-PLASTIC DAMPER

Consider the situation where the damper acts on a discrete system with multiple degrees of freedom. The governing equation of motion with negligible structural damping takes the form

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = -\mathbf{w}F(t) \quad (9)$$

where the nodal displacements are contained in the vector $\mathbf{q}$, while $\mathbf{M}$ and $\mathbf{K}$ are the mass and stiffness matrices, respectively. The location of the single damper force $F(t)$ follows from the influence vector $\mathbf{w}$ describing the points of the structure connected by the damper. Due to the absence of external loads the present equation of motion describes the damped modal vibrations of the structure. The dynamic response of a flexible structure is typically dominated by resonance and robust design of the damper should be based on a sufficiently large damping ratio for the critical vibration mode(s).

3.1. Damping of reduced system

Modal vibrations are investigated by assuming solutions of the form,

$$\mathbf{q}(t) = \mathbf{u} \exp(i\omega t) \quad (10)$$

where $\mathbf{u}$ is the mode shape and ω is the corresponding natural frequency of a particular mode. A subscript denoting the mode number is omitted in the present section to keep the notation simple. Due to the presence of the external damper both $\mathbf{u}$ and ω are complex valued. The actual vibration shape is given by the real part of $\mathbf{u}$, while the imaginary part represents the phase delay introduced by the damper.

A simple, yet accurate, expression for the natural frequency follows from the two-component representation by Main & Krenk [25], where the damped vibration mode $\mathbf{u}$ is described by a linear combination of the two limiting mode shapes of the structure without the damper, $\mathbf{u}_0$, and with the damper fully locked, $\mathbf{u}_\infty$, respectively. Note that locking of the damper link results in $\mathbf{w}^T \mathbf{u}_\infty = 0$. The mode shapes are normalized to unit modal mass,

$$\mathbf{u}_0^T \mathbf{M} \mathbf{u}_0 = 1 \quad , \quad \mathbf{u}_\infty^T \mathbf{M} \mathbf{u}_\infty = 1 \quad (11)$$

whereby the corresponding natural frequencies are given as

$$\mathbf{u}_0^T \mathbf{K} \mathbf{u}_0 = \omega_0^2 \quad , \quad \mathbf{u}_\infty^T \mathbf{K} \mathbf{u}_\infty = \omega_\infty^2 \quad (12)$$

The limiting mode shapes and natural frequencies are typically available from the numerical structural model, e.g. of finite element format. Figure 4 shows the two limiting mode 1 vibration forms of a 4-storey shear frame where the damper acts on the motion between storeys 2 and 3.

Representation of the damped vibration mode in terms of $\mathbf{u}_0$ and $\mathbf{u}_\infty$ leads to the introduction of the degrees of freedom p_0 and p_∞ ,

$$\mathbf{u} = \mathbf{u}_0 p_0 + \mathbf{u}_\infty p_\infty = [\mathbf{u}_0 \quad \mathbf{u}_\infty] \begin{bmatrix} p_0 \\ p_\infty \end{bmatrix} \quad (13)$$

FIGURE 4. Limiting vibration modes $\mathbf{u}_0$ (left) and $\mathbf{u}_\infty$ (right).

The frequency equation for the reduced system is found by substitution of (13) into (10), followed by substitution of this representation into the equation of motion (9). The damper force is $F(t) = k_1 x_1 f(t)$, where $f(t)$ is given in (4). The equation of motion in the reduced system sub-space is obtained by pre-multiplication with $[\mathbf{u}_0 \ \mathbf{u}_\infty]^T$. Evaluation of the various terms leads to the determinant equation

$$\begin{bmatrix} (\omega_0^2 - \omega^2) + k_1(\kappa + i\beta)(\mathbf{w}^T \mathbf{u}_0)^2 & (1 - \gamma)(\omega_0^2 - \omega^2) \\ (1 - \gamma)(\omega_0^2 - \omega^2) & \omega_\infty^2 - \omega^2 \end{bmatrix} \begin{bmatrix} p_0 \\ p_\infty \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (14)$$

where $(1 - \gamma) = \mathbf{u}_\infty^T \mathbf{M} \mathbf{u}_0$ represents the coupling between the limiting mode shapes. The damper transfer relation, $\kappa + i\beta$, follows directly from (7) and (8). Setting the determinant of the system matrix equal to zero gives a characteristic polynomial that is quartic in ω . This equation can be re-arranged into an implicit format that is suitable for numerical solution by iteration [25].

In cases where the presence of the damper only leads to a small change in the vibration mode relative to the undamped mode, as in Fig. 4, the coupling parameter γ becomes small and $1 - \gamma \simeq 1$. Cancellation of the terms containing $1 - \gamma$ leads to the explicit frequency solution

$$\frac{\Delta\omega}{\Delta\omega_\infty} \simeq \frac{\nu(\kappa + i\beta)}{1 + \nu(\kappa + i\beta)} \quad (15)$$

where the changes in natural frequency relative to the undamped case are defined as

$$\Delta\omega = \omega - \omega_0, \quad \Delta\omega_\infty = \omega_\infty - \omega_0 \quad (16)$$

and where the elastic stiffness k_1 is absorbed by the non-dimensional gain factor

$$\nu = k_1 \frac{(\mathbf{w}^T \mathbf{u}_0)^2}{\omega_\infty^2 - \omega_0^2} \quad (17)$$

