The Cauchy problem for the Benjamin-Ono equation in L^2 revisited

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THE CAUCHY PROBLEM FOR THE BENJAMIN-ONO EQUATION IN L^2 REVISITED

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Abstract. In a recent work [12], Ionescu and Kenig proved that the Cauchy problem associated to the Benjamin-Ono equation is well-posed in $L^2(\mathbb{R})$. In this paper we give a simpler proof of Ionescu and Kenig's result, which moreover provides stronger uniqueness results. In particular, we prove unconditional well-posedness in $H^s(\mathbb{R})$, for $s > \frac{1}{4}$.

1. Introduction

The Benjamin-Ono equation is one of the fundamental equation describing the evolution of weakly nonlinear internal long waves. It has been derived by Benjamin [3] as an approximate model for long-crested unidirectional waves at the interface of a two-layer system of incompressible inviscid fluids, one being infinitely deep. In nondimensional variables, the initial value problem (IVP) associated to the Benjamin-Ono equation (BO) writes as

$$\begin{cases} \partial_t u + \mathcal{H}\partial_x^2 u = u\partial_x u \\ u(x,0) = u_0(x), \end{cases} \tag{1.1}$$

where $x \in \mathbb{R}$ or $\mathbb{T}$, $t \in \mathbb{R}$, u is a real-valued function, and $\mathcal{H}$ is the Hilbert transform, *i.e.*

$$\mathcal{H}f(x) = \text{p.v.}\, \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(y)}{x-y}dy. \tag{1.2}$$

The Benjamin-Ono equation is, at least formally, completely integrable [2] and thus possesses an infinite number of conservation laws. For example, the momentum and the energy, respectively given by

$$M(u) = \int u^2 dx, \quad \text{and} \quad E(u) = \frac{1}{2}\int \left|D_x^{\frac{1}{2}} u\right|^2 dx + \frac{1}{6}\int u^3 dx, \tag{1.3}$$

are conserved by the flow of (1.1).

The IVP associated to the Benjamin-Ono equation presents interesting mathematical difficulties and has been extensively studied in the recent years. In the continuous case, well-posedness in $H^s(\mathbb{R})$ for $s > \frac{3}{2}$ was proved by Iorio [13] by using purely hyperbolic energy methods (see also [1] for global well-posedness in the same rage of s). Then, Ponce [25] derived a local smoothing effect associated

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to the dispersive part of the equation, which combined to compactness methods, enable to reach $s = \frac{3}{2}$. This technique was refined by Koch and Tzvetkov [17] and Kenig and Koenig [14] who reach respectively $s > \frac{5}{4}$ and $s > \frac{9}{8}$. On the other hand Molinet, Saut and Tzvetkov [23] proved that the flow map associated to BO, when it exists, fails to be C^2 in any Sobolev space $H^s(\mathbb{R})$, $s \in \mathbb{R}$. This results is based on the fact that the dispersive smoothing effects of the linear part of BO are not strong enough to control the low-high frequency interactions appearing in the nonlinearity of (1.1). It was improved by Koch and Tzvetkov [14] who showed that the flow map fails even to be uniformly continuous in $H^s(\mathbb{R})$ for $s > 0$ (see [4] for the same result in the case $s < -1/2$.) As the consequence of those results, one cannot solve the Cauchy problem for the Benjamin-Ono by a Picard iterative method implemented on the integral equation associated to (1.1) for initial data in the Sobolev space $H^s(\mathbb{R})$, $s \in \mathbb{R}$. In particular, the methods introduced by Bourgain [6] and Kenig, Ponce and Vega [15], [16] for the Korteweg- de Vries equation do not apply directly to the Benjamin-Ono equation.

Therefore, the problem to obtain well-posedness in less regular Sobolev spaces turns out to be far from trivial. Due to the conservations laws (1.3), $L^2(\mathbb{R})$ and $H^{\frac{1}{2}}(\mathbb{R})$ are two natural spaces where well-posedness is expected. In this direction, a decisive breakthrough was achieved by Tao [26]. By combining a complex variant of the Cole-Hopf transform (which linearizes Burgers equation) with Strichartz estimates, he proved well-posedness in $H^1(\mathbb{R})$. More precisely, to obtain estimates at the H^1-level, he introduced the new unknown

$$w = \partial_x P_{+hi}\big(e^{-\frac{i}{2}F}\big), \tag{1.4}$$

where F is some spatial primitive of u and P_{+hi} denotes the projection on high positive frequencies. Then w satisfies an equation on the form

$$\partial_t w - i\partial_x^2 w = -\partial_x P_{+hi}\big(\partial_x^{-1} w P_- \partial_x u\big) + \text{negligible terms}. \tag{1.5}$$

Observe that, thanks to the frequency projections, the nonlinear term appearing on the right-hand side of (1.5) does not exhibit any low-high frequency interaction terms. Finally, to inverse this gauge transformation, one gets an equation on the form

$$u = 2ie^{\frac{i}{2}F} w + \text{negligible terms}. \tag{1.6}$$

Very recently, Burq and Planchon [7], and Ionescu and Kenig [12] were able to use Tao's ideas in the context of Bourgain's spaces to prove well-posedness for the Benjamin-Ono equation in $H^s(\mathbb{R})$ for respectively $s > \frac{1}{4}$ and $s \geq 0$. The main difficulty arising here is that Bourgain's spaces do not enjoy an algebra property, so that one is loosing regularity when estimating u in terms of w via equation (1.6). Burq and Planchon first paralinearized the equation and then used a localized version of the gauge transformation on the worst nonlinear term. On the other hand, Ionescu and Kenig decomposed the solution in two parts: the first one is the smooth solution of BO evolving from the low frequency part of the initial data, while the second one solves a dispersive system renormalized by a gauge transformation involving the first part. The authors then were able to solve the system via a fixed point argument in a dyadic version of Bourgain's spaces (already used in the context of wave maps [27]) with a special structure in low frequencies. It is worth noticing that their result only ensures the uniqueness in the class of limits of smooth solutions, while Burq and Planchon obtained a stronger uniqueness result.

Indeed, by applying their approach to the equation satisfied by the difference of two solutions, they succeed in proving that the flow map associated to BO is Lipschitz in a weaker topology when the initial data belongs to $H^s(\mathbb{R})$, $s > \frac{1}{4}$.

In the periodic setting, Molinet [20], [21] proved well-posedness in $H^s(\mathbb{T})$ for successively $s \geq \frac{1}{2}$ and $s \geq 0$. Once again, these works combined Tao's gauge transformation with estimates in Bourgain's spaces. It should be pointed out that in the periodic case, one can assume that u has mean value zero to define a primitive. Then, it is easy to check by the mean value theorem that the gauge transformation in (1.4) is Lipschitz from L^2 into L^∞. This property, which is not true in the real line, is crucial to prove the uniqueness and the Lipschitz property of the flow map.

The aim of this paper is to give a simpler proof of Ionescu and Kenig's result, which also provides a stronger uniqueness result for the solutions at the L^2- level. It is worth noticing that to reach L^2 in [12] or [21] the authors substituted u in (1.4) by the formula given in (1.6). The good side of this substitution is that now u will not appear anymore in (1.4). On the other hand, it introduces new technical difficulties to handle the multiplication by $e^{\mp iF/2}$ in Bourgain's spaces. In the present paper we are able to avoid this substitution which will really simplify the proof. Our main result is the following

Theorem 1.1. *Let $s \geq 0$ be given.*
Existence : For all $u_0 \in H^s(\mathbb{R})$ and all $T > 0$, there exists a solution

$$u \in C([0,T]; H^s(\mathbb{R})) \cap X_T^{s-1,1} \cap L_T^4 W_x^{s,4} \tag{1.7}$$

of (1.1) *such that*

$$w = \partial_x P_{+hi}\big(e^{-\frac{i}{2}F[u]}\big) \in Y_T^s. \tag{1.8}$$

where $F[u]$ is some primitive of u defined in (3.2).

Uniqueness : This solution is unique in the following classes :

i) $u \in L^\infty(]0,T[; L^2(\mathbb{R})) \cap L^4(]0,T[\times\mathbb{R})$ *and* $w \in X_T^{0,\frac{1}{2}}$.
ii) $u \in L^\infty(]0,T[; H^s(\mathbb{R})) \cap L_T^4 W_x^{s,4}$ *whenever* $s > 0$.
iii) $u \in L^\infty(]0,T[; H^s(\mathbb{R}))$ *whenever* $s > \frac{1}{4}$.

Moreover, $u \in C_b(\mathbb{R}; L^2(\mathbb{R}))$ and the flow map data-solution : $u_0 \mapsto u$ is continuous from $H^s(\mathbb{R})$ into $C([0,T]; H^s(\mathbb{R}))$.

Note that above $H^s(\mathbb{R})$ denotes the space of all real-valued functions with the usual norm, $X_T^{s,b}$ and Y_T^s are Bourgain spaces defined in Subsection 2.2, while the primitive $F[u]$ of u is defined in Subsection 3.1.

Remark 1.2. Since the function spaces in the uniqueness class i) are reflexive and since $\partial_x P_{+hi}\big(e^{-\frac{i}{2}F[u_n]}\big)$ converges to $\partial_x P_{+hi}\big(e^{-\frac{i}{2}F[u]}\big)$ in $L^\infty(]-T,T[; L^2(\mathbb{R}))$ whenever u_n converges to u in $L^\infty(]-T,T[; L^2(\mathbb{R}))$, our result clearly implies the uniqueness in the class of $L^\infty(]-T,T[; L^2(\mathbb{R}))$-limits of smooth solutions.

Remark 1.3. It is worth noticing that for $s > 0$ we get a uniqueness class without condition on w (see [7] for the case $s > \frac{1}{4}$).

Remark 1.4. According to *iii)* we get unconditional well-posedness in $H^s(\mathbb{R})$ for $s > \frac{1}{4}$. This implies in particular the uniqueness of the (energy) weak solutions that belong to $L^\infty(\mathbb{R}; H^{1/2}(\mathbb{R}))$. These solutions are constructed by regularizing the equation and passing to the limit as the regularizing coefficient goes to 0 (taking into account some energy estimate for the regularizing equation related to the energy conservation of (1.1)) .

Our proof also combines Tao's ideas with the use of Bourgain's spaces. Actually, it follows closely the strategy introduced by the first author in [20]. The main new ingredient is a bilinear estimate for the nonlinear term appearing in (1.5), which allows to recover one derivative at the L^2-level. It is interesting to note that, at the H^s-level with $s > 0$, this estimate follows from the Cauchy-Schwarz method introduced by Kenig, Ponce and Vega in [16] (see the appendix for the use of this method in some region of integration). To reach L^2, one of the main difficulty is that we cannot substitute the Fourier transform of u by its modulus in the bilinear estimate since we are not able to prove that $\mathcal{F}^{-1}(|\hat{u}|)$ belongs $L^4_{x,t}$ but only that u belongs to $L^4_{x,t}$. To overcome this difficulty we use a Littlewood-Paley decomposition of the functions and carefully divide the domain of integration into suitable disjoint subdomains.

To obtain our uniqueness result, following the same method as in the periodic setting, we derive a Lipschitz bound for the gauge transformation from some affine subspaces of $L^2(\mathbb{R})$ into $L^\infty(\mathbb{R})$. Recall that this is clearly not possible for general initial data since it would imply the uniform continuity of the flow-map. The main idea is to notice that such Lipschitz bound holds for solutions emanating from initial data having the same low frequency part and this is sufficient for our purpose.

Let us point out some applications. First our uniqueness result allows to really simplify the proof of the continuity of the flow map associated to the Benjamin-Ono equation for the weak topology of $L^2(\mathbb{R})$. This result was recently proved by Cui and Kenig [9].

It is also interesting to observe that the method of proof used here still works in the periodic setting, and thus, we reobtain the well-posedness result [21] in a simpler way. Moreover, as in the continuous case, we also prove new uniqueness results (see Theorem 7.1 below). In particular, we get unconditional well-posedness in $H^s(\mathbb{T})$ as soon as $s \geq \frac{1}{2}$.

Finally, we believe that this technique may be useful for another nonlinear dispersive equations presenting the same kind of difficulties as the Benjamin-Ono equation. For example, consider the higher-order Benjamin-Ono equation

$$\partial_t v - b\mathcal{H}\partial_x^2 v + a\partial_x^3 v = cv\partial_x v - d\partial_x(v\mathcal{H}\partial_x v + \mathcal{H}(v\partial_x v)), \tag{1.9}$$

where $x,\, t \in \mathbb{R}$, v is a real-valued function, $a \in \mathbb{R}$, b, c and d are positive constants. The equation above corresponds to a second order approximation model of the same phenomena described by the Benjamin-Ono equation. It was derived by Craig, Guyenne and Kalisch [8] using a Hamiltonian perturbation theory, and possesses an energy at the H^1-level. As for the Benjamin-Ono equation, the flow map associated to (1.9) fails to be smooth in any Sobolev space $H^s(\mathbb{R})$, $s \in \mathbb{R}$ [24]. Recently, the Cauchy problem associated to (1.9) was proved to be well-posed in $H^2(\mathbb{R})$ [19]. In a forthcoming paper, the authors will show that it is actually well-posed in the energy space $H^1(\mathbb{R})$.

This paper is organized as follows: in the next section, we introduce the notations, define the function spaces and recall some classical linear estimates. Section 3 is devoted to the key nonlinear estimates, which are used in Section 4 to prove the main part of Theorem 1.1, while the assertions *i*) and *ii*) are proved in Section 5.

In Section 6, we give a simple proof of the continuity of the flow-map for the weak $L^2(\mathbb{R})$-topology whereas Section 7 is devoted to some comments and new results in the periodic case. Finally, in the appendix we prove the bilinear estimate used in Section 5.

2. Notation, function spaces and preliminary estimates

2.1. **Notation.** For any positive numbers a and b, the notation $a \lesssim b$ means that there exists a positive constant c such that $a \le cb$. We also denote $a \sim b$ when $a \lesssim b$ and $b \lesssim a$. Moreover, if $\alpha \in \mathbb{R}$, α_+, respectively α_-, will denote a number slightly greater, respectively lesser, than α.