The solution (15) is a parametric representation of the complex-valued natural frequency. It is common practice to represent the natural frequency in terms of the damping ratio ζ ,

$$\omega = |\omega|(\sqrt{1 - \zeta^2} + i\zeta), \quad \zeta = \frac{\text{Im}[\omega]}{|\omega|} = \frac{\text{Im}[\Delta\omega]}{|\omega|} \quad (18)$$

For moderate damping it is reasonable to assume that $|\omega| \simeq \omega_0$, whereby the damping ratio follows directly from (15) as

$$\zeta \simeq \frac{\text{Im}[\Delta\omega]}{\omega_0} \simeq \frac{\Delta\omega_\infty}{\omega_0} \frac{\nu\beta}{(1 + \nu\kappa)^2 + (\nu\beta)^2} \quad (19)$$

Provided that ν is constant for the given structure the last factor is independent of frequency and thereby also independent of the mode number.

3.2. Optimal tuning of elasto-plastic damper

The optimal tuning of the elasto-plastic damper follows from the maximization of the damping ratio in (19) with respect to the tunable parameters. In the analysis μ is assumed known while the damping ratio is maximized with respect to ν and θ , respectively. Maximum of the damping ratio follows from the variation

$$\delta\zeta = \frac{\partial\zeta}{\partial\nu}\delta\nu + \frac{\partial\zeta}{\partial\theta}\delta\theta = 0 \quad (20)$$

Cancellation of the first term, $\partial\zeta/\partial\nu = 0$, leads to the quadratic equation,

$$(\nu\kappa)^2 + (\nu\beta)^2 = 1 \quad (21)$$

where ν may be isolated. The quadratic form of this equation suggests a polar representation in terms of the phase angle φ ,

$$\cos \varphi = \nu\kappa, \quad \sin \varphi = \nu\beta \quad (22)$$

The quadratic equation (21) describes the unit circle, as shown in Fig. 5a, where φ represents the actual phase angle in state-space, as illustrated in Fig. 5b.

FIGURE 5. (a) Polar representation of (22) and (b) phase-plane representation.

The phase angle is independent of the gain ν ,

$$\tan \varphi = \frac{\beta}{\kappa} \quad (23)$$

and substitution of (22) into (19) leads to an expression for the damping ratio in terms of the phase angle,

$$\zeta_* \simeq \frac{1}{2} \frac{\Delta\omega_\infty}{\omega_0} \frac{\sin \varphi}{1 + \cos \varphi} = \frac{1}{2} \frac{\Delta\omega_\infty}{\omega_0} \tan \frac{1}{2} \varphi \quad (24)$$

Subscript * merely indicates that this expression is valid only when (21) is satisfied. Maximizing ζ_* means maximizing the phase angle φ and hence maximizing $\tan \frac{1}{2} \varphi$. However, it turns out to be more convenient to maximize $\tan \varphi$ in (23). Maximizing $\tan \varphi$ is equivalent to maximizing $\tan \frac{1}{2} \varphi$ since both β and κ are non-negative, whereby $\varphi \leq \frac{1}{2} \pi$. Substitution of (7) and (8) into (23) gives

$$\tan \varphi = \frac{\mu(1 - \cos 2\theta)}{2\pi(1 - \mu) + \mu(2\theta - \sin 2\theta)}, \quad a > 1 \quad (25)$$

Figure 6 shows a surface plot of the phase angle based on (25). The optimal combination of μ and $1/a$ is defined by the ridge of the surface with maximum in the limit $\mu = 1$ and $1/a \rightarrow 0$. The maximum of $\tan \varphi$ with respect to θ is most easily found in terms of $\partial(\tan \varphi)/\partial(2\theta) = 0$. This leads to a simple relation between θ and μ ,

$$\frac{\tan \theta - \theta}{\pi} = \frac{1 - \mu}{\mu} \quad (26)$$

For a given value of μ this equation is solved numerically by iteration, resulting in the optimal value for θ . The corresponding value of $1/a$ follows from (6) and the optimal relation between $1/a$ and θ is shown Fig. 7.

FIGURE 6. Surface plot of phase angle.

FIGURE 7. Optimal relation between μ and $1/a$.

Substitution of θ into (7) and (8) gives the optimal values for the parameters β and κ . They are shown in Fig. 8 as function of μ . It is seen that stiffness κ decreases with increasing μ , while the viscosity β is zero in the limits and attains its maximum around $\mu = 0.78$.

The optimal value of ν is obtained from the quadratic equation (21), and the maximum attainable damping ratio follows from (24). Figure 9 shows the maximum damping ratio in normalized form $\zeta\omega_0/\Delta\omega_\infty$. It is seen that the damping ratio attains its maximum value $\zeta^{max} = \frac{1}{2}\Delta\omega_\infty/\omega_0$ for $\mu = 1$.