For $u = u(x,t) \in \mathcal{S}(\mathbb{R}^2)$, $\mathcal{F}u = \widehat{u}$ will denote its space-time Fourier transform, whereas $\mathcal{F}_x u = (u)^{\wedge_x}$, respectively $\mathcal{F}_t u = (u)^{\wedge_t}$, will denote its Fourier transform in space, respectively in time. For $s \in \mathbb{R}$, we define the Bessel and Riesz potentials of order $-s$, J^s_x and D^s_x, by

$$J^s_x u = \mathcal{F}_x^{-1}\big((1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}_x u\big) \quad \text{and} \quad D^s_x u = \mathcal{F}_x^{-1}\big(|\xi|^s \mathcal{F}_x u\big).$$

Throughout the paper, we fix a cutoff function η such that

$$\eta \in C_0^\infty(\mathbb{R}), \quad 0 \le \eta \le 1, \quad \eta_{|_{[-1,1]}} = 1 \quad \text{and} \quad \operatorname{supp}(\eta) \subset [-2,2].$$

We define

$$\phi(\xi) := \eta(\xi) - \eta(2\xi) \quad \text{and} \quad \phi_{2^l}(\xi) := \phi(2^{-l}\xi).$$

Any summations over capitalized variables such as N are presumed to be dyadic with $N \ge 1$, *i.e.*, these variables range over numbers of the form 2^n, $n \in \mathbb{Z}_+$. Then, we have that

$$\sum_N \phi_N(\xi) = 1 - \eta(2\xi), \ \forall \xi \ne 0 \quad \text{and} \quad \operatorname{supp}(\phi_N) \subset \{\frac{N}{2} \le |\xi| \le 2N\}.$$

Let us define the Littlewood-Paley multipliers by

$$P_N u = \mathcal{F}_x^{-1}\big(\phi_N \mathcal{F}_x u\big), \quad \text{and} \quad P_{\ge N} := \sum_{K \ge N} P_K.$$

Moreover, we also define the operators P_{hi}, P_{HI}, P_{lo} and P_{LO} by

$$P_{hi} = \sum_{N \ge 2} P_N, \quad P_{HI} = \sum_{N \ge 8} P_N, \quad P_{lo} = 1 - P_{hi}, \quad \text{and} \quad P_{LO} = 1 - P_{HI}.$$

Let P_+ and P_- the projection on respectively the positive and the negative Fourier frequencies. Then

$$P_\pm u = \mathcal{F}_x^{-1}\big(\chi_{\mathbb{R}_\pm}\mathcal{F}_x u\big),$$

and we also denote $P_{\pm hi} = P_\pm P_{hi}$, $P_{\pm HI} = P_\pm P_{HI}$, $P_{\pm lo} = P_\pm P_{lo}$ and $P_{\pm LO} = P_\pm P_{LO}$. Observe that P_{hi}, P_{HI}, P_{lo} and P_{LO} are bounded operators on $L^p(\mathbb{R})$ for $1 \le p \le \infty$, while $P_\pm$ are only bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$. We also note that

$$\mathcal{H} = -iP_+ + iP_-.$$

Finally, we denote by $U(\cdot)$ the free group associated with the linearized Benjamin-Ono equation, which is to say,

$$\mathcal{F}_x\big(U(t)f\big)(\xi) = e^{-it|\xi|\xi}\mathcal{F}_x f(\xi).$$

2.2. **Function spaces.** For $1 \le p \le \infty$, $L^p(\mathbb{R})$ is the usual Lebesgue space with the norm $\|\cdot\|_{L^p}$, and for $s \in \mathbb{R}$, the real-valued Sobolev spaces $H^s(\mathbb{R})$ and $W^{s,p}(\mathbb{R})$ denote the spaces of all real-valued functions with the usual norms

$$\|f\|_{H^s} = \|J^s u\|_{L^2} \quad \text{and} \quad \|f\|_{W^{s,p}} = \|J^s_x f\|_{L^p}.$$

For $1 < p < \infty$, we define the space $\tilde{L}^p$

$$\|f\|_{\tilde{L}^p} = \|P_{lo} f\|_{L^p} + \Big(\sum_N \|P_N f\|_{L^p}^2\Big)^{\frac12}.$$

Observe that when $p \ge 2$, the Littlewood-Paley theorem on the square function and Minkowski's inequality imply that the injection $\tilde{L}^p \hookrightarrow L^p$ is continuous. Moreover, if $u = u(x,t)$ is a real-valued function defined for $x \in \mathbb{R}$ and t in the time interval $[0,T]$, with $T > 0$, if B is one of the spaces defined above and $1 \le p \le \infty$, we will define the mixed space-time spaces $L^p_T B_x$, respectively $L^p_t B_x$, by the norms

$$\|u\|_{L^p_T B_x} = \Big(\int_0^T \|u(\cdot,t)\|_B^p dt\Big)^{\frac1p} \quad \text{respectively} \quad \|u\|_{L^p_t B_x} = \Big(\int_{\mathbb{R}} \|u(\cdot,t)\|_B^p dt\Big)^{\frac1p}.$$

For $s,\ b \in \mathbb{R}$, we introduce the Bourgain spaces $X^{s,b}$ and $Z^{s,b}$ related to the Benjamin-Ono equation as the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ under the norms

$$\|u\|_{X^{s,b}} := \Big(\int_{\mathbb{R}^2} \langle \tau + |\xi|\xi\rangle^{2b} \langle \xi\rangle^{2s} |\widehat{u}(\xi,\tau)|^2 d\xi d\tau\Big)^{1/2}, \tag{2.1}$$

$$\|u\|_{Z^{s,b}} := \Big(\int_{\mathbb{R}} \Big(\int_{\mathbb{R}} \langle \tau + |\xi|\xi\rangle^{b} \langle \xi\rangle^{s} |\widehat{u}(\xi,\tau)| d\tau\Big)^2 d\xi\Big)^{1/2}, \tag{2.2}$$

$$\|u\|_{\widetilde{Z}^{s,b}} = \|P_{lo} u\|_{Z^{s,b}} + \Big(\sum_N \|P_N u\|_{Z^{s,b}}^2\Big)^{\frac12}, \tag{2.3}$$

and

$$\|u\|_{Y^s} = \|u\|_{X^{s,\frac12}} + \|u\|_{\widetilde{Z}^{s,0}}, \tag{2.4}$$

where $\langle x\rangle := 1 + |x|$. We will also use the localized (in time) version of these spaces. Let $T > 0$ be a positive time and $\|\cdot\|_B = \|\cdot\|_{X^{s,b}}$, $\|\cdot\|_{\widetilde{Z}^{s,b}}$ or $\|\cdot\|_{Y^s}$. If $u : \mathbb{R} \times [0,T] \to \mathbb{C}$, then

$$\|u\|_{B_T} := \inf\{\|\tilde{u}\|_B \mid \tilde{u} : \mathbb{R}\times\mathbb{R} \to \mathbb{C}, \tilde{u}|_{\mathbb{R}\times[0,T]} = u\}.$$

it is worth recalling that

$$Y^s_T \hookrightarrow Z^{s,0}_T \hookrightarrow C([0,T]; H^s(\mathbb{R})).$$

2.3. **Linear estimates.** In this subsection, we recall some linear estimates in Bourgain's spaces which will be needed later. The first ones are well-known (cf. [10] for example).

Lemma 2.1 (Homogeneous linear estimate)**.** *Let $s \in \mathbb{R}$. Then*

$$\|\eta(t) U(t) f\|_{Y^s} \lesssim \|f\|_{H^s}. \tag{2.5}$$

Lemma 2.2 (Non-homogeneous linear estimate)**.** *Let $s \in \mathbb{R}$. Then for any $0 < \delta < 1/2$,*

$$\big\|\eta(t)\int_0^t U(t-t')g(t')dt'\big\|_{X^{s,\frac12+\delta}} \lesssim \|g\|_{X^{s,-\frac12+\delta}} \tag{2.6}$$

and

$$\big\|\eta(t)\int_0^t U(t-t')g(t')dt'\big\|_{Y^s} \lesssim \|g\|_{X^{s,-\frac12}} + \|g\|_{\widetilde{Z}^{s,-1}}. \tag{2.7}$$

Proof of Lemmas 2.1 and 2.2. The proof of Lemmas 2.1 and 2.2 is a direct consequence of the classical linear estimates for $X^{s,b}$ and $Z^{s,b}$ and the fact that

$$\|u\|_{X^{s,b}} = \|P_{lo}u\|_{X^{s,b}} + \big(\sum_N \|P_N u\|_{X^{s,b}}^2\big)^{1/2}.$$

□

Lemma 2.3. *For any $T > 0$, $s \in \mathbb{R}$ and for all $-\frac12 < b' \le b < \frac12$, it holds*

$$\|u\|_{X_T^{s,b'}} \lesssim T^{b-b'}\|u\|_{X_T^{s,b}}. \tag{2.8}$$

The following Bourgain-Strichartz estimates will also be useful.

Lemma 2.4. *It holds that*

$$\|u\|_{L^4_{x,t}} \lesssim \|u\|_{\widetilde{L}^4_{x,t}} \lesssim \|u\|_{X^{0,\frac38}} \tag{2.9}$$

and for any $T > 0$ and $\frac38 \le b \le \frac12$,

$$\|u\|_{L^4_{x,T}} \lesssim T^{b-\frac38}\|u\|_{X_T^{0,b}}. \tag{2.10}$$

Proof. Estimate (2.9) follows directly by applying the estimate

$$\|u\|_{L^4_{x,t}} \lesssim \|u\|_{X^{0,\frac38}},$$

proved in the appendix of [20], to each dyadic block on the left-hand side of (2.9).

To prove (2.10), we choose an extension $\tilde{u} \in X^{0,b}$ of u such that $\|\tilde{u}\|_{X^{0,b}} \le 2\|u\|_{X_T^{0,b}}$. Therefore, it follows from (2.8) and (2.9) that

$$\|u\|_{L^4_{x,T}} \le \|\tilde{u}\|_{L^4_{x,t}} \lesssim \|\tilde{u}\|_{X^{0,\frac38}} \lesssim T^{b-\frac38}\|u\|_{X_T^{0,b}}.$$

□

2.4. **Fractional Leibniz's rules.** First we state the classical fractional Leibniz rule estimate derived by Kenig, Ponce and Vega (See Theorems A.8 and A.12 in [15]).

Proposition 2.5. *Let $0 < \alpha < 1$, $p,\ p_1,\ p_2 \in (1,+\infty)$ with $\frac{1}{p_1} + \frac{1}{p_2} = \frac1p$ and $\alpha_1,\ \alpha_2 \in [0,\alpha]$ with $\alpha = \alpha_1 + \alpha_2$. Then,*

$$\big\|D_x^\alpha(fg) - fD_x^\alpha g - gD_x^\alpha f\big\|_{L^p} \lesssim \|D_x^{\alpha_1}g\|_{L^{p_1}}\|D_x^{\alpha_2}f\|_{L^{p_2}}. \tag{2.11}$$

Moreover, for $\alpha_1 = 0$, the value $p_1 = +\infty$ is allowed.

The next estimate is a frequency localized version of estimate (2.11) in the same spirit as Lemma 3.2 in [26]. It allows to share most of the fractional derivative in the first term on the right-hand side of (2.12).

Lemma 2.6. *Let $\alpha \geq 0$ and $1 < q < \infty$. Then,*

$$\big\|D_x^\alpha P_+\big(fP_-\partial_x g\big)\big\|_{L^q} \lesssim \|D_x^{\alpha_1} f\|_{L^{q_1}}\|D_x^{\alpha_2} g\|_{L^{q_2}}, \tag{2.12}$$

with $1 < q_i < \infty$, $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{q}$ and $\alpha_1 \geq \alpha$, $\alpha_2 \geq 0$ and $\alpha_1+\alpha_2 = 1+\alpha$.

Proof. See Lemma 3.2 in [20]. □

Finally, we derive an estimate to handle the multiplication by a term on the form $e^{\pm\frac{i}{2}F}$, where F is a real-valued function, in fractional Sobolev spaces.

Lemma 2.7. *Let $2 \leq q < \infty$ and $0 \leq \alpha \leq \frac{1}{q}$. Consider F_1 and F_2 two real-valued functions such that $u_j = \partial_x F_j$ belongs to $L^2(\mathbb{R})$ for $j = 1, 2$. Then, it holds that*

$$\|J_x^\alpha\big(e^{\pm\frac{i}{2}F_1}g\big)\|_{L^q} \lesssim (1+\|u_1\|_{L^2})\|J_x^\alpha g\|_{L^q}, \tag{2.13}$$

and

$$\begin{aligned}&\|J_x^\alpha\big(\big(e^{\pm\frac{i}{2}F_1}-e^{\pm\frac{i}{2}F_2}\big)g\big)\|_{L^q}\\ &\quad\lesssim \Big(\|u_1-u_2\|_{L^2}+\|e^{\pm\frac{i}{2}F_1}-e^{\pm\frac{i}{2}F_2}\|_{L^\infty}(1+\|u_1\|_{L^2})\Big)\|J_x^\alpha g\|_{L^q}.\end{aligned} \tag{2.14}$$

Proof. In the case $\alpha = 0$, we deduce from Hölder's inequality that

$$\|e^{\pm\frac{i}{2}F_1}g\|_{L^q} \leq \|g\|_{L^q}, \tag{2.15}$$

since F_1 is real-valued. Therefore we can assume that $0 < \alpha \leq \frac{1}{q}$ and it is enough to bound $\|D_x^\alpha\big(e^{\pm\frac{i}{2}F_1}g\big)\|_{L^q}$. First, we observe that

$$\|D_x^\alpha\big(e^{\pm\frac{i}{2}F_1}g\big)\|_{L^q} \leq \|D_x^\alpha\big(P_{lo}e^{\pm\frac{i}{2}F_1}g\big)\|_{L^q}+\|D_x^\alpha\big(P_{hi}e^{\pm\frac{i}{2}F_1}g\big)\|_{L^q} \tag{2.16}$$

Estimate (2.11) and Bernstein's inequality imply that

$$\begin{aligned}&\|D_x^\alpha\big(P_{lo}e^{\pm\frac{i}{2}F_1}g\big)\|_{L^q}\\ &\quad\lesssim \|P_{lo}e^{\pm\frac{i}{2}F_1}\|_{L^\infty}\|D_x^\alpha g\|_{L^q}+\|D_x^\alpha P_{lo}e^{\pm\frac{i}{2}F_1}\|_{L^\infty}\|g\|_{L^q} \lesssim \|J_x^\alpha g\|_{L^q}.\end{aligned} \tag{2.17}$$

On the other hand, by using again estimate (2.11), we get that

$$\|D_x^\alpha\big(P_{hi}e^{\pm\frac{i}{2}F_1}g\big)\|_{L^q} \lesssim \|P_{hi}e^{\pm\frac{i}{2}F_1}\|_{L^\infty}\|D_x^\alpha g\|_{L^q}+\|g\|_{L^{q_1}}\|D_x^\alpha P_{hi}e^{\pm\frac{i}{2}F_1}\|_{L^{q_2}},$$

with $\frac{1}{q_1}=\frac{1}{q}-\alpha$, $\frac{1}{q_2}=\alpha$, so that $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{q}$. Then, it follows from the facts that F_1 is real-valued, $\partial_x F_1 = u_1$ and the Sobolev embedding that

$$\begin{aligned}\|D_x^\alpha\big(P_{hi}e^{\pm\frac{i}{2}F_1}g\big)\|_{L^q} &\lesssim \|D_x^\alpha g\|_{L^q}+\|J_x^\alpha g\|_{L^q}\|D_x^{\alpha+\frac12}P_{hi}e^{\pm\frac{i}{2}F_1}\|_{L^2}\\ &\lesssim \|J_x^\alpha g\|_{L^q}\big(1+\|\partial_x e^{\pm\frac{i}{2}F_1}\|_{L^2}\big)\\ &\lesssim \|J_x^\alpha g\|_{L^q}\big(1+\|u_1\|_{L^2}\big).\end{aligned} \tag{2.18}$$

The proof of estimate (2.13) is concluded gathering (2.15)–(2.18).

Estimate (2.14) can be obtained exactly in the same way, using that

$$\|\partial_x\big(e^{\pm\frac{i}{2}F_1}-e^{\pm\frac{i}{2}F_2}\big)\|_{L^2} \lesssim \|u_1-u_2\|_{L^2}+\|e^{\pm\frac{i}{2}F_1}-e^{\pm\frac{i}{2}F_2}\|_{L^\infty}\|u_1\|_{L^2}. \tag{2.19}$$

□

3. A priori estimates in $H^s(\mathbb{R})$ for $s \geq 0$

In this section we will derive *a priori* estimates on a solution u to (1.1) at the H^s-level, for $s \geq 0$. First, following Tao in [26], we perform a nonlinear transformation on the equation to weaken the high-low frequency interaction in the nonlinearity. Furthermore, since we want to reach L^2, we will need to use Bourgain spaces. This requires a new bilinear estimate which will be derive in Subsection 3.2.

3.1. **The gauge transformation.** Let u be a solution to the equation in (1.1). First, we construct a spatial primitive $F = F[u]$ of u, *i.e.* $\partial_x F = u$, that satisfies the equation :

$$\partial_t F = -\mathcal{H}\partial_x^2 F + \frac{1}{2}(\partial_x F)^2. \tag{3.1}$$

It is worth noticing that these two properties defined F up to a constant. In order to construct F for u with low regularity, we use the construction of Burq and Planchon in [7]. Consider $\psi \in C_0^\infty(\mathbb{R})$ such that $\int_{\mathbb{R}} \psi(y)dy = 1$ and define

$$F(x,t) = \int_{\mathbb{R}} \psi(y)\Big(\int_y^x u(z,t)dz\Big)dy + G(t), \tag{3.2}$$

as a mean of antiderivatives of u. Obviously, $\partial_x F = u$ and

$$\begin{aligned}
\partial_t F(x,t) &= \int_{\mathbb{R}} \psi(y)\Big(\int_y^x \partial_t u(z,t)dz\Big)dy + G'(t) \\
&= \int_{\mathbb{R}} \psi(y)\Big(\int_y^x \big(-\mathcal{H}\partial_z^2 u(z,t) + \frac{1}{2}\partial_z(u(z,t)^2)\big)dz\Big)dy + G'(t) \\
&= -\mathcal{H}\partial_x u(x,t) + \frac{1}{2}u(x,t)^2 + \int_{\mathbb{R}} \big(\mathcal{H}\psi'(y)u(y,t) - \psi(y)\frac{1}{2}u(y,t)^2\big)dy + G'(t).
\end{aligned}$$

Therefore we choose G as

$$G(t) = \int_0^t \int_{\mathbb{R}} \big(-\mathcal{H}\psi'(y)u(y,s) + \psi(y)\frac{1}{2}u(y,s)^2\big)dyds,$$

to ensure that (3.1) is satisfied. Observe that this construction makes sense for $u \in L^2_{\text{loc}}(\mathbb{R}^2)$. Next, we introduce the new unknown

$$W = P_{+hi}\big(e^{-\frac{i}{2}F}\big) \quad \text{and} \quad w = \partial_x W = -\frac{i}{2}P_{+hi}\big(e^{-\frac{i}{2}F}u\big). \tag{3.3}$$

Then, it follows from (3.1) and the identity $\mathcal{H} = -i(P_+ - P_-)$ that

$$\begin{aligned}
\partial_t W + \mathcal{H}\partial_x^2 W = \partial_t W - i\partial_x^2 W &= -\frac{i}{2}P_{+hi}\big(e^{-\frac{i}{2}F}(\partial_t F - i\partial_x^2 F - \frac{1}{2}(\partial_x F)^2)\big) \\
&= -P_{+hi}\big(WP_-\partial_x u\big) - P_{+hi}\big(P_{lo}e^{-\frac{i}{2}F}P_-\partial_x u\big),
\end{aligned}$$

since the term $-P_{+hi}\big(P_{-hi}e^{-\frac{i}{2}F}P_-\partial_x u\big)$ cancels due to the frequency localization. Thus, it follows differentiating that

$$\partial_t w - i\partial_x^2 w = -\partial_x P_{+hi}\big(WP_-\partial_x u\big) - \partial_x P_{+hi}\big(P_{lo}e^{-\frac{i}{2}F}P_-\partial_x u\big). \tag{3.4}$$

On the other hand, one can write u as

$$\begin{aligned}
u = F_x &= e^{\frac{i}{2}F}e^{-\frac{i}{2}F}F_x = 2ie^{\frac{i}{2}F}\partial_x\big(e^{-\frac{i}{2}F}\big) \\
&= 2ie^{\frac{i}{2}F}w - e^{\frac{i}{2}F}P_{lo}(e^{-\frac{i}{2}F}u) - e^{\frac{i}{2}F}P_{-hi}(e^{-\frac{i}{2}F}u),
\end{aligned} \tag{3.5}$$

so that it follows from the frequency localization

$$\begin{aligned} P_{+HI}u &= 2iP_{+HI}\big(e^{\frac{i}{2}F}w\big) - P_{+HI}\big(P_{+hi}e^{\frac{i}{2}F}P_{lo}(e^{-\frac{i}{2}F}u)\big) \\ &\quad + 2iP_{+HI}\big(P_{+HI}e^{\frac{i}{2}F}\partial_x P_{-hi}e^{-\frac{i}{2}F}\big). \end{aligned} \tag{3.6}$$

Remark 3.1. Note that the use of P_{+HI} allows to replace $e^{\frac{i}{2}F}$ by $P_{+hi}e^{\frac{i}{2}F}$ in the second term on the right-hand side of (3.6). This fact will be useful to obtain at least a quadratic term in $\|u\|_{L^\infty_T L^2_x}$ on the right-hand side of estimate (3.8) in Proposition 3.2.

Then, we have the following *a priori* estimates on u in terms of w.

Proposition 3.2. *Let $0 \le s \le 1$, $0 < T \le 1$, $0 \le \theta \le 1$ and u be a solution to (1.1) in the time interval $[0,T]$. Then, it holds that*

$$\|u\|_{X^{s-\theta,\theta}_T} \lesssim \|u\|_{L^\infty_T H^s_x} + \|u\|_{L^4_{T,x}}\|J^s_x u\|_{L^4_{T,x}}. \tag{3.7}$$

Moreover, if $0 \le s \le \frac14$, it holds that

$$\|J^s_x u\|_{L^p_T L^q_x} \lesssim \|u_0\|_{L^2} + \big(1 + \|u\|_{L^\infty_T L^2_x}\big)\big(\|w\|_{Y^s_T} + \|u\|^2_{L^\infty_T L^2_x}\big), \tag{3.8}$$

for $(p,q) = (\infty,2)$ or $(4,4)$.

Remark 3.3. It is worth notice that (3.8) could be rewritten in a convenient form for $s \ge \frac14$ (cf. [20]).

Proof. We begin with the proof of estimate (3.7) and construct a suitable extension in time $\tilde u$ of u. First, we consider $v(t) = U(-t)u(t)$ on the time interval $[0,T]$ and extend v on $[-2,2]$ by setting $\partial_t v = 0$ on $[-2,2] \setminus [0,T]$. Then, it is pretty clear that

$$\|\partial_t v\|_{L^2_{[-2,2]}H^r_x} = \|\partial_t v\|_{L^2_T H^r_x}, \quad \text{and} \quad \|v\|_{L^2_{[-2,2]}H^r_x} \lesssim \|v\|_{L^\infty_T H^r_x},$$

for all $r \in \mathbb{R}$. Now, we define $\tilde u(x,t) = \eta(t)U(t)v(t)$. Obviously, it holds

$$\|\tilde u\|_{X^{s-1,1}} \lesssim \|\partial_t v\|_{L^2_{[-2,2]}H^{s-1}_x} + \|v\|_{L^2_{[-2,2]}H^{s-1}_x} \lesssim \|\partial_t v\|_{L^2_T H^{s-1}_x} + \|v\|_{L^\infty_T H^{s-1}_x}, \tag{3.9}$$

and

$$\|\tilde u\|_{X^{s,0}} \lesssim \|v\|_{L^2_{[-2,2]}H^s_x} \lesssim \|v\|_{L^\infty_T H^s_x} = \|u\|_{L^\infty_T H^s_x}. \tag{3.10}$$

Then, it is deduced interpolating between (3.9) and (3.10) and using the identity

$$\partial_t v = \mathcal{H}\partial_x^2 U(-t)u + U(-t)\partial_t u = U(-t)\Big[\mathcal{H}\partial_x^2 u + \partial_t u\Big],$$

that

$$\|\tilde u\|_{X^{s-\theta,\theta}} \lesssim \|\partial_t u + \mathcal{H}\partial_x^2 u\|_{L^2_T H^{s-1}_x} + \|u\|_{L^\infty_T H^s_x}, \tag{3.11}$$

for all $0 \le \theta \le 1$. Therefore, the fact that u is a solution to (1.1) and the fractional Leibniz rule (cf. [15]) yield

$$\|\tilde u\|_{X^{s-\theta,\theta}} \lesssim \|u\|_{L^\infty_T H^s_x} + \|u\|_{L^4_{x,T}}\|J^s_x u\|_{L^4_{x,T}},$$

which concludes the proof of (3.7) since $\tilde u$ extends u outside of $[0,T]$.

Next, we turn to the proof of (3.8). Let $0 \le T \le 1$, $0 \le s \le \frac14$, $(p,q) = (\infty,2)$ or $(4,4)$ and u a smooth solution to the equation in (1.1). Since u is real-valued, it holds $P_- u = \overline{P_+ u}$, so that

$$\|J^s_x u\|_{L^p_T L^q_x} \lesssim \|P_{LO}u\|_{L^p_T L^q_x} + \|D^s_x P_{+HI}u\|_{L^p_T L^q_x}. \tag{3.12}$$

To estimate the second term on the right-hand side of (3.12), we use (3.6) to deduce that

$$\begin{aligned}\|D_x^sP_{+HI}u\|_{L^p_TL^q_x} &\lesssim \big\|D_x^sP_{+HI}\big(e^{\frac{i}{2}F}w\big)\big\|_{L^p_TL^q_x}+\big\|D_x^sP_{+HI}\big(P_{+hi}e^{\frac{i}{2}F}P_{lo}(e^{-\frac{i}{2}F}u)\big)\big\|_{L^p_TL^q_x}\\ &\quad+\big\|D_x^sP_{+HI}\big(P_{+HI}e^{\frac{i}{2}F}\partial_xP_{-hi}e^{-\frac{i}{2}F}\big)\big\|_{L^p_TL^q_x}\\ &:= I+II+III.\end{aligned}$$

Estimates (2.10) and (2.13) yield

$$I\lesssim (1+\|u\|_{L^\infty_TL^2_x})\|J_x^sw\|_{L^p_TL^q_x}\lesssim (1+\|u\|_{L^\infty_TL^2_x})\|w\|_{Y^s_T}. \tag{3.13}$$

On the other hand the fractional Leibniz rule (cf. Lemma 2.5), Hölder's inequality in time and the Sobolev embedding imply that

$$\begin{aligned}II &\lesssim \|D_x^sP_{+hi}e^{\frac{i}{2}F}\|_{L^p_TL^q_x}\|P_{+lo}\big(ue^{-\frac{i}{2}F}\big)\|_{L^\infty_{T,x}}\\ &\quad+\|P_{+hi}e^{\frac{i}{2}F}\|_{L^\infty_{T,x}}\|D_x^sP_{+lo}\big(ue^{-\frac{i}{2}F}\big)\|_{L^p_TL^q_x}\\ &\lesssim \|\partial_xP_{+hi}e^{\frac{i}{2}F}\|_{L^p_TL^2_x}\|P_{+lo}\big(ue^{-\frac{i}{2}F}\big)\|_{L^\infty_TL^2_x}\\ &\lesssim T^{\frac1p}\|u\|^2_{L^\infty_TL^2_x}.\end{aligned} \tag{3.14}$$

Finally estimate (2.12) with $\alpha_1=\alpha_2=(1+s)/2$ and $q_1=q_2=q$, Hölder's inequality in time and the Sobolev embedding lead to

$$\begin{aligned}III &\lesssim \|D_x^{(1+s)/2}P_{+HI}e^{\frac{i}{2}F}\|_{L^{2p}_TL^{2q}_x}\|D_x^{(1+s)/2}P_{-hi}e^{-\frac{i}{2}F}\|_{L^{2p}_TL^{2q}_x}\\ &\lesssim T^{\frac1p}\|D_x^{1+\frac{s}{2}-\frac{1}{2q}}P_{+HI}e^{\frac{i}{2}F}\|_{L^\infty_TL^2_x}\|D_x^{1+\frac{s}{2}-\frac{1}{2q}}P_{-hi}e^{-\frac{i}{2}F}\|_{L^\infty_TL^2_x}\\ &\lesssim T^{\frac1p}\|\partial_xP_{+HI}e^{\frac{i}{2}F}\|_{L^\infty_TL^2_x}\|\partial_xP_{-hi}e^{-\frac{i}{2}F}\|_{L^\infty_TL^2_x}\\ &\lesssim T^{\frac1p}\|u\|^2_{L^\infty_TL^2_x},\end{aligned} \tag{3.15}$$

since $0\le s\le \frac1q$. Therefore, we deduce gathering (3.13)–(3.15) that

$$\|D_x^sP_{+HI}u\|_{L^p_TL^q_x}\lesssim \big(1+\|u\|_{L^\infty_TL^2_x}\big)\big(\|w\|_{Y^s_T}+T^{\frac1p}\|u\|^2_{L^\infty_TL^2_x}\big). \tag{3.16}$$

Next we turn to the first term on the right-hand side of (3.12) and consider the integral equation satisfied by $P_{LO}u$,

$$P_{LO}u=U(t)P_{LO}u_0+\int_0^tU(t-\tau)P_{LO}\partial_x(u^2)(\tau)d\tau. \tag{3.17}$$

First, observe that

$$\|P_{LO}u\|_{L^p_TL^q_x}\lesssim T^{\frac1p}\|P_{LO}u\|_{L^\infty_TL^2_x}$$

Then, we deduce from (3.17), using the fact that U is a unitary group in L^2 and Bernstein's inequality, that

$$\begin{aligned}\|P_{LO}u\|_{L^p_TL^q_x} &\lesssim T^{\frac1p}\|u_0\|_{L^2_x}+T^{1+\frac1p}\|\partial_xP_{LO}(u^2)\|_{L^\infty_TL^2_x}\\ &\lesssim T^{\frac1p}\|u_0\|_{L^2_x}+T^{1+\frac1p}\|P_{LO}(u^2)\|_{L^\infty_TL^1_x}\\ &\lesssim \|u_0\|_{L^2_x}+\|u\|^2_{L^\infty_TL^2_x},\end{aligned} \tag{3.18}$$

since $0\le T\le 1$.