In connection with semi-active control of hysteretic dampers the yield level, or friction force, is often a convenient control parameter, as discussed in e.g. [7, 9]. The equivalent yield force in (1) can be written as $F_y = (k_1 - k_2)x_1 = \mu k_1 x_1$. Elimination of k_1 by (17) and x_1 by (2c) implies that F_y is proportional to the non-dimensional ratio $\nu\mu/a$,

$$F_y = (k_1 - k_2)x_1 = \left(\frac{\nu\mu}{a}\right) \frac{\omega_\infty^2 - \omega_0^2}{(\mathbf{w}^T \mathbf{u}_0)^2} A \quad (27)$$

Figure 10 shows the factor $\nu\mu/a$ as function of μ . It turns out that for maximum attainable damping at $\mu = 1$ the optimal elasto-plastic damper appears as an ideal friction damper, where $\nu \rightarrow \infty$ and $1/a \rightarrow 0$ in such a way that $\nu\mu/a = \pi/4$. The optimal friction force level is therefore given as

$$F_{fric} = \frac{\pi}{4} \frac{\omega_\infty^2 - \omega_0^2}{(\mathbf{w}^T \mathbf{u}_0)^2} A \quad (28)$$

FIGURE 8. Optimal $\eta\omega$ (left) and κ (right).

FIGURE 9. Maximum attainable damping ratio.

This expression may be used in connection with semi-active control of for instance MR-dampers, which almost exhibit ideal friction characteristics, see [9]. The adaptive tuning procedure presented in the following section could therefore very well be implemented for MR-dampers with (28) as the desired force-amplitude relation.

4. ADAPTIVE TUNING PROCEDURE

In connection with the application of semi-active dampers with hysteretic effects it is desirable to formulate a control procedure that specifically utilizes the frequency independent properties for robust, autonomous and effective damping of several vibration modes. In this section a simple adaptive tuning procedure is formulated which aims at controlling the amplitude dependence of the elasto-plastic damper by maintaining the optimal yield level to amplitude ratio in (27). For a given value of μ the optimal parameters for the elasto-plastic damper are determined by the following steps:

- 1) θ from (26)
- 2) $1/a$ from (6)
- 3) β from (7) and κ from (8)
- 4) ν from (21)
- 5) F_y/A from (27)

FIGURE 10. Optimal yield force.

FIGURE 11. Damper response: (a) Force-displacement trajectory and (b) displacement record.

These expressions have been obtained for pure sinusoidal damper motion. This leaves the question of how to interpret and evaluate the amplitude A in the case of irregular response.

4.1. Interpretation of amplitude

Figure 11 shows a part of an irregular damper response history, where Fig. 11a shows the force-displacement trajectory and Fig. 11a shows the displacement response. Dissipated energy per period corresponds to the area inside the various hysteresis loops, which is given in (3) for pure sinusoidal motion. In the case of irregular response the amplitude A may be evaluated as the range R of the various closed loops. In Fig. 11 two loops are formed. The displacement ranges are denoted as R_j and R_{j+1} . The identification of ranges in terms of closed loops may be related to fatigue analysis where stress ranges are evaluated in terms of closed loops by for instance the rain flow counting method. In rain flow counting stress ranges are evaluated in terms of displacement ranges based on the hysteresis analogy of Masing, stating that loops are perfectly closed.

For flexible and lightly damped structures the response is typically dominated by resonance with a fairly narrow bandwidth. In that case the envelope function of the response is slowly varying, and the change in displacement range R between adjacent loops is relatively small. This is illustrated in Fig. 12 where the extremes of the displacement history are denoted X_j . For a narrow-band response the magnitude of an extremum is approximately equal to the magnitude of the previous extremum: $|X_{j+1}| \simeq |X_j|$. Consequently, the displacement range of the loop is approximately equal to the magnitude of the difference between the two adjacent extremes: $R_j \simeq |X_{j+1} - X_j|$. The apparent amplitude of the loop is therefore evaluated as half the displacement range: $A_j \simeq \frac{1}{2} R_j$. This means that the amplitude of loop j can be estimated directly in terms of the absolute value of associated extremum X_j ,

$$A_j = |X_j| \quad (29)$$

Thus, when a local extremum of the damper response is reached the apparent amplitude of the present loop is estimated as the absolute value of the particular extremum. This sequential representation of the amplitude is the basis of the adaptive tuning of the yield level F_y , as discussed next.

4.2. Adjustment of yield force

Narrow-band response is often adequately described by the sequence of extremes (minima and maxima), as shown in Fig. 12. The adaptive tuning procedure adjusts the yield level of the elasto-plastic damper at each local extremum. At a given extremum X_j the yield force of the elasto-plastic damper is therefore adjusted based on the prediction of the amplitude in (29). The

FIGURE 12. Sequence of extremes.

corresponding yield force is found by substitution into (27),

$$F_y = \left(\frac{\nu\mu}{a} \right) \frac{\omega_\infty^2 - \omega_0^2}{(\mathbf{w}^T \mathbf{u}_0)^2} |X_j| \quad (30)$$

The elasto-plastic model contains an upper and a lower yield level. In the present procedure the upper yield force F_y^+ and the lower yield force F_y^- are adjusted separately. The upper yield force F_y^+ is adjusted when X_j is a minimum, while the lower yield force F_y^- is adjusted when X_j is a maximum. This separation of the control of the two yield levels reduces the possibility of transients and energy spill-over, because the yield limit associated with the current extremum X_j is unchanged.

The steps of the adaptive tuning procedure are summarized in Table 1, considering the particular case of a local maximum. Initially the procedure checks whether the new maximum exceeds the present maximum. If that is the case the maximum X_{max} is update in 1), and the amplitude A is determined as the absolute value of the maximum in 2). Since a maximum is considered, the lower yield level F_y^- is adjusted in 3). Finally, X_{min} is stored so that the procedure is ready for when $X(t)$ reaches a minimum, where the procedure is similar to Table 1.