Thus, estimate (3.8) follows combining (3.12), (3.16) and (3.18). This concludes the proof of Proposition 3.2. □

3.2. **Bilinear estimates.** The aim of this subsection is to derive the following estimate on $\|w\|_{Y^s_T}$:

Proposition 3.4. *Let $0 < T \le 1$, $0 \le s \le \frac{1}{2}$ and u be a solution to* (1.1) *on the time interval* $[0,T]$. *Then it holds that*

$$\begin{aligned} \|w\|_{Y^s_T} &\lesssim \big(1+\|u_0\|_{L^2}\big)\|u_0\|_{H^s} + \|u\|^2_{L^4_{x,T}} \\ &\quad + \|w\|_{X^{s,1/2}_T}\big(\|u\|_{L^\infty_T L^2_x} + \|u\|_{L^4_{x,T}} + \|u\|_{X^{-1,1}_T}\big). \end{aligned} \tag{3.19}$$

The main tool to prove Proposition 3.4 is the following crucial bilinear estimates.

Proposition 3.5. *Let $s \ge 0$. Then we have that*

$$\begin{aligned} &\|\partial_x P_{+hi}\big(\partial_x^{-1} w P_- \partial_x u\big)\|_{X^{s,-\frac12}} \\ &\qquad \lesssim \|w\|_{X^{s,\frac12}}\big(\|u\|_{L^2_{x,t}} + \|u\|_{L^4_{x,t}} + \|u\|_{X^{-1,1}}\big), \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} &\|\partial_x P_{+hi}\big(\partial_x^{-1} w P_- \partial_x u\big)\|_{\widetilde{Z}^{s,-1}} \\ &\qquad \lesssim \|w\|_{X^{s,\frac12}}\big(\|u\|_{L^2_{x,t}} + \|u\|_{L^4_{x,t}} + \|u\|_{X^{-1,1}}\big). \end{aligned} \tag{3.21}$$

Remark 3.6. Note that $\partial_x^{-1} w$ is well defined since w is localized in high frequencies.

Proof. We will only give the proof in the case of $s = 0$, since the case $s > 0$ can be deduced by using similar arguments. By duality to prove (3.20) is equivalent to prove that

$$\big|I\big| \lesssim \|h\|_{L^2_{x,t}}\|w\|_{X^{0,\frac12}}\big(\|u\|_{L^2_{x,t}} + \|u\|_{L^4_{x,t}} + \|u\|_{X^{-1,1}}\big), \tag{3.22}$$

where

$$I = \int_{\mathcal{D}} \frac{\xi}{\langle\sigma\rangle^{\frac12}}\widehat{h}(\xi,\tau)\xi_1^{-1}\widehat{w}(\xi_1,\tau_1)\xi_2\widehat{u}(\xi_2,\tau_2)d\nu, \tag{3.23}$$

$$d\nu = d\xi d\xi_1 d\tau d\tau_1, \quad \xi_2 = \xi - \xi_1, \quad \tau_2 = \tau - \tau_1, \quad \sigma_i = \tau_i + \xi_i|\xi_i|, \ i = 1,2, \tag{3.24}$$

and

$$\mathcal{D} = \big\{(\xi,\xi_1,\tau,\tau_1) \in \mathbb{R}^4 \mid \xi \ge 1, \ \xi_1 \ge 1 \text{ and } \xi_2 \le 0\big\}. \tag{3.25}$$

Observe that we always have in $\mathcal{D}$ that

$$\xi_1 \ge \xi \ge 1 \quad \text{and} \quad \xi_1 \ge |\xi_2|. \tag{3.26}$$

In the case where $|\xi_2| \le 1$, we have by using Hölder's inequality and estimate (2.9) that

$$\begin{aligned} \big|I\big| &\lesssim \int_{\mathbb{R}^4} \frac{|\widehat{h}|}{\langle\sigma\rangle^{\frac12}}|\widehat{w}(\xi_1,\tau_1)||\widehat{u}(\xi_2,\tau_2)|d\nu \\ &\lesssim \Big\|\Big(\frac{|\widehat{h}|}{\langle\sigma\rangle^{\frac12}}\Big)^\vee\Big\|_{L^4_{x,t}}\|(|\widehat{w}|)^\vee\|_{L^4_{x,t}}\|u\|_{L^2_{x,t}} \\ &\lesssim \|h\|_{L^2_{x,t}}\|w\|_{X^{\frac38}}\|u\|_{L^2_{x,t}}. \end{aligned}$$

Then, from now on we will assume that $|\xi_2| \ge 1$ in $\mathcal{D}$.

By using a dyadic decomposition in space-frequency for the functions h, w and u one can rewrite I as

$$I = \sum_{N,N_1,N_2} I_{N,N_1,N_2} \tag{3.27}$$

with

$$I_{N,N_1,N_2} := \int_{\mathcal{D}} \frac{\xi}{\langle\sigma\rangle^{\frac12}} \widehat{P_N h}(\xi,\tau)\xi_1^{-1}\widehat{P_{N_1}w}(\xi_1,\tau_1)\xi_2\widehat{P_{N_2}u}(\xi_2,\tau_2)d\nu,$$

and the dyadic numbers N, N_1 and N_2 ranging from 1 to $+\infty$. Moreover, the resonance identity

$$\sigma_1+\sigma_2-\sigma = \xi_1^2+(\xi-\xi_1)|\xi-\xi_1|-\xi^2 = -2\xi\xi_2 \tag{3.28}$$

holds in $\mathcal{D}$. Therefore, to calculate I_{N,N_1,N_2}, we split the integration domain $\mathcal{D}$ in the following disjoint regions

$$\begin{aligned}
&\mathcal{A}_{N,N_2} = \big\{(\xi,\xi_1,\tau,\tau_1)\in\mathcal{D} \mid |\sigma| \ge \tfrac16 NN_2\big\},\\
&\mathcal{B}_{N,N_2} = \big\{(\xi,\xi_1,\tau,\tau_1)\in\mathcal{D} \mid |\sigma_1| \ge \tfrac16 NN_2\ ,\ |\sigma| < \tfrac16 NN_2\big\},\\
&\mathcal{C}_{N,N_2} = \big\{(\xi,\xi_1,\tau,\tau_1)\in\mathcal{D} \mid |\sigma| < \tfrac16 NN_2,\ |\sigma_1| < \tfrac16 NN_2\ ,|\sigma_2| \ge \tfrac16 NN_2\big\},
\end{aligned} \tag{3.29}$$

and denote by $I^{\mathcal{A}_{N,N_2}}_{N,N_1,N_2}$, $I^{\mathcal{B}_{N,N_2}}_{N,N_1,N_2}$, $I^{\mathcal{C}_{N,N_2}}_{N,N_1,N_2}$ the restriction of I_{N,N_1,N_2} to each of these regions. Then, it follows that

$$I_{N,N_1,N_2} = I^{\mathcal{A}_{N,N_2}}_{N,N_1,N_2} + I^{\mathcal{B}_{N,N_2}}_{N,N_1,N_2} + I^{\mathcal{C}_{N,N_2}}_{N,N_1,N_2}$$

and thus

$$\big|I\big| \le \big|I_{\mathcal{A}}\big| + \big|I_{\mathcal{B}}\big| + \big|I_{\mathcal{C}}\big|, \tag{3.30}$$

where

$$I_{\mathcal{A}} := \sum_{N,N_1,N_2} I^{\mathcal{A}_{N,N_2}}_{N,N_1,N_2},\ I_{\mathcal{B}} := \sum_{N,N_1,N_2} I^{\mathcal{B}_{N,N_2}}_{N,N_1,N_2} \text{ and } I_{\mathcal{C}} := \sum_{N,N_1,N_2} I^{\mathcal{C}_{N,N_2}}_{N,N_1,N_2}\ .$$

Therefore, it suffices to bound $\big|I_{\mathcal{A}}\big|$, $\big|I_{\mathcal{B}}\big|$ and $\big|I_{\mathcal{C}}\big|$. Note that one of the two following cases holds:

(1) high-low interaction: $N_1 \sim N$ and $N_2 \le N_1$
(2) high-high interaction: $N_1 \sim N_2$ and $N \le N_1$.

Estimate for $\big|I_{\mathcal{A}}\big|$. In the first case, we observe from the Cauchy-Schwarz inequality that

$$\begin{aligned}
\big|I_{\mathcal{A}}\big| &\sim \Big|\int_{\mathbb{R}^2}\widehat{h}\sum_{N_1}\sum_{j=0}^{\frac{\ln(N_1)}{\ln(2)}} \phi_{N_1}\xi\langle\sigma\rangle^{-\frac12}\chi_{\{|\sigma|\ge\frac16 N_1{}^2 2^{-j}\}}\mathcal{F}\big(P_+\big(\partial_x^{-1}P_{N_1}wP_-\partial_xP_{2^{-j}N_1}u\big)\big)d\xi d\tau\Big|\\
&\lesssim \|\widehat{h}\|_{L^2_{\xi,\tau}}\Big\|\sum_{N_1}\sum_{j\ge0}N_1{}^2(N_1^2 2^{-j})^{-1}\phi_{N_1}\Big|\mathcal{F}\big(P_+\big(\partial_x^{-1}P_{N_1}wP_-\partial_xP_{2^{-j}N_1}u\big)\big)\Big|\Big\|_{L^2_{\xi,\tau}}.
\end{aligned}$$

Then, the Plancherel identity and the triangular inequality imply that

$$\big|I_{\mathcal{A}}\big| \lesssim \|h\|_{L^2_{x,t}}\sum_{j\ge0}\Big(\sum_{N_1}2^j\Big\|P_{N_1}\big(\partial_x^{-1}P_{N_1}wP_-\partial_xP_{2^{-j}N_1}u\big)\Big\|^2_{L^2_{x,t}}\Big)^{\frac12}.$$

By using the Hölder and Bernstein inequalities, we deduce that

$$\begin{aligned}
\big|I_{\mathcal{A}}\big| &\lesssim \|h\|_{L^2_{x,t}}\sum_{j\ge0}\Big(\sum_{N_1}2^{-j}\|P_{N_1}w\|^2_{L^4_{x,t}}\|P_{2^{-j}N_1}u\|^2_{L^4_{x,t}}\Big)^{\frac12}\\
&\lesssim \|h\|_{L^2_{x,t}}\Big(\sum_{N}\|P_{N_1}w\|^2_{L^4_{x,t}}\Big)^{\frac12}\|u\|_{L^4_{x,t}}.
\end{aligned} \tag{3.31}$$

In the second case, it follows using the same strategy as in the first case, that

$$\begin{aligned}\big|I_{\mathcal{A}}\big| &\lesssim \|h\|_{L^2_{x,t}} \\ &\times \sum_{j\ge 0}\Big(\sum_{N_1}(2^{-j}N_1)^2(2^{-j}N_1N_1)^{-1}\Big\|P_{2^{-j}N_1}\big(\partial_x^{-1}P_{N_1}wP_-\partial_xP_{N_1}u\big)\Big\|^2_{L^2_{x,t}}\Big)^{\frac12},\end{aligned}$$

which implies using the Hölder and Bernstein inequalities

$$\begin{aligned}\big|I_{\mathcal{A}}\big| &\lesssim \|h\|_{L^2_{x,t}}\sum_{j\ge 0}\Big(\sum_{N_1}2^{-j}\|P_{N_1}w\|^2_{L^4_{x,t}}\|P_{N_1}u\|^2_{L^4_{x,t}}\Big)^{\frac12} \\ &\lesssim \|h\|_{L^2_{x,t}}\Big(\sum_{N_1}\|P_{N_1}w\|^2_{L^4_{x,t}}\Big)^{\frac12}\|u\|_{L^4_{x,t}}.\end{aligned} \tag{3.32}$$

Therefore, we deduce gathering (3.31)–(3.32) and using estimate (2.9) that

$$\big|I_{\mathcal{A}}\big| \le \|h\|_{L^2_{x,t}}\|w\|_{X^{0,\frac38}}\|u\|_{L^4_{x,t}}. \tag{3.33}$$

Estimate for $\big|I_{\mathcal{B}}\big|$. By using again the triangular and the Cauchy-Schwarz inequalities, we have in the first case that

$$\begin{aligned}\big|I_{\mathcal{B}}\big| &\le \|w\|_{X^{0,\frac12}} \\ &\times \sum_{j\ge 0}\left(\sum_{N_1}N_1^{-2}(N_12^{-j}N_1)^{-1}\Big\|P_{N_1}\Big(\partial_xP_{+hi}P_{N_1}\Big(\frac{\widehat{h}}{\langle\sigma\rangle^{\frac12}}\Big)^{\vee}P_+\partial_xP_{2^{-j}N_1}\tilde{u}\Big)\Big\|^2_{L^2_{x,t}}\right)^{\frac12},\end{aligned}$$

where $\tilde{u}(x,t)=u(-x,-t)$. Thus it follows from the Bernstein and Hölder inequalities that

$$\begin{aligned}\big|I_{\mathcal{B}}\big| &\lesssim \|w\|_{X^{0,\frac12}}\sum_{j\ge 0}\Big(\sum_{N_1}2^{-j}\Big\|P_{N_1}\Big(\frac{\widehat{h}}{\langle\sigma\rangle^{\frac12}}\Big)^{\vee}\Big\|^2_{L^4_{x,t}}\|P_{2^{-j}N_1}u\|^2_{L^4_{x,t}}\Big)^{\frac12} \\ &\lesssim \|w\|_{X^{0,\frac12}}\Big(\sum_{N_1}\Big\|P_{N_1}\Big(\frac{\widehat{h}}{\langle\sigma\rangle^{\frac12}}\Big)^{\vee}\Big\|^2_{L^4_{x,t}}\Big)^{\frac12}\|u\|_{L^4_{x,t}}.\end{aligned} \tag{3.34}$$

In the second case, we bound $\big|I_{\mathcal{B}}\big|$ as follows,

$$\begin{aligned}\big|I_{\mathcal{B}}\big| &\le \|w\|_{X^{0,\frac12}} \\ &\times \sum_{j\ge 0}\left(\sum_{N_1}N_1^{-2}(2^{-j}N_1N_1)^{-1}\Big\|P_{N_1}\Big(\partial_xP_{+hi}P_{2^{-j}N_1}\Big(\frac{\widehat{h}}{\langle\sigma\rangle^{\frac12}}\Big)^{\vee}P_+\partial_xP_{N_1}\tilde{u}\Big)\Big\|^2_{L^2_{x,t}}\right)^{\frac12},\end{aligned}$$

so that

$$\begin{aligned}\big|I_{\mathcal{B}}\big| &\lesssim \|w\|_{X^{0,\frac12}}\sum_{j\ge 0}\Big(\sum_{N_1}2^{-j}\Big\|P_{2^{-j}N_1}P_{+hi}\Big(\frac{\widehat{h}}{\langle\sigma\rangle^{\frac12}}\Big)^{\vee}\Big\|^2_{L^4_{x,t}}\|P_{N_1}u\|^2_{L^4_{x,t}}\Big)^{\frac12} \\ &\lesssim \|w\|_{X^{0,\frac12}}\sum_{j\ge 0}2^{-\frac{j}{2}}\Big(\sum_{N_1}\Big\|P_{2^{-j}N_1}P_{+hi}\Big(\frac{\widehat{h}}{\langle\sigma\rangle^{\frac12}}\Big)^{\vee}\Big\|^2_{L^4_{x,t}}\Big)^{\frac12}\|u\|_{L^4_{x,t}} \\ &\lesssim \|w\|_{X^{0,\frac12}}\Big(\sum_{N_1}\Big\|P_{N_1}\Big(\frac{\widehat{h}}{\langle\sigma\rangle^{\frac12}}\Big)^{\vee}\Big\|^2_{L^4_{x,t}}\Big)^{\frac12}\|u\|_{L^4_{x,t}}.\end{aligned} \tag{3.35}$$

In conclusion, we obtain gathering (3.34)–(3.35) and using estimate (2.9) that

$$\big|I_{\mathcal{B}}\big| \le \|h\|_{L^2_{x,t}}\|w\|_{X^{0,\frac12}}\|u\|_{L^4_{x,t}}. \tag{3.36}$$

Estimate for $\big|I_{\mathcal{C}}\big|$. First observe that

$$\big|I_{\mathcal{C}}\big| \lesssim \int_{\widetilde{\mathcal{C}}} \frac{|\xi|}{\langle\sigma\rangle^{\frac12}} |\widehat{h}(\xi,\tau)||\xi_1|^{-1}|\widehat{w}(\xi_1,\tau_1)|\frac{|\xi_2|^2}{\langle\sigma_2\rangle}\frac{\langle\sigma_2\rangle}{|\xi_2|}|\widehat{u}(\xi_2,\tau_2)|d\nu, \tag{3.37}$$

where

$$\widetilde{\mathcal{C}} = \big\{(\xi,\xi_1,\tau,\tau_1)\in\mathcal{D} \mid (\xi,\xi_1,\tau,\tau_1)\in \bigcup_{N,N_2}\mathcal{C}_{N,N_2}\big\}.$$

Since $|\sigma_2|>|\sigma|$ and $|\sigma_2|>|\sigma_1|$ in $\widetilde{\mathcal{C}}$, it follows from (3.28) that $|\sigma_2|\gtrsim|\xi\xi_2|$. Then,

$$|\xi\xi_1^{-1}\xi_2^2\langle\sigma_2\rangle^{-1}|\lesssim 1 \tag{3.38}$$

holds in $\widetilde{\mathcal{C}}$, so that

$$\begin{aligned}\big|I_{\mathcal{C}}\big| &\lesssim \int_{\widetilde{\mathcal{C}}} \frac{|\widehat{h}(\xi,\tau)|}{\langle\sigma\rangle^{\frac12}}|\widehat{w}(\xi_1,\tau_1)|\frac{\langle\sigma_2\rangle}{|\xi_2|}|\widehat{u}(\xi_2,\tau_2)|d\nu\\ &\lesssim \Big\|\Big(\frac{|\widehat{h}|}{\langle\sigma\rangle^{\frac12}}\Big)^{\vee}\Big\|_{L^4_{x,t}}\|(|\widehat{w}|)^{\vee}\|_{L^4_{x,t}}\|u\|_{X^{-1,1}}\\ &\lesssim \|h\|_{L^2_{x,t}}\|w\|_{X^{\frac38}}\|u\|_{X^{-1,1}}\end{aligned} \tag{3.39}$$

is deduced by using Hölder's inequality and estimate (2.9).

Therefore, estimates (3.30), (3.33), (3.36) and (3.39) imply estimate (3.22), which concludes the proof of estimate (3.20).

To prove estimate (3.21), we also proceed by duality. Then it is sufficient to show that

$$\big|J\big| \lesssim \Big(\sum_N \|g_N\|^2_{L^2_\xi L^\infty_\tau}\Big)^{\frac12}\|w\|_{X^{0,\frac12}}\big(\|u\|_{L^2_{x,t}}+\|u\|_{L^4_{x,t}}+\|u\|_{X^{-1,1}}\big), \tag{3.40}$$

where

$$J=\sum_N\int_{\mathcal{D}}\frac{\xi}{\langle\sigma\rangle}g_N(\xi,\tau)\phi_N(\xi)\xi_1^{-1}\widehat{w}(\xi_1,\tau_1)\xi_2\widehat{u}(\xi_2,\tau_2)d\nu,$$

and $d\nu$ and $\mathcal{D}$ are defined in (3.24) and (3.25). As in the case of I, we can also assume that $|\xi_2|\ge 1$. By using dyadic decompositions as in (3.27), J can be rewritten as

$$J=\sum_{N,N_1,N_2} J_{N,N_1,N_2},$$

where

$$J_{N,N_1,N_2} := \int_{\mathcal{D}}\frac{\xi}{\langle\sigma\rangle}\phi_N(\xi)g_N(\xi,\tau)\xi_1^{-1}\widehat{P_{N_1}w}(\xi_1,\tau_1)\xi_2\widehat{P_{N_2}u}(\xi_2,\tau_2)d\nu,$$

and the dyadic numbers N, N_1 and N_2 range from 1 to $+\infty$. Moreover, we will denote by $J^{\mathcal{A}_{N,N_2}}_{N,N_1,N_2}$, $J^{\mathcal{B}_{N,N_2}}_{N,N_1,N_2}$, $J^{\mathcal{C}_{N,N_2}}_{N,N_1,N_2}$ the restriction of J_{N,N_1,N_2} to the regions $\mathcal{A}_{N,N_2}$, $\mathcal{B}_{N,N_2}$ and $\mathcal{C}_{N,N_2}$ defined in (3.28). Then, it follows that

$$\big|J\big|\le\big|J_{\mathcal{A}}\big|+\big|J_{\mathcal{B}}\big|+\big|J_{\mathcal{C}}\big|, \tag{3.41}$$

where

$$J_{\mathcal{A}} := \sum_{N,N_1,N_2} J^{\mathcal{A}_{N,N_2}}_{N,N_1,N_2},\ J_{\mathcal{B}} := \sum_{N,N_1,N_2} J^{\mathcal{B}_{N,N_2}}_{N,N_1,N_2} \text{ and } J_{\mathcal{C}} := \sum_{N,N_1,N_2} J^{\mathcal{C}_{N,N_2}}_{N,N_1,N_2},$$

so that it suffices to estimate $\big|J_{\mathcal{A}}\big|$, $\big|J_{\mathcal{B}}\big|$ and $\big|J_{\mathcal{C}}\big|$.

Estimate for $\big|J_{\mathcal{A}}\big|$. To estimate $\big|J_{\mathcal{A}}\big|$, we divide each region $\mathcal{A}_{N,N_2}$ into disjoint subregions

$$\mathcal{A}^q_{N,N_2}=\big\{(\xi,\xi_1,\tau,\tau_1)\in\mathcal{A}_{N,N_2}\ |\ 2^{q-3}NN_2\le|\sigma|<2^{q-2}NN_2\big\},$$

for $q\in\mathbb{Z}_+$. Thus if $J^{\mathcal{A}^q_{N,N_2}}_{N,N_1,N_2}$ denote the restriction of $J^{\mathcal{A}_{N,N_2}}_{N,N_1,N_2}$ to each of these regions, we have that $J_{\mathcal{A}}=\sum_{q\ge0}\sum_{N,N_1,N_2}J^{\mathcal{A}^q_{N,N_2}}_{N,N_1,N_2}$. In the case of high-low interactions, we deduce by using the Plancherel identity Cauchy-Schwarz and Minkowski inequalities that

$$\begin{aligned}|J_{\mathcal{A}}|\le\sum_{q\ge0}\sum_{N_1}\sum_{N_2\le N_1}&\|g_{N_1}\chi_{\{|\sigma|\sim2^qN_1N_2\}}\|_{L^2_{\xi,\tau}}\\&\times(2^qN_1N_2)^{-1}N_1\Big\|\partial_x^{-1}P_{N_1}wP_-\partial_xP_{N_2}u\Big\|_{L^2_{x,t}}.\end{aligned}$$

Moreover, we get from Hölder's inequality

$$\|g_{N_1}\chi_{\{|\sigma|\sim2^qN_1N_2\}}\|_{L^2_{\xi,\tau}}\lesssim(2^qNN_2)^{\frac12}\|g_{N_1}\|_{L^2_\xi L^\infty_\tau},$$

so that, the Cauchy-Schwarz inequality yields

$$\begin{aligned}|J_{\mathcal{A}}|&\lesssim\sum_{N_1}\sum_{N_2\le N_1}(N_2N_1^{-1})^{\frac12}\|g_{N_1}\|_{L^2_\xi L^\infty_\tau}\|P_{N_1}w\|_{L^4_{x,t}}\|P_{N_2}u\|_{L^4_{x,t}}\\&\lesssim\|u\|_{L^4_{x,t}}\sum_{N_1}\|g_{N_1}\|_{L^2_\xi L^\infty_\tau}\|P_{N_1}w\|_{L^4_{x,t}}\\&\lesssim\big(\sum_{N_1}\|g_{N_1}\|^2_{L^2_\xi L^\infty_\tau}\big)^{\frac12}\|w\|_{\tilde{L}^4_{x,t}}\|u\|_{L^4_{x,t}}.\end{aligned}\tag{3.42}$$

In the high-high interaction case, it follows from the Minkowski and Cauchy-Schwarz inequalities that

$$\begin{aligned}|J_{\mathcal{A}}|\le\sum_{q\ge0}\sum_{N_1}\sum_{N\le N_1}&\|g_N\chi_{\{|\sigma|\sim2^qNN_1\}}\|_{L^2_{\xi,\tau}}\\&\times(2^qNN_1)^{-1}N\Big\|\partial_x^{-1}P_{N_1}wP_-\partial_xP_{N_1}u\Big\|_{L^2_{x,t}}.\end{aligned}$$

Moreover, we deduce from Hölder's inequality that

$$\|g_N\chi_{\{|\sigma|\sim2^qNN_1\}}\|_{L^2_{\xi,\tau}}\lesssim(2^qNN_1)^{\frac12}\|g_N\|_{L^2_\xi L^\infty_\tau}.$$

Then, the Cauchy-Schwarz inequality implies that

$$\begin{aligned}|J_{\mathcal{A}}|&\lesssim\sum_{j\ge0}\sum_{N_1}(N_1^{-1}2^{-j}N_1)^{\frac12}\|g_{2^{-j}N_1}\|_{L^2_\xi L^\infty_\tau}\|P_{N_1}w\|_{L^4_{x,t}}\|P_{N_1}u\|_{L^4_{x,t}}\\&\lesssim\sum_{j\ge0}2^{-\frac{j}{2}}\big(\sum_{N_1}\|g_{2^{-j}N_1}\|^2_{L^2_\xi L^\infty_\tau}\big)^{\frac12}\big(\sum_{N_1}\|P_{N_1}w\|^2_{L^4_{x,t}}\big)^{\frac12}\|u\|_{L^4_{x,t}}\\&\lesssim\big(\sum_{N_1}\|g_{N_1}\|^2_{L^2_\xi L^\infty_\tau}\big)^{\frac12}\|w\|_{\tilde{L}^4_{x,t}}\|u\|_{L^4_{x,t}}.\end{aligned}\tag{3.43}$$

Then estimates (2.9), (3.42) and (3.43) yield

$$|J_{\mathcal{A}}|\lesssim\big(\sum_N\|g_N\|^2_{L^2_\xi L^\infty_\tau}\big)^{\frac12}\|w\|_{X^{0,\frac38}}\|u\|_{L^4_{x,t}}.\tag{3.44}$$

Estimate for $|J_{\mathcal{B}}|$ *and* $|J_{\mathcal{C}}|$. Arguing as in the proof of (3.20), it is deduced that

$$|J_{\mathcal{B}}| + |J_{\mathcal{C}}| \lesssim \Big(\Big\| \Big(\frac{g}{\langle\sigma\rangle} \Big)^{\vee} \Big\|_{\widetilde{L}^4_{x,t}} + \Big\| \Big(\frac{|g|}{\langle\sigma\rangle} \Big)^{\vee} \Big\|_{\widetilde{L}^4_{x,t}} \Big) \|w\|_{X^{0,\frac12}} \big(\|u\|_{L^4_{x,t}} + \|u\|_{X^{-1,1}} \big),$$

where $g = \sum_N \phi_N g_N$. Moreover, estimate (2.9) and Hölder's inequality imply

$$\begin{aligned} \Big\| \Big(\frac{g}{\langle\sigma\rangle} \Big)^{\vee} \Big\|_{\widetilde{L}^4_{x,t}} + \Big\| \Big(\frac{|g|}{\langle\sigma\rangle} \Big)^{\vee} \Big\|_{\widetilde{L}^4_{x,t}} &\lesssim \|\langle\sigma\rangle^{-\frac58} \sum_N \phi_N g_N\|_{L^2_{\xi,\tau}} \\ &\lesssim \big(\sum_N \|\langle\sigma\rangle^{-\frac58} g_N\|^2_{L^2_{\xi,\tau}} \big)^{\frac12} \\ &\lesssim \big(\sum_N \|g_N\|^2_{L^2_\xi L^\infty_\tau} \big)^{\frac12}, \end{aligned}$$

so that

$$|J_{\mathcal{B}}| + |J_{\mathcal{C}}| \lesssim \big(\sum_N \|g_N\|^2_{L^2_\xi L^\infty_\tau} \big)^{\frac12} \|w\|_{X^{0,\frac12}} \big(\|u\|_{L^4_{x,t}} + \|u\|_{X^{-1,1}} \big). \tag{3.45}$$

Finally (3.41), (3.44) and (3.45) imply (3.40), which concludes the proof of estimate (3.21). □