Figure 13 shows a part of the damper response when the adaptive tuning procedure is implemented. The force-displacement trajectory in Fig. 13a contains two hysteresis loops, i.e. a small and a large loop. The displacement record in Fig. 13b contains the corresponding four extremes. The thin vertical lines in Fig. 13b represent twice the prediction of the ranges based on the various extremes. These ranges are also indicated in Fig. 13a by the thin horizontal lines. It is seen that the quality of the predictions of the actual closed loop ranges are modest. However, for narrow-band response the predictions of amplitude will be more accurate, and the adaptive tuning procedure will consequently perform better. This is demonstrated by the numerical examples in Section 5.

TABLE 1. Control logic for elasto-plastic damper

For $X(t) = \text{maximum}$
if $X(t) > 0$ & $X(t) > X_{max}$
1) Define maximum: $X_{max} = X(t)$
2) Predict amplitude: $A = X_{max} $
3) Adjust lower yield level F_y^- by (30)
4) Reset minimum: $X_{min} = 0$

FIGURE 13. Damper response with adaptive tuning.

Compared to the passive case in Fig. 11 it is seen that the yield level in Fig. 13a changes with the magnitude of the extremes. The aim of the adaptive tuning procedure is to maintain a constant and optimal shape of all closed loops based on the desired yield level to amplitude ratio given in (27). It is seen that the shape of the two loops in Fig. 13 are nearly the same, indicating that the amplitude dependence of the elasto-plastic damper is almost cancelled. A similar effect has been observed in terms of the semi-active concept for the Reid spring in [6].

5. NUMERICAL EXAMPLES

The purpose of the following numerical examples is to demonstrate the ability of the adaptive tuning procedure to control the amplitude dependence of the elasto-plastic model. Two structural models are considered: (a) a discrete model of a taut cable and (b) a simple shear frame. Elasto-plastic dampers with different values of μ are introduced, and equivalent linear and optimal viscous dampers are considered for comparison. The applied load is a distributed stochastic vector process with well defined, and fairly realistic, spatial and temporal correlation.

5.1. Discrete structural models

A discrete structural model is governed by an equation of motion similar to (9),

$$\mathbf{M}\ddot{\mathbf{q}}(t) + \mathbf{K}\mathbf{q}(t) = -\mathbf{w}F(t) + \mathbf{F}_{ext}(t) \quad (31)$$

where $\mathbf{F}_{ext}(t)$ in this case represents the distributed stochastic load process.

Semi-active dampers are often used and designed for damping of wind induced vibrations of stay-cables [9–11] and high-rise buildings [7,8]. Figure 14 shows discrete models of a taut cable and a shear frame with n degrees of freedom and with the damper attached to node 1 of the structure.

FIGURE 14. Discrete models: (a) taut cable, (b) Shear frame. Damper at node 1.

This means that the influence vector is $\mathbf{w}^T = [1, 0, \dots, 0]$. When using linear shape functions the mass and stiffness matrix for the cable are given as

$$\mathbf{M} = m \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}, \quad \mathbf{K} = k \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & & \\ & & \ddots & -1 \\ & & -1 & 2 \end{bmatrix} \quad (32)$$

where m is the lumped nodal mass and k is the element stiffness. For the shear frame node n represents a free end, which means that $\mathbf{K}(n, n) = k$ instead of $2k$. Consider a continuous cable with length l , mass per unit length ρ and tension T . Discretization into $n + 1$ elements, each of length $l_e = l/(n + 1)$, gives stiffness $k = T/l_e$ and nodal mass $m = \rho l_e$. The undamped natural frequency of mode j for the cable is given as, see [30],

$$\omega_{0j} = 2\sqrt{k/m} \sin\left(\frac{\pi}{2} \frac{j}{n+1}\right) \quad (33)$$

For the shear frame the natural frequencies are given by a similar expression with $(\pi/2)(2j - 1)/(2n + 1)$ as the argument of the sine-function. In this section the subscript j refers to the mode number.

The cable is discretized into 50 elements, while the shear frame has 10 storeys. For the cable $l_e/l = 0.02$. For an optimal viscous damper this leads to around 1% of maximum critical damping, [27]. In the following $m = 1$ and $\omega_{01} = 2\pi$ for both cable and shear frame. This implies that $k = 1000$ for the cable and $k = 1767$ for the shear frame. The governing modal parameters are summarized in Table 2. It is seen that $(\mathbf{w}^T \mathbf{u}_{0j})^2 / (\omega_{\infty j}^2 - \omega_{0j}^2)$, which governs the tuning of the damper, and $\Delta\omega_{\infty j} / \omega_{0j}$, which represents the attainable damping, are virtually independent of mode number j .

5.2. Damper properties

Three configurations of the elasto-plastic damper are considered with $\mu = 0.5, 0.75$ and 0.95 , respectively. The optimal parameters, including the relation between F_y and A , are determined by the steps summarized at the beginning of Section 4. The modal parameters $(\mathbf{w}^T \mathbf{u}_{0j})^2 / (\omega_{\infty j}^2 - \omega_{0j}^2)$ and $\Delta\omega_{\infty j} / \omega_{0j}$ are chosen for $j = 1$, see the values in Table 2. The optimal values for F_y/A are given in Table 3. Moreover, the damping ratios are found by (19) and given in Table 3 for modes 1 to 3.

The elasto-plastic damper is compared to the equivalent linear damper. The parameters are

$$c_{eq} \omega_{0j} = k_1 \beta, \quad k_{eq} = k_1 \kappa \quad (34)$$

where β and κ are given in (7) and (8). This means that c_{eq} depends on the undamped natural frequency ω_{0j} . The equivalent parameters are given in Table 3 for both cable and shear frame.

TABLE 2. Modal properties of taut cable and shear frame.

mode j	taut cable			shear frame		
	1	2	3	1	2	3
$\omega_{0j}/2\pi$	1.000	1.999	2.996	1.000	2.978	4.889
$\omega_{\infty j}/2\pi$	1.020	2.040	3.057	1.105	3.285	5.375
$\frac{(\mathbf{w}^T \mathbf{u}_{0j})^2}{\omega_{\infty j}^2 - \omega_{0j}^2} \cdot 10^4$	0.969	0.967	0.964	4.847	4.720	4.470
$\frac{\Delta\omega_{\infty j}}{\omega_{0j}}$	0.0204	0.0204	0.0203	0.105	0.103	0.100

TABLE 3. Parameters for elasto-plastic and optimal viscous damper. Damping ratios (in percent) for modes $j = 1 - 3$.

elasto-plastic damper														
taut cable							shear frame							
μ	F_y/A	c_{eq}	ω_{0j}	k_{eq}	ζ_1	ζ_2	ζ_3	F_y/A	c_{eq}	ω_{0j}	k_{eq}	ζ_1	ζ_2	ζ_3
0.50	2893	2242	10072	0.11	0.11	0.11		578	448	2014	0.58	0.57	0.55	
0.75	4757	4287	9386	0.22	0.22	0.22		951	857	1876	1.14	1.12	1.08	
0.95	6919	7685	6885	0.46	0.46	0.45		1383	1537	1377	2.35	2.31	2.31	
optimal viscous damper														
$c_{opt} = 1642$				1.02	0.82	0.61		$c_{opt} = 328$				5.25	3.18	2.10

Finally, the elasto-plastic damper is compared to a pure viscous damper that is optimal with respect to mode 1. The optimal viscous parameter c_{opt} follows from (21) for $\nu\omega_{01}c_{opt} = 1$.

$$c_{opt} = \frac{\omega_{\infty 1}^2 - \omega_{01}^2}{(\mathbf{w}^T \mathbf{u}_{01})^2 \omega_{01}} \quad (35)$$

The optimal viscous parameter and the associated modal damping ratios are given in the last row of Table 3.

5.3. Stochastic load process

The distributed load is modelled as a stochastic vector process with a well defined spatial and temporal correlation. The desired spatial correlation of the stochastic field is introduced via an n -term expansion of Karhunen-Loeve type, see [31],

$$\mathbf{F}_{ext}(t) = \boldsymbol{\mu}_F + \sum_{i=1}^n \mathbf{v}_i p_i(t) \quad (36)$$

The mean value $\boldsymbol{\mu}_F = \mathbf{0}$ in the present study. The vector $\mathbf{v}_i$ represents the i 'th eigenvector of the spatial covariance matrix $\boldsymbol{\Sigma}$ governed by the eigenvalue problem

$$\boldsymbol{\Sigma} \mathbf{v}_i = \lambda_i \mathbf{v}_i \quad (37)$$

The solutions to the eigenvalue problem are ordered with descending magnitude of λ_i , and the eigenvectors are normalized so that $\mathbf{v}_i^T \mathbf{v}_i = \lambda_i$. Hereby the covariance matrix can be recovered as $\boldsymbol{\Sigma} = \sum_i \mathbf{v}_i \mathbf{v}_i^T$. The spatial correlation is often considered purely exponential: $\rho(s) = \exp(-\gamma s)$, where s is the non-dimensional spatial separation. Note that in this section γ is the wave number of the load process. The use of an exponential correlation function leads to a finite probability of a positive mean value over an infinite plane, if a positive value is observed at one point. For most physical phenomena this is inconsistent. The inconsistency can be removed by introducing the factor $(1 - \frac{1}{2}\gamma s)$, which leads to the modified exponential format, see [32],

$$\rho(\gamma s) = (1 - \frac{1}{2}\gamma s) \exp(-\gamma s) \quad (38)$$

This format provides a change of sign at $\gamma s = 2$ and a negative minimum at $\gamma s = 3$, as illustrated in Fig. 15. The covariance matrix can now be written componentwise as

$$\Sigma_{ij} = \sigma_p^2 \left(1 - \frac{1}{2}\gamma \frac{|i-j|}{n} \right) \exp \left(-\gamma \frac{|i-j|}{n} \right) \quad (39)$$

where σ_p is the collocated intensity of the process.

The n individual stochastic processes $p_i(t)$, used in the series representation (36), are realized by passing filtered Gaussian processes $a_1(t)$ and $a_2(t)$ through a trigonometric envelope functions with characteristic frequency ω_p , see [33],

$$p(t) = p_0 [a_1(t) \cos(\omega_p t) + a_2(t) \sin(\omega_p t)] \quad (40)$$

FIGURE 15. Spatial correlation for modified exponential format.