Lemma 3.7. *Let* $0 < T \le 1$, $s \ge 0$, u_1, $u_2 \in L^\infty(\mathbb{R}; L^2(\mathbb{R})) \cap L^4(\mathbb{R}^2)$ *supported in the time interval* $[-2T, 2T]$, *and* F_1, F_2 *be some spatial primitive of respectively* u_1 *and* u_2. *Then*

$$\begin{aligned} &\big\| \partial_x P_{+hi} \big(P_{lo} e^{-\frac{i}{2}F_1} P_- \partial_x u_1 \big) \big\|_{\widetilde{Z}^{s,-1}} \\ &+ \big\| \partial_x P_{+hi} \big(P_{lo} e^{-\frac{i}{2}F_1} P_- \partial_x u_1 \big) \big\|_{X^{s,-\frac12}} \lesssim \|u_1\|^2_{L^4_{x,t}}, \end{aligned} \tag{3.46}$$

and

$$\begin{aligned} &\big\| \partial_x P_{+hi} \big(P_{lo} \big(e^{-\frac{i}{2}F_1} - e^{-\frac{i}{2}F_2} \big) P_- \partial_x u_2 \big) \big\|_{\widetilde{Z}^{s,-1}} \\ &+ \big\| \partial_x P_{+hi} \big(P_{lo} \big(e^{-\frac{i}{2}F_1} - e^{-\frac{i}{2}F_2} \big) P_- \partial_x u_2 \big) \big\|_{X^{s,-\frac12}} \\ &\lesssim \Big(\|u_1 - u_2\|_{L^\infty_t L^2_x} + \|e^{-\frac{i}{2}F_1} - e^{-\frac{i}{2}F_2}\|_{L^\infty_{x,t}} \|u_2\|_{L^\infty_t L^2_x} \Big) \|u_2\|_{L^4_{x,t}}. \end{aligned} \tag{3.47}$$

Proof. We deduce from the Cauchy-Schwarz inequality, the Sobolev embedding $\|f\|_{H_t^{-\frac12+\epsilon}} \lesssim \|f\|_{L_t^{1+\epsilon'}}$ with $1+\epsilon' = \frac{1}{1-\epsilon}$, and the Minkowski inequality that

$$\begin{aligned} \|f\|_{\widetilde{Z}^{s,-1}} + \|f\|_{X^{s,-\frac12}} \lesssim \|f\|_{X^{s,-\frac12+\epsilon}} &= \Big\| \big\| \big(J^s_x U(-t) f \big)^{\wedge_x}(\xi) \big\|_{H_t^{-\frac12+\epsilon}} \Big\|_{L^2_\xi} \\ &\lesssim \Big\| \big\| \big(J^s_x U(-t) f \big)^{\wedge_x}(\xi) \big\|_{L_t^{1+\epsilon'}} \Big\|_{L^2_\xi} \lesssim \|f\|_{L_t^{1+\epsilon'} H^s_x}. \end{aligned} \tag{3.48}$$

On the other hand, it follows from the frequency localization that

$$\partial_x P_{+hi} \big(P_{lo} e^{-\frac{i}{2}F} P_- \partial_x u \big) = \partial_x P_{+LO} \big(P_{lo} e^{-\frac{i}{2}F} P_{-LO} \partial_x u \big).$$

Therefore, by using (3.48), Bernstein's inequalities and estimate (2.12), we can bound the left-hand side of (3.46) by

$$\Big\| P_{+LO} \Big(P_{lo} e^{-\frac{i}{2}F} P_{-LO} \partial_x u \Big) \Big\|_{L_t^{1+\epsilon'} L^2_x} \lesssim T^\gamma \|\partial_x e^{-\frac{i}{2}F}\|_{L^4_{x,t}} \|u\|_{L^4_{x,t}}, \tag{3.49}$$

with $\frac{1}{\gamma} = \frac12 - \epsilon'$, which concludes the proof of estimate (3.46) recalling that $\partial_x F = u$ and $0 < T \le 1$. Estimate (3.47) can be proved exactly as above recalling (2.19). □

A proof of Proposition 3.4 is now in sight.

Proof of Proposition 3.4. Let $0 \le s \le \frac{1}{2}$, $0 < T \le 1$ and let $\tilde{u}$ and $\tilde{w}$ be extensions of u and w such that $\|\tilde{u}\|_{X^{-1,1}} \le 2\|u\|_{X_T^{-1,1}}$ and $\|\tilde{w}\|_{X^{s,1/2}} \le 2\|w\|_{X_T^{s,1/2}}$. By the Duhamel principle, the integral formulation associated to (3.4) reads

$$\begin{aligned} w(t) = \eta(t)w(0) - \eta(t)\int_0^t U(t-t')\partial_x P_{+hi}\big(\eta_T \partial_x^{-1}\tilde{w}P_-\big(\eta_T\partial_x u\big)\big)(t')dt' \\ - \eta(t)\int_0^t \partial_x P_{+hi}\big(P_{lo}\big(\eta_T e^{-\frac{i}{2}\tilde{F}}\big)P_-\big(\eta_T\partial_x\tilde{u}\big)\big)(t')dt', \end{aligned}$$

for $0 < t \le T \le 1$. Therefore, we deduce gathering estimates (2.5), (2.7), (3.20), (3.21) and (3.46) that

$$\|w\|_{Y_T^s} \lesssim \|w(0)\|_{H^s} + \|u\|_{L^4_{x,T}}^2 + \|w\|_{X_T^{s1/2}}\|\big(\|u\|_{L^\infty_T L^2_x} + \|u\|_{L^4_{x,T}} + \|u\|_{X_T^{-1,1}}\big).$$

This concludes the proof of estimate (3.19), since

$$\|w(0)\|_{H^s} \lesssim \big\|J_x^s\big(e^{-\frac{i}{2}F(\cdot,0)}u_0\big)\big\|_{L^2} \lesssim \big(1+\|u_0\|_{L^2}\big)\|u_0\|_{H^s}, \tag{3.50}$$

follows from estimate (2.13) and the fact that $0 \le s \le \frac{1}{2}$. □

4. Proof of Theorem 1.1

First it is worth noticing that we can always assume that we deal with data that have small $L^2(\mathbb{R})$-norm. Indeed, if u is a solution to the IVP (1.1) on the time interval $[0,T]$ then, for every $0 < \lambda < \infty$, $u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$ is also a solution to the equation in (1.1) on the time interval $[0, \lambda^{-2}T]$ with initial data $u_{0,\lambda} = \lambda u_0(\lambda\cdot)$. For $\varepsilon > 0$ let us denote by B_ε the ball of $L^2(\mathbb{R})$, centered at the origin with radius ε. Since $\|u_\lambda(\cdot,0)\|_{L^2} = \lambda^{\frac{1}{2}}\|u_0\|_{L^2}$, we see that we can force $u_{0,\lambda}$ to belong to B_ϵ by choosing $\lambda \sim \min(\varepsilon^2\|u_0\|_{L^2}^{-2}, 1)$. Therefore the existence and uniqueness of a solution of (1.1) on the time interval $[0,1]$ for small $L^2(\mathbb{R})$-initial data will ensure the existence of a unique solution u to (1.1) for arbitrary large $L^2(\mathbb{R})$-initial data on the time interval $T \sim \lambda^2 \sim \min(\|u_0\|_{L^2}^{-4}, 1)$. Using the conservation of the $L^2(\mathbb{R})$-norm, this will lead to global well-posedness in $L^2(\mathbb{R})$.

4.1. **Uniform bound for small initial data.** First, we begin by deriving *a priori* estimates on smooth solutions associated to initial data $u_0 \in H^s(\mathbb{R})$ that is small in $L^2(\mathbb{R})$. It is known from the classical well-posedness theory (cf. [13]) that such an initial data gives rise to a global solution $u \in C(\mathbb{R}; H^\infty(\mathbb{R}))$ to the Cauchy problem (1.1). Setting for $0 < T \le 1$,

$$N_T^s(u) := \max\Big(\|u\|_{L^\infty_T H^s_x}, \|J^s_x u\|_{L^4_{x,T}}, \|w\|_{X_T^{s,\frac{1}{2}}}\Big), \tag{4.1}$$

it follows from the smoothness of u that $T \mapsto N_T^s(u)$ is continuous and non decreasing on $\mathbb{R}_+^*$. Moreover, from (3.4), the linear estimate (2.7), (3.50) and (3.7) we infer that $\lim_{T\to 0+} N_T^s(u) \lesssim (1+\|u_0\|_{L^2})\|u_0\|_{H^s}$. On the other hand, combining (3.7)-(3.8) and (3.19) and the conservation of the L^2-norm we infer that

$$N_T^0(u) \lesssim (1+\|u_0\|_{L^2})\|u_0\|_{L^2} + (N_T^0(u))^2 + (N_T^0(u))^3 .$$

By continuity, this ensures that there exists $\varepsilon_0 > 0$ and $C_0 > 0$ such that $N_1^0(u) \le C_0\varepsilon$ provided $\|u_0\|_{L^2} \le \varepsilon \le \varepsilon_0$. Finally, using again (3.7)-(3.8) and (3.19), this leads to $N_1^s(u) \lesssim \|u_0\|_{H^s}$ provided $\|u_0\|_{L^2} \le \varepsilon \le \varepsilon_0$.

4.2. **Lipschitz bound for initial data having the same low frequency part.** To prove the uniqueness as well as the continuity of the solution we will derive a Lipschitz bound on the solution map on some affine subspaces of $H^s(\mathbb{R})$ with values in $L^\infty_T H^s(\mathbb{R})$. We know from [17] that such Lipschitz bound does not exist in general in $H^s(\mathbb{R})$. Here we will restrict ourself to solutions emanating from initial data having the same low frequency part. This is clearly sufficient to get uniqueness and it will turn out to be sufficient to get the continuity of the solution as well as the continuity of the flow-map.

Let φ_1, $\varphi_2 \in B_\epsilon \cap H^s(\mathbb{R})$, $s \ge 0$, such that $P_{LO}\varphi_1 = P_{LO}\varphi_2$ and let u_1, u_2 be two solutions to (1.1) emanating respectively from φ_1, and φ_2 that satisfy (7.1) on the time interval $[0,T]$, $0 < T < 1$. We also assume that the primitives $F_1 := F[u_1]$ and $F_2 := F[u_2]$ of respectively u_1 and u_2 are such that the associated gauge functions W_1, w_1, respectively W_2, w_2, constructed in Subsection 3.1, satisfy (7.2). Finally, we assume that

$$N_T^0(u_i) \le C_0\varepsilon \le C_0\varepsilon_0. \tag{4.2}$$

First, by construction, we observe that since $F(x) - F(y) = \int_x^y u(z)\,dz$, it holds $P_{LO}\int_y^x udz = P_{LO}\Big(F(x)-F(y)\Big) = P_{LO}F(x) - F(y)$. On the other hand, since P_{LO} and ∂_x do commute, we have $\partial_x P_{LO}F = P_{LO}u$ and, by integrating, $\int_y^x P_{LO}udz = P_{LO}F(x) - P_{LO}F(y)$. Gathering these two identities, we get

$$\int_y^x P_{LO}udz - P_{LO}\int_y^x udz = F(y) - P_{LO}F(y) = P_{HI}F(y),$$

which leads to

$$P_{lo}\int_y^x udz = P_{lo}\int_y^x P_{LO}udz.$$

We thus infer that

$$\begin{aligned} P_{lo}(F_1-F_2)(x,0) &= \int_{\mathbb{R}} \psi(y)P_{lo}\int_y^x (u_1-u_2)(z,0)dzdy \\ &= \int_{\mathbb{R}} \psi(y)P_{lo}\int_y^x P_{LO}(\varphi_1(z)-\varphi_2(z))(z,0)dzdy = 0. \end{aligned} \tag{4.3}$$

Then, we set $v = u_1 - u_2$, $Z = W_1 - W_2$ and $z = w_1 - w_2$. Obviously, z satisfies

$$\begin{aligned} \partial_t z - i\partial_x^2 z = &-\partial_x P_{+hi}\big(W_1 P_-\partial_x v\big) - \partial_x P_{+hi}\big(Z P_-\partial_x u_2\big) \\ &- \partial_x P_{+hi}\big(P_{lo}e^{-\frac{i}{2}F_1}P_-\partial_x v\big) - \partial_x P_{+hi}\big(P_{lo}\big(e^{-\frac{i}{2}F_1} - e^{-\frac{i}{2}F_2}\big)P_-\partial_x u_2\big). \end{aligned}$$

Thus, we deduce gathering estimates (2.7), (3.20), (3.21), (3.46) and (3.47) that

$$\begin{aligned} \|z\|_{Y_1^s} \lesssim{}& \|z(0)\|_{H^s} + \|w_1\|_{X_1^{s,1/2}}\big(\|v\|_{X_1^{-1,1}} + \|v\|_{L^4_{x,1}} + \|v\|_{L^\infty_1 L^2_x}\big) + \|v\|^2_{L^4_{x,1}} \\ &+ \|z\|_{X_1^{s,1/2}}\big(\|u_2\|_{X_1^{-1,1}} + \|u_2\|_{L^4_{x,1}} + \|u_2\|_{L^\infty_1 L^2_x}\big) \\ &+ \big(\|v\|_{L^\infty_1 L^2_x} + \|e^{-\frac{i}{2}F_1} - e^{-\frac{i}{2}F_2}\|_{L^\infty_{x,1}}\big)\|u_2\|_{L^4_{x,1}}, \end{aligned}$$

which implies recalling (4.1) and (4.2) that

$$\|z\|_{Y_1^s} \lesssim \|z(0)\|_{H^s} + \varepsilon\big(\|v\|_{X_1^{-1,1}} + \|v\|_{L^4_{x,1}} + \|v\|_{L^\infty_1 L^2_x}\big) + \varepsilon\|e^{-\frac{i}{2}F_1} - e^{-\frac{i}{2}F_2}\|_{L^\infty_{x,1}}. \tag{4.4}$$

where, by the mean-value theorem,

$$\begin{aligned}\|z(0)\|_{H^s} &\lesssim \|\varphi_1-\varphi_2\|_{H^s}\Big(1+\|\varphi_1\|_{H^s}+\|\varphi_2\|_{L^2}\Big)\\ &\quad +\|e^{-iF_1(0)/2}-e^{-iF_2(0)/2}\|_{L^\infty}\|\varphi_1\|_{H^s}(1+\|\varphi_1\|_{L^2})\\ &\lesssim \|\varphi_1-\varphi_2\|_{H^s}+\|F_1(0)-F_2(0)\|_{L^\infty}.\end{aligned}$$

On the other hand, the equation for $v=u_1-u_2$ reads

$$\partial_t v+\mathcal{H}\partial_x^2 v=\frac{1}{2}\partial_x\big((u_1+u_2)v\big),$$

so that it is deduced from (3.11), (4.1) and the fractional Leibniz rule that

$$\|v\|_{X_1^{-1,1}}\lesssim \|\partial_t v+\mathcal{H}\partial_x^2 v\|_{L^2_1H^{-1}_x}+\|v\|_{L^\infty_T L^2_x}\lesssim \varepsilon\|v\|_{L^4_{x,1}}+\|v\|_{L^\infty_1L^2_x}. \tag{4.5}$$

Next, proceeding as in (3.6), we infer that

$$\begin{aligned}P_{+HI}v&=2iP_{+HI}\big(e^{\frac{i}{2}F_1}z\big)+2iP_{+HI}\big((e^{\frac{i}{2}F_1}-e^{\frac{i}{2}F_2})w_2\big)\\ &\quad+2iP_{+HI}\big(P_{+hi}e^{\frac{i}{2}F_1}\partial_xP_{+lo}(e^{-\frac{i}{2}F_1}-e^{-\frac{i}{2}F_2})\big)\\ &\quad+2iP_{+HI}\big(P_{+hi}(e^{\frac{i}{2}F_1}-e^{\frac{i}{2}F_2})\partial_xP_{+lo}e^{-\frac{i}{2}F_2}\big)\\ &\quad+2iP_{+HI}\big(P_{+HI}e^{\frac{i}{2}F_1}\partial_xP_{-}(e^{-\frac{i}{2}F_1}-e^{-\frac{i}{2}F_2})\big)\\ &\quad+2iP_{+HI}\big(P_{+HI}(e^{\frac{i}{2}F_1}-e^{\frac{i}{2}F_2})\partial_xP_{-}e^{-\frac{i}{2}F_2}\big).\end{aligned}$$

Thus, we deduce using estimates (2.14), (2.19) and arguing as in the proof of Proposition 3.2 that

$$\begin{aligned}\|J^s_xv\|_{L^p_1L^q_x}&\lesssim \big(\|u_1\|_{L^\infty_1L^2_x}+\|u_2\|_{L^\infty_1L^2_x}\big)\|v\|_{L^\infty_1L^2_x}+(1+\|u_1\|_{L^\infty_1L^2_x})\|z\|_{Y^s_1}\\ &\quad+\big(\|v\|_{L^\infty_1L^2_x}+\|e^{\frac{i}{2}F_1}-e^{\frac{i}{2}F_2}\|_{L^\infty_{x,1}}(1+\|u_1\|_{L^\infty_1L^2_x})\big)\|w_2\|_{Y^s_1}\\ &\quad+\|u_1\|_{L^\infty_1L^2_x}\big(\|v\|_{L^\infty_1L^2_x}+\|e^{-\frac{i}{2}F_1}-e^{-\frac{i}{2}F_2}\|_{L^\infty_{x,1}}\|u_1\|_{L^\infty_1L^2_x}\big)\\ &\quad+\|u_2\|_{L^\infty_1L^2_x}\big(\|v\|_{L^\infty_1L^2_x}+\|e^{\frac{i}{2}F_1}-e^{\frac{i}{2}F_2}\|_{L^\infty_{x,1}}\|u_1\|_{L^\infty_1L^2_x}\big),\end{aligned}$$

for $(p,q)=(\infty,2)$ or $(p,q)=(4,4)$, which implies recalling (4.2) that

$$\|J^s_xv\|_{L^\infty_1L^2_x}+\|J^s_xv\|_{L^4_{x,1}}\lesssim \|z\|_{Y^s_1}+\varepsilon\|e^{-\frac{i}{2}F_1}-e^{-\frac{i}{2}F_2}\|_{L^\infty_{x,1}}+\varepsilon\|e^{\frac{i}{2}F_1}-e^{\frac{i}{2}F_2}\|_{L^\infty_{x,1}}. \tag{4.6}$$

Finally, we use the mean value theorem to get the bound

$$\|e^{\pm\frac{i}{2}F_1}-e^{\pm\frac{i}{2}F_2}\|_{L^\infty_{x,1}}\lesssim \|F_1-F_2\|_{L^\infty_{x,1}}. \tag{4.7}$$

The following crucial lemma gives an estimate for the right-hand side of (4.7).

Lemma 4.1. *It holds that*

$$\|F_1(0)-F_2(0)\|_{L^\infty}\lesssim \|\varphi_1-\varphi_2\|_{L^2}. \tag{4.8}$$

and

$$\|F_1-F_2\|_{L^\infty_{x,1}}\lesssim \|v\|_{L^\infty_1L^2_x}. \tag{4.9}$$

Proof. (4.8) clearly follows from (4.3) together with Bernstein inequality. To prove (4.9) we set $G=F_1-F_2$, $G_{lo}=P_{lo}G$ and $G_{hi}=P_{hi}G$. Then,

$$\|G\|_{L^\infty_{x,1}}\le \|G_{lo}\|_{L^\infty_{x,1}}+\|G_{hi}\|_{L^\infty_{x,1}}. \tag{4.10}$$

Observe that, from the Duhamel principle and (4.3), G_{lo} satisfies

$$G_{lo} = \frac{1}{2}\int_0^t U(t-\tau)P_{lo}\big((u_1+u_2)v\big)(\tau)d\tau$$

Therefore, it follows using Bernstein and Hölder's inequality that

$$\|G_{lo}\|_{L^\infty_{x,1}} \lesssim \|(u_1+u_2)v\|_{L^\infty_1 L^1_x} \lesssim \big(\|u_1\|_{L^\infty_1 L^2_x} + \|u_2\|_{L^\infty_1 L^2_x}\big)\|v\|_{L^\infty_1 L^2_x}. \tag{4.11}$$

On the other hand, the Bernstein inequality ensures that

$$\|G_{hi}\|_{L^\infty_{x,1}} \lesssim \|\partial_x G_{hi}\|_{L^\infty_1 L^2_x} \lesssim \|v\|_{L^\infty_1 L^2_x}, \tag{4.12}$$

since $\partial_x G = v$. The proof of Lemma 4.1 is concluded gathering (4.2), (4.10)–(4.12). $\square$

Finally, estimates (4.4)–(4.9) lead to

$$\begin{aligned}
&\|z\|_{Y^s_1} + \|v\|_{X^{s-1,1}_1} + \|v\|_{L^\infty_1 H^s_x} + \|J^s_x v\|_{L^4_{x,1}} \\
&\qquad \lesssim \|\varphi_1 - \varphi_2\|_{H^s} + \varepsilon\big(\|z\|_{Y^s_1} + \|v\|_{X^{s-1,1}_1} + \|v\|_{L^\infty_1 H^s_x} + \|J^s_x v\|_{L^4_{x,1}}\big),
\end{aligned}$$

Therefore we conclude that there exists $0<\varepsilon_1\le\varepsilon_0$ such that

$$\|z\|_{Y^s_1} + \|v\|_{X^{s-1,1}_1} + \|v\|_{L^\infty_1 H^s_x} + \|J^s_x v\|_{L^4_{x,1}} \lesssim \|\varphi_1-\varphi_2\|_{H^s} \tag{4.13}$$

provided u_1 and u_2 satisfy (4.2) with $0<\varepsilon\le\varepsilon_1$.

4.3. **Well-posedness.** Let $u_0 \in B_{\varepsilon_1}\cap H^s(\mathbb{R})$ and consider the sequence of initial data $\{u_0^j\}\subset H^\infty(\mathbb{R})$, defined by

$$u_0^j = \mathcal{F}_x^{-1}\big(\chi_{|[-j,j]}\mathcal{F}_x u_0\big), \quad \forall\, j\ge 20. \tag{4.14}$$

Clearly, $\{u_0^j\}$ converges to u_0 in $H^s(\mathbb{R})$. By the classical well-posedness theory, the associated sequence of solutions $\{u^j\}$ is a subset of $C([0,1];H^\infty(\mathbb{R}))$ and according to Subsection 4.1, it satisfies $N^s_1(u^j)\le C_0\varepsilon_1$. Moreover, since $P_{LO}u_0^j = P_{LO}u_0$ for all $j\ge 20$, it follows from the preceding subsection that

$$\|u^j - u^{j'}\|_{L^\infty_1 H^s_x} + \|u^j - u^{j'}\|_{L^4_1 W^{s,4}_x} + \|w^j - w^{j'}\|_{X^{0,1/2}} \lesssim \|u^j_0 - u^{j'}_0\|_{H^s_x}. \tag{4.15}$$

Therefore the sequence $\{u^j\}$ converges strongly in $L^\infty_1 H^s(\mathbb{R})\cap L^4_1 W^{s,4}$ to some function $u\in C([0,1];H^s(\mathbb{R})$ and $\{w_j\}_{j\ge 4}$ converges strongly to some function w in $X^{s,1/2}$. Thanks to these strong convergences it is easy to check that u is a solution to (1.1) emanating from u_0 and that $w = P_{hi}(\partial_x(e^{-i\partial_x^{-1}u/2}))$. Moreover from the conservation of the $L^2(\mathbb{R})$-norm, $u\in C_b(\mathbb{R};L^2(\mathbb{R}))\cap C(\mathbb{R};H^s(\mathbb{R}))$.

Now let $\tilde{u}$ be another solution of (1.1) on $[0,T]$ emanating from u_0 belonging to the same class of regularity as u. By using again the scaling argument we can always assume that $\|\tilde u\|_{L^\infty_T L^2_x} + \|\tilde u\|_{L^4_{x,T}} \le C_0\varepsilon_1$. Moreover, setting $\tilde w := P_{+hi}(e^{-iF[\tilde u]})$, by the Lebesgue monotone convergence theorem, there exists $N>0$ such that $\|P_{\ge N}\tilde w\|_{X^{0,\frac12}_T} \le C_0\varepsilon_1/2$. On the other hand, using Lemma 2.1-2.2, it is easy to check that

$$\begin{aligned}
\|(1-P_{\ge N})\tilde w\|_{X^{0,\frac12}_T} &\lesssim \|u_0\|_{L^2} + NT^{\frac14}\|\tilde u\|_{L^4_{x,T}}\|\tilde w\|_{L^4_{x,T}} + \|\tilde u\|^2_{L^4_{x,T}} \\
&\lesssim \|u_0\|_{L^2} + NT^{\frac14}\|\tilde w\|_{X^{0,\frac12}_T}\|\tilde u\|_{L^4_{x,T}} + \|\tilde u\|^2_{L^4_{x,T}} \quad .
\end{aligned}$$

Therefore, for $T > 0$ small enough we can require that $\tilde{u}$ satisfies the smallness condition (4.2) with ε_1 and thus by (4.13), $\tilde{u} \equiv u$ on $[0,T]$. This proves the uniqueness result for initial data belonging to B_{ε_1}.

Next, we turn to the continuity of the flow map. Fix $u_0 \in B_{\epsilon_1}$ and $\lambda > 0$ and consider the emanating solution $u \in C([0,1]; H^s(\mathbb{R}))$. We will prove that if $v_0 \in B_{\epsilon_1}$ satisfies $\|u_0 - v_0\|_{H^s} \le \delta$, where δ will be fixed later, then the solution v emanating from v_0 satisfies

$$\|u - v\|_{L^\infty_1 H^s_x} \le \lambda. \tag{4.16}$$

For $j \ge 1$, let u_0^j and v_0^j be constructed as in (4.14), and denote by u^j and v^j the solutions emanating from u_0^j and v_0^j. Then, it follows by the triangular inequality that

$$\|u - v\|_{L^\infty_1 H^s_x} \le \|u - u^j\|_{L^\infty_1 H^s_x} + \|u^j - v^j\|_{L^\infty_1 H^s_x} + \|v - v^j\|_{L^\infty_1 H^s_x}. \tag{4.17}$$

First, according to (4.15), we can choose j_0 large enough so that

$$\|u - u^{j_0}\|_{L^\infty_1 H^s_x} + \|v - v^{j_0}\|_{L^\infty_1 H^s_x} \le 2\lambda/3 \ .$$

Second, from the definition of u_0^j and v_0^j in (4.14) we infer that

$$\|u_0^j - v_0^j\|_{H^3} \le j^{3-s}\|u_0 - v_0\|_{H^s} \le j^{3-s}\delta.$$

Therefore, by using the continuity of the flow map for smooth initial data, we can choose $\delta > 0$, such that

$$\|u^{j_0} - v^{j_0}\|_{L^\infty_1 H^s_x} \le \frac{\lambda}{3},$$

This concludes the proof of Theorem 1.1.

5. Improvement of the uniqueness result for $s > 0$

In this section we prove that uniqueness holds for initial data $u_0 \in H^s(\mathbb{R})$, $s > 0$, in the class $u \in L^\infty_T H^s_x \cap L^4_T W^{s,4}_x$. The great interest of this result is that we do not assume any condition on the gauge transform of u anymore. Moreover, when $s > \frac{1}{4}$, the Sobolev embedding $L^\infty_T H^s_x \hookrightarrow L^4_T W^{0+,4}_x$ ensures that uniqueness holds in $L^\infty_T H^s_x$, and thus the Benjamin-Ono equation is unconditionally well-posed in $H^s(\mathbb{R})$ for $s > \frac{1}{4}$.

According to the uniqueness result *i)* of Theorem 1.1, it suffices to prove that for any solution u to (1.1) that belongs to $L^\infty_T H^s_x \cap L^4_T W^{s,4}$, the associated gauge function $w = \partial_x P_{hi}(e^{-\frac{i}{2}F[u]})$ belongs to $X_T^{0,\frac{1}{2}}$. The proof is based on the following bilinear estimate that is shown in the appendix :

Proposition 5.1. *Let $s > 0$. Then, there exists $0 < \delta < \frac{s}{10}$ and $\theta \in (\frac{1}{2}, 1)$, let us say $\theta = \frac{1}{2} + \delta$, such that*

$$\begin{aligned} &\|P_{+hi}(W P_- \partial_x u)\|_{X^{\frac{1}{2},-\frac{1}{2}+2\delta}} \\ &\qquad \lesssim \|W\|_{X^{\frac{1}{2},\frac{1}{2}+\delta}} \big(\|J^s u\|_{L^2_{x,t}} + \|J^s u\|_{L^4_{x,t}} + \|u\|_{X^{s-\theta,\theta}}\big). \end{aligned} \tag{5.1}$$

First note that by the same scaling argument as in Section 4.3, for any given $\varepsilon > 0$, we can always assume that $\|J^s u\|_{L^\infty_T L^2_x} + \|J^s u\|_{L^4_{Tx}} \le \varepsilon$ and by (3.7) it follows that $\|u\|_{X_T^{s-\theta,\theta}} \lesssim \varepsilon$ for $0 \le \theta \le 1$.