The covariance function of this process is $R_p(t) = p_0^2 \cos \omega_p t$. Both a_1 and a_2 are filtered white-noise processes generated by the following first order filter with corner frequency ω_a ,

$$\dot{a}(t) + \omega_a a(t) = \omega_a n(t) \quad (41)$$

Here $n(t)$ is a unit white noise process. Sample points a_i with time separation Δt are computed by the one-step auto-regressive formula, see [34],

$$a_{i+1} = a_i e^{-\omega_a \Delta t} + n_i \sqrt{\frac{1}{2} \omega_a [1 - e^{-2\omega_a \Delta t}]} \quad (42)$$

where n_i is a sequence of independent normal variables. The process is normalized so that $p_0 = \sqrt{2/\beta}$, which corresponds to a unit standard deviation for $p(t)$. The double sided spectral density function of the process $p(t)$ is given as,

$$S_p(\omega) = \frac{p_0^2}{4\pi} \left[\frac{(\omega_a/\omega_p)^2}{(\omega_a/\omega_p)^2 + (\omega/\omega_p - 1)^2} + \frac{(\omega_a/\omega_p)^2}{(\omega_a/\omega_p)^2 + (\omega/\omega_p + 1)^2} \right] \quad (43)$$

where ω_p determines the peak frequency while ω_a controls the bandwidth. Figure 16 shows one half of the two-sided spectral density function for $\omega_a/\omega_p = 0.2, 0.5$ and 1.0 , respectively. The dots represent simulations with 10^6 sample points and $\Delta t = 0.01$. Good agreement is observed between simulations and the analytical solutions.

In the following two load cases are applied for both the cable and the shear frame. The governing parameters of these load cases are summarized in Table 4. Load cases 1 and 2 primarily excite modes 1 and 2, respectively.

FIGURE 16. Spectral density function for $\omega_a/\omega_p = 0.2, 0.5$ and 1 .

TABLE 4. Properties of load case 1 and 2

case	taut cable				shear frame			
	γ	$\omega_p/2\pi$	ω_a/ω_p	p_0	γ	$\omega_p/2\pi$	ω_a/ω_p	p_0
1	1.0	1.000	0.2	1.26	1.0	1.000	0.2	1.26
2	6.0	1.999	0.2	0.89	9.0	2.978	0.2	0.73

5.4. Results of simulations

The response of the cable and the shear frame are obtained by direct numerical time integration by means of the energy conserving algorithm, briefly summarized in Appendix B. Each simulated response record contains 10^6 time steps with time increment $\Delta t = 0.01$. For mode 1 response this corresponds to 10.000 periods. Due to the fairly narrow bandwidth of the load process the structural response is dominated by a single frequency component and a slowly varying envelope function. The latter is illustrated in Fig. 17 showing a part of the response for the cable with an elasto-plastic damper ($\mu = 0.5$). The cable is excited by load case 1 and the response is hence dominated by mode 1. A close-up is provided in Fig. 18 for the elasto-plastic damper with $\mu = 0.5$ and 0.95, respectively. It is seen that over a couple of periods the magnitude of the extremes are virtually unchanged. This indicates that the prediction of the amplitude based on (29) is sufficiently accurate.

Let σ_i^2 be the variance of node i of the structural model. The response measure σ is introduced by summation over all nodes,

$$\sigma = \sqrt{\sum_{i=1}^n \sigma_i^2} \quad (44)$$

The performance of the elasto-plastic damper with adaptive tuning is compared to the equivalent linear models and the optimal viscous damper in terms of σ for load cases 1 and 2, respectively. The response measure σ is given in Tables 5 for the cable and in Table 6 for the shear frame.

The first row gives σ for the elasto-plastic damper with adaptive tuning. The second row represents the equivalent linear damper with viscosity $c_{eq}\omega_{0j}$ and stiffness k_{eq} , see Table 3. For load cases 1 and 2 the viscosity c_{eq} is determined with respect to ω_{01} and ω_{02} , respectively. In row three the stiffness of the equivalent model is omitted, thereby leaving an equivalent linear viscous damper with viscous parameter c_{eq} . The fourth and final row gives σ for the optimal viscous damper with c_{opt} given in (35).

FIGURE 17. Response of damper on cable for $\mu = 0.5$.

FIGURE 18. Response of damper on cable: $\mu = 0.5$ (left) and 0.95 (right).

It is seen that the magnitude of the response for the elasto-plastic damper almost corresponds to that of the equivalent linear damper with viscosity and stiffness. This indicates that the adaptive tuning procedure is able to almost cancel the amplitude dependence of the elasto-plastic damper. The small deviations may be due to the assumption of sinusoidal motion leading to the equivalent linear model. As observed in Fig. 18 the damper response is clearly not sinusoidal. However, in general there appears to be a good agreement between the performance of the adaptively tuned elasto-plastic damper and the equivalent linear model.

The response measure σ for the equivalent viscous damper in row three is significantly smaller than for the elasto-plastic damper. This illustrates the deteriorating effect of damper stiffness. However, for increasing μ the influence of stiffness decreases, and it is seen that the efficiency of the elasto-plastic damper (row 1) and the pure equivalent viscous damper (row 3) converge. The influence of damper stiffness also becomes evident when comparing the performance of the elasto-plastic damper and the optimal viscous damper (row 4). For load case 1 the optimal viscous damper is significantly better than any of the elasto-plastic dampers. The superiority of the optimal viscous damper is however less evident for load case 2, where mode 2 is dominant. This is because the optimal viscous damper is optimal with respect to mode 1, whereas the efficiency of the elasto-plastic damper is practically independent of frequency. It is seen that the elasto-plastic damper with $\mu = 0.95$ becomes almost as effective as the optimal viscous damper for load case 2.

6. CONCLUSIONS

The paper considers damping of flexible structures by an elasto-plastic damper. An equivalent linear model for the elasto-plastic damper has been obtained by harmonic averaging [13]. An expression for the complex-valued natural frequency of the damped vibration mode is found in

TABLE 5. Response magnitude σ for taut cable.

load case μ	1			2		
	0.5	0.75	0.95	0.5	0.75	0.95
elasto-plastic	3.99	2.75	1.93	0.73	0.52	0.38
equivalent viscous & stiffness	4.28	3.08	2.15	0.79	0.57	0.42
equivalent viscous	2.26	1.72	1.47	0.41	0.33	0.31
optimal viscous (mode 1)	1.44			0.31		

TABLE 6. Response magnitude σ for shear frame

load case μ	1			2		
	0.5	0.75	0.95	0.5	0.75	0.95
elasto-plastic	0.70	0.47	0.33	0.073	0.051	0.037
equivalent viscous & stiffness	0.78	0.55	0.38	0.082	0.059	0.043
equivalent viscous	0.42	0.31	0.26	0.045	0.035	0.032
optimal viscous (mode 1)	0.25			0.033		

terms of a two-component system reduction technique introduced in [25]. Optimal tuning of the elasto-plastic damper corresponds to the maximum of the modal damping ratio. This leads to a non-linear relation between the stiffness and amplitude ratios, and consequently to an optimal relation between the yield level and the apparent amplitude of the damper response. Efficient and robust damping requires that the magnitude of the damper response is controlled effectively. This is realized by an adaptive tuning procedure where 1) the amplitude is estimated from the most recent maximum or minimum of the damper response history and 2) the yield level is determined from this estimate and the optimal yield level to amplitude ratio. The efficiency of this procedure is assessed by numerical examples, which show good agreement between the adaptively tuned elasto-plastic damper and the equivalent linear damper model. The numerical examples also demonstrate that the elasto-plastic maintains its efficiency for several vibration modes.

The present study demonstrates that effective and robust damping can be introduced semi-actively for elasto-plastic dampers by adjusting the yield force level relative to the amplitude level. In the case of MR-dampers, which have a force-displacement relation almost similar to that of the ideal friction damper, this type of control may be very attractive since the friction force level of MR-dampers is directly controlled by the input current [9]. And in contrast to linear control laws the present approach is efficient for several modes due to the hysteretic nature of the MR-damper. Moreover, semi-active amplitude control potentially eliminates the possibility of sticking, and thereby locking of the structure at the location of the damper.

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APPENDIX A. COMPUTATION OF DAMPER FORCE

At current time t_n the damper displacement X_n and damper force F_n are known. At the next time instant $t_{n+1} = t_n + \Delta t$ the damper force F_{n+1} is computed based on the new displacement $X_{n+1} = X_n + \Delta X$. The upper and lower yield limits of the elasto-plastic model are described by $\pm F_y + k_2 X$. Inside these yield limits the damper behaves elastically with incremental stiffness k_1 .

The procedure for calculating the new damper force is similar to the incremental procedure for numerical plasticity analysis. The present incremental formulation for the elasto-plastic damper is fully described by a combination of the three incremental cases depicted in Fig. A.1. Case (1) is pure elastic loading or unloading, case 2) is transition from elastic to plastic deformation, and (3) is pure plastic loading. Reversal points at local maxima or minima are determined by requiring zero velocity. When the velocity is interpolated linearly between $\dot{X}_n$ and $\dot{X}_{n+1}$ the maximum deformation X_{max} can be found explicitly. The incremental approach has been employed because it is easily extended to more general model with different rules of plasticity, and because the proposed adaptive tuning procedure in Table 1 is implemented in a straightforward manner. The pseudo-code for the updating of the damper force is given in Table A.1. Reference to the three incremental cases in Fig. A.1 are given in the right margin of the table. The pseudo-code is divided into two parts: elastic or plastic initial state (X_n, F_n) , respectively. Each part is divided into two sub-parts: pure loading or unloading, and the case where a maximum or minimum occurs, respectively. Every time a new force-displacement state, (X_{n+1}, F_{n+1}) or (X_{max}, F_{max}) , is computed it is verified whether this state exceeds the yield limit. If that is the case the state is pulled explicitly back on to the yield limit by e.g. $F_{n+1} = k_2 X_{n+1} \pm F_y$ where '+' and '-' represent upper and lower yield limits, respectively.

FIGURE A.1. Incremental formulation by (1) elastic loading or unloading, (2) from elastic to plastic regime and (3) plastic loading.

APPENDIX B. DIRECT TIME INTEGRATION PROCEDURE

The time-stepping algorithm is conveniently described with the displacement as the variable to be advanced. Let therefore

$$\Delta \mathbf{q} = \mathbf{q}_{n+1} - \mathbf{q}_n \quad (\text{B.1})$$

denote the displacement increment from time $t_n = n\Delta t$ to $t_{n+1} = (n+1)\Delta t$. When the displacement increment is expressed as the central difference of the corresponding velocities, $\Delta \mathbf{q} = \frac{1}{2}\Delta t(\dot{\mathbf{q}}_n + \dot{\mathbf{q}}_{n+1})$, the velocity increment is given as

$$\Delta \dot{\mathbf{q}} = (2/\Delta t)\Delta \mathbf{q} - 2\dot{\mathbf{q}}_n \quad (\text{B.2})$$

TABLE A.1. Pseudo-code for calculation of elasto-plastic force

Elastic initial state (X_n, F_n)	
if $ F_n - k_2 X_n < F_y$	
Pure loading or unloading	
if $\dot{X}_{n+1} \dot{X}_n \geq 0$	
$F_{n+1} = F_n + k_1(X_{n+1} - X_n)$	(1)
if $ F_{n+1} - k_2 X_{n+1} > F_y$	
$F_{n+1} = k_2 X_{n+1} \pm F_y$	(2)
Reversal at state (X_{max}, F_{max})	
if $\dot{X}_{n+1} \dot{X}_n < 0$	
$X_{max} = X_n + \frac{1}{2} \dot{X}_n^2 / (\dot{X}_n - \dot{X}_{n+1})$	
$F_{max} = F_n + k_1(X_{max} - X_n)$	(1)
if $ F_{max} - k_2 X_{max} > F_y$	
$F_{max} = k_2 X_{max} \pm F_y$	(2)
$F_{n+1} = F_{max} + k_1(X_{n+1} - X_{max})$	(1)
if $ F_{n+1} - k_2 X_{n+1} > F_y$	
$F_{n+1} = k_2 X_{n+1} \pm F_y$	(2)
Plastic initial state (X_n, F_n)	
if $ F_n - k_2 X_n = F_y$	
Pure loading or unloading	
if $\dot{X}_{n+1} \dot{X}_n \geq 0$	
if $(F_n - k_2 X_n)(X_{n+1} - X_n) \geq 0$	
$F_{n+1} = F_n + k_2(X_{n+1} - X_n)$	(3)
else	
$F_{n+1} = F_n + k_1(X_{n+1} - X_n)$	(1)
if $ F_{n+1} - k_2 X_{n+1} > F_y$	
$F_{n+1} = k_2 X_{n+1} \pm F_y$	(2)
Reversal at state (X_{max}, F_{max})	
if $\dot{X}_{n+1} \dot{X}_n < 0$	
$X_{max} = X_n + \frac{1}{2} \dot{X}_n^2 / (\dot{X}_n - \dot{X}_{n+1})$	
$F_{max} = F_n + k_2(X_{max} - X_n)$	(3)
$F_{n+1} = F_{max} + k_1(X_{n+1} - X_{max})$	(1)
if $ F_{n+1} - k_2 X_{n+1} > F_y$	
$F_{n+1} = k_2 X_{n+1} \pm F_y$	(2)

Equilibrium has been obtained at time t_n for the dynamic state $\mathbf{q}_n$ and $\dot{\mathbf{q}}_n$. The aim of the integration procedure is to obtain equilibrium at time $t_{n+1} = t_n + \Delta t$ by the following iterative procedure. First a predictor is determined by

$$\Delta \mathbf{q} = \Delta t \dot{\mathbf{q}}_n \quad (\text{B.3})$$

If the predictor does not lead to equilibrium a residual arises between the external and the internal forces. This residual may be expressed as the out of balance force with respect to the predicted

displacement state $\mathbf{q}_n + \Delta\mathbf{q}$,

$$\mathbf{r} = \mathbf{F}_n^{ext} + \mathbf{F}_{n+1}^{ext} - \mathbf{w}_r(F_n + F_{n+1}) - [(2/\Delta t)^2\mathbf{M} + (2/\Delta t)\mathbf{C}]\Delta\mathbf{q} + (4/\Delta t)\mathbf{M}\dot{\mathbf{q}}_n \quad (\text{B.4})$$

where the external force is constant over the interval. The internal force F represents the force produced by the elasto-plastic damper. This force is non-linear in the displacement increment and is determined explicitly on an incremental basis from the bilinear elasto-plastic relation as described in Appendix A. The residual may be reduced by means of Newton-Raphson iterations with the tangent stiffness matrix given as

$$\mathbf{K}_t = [\mathbf{K} + (2/\Delta t)^2\mathbf{M} + (2/\Delta t)\mathbf{C} + k_j\mathbf{w}\mathbf{w}^T] \quad (\text{B.5})$$

The last term represents the stiffness from the elasto-plastic damper whereby k_j is k_1 or k_2 depending whether the current state of the damper is inside or on the yield limit, respectively. The sub-incremental displacement correction is then found by

$$\delta\mathbf{q} = \mathbf{K}_t^{-1}\mathbf{r} \quad (\text{B.6})$$

and the displacement increment is updated by

$$\Delta\mathbf{q} = \Delta\mathbf{q} + \delta\mathbf{q} \quad (\text{B.7})$$

The residual is reduced below a predefined tolerance level by forming an iteration loop over equations (B.4) to (B.7). When convergence is reached the new displacement $\mathbf{q}_{n+1}$ is found by (B.1) and the corresponding velocity is updated by (B.2). A new step is taken by advancing time by Δt and computing a new predictor from (B.3).