Now, since $u \in L^\infty([0,T]; H^s(\mathbb{R})) \cap L^4_T W^{s,4}_x$ and satisfies (1.1), it follows that $u_t \in L^\infty([0,T]; H^{s-2}(\mathbb{R}))$. Therefore $F := F[u] \in L^\infty([0,T]; H^{s+1}_{loc})$ and $\partial_t F \in L^\infty([0,T]; H^{s-1}_{loc})$. It ensures that

$$W := P_{hi}(e^{-\frac{i}{2}F}) \in L^\infty([0,T]; H^{s+1}(\mathbb{R})) \cap L^4_T W^{s+1,4}_x \hookrightarrow X^{1,0}, \tag{5.2}$$

$e^{-\frac{i}{2}F} \in L^\infty([0,T]; H^{s+1}_{loc})$ and the following calculations are thus justified:

$$\begin{aligned} \partial_t W = \partial_t P_+(e^{-\frac{i}{2}F}) &= -\frac{i}{2} P_{hi}(F_t e^{-\frac{i}{2}F}) \\ &= -\frac{i}{2} P_{hi}\Big(e^{-\frac{i}{2}F}(-\mathcal{H}F_{xx} + \frac{1}{2}F_x^2)\Big) \end{aligned}$$

and

$$\partial_{xx} W = \partial_{xx} P_{hi}(e^{-\frac{i}{2}F}) = P_{hi}\Big(e^{-\frac{i}{2}F}(-\frac{1}{4}F_x^2 - \frac{i}{2}F_{xx})\Big) \quad .$$

It follows that W satisfies at least in a distributional sense,

$$\left\{ \begin{array}{l} \partial_t W - i\partial_x^2 W = -P_{+hi}\big(W P_- \partial_x u\big) - P_{+hi}\big(P_{lo} e^{-\frac{i}{2}F} P_- \partial_x u\big) \\ W(\cdot,0) = P_{+hi}(e^{-\frac{i}{2}F[u_0]}) \quad . \end{array} \right. \tag{5.3}$$

From (5.2) and Lemma 2.6 we thus deduce that $W \in X^{s,1}_T$, so that, by interpolation with (5.2), $W \in X^{1/2,1/2+}_T$. But, u being given in $L^\infty_T H^s_x \cap L^4_T W^{s,4}_x \cap X^{s-\theta,\theta}_T$, on one hand gathering (2.6), the bilinear estimate (5.1) and (3.49), we infer that there exists only one solution to (5.3) in $X^{1/2,\frac{1}{2}+}_T$. Hence, $w = \partial_x W$ belongs to $X^{-1/2,\frac{1}{2}+}_T$ and is the unique solution to (3.4) in $X^{-1/2,\frac{1}{2}+}_T$ emanating from the initial data $w_0 = \partial_x P_{hi}(e^{-\frac{i}{2}F[u_0]}) \in L^2(\mathbb{R})$. On the other hand, according to Proposition 3.4, one can construct a solution to (3.4) emanating from w_0 and belonging to Y^s_T, by using a Picard iterative scheme. Moreover, using (1.1) and Lemma 2.6 we can easily check that this solution belongs to $X^{-1,1}_T$ and thus by interpolation to $X^{s-,\frac{1}{2}+}_T \hookrightarrow X^{-1/2,\frac{1}{2}+}_T$. This ensures that $w = \partial_x P_{hi}(e^{-iF/2})$ belongs to $Y^s_T \hookrightarrow X^{0,1/2}_T$ which concludes the proof.

6. Continuity of the flow-map for the weak L^2-topology

In [9] it is proven that, for any $t \geq 0$, the flow-map $u_0 \mapsto u(t)$ associated to the Benjamin-Ono equation is continuous from $L^2(\mathbb{R})$ equipped with the weak topology into itself. In this section, we explain how the uniqueness part of Theorem 1.1 enables to really simplify the proof of this result by following the approach developed in [11].

Let $\{u_{0,n}\}_n \subset L^2(\mathbb{R})$ be a sequence of initial data that converges weakly to u_0 in $L^2(\mathbb{R})$ and let u be the solution emanating from u_0 given by Theorem 1.1. From the Banach-Steinhaus theorem, we know that $\{u_{0,n}\}_n$ is bounded in $L^2(\mathbb{R})$ and from Theorem 1.1 we know that $\{u_{0,n}\}_n$ gives rise to a sequence $\{u_n\}_n$ of solutions to (1.1) bounded in $C([0,1]; L^2(\mathbb{R})) \cap L^4(]0,1[\times\mathbb{R})$ with an associated sequence of gauge functions $\{w_n\}_n$ bounded in $X^{0,1/2}_1$. Therefore there exist $v \in L^\infty(]0,1[; L^2(\mathbb{R})) \cap X^{-1,1}_1 \cap L^4(]0,1[\times\mathbb{R})$ and $z \in X^{0,1/2}_1$ such that, up to the extraction of a subsequence, $\{u_n\}_n$ converges to v weakly in $L^4(]0,1[\times\mathbb{R})$ and weakly star in $L^\infty(]0,1[\times\mathbb{R})$ and $\{w_n\}_n$ converges to z weakly in $X^{0,1/2}_1$. We now need some compactness on $\{u_n\}_n$ to ensure that z is the gauge transform of v. In this direction, we first notice, since $\{w_n\}_n$ is bounded in $X^{0,1/2}_1$ and by using the

Kato's smoothing effect injected in Bourgain's spaces framework, that $\{D_x^{\frac14} w_n\}_n$ is bounded in $L^4_x L^2_1$. Let $\eta_R(\cdot) := \eta(\cdot/R)$. Using (3.6) and Lemma 2.6 we infer that

$$
\begin{aligned}
\|D_x^{\frac14} P_{+HI} u_n\|_{L^2(]0,1[\times]-R,R[} &\lesssim \|D_x^{\frac14} P_{+HI}\big(e^{\frac{i}{2}F[u_n]} w_n \eta_R\big)\|_{L^2_{1,x}} \\
&+ \|D_x^{\frac14} P_{+HI}\big(P_{+hi} e^{\frac{i}{2}F[u_n]} \partial_x P_{lo} e^{-\frac{i}{2}F[u_n]}\big)\|_{L^2_{1,x}} \\
&+ \|D_x^{\frac14} P_{+HI}\big(P_{+HI} e^{\frac{i}{2}F[u_n]} \partial_x P_{-hi} e^{-\frac{i}{2}F[u_n]}\big)\|_{L^2_{1,x}} \\
&\lesssim \|D_x^{\frac14}(w_n\eta_R)\|_{L^2_x L^2_1} + \|D_x^{\frac14} e^{iF[u_n]}\|_{L^8_{1,x}} \|w_n\|_{L^{\frac83}_{1,x}} + \|u_n\|^2_{L^4_{1,x}}
\end{aligned}
$$

But clearly,

$$
\|D_x^{\frac14}(w_n\eta_R)\|_{L^2_x L^2_1} \lesssim C(R)(\|D_x^{\frac14} w_n\|_{L^4_x L^2_1} + \|w_n\|_{L^2_{1,x}})
$$

and by interpolation $\|D_x^{\frac14} e^{iF[u_n]}\|_{L^8_{1,x}} \lesssim \|u_n\|^{\frac34}_{L^2_{1,x}}$. Therefore, recalling that the u_n are real-valued functions, it follows that $\{u_n\}_n$ is bounded in $L^2_1 H^{\frac14}(]-R,R[)$.

Since, according to the equation (1.1), $\{\partial_t u_n\}_n$ is bounded in $L^2_1 H^{-2}_x$, Aubin-Lions compactness theorem and standard diagonal extraction arguments ensure that there exists an increasing sequence of integer $\{n_k\}_k$ such that $u_{n_k} \to v$ a.e. in $]0,1[\times\mathbb{R}$ and $u^2_{n_k} \rightharpoonup v^2$ in $L^2(]0,1[\times\mathbb{R})$. In view of our construction of the primitive $F[u_n]$ of u_n (see Section 3.1), it is then easy to check that $F[u_{n_k}]$ converges to the primitive $F[v]$ of v a.e. in $]0,1[\times\mathbb{R}$. This ensures that $P_{+hi}(e^{-\frac{i}{2}F[u_{n_k}]})$ converges weakly to $P_{+hi}(e^{-\frac{i}{2}F[v]})$ in $L^2(]0,1[\times\mathbb{R})$ and thus z is the gauge transform of v . Passing to the limit in the equation , we conclude that v satisfies (1.1) and belong the class of uniqueness of Theorem 1.1.

Moreover, setting $(\cdot,\cdot)$ for the L^2_x scalar product, by (1.1) and the bounds above, it is easy to check that, for any smooth space function ϕ with compact support, the family $\{t \mapsto (u_{n_k}(t),\phi)\}$ is uniformly equi-continuous on $[0,1]$. Ascoli's theorem then ensures that $(u_{n_k}(\cdot),\phi)$ converges to $(v(\cdot),\phi)$ uniformly on $[0,1]$ and thus $v(0) = u_0$. By uniqueness, it follows that $v \equiv u$ which ensures that the whole sequence $\{u_n\}$ converges to v in the sense above and not only a subsequence. Finally, from the above convergence result, it results that $u_n(t) \rightharpoonup u(t)$ in L^2_x for all $t \in [0,1]$. □

7. The periodic case

In this section we explain how the bilinear estimate proved in Proposition 3.5 can lead to a great simplification of the global well-posedness result in $L^2(\mathbb{T})$ derived in [21] and to new uniqueness results in $H^s(\mathbb{T})$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. With the notations of [20] these new results lead to the following global well-posedness theorem :

Theorem 7.1. *Let $s \geq 0$ be given.*
<u>Existence :</u> For all $u_0 \in H^s(\mathbb{T})$ and all $T > 0$, there exists a solution

$$
u \in C([0,T]; H^s(\mathbb{T})) \cap X^{s-1,1}_T \cap L^4_T W^{s,4}(\mathbb{T}) \tag{7.1}
$$

of (1.1) *such that*

$$
w = \partial_x P_{+hi}\big(e^{-\frac{i}{2}\partial_x^{-1}\tilde{u}}\big) \in Y^s_T. \tag{7.2}
$$

where

$$
\tilde{u} := u(t, x - t \textstyle\fint u_0) - \fint u_0 \quad \textit{and} \quad \widehat{\partial_x^{-1}} := \dfrac{1}{i\xi},\ \xi \in \mathbb{Z}^* \quad .
$$

Uniqueness : This solution is unique in the following classes :

i)	$u\in L^\infty(]0,T[;L^2(\mathbb{T})\cap L^4(]0,T[\times\mathbb{T})$ *and* $w\in X_T^{0,\frac12}$.	
ii)	$u\in L^\infty(]0,T[;H^{\frac14}(\mathbb{T})\cap L^4_T W^{\frac14,4}(\mathbb{T})$	*whenever* $s\geq\frac14$.
iii)	$u\in L^\infty(]0,T[;H^{\frac12}(\mathbb{T})$	*whenever* $s\geq\frac12$.

Moreover, $u\in C_b(\mathbb{R};L^2(\mathbb{T}))$ *and the flow map data-solution* : $u_0\mapsto u$ *is continuous from* $H^s(\mathbb{T})$ *into* $C([0,T];H^s(\mathbb{T}))$.

Sketch of the proof. In the periodic case, following [20], the gauge transform is defined as follows : Let u be a smooth 2π-periodic solution of (BO) with initial data u_0. In the sequel, we will assume that $u(t)$ has mean value zero for all time. Otherwise we do the change of unknown :

$$\tilde u(t,x):=u(t,x-t\fint u_0)-\fint u_0 \quad , \tag{7.3}$$

where $\fint u_0 := \frac{1}{2\pi}\int_{\mathbb{T}} u_0$ is the mean value of u_0. It is easy to see that $\tilde u$ satisfies (BO) with $u_0-\fint u_0$ as initial data and since $\fint\tilde u$ is preserved by the flow of (BO), $\tilde u(t)$ has mean value zero for all time. We take for the primitive of u the unique periodic, zero mean value, primitive of u defined by

$$\hat F(0)=0 \quad\text{ and } \widehat F(\xi)=\frac{1}{i\xi}\hat u(\xi), \quad \xi\in\mathbb{Z}^* \quad .$$

The gauge transform is then defined by

$$W:=P_+(e^{-iF/2}) \quad . \tag{7.4}$$

Since F satisfies

$$F_t+\mathcal{H}F_{xx}=\frac{F_x^2}{2}-\frac12\fint F_x^2=\frac{F_x^2}{2}-\frac12 P_0(F_x^2) \quad ,$$

we finally obtain that $w:=W_x=-\frac{i}{2}P_{+hi}(e^{-iF/2}F_x)=-\frac{i}{2}P_+(e^{-iF/2}u)$ satisfies

$$\begin{aligned} w_t-iw_{xx} &= -\partial_x P_{hi}\Big[e^{-iF/2}\Big(P_-(F_{xx})-\frac{i}{4}P_0(F_x^2)\Big)\Big] \\ &= -\partial_x P_{+hi}\Big(WP_-(u_x)\Big)+\frac{i}{4}P_0(F_x^2)w \; . \end{aligned} \tag{7.5}$$

Clearly the second term is harmless and the first one has exactly the same structure as the one that we estimated in Proposition 3.5 . Following carefully the proof of this proposition, it is not too hard to check that it also holds in the periodic case independently of the period $\lambda\geq 1$. Note in particular that (2.9) also holds with $L^4_{x,t}$ and $X^{0,\frac38}$ respectively replaced by $L^4_{t,\lambda}$ and $X_\lambda^{0,\frac38}$, $\lambda\geq1$, where the subscript λ denotes spaces of functions with space variable on the torus $\mathbb{R}/2\pi\lambda\mathbb{Z}$ (see [5] and also [20]). This leads to a great simplification of the proof the global well-posedness in $L^2(\mathbb{T})$ proved in [21].

Now to derive the new uniqueness result we proceed exactly as in Section 5 except that Proposition 5.1 does not hold on the torus. Actually, on the torus it should be replaced by

Proposition 7.2. *For $s \geq \frac{1}{4}$ and all $\lambda \geq 1$ it holds*

$$\begin{aligned}(7.6)\qquad &\|P_{+hi}(WP_-\partial_x u)\|_{X_\lambda^{s+\frac{1}{2},-\frac{1}{2}}} \\ &\qquad \lesssim \|W\|_{X_\lambda^{s+\frac{1}{2},\frac{1}{2}}}\big(\|J_x^s u\|_{L^2_{T,\lambda}} + \|J_x^s u\|_{L^4_{T,\lambda}} + \|u\|_{X_\lambda^{s-1,1}}\big).\end{aligned}$$

Going back to the proof of the bilinear estimate it easy to be convinced that the above estimate works at the level $s = 0+$ in the regions $\mathcal{A}$ and $\mathcal{B}$ (see the proof of Proposition 5.1), whereas in the region $\mathcal{C}$ we are clearly in trouble. Indeed, when $s = 0$, (3.38) has then to be replaced by

$$|k^{\frac{1}{2}} k_1^{-\frac{1}{2}} k_2^2 \langle \sigma_2\rangle^{-1}| \sim |k^{-\frac{1}{2}} k_1^{-\frac{1}{2}} k_2|$$

which cannot be bound when $|k_2| >> k$. On the other hand at the level $s = \frac{1}{4}$ it becomes

$$|k^{\frac{3}{4}} k_1^{-\frac{3}{4}} k_2^{\frac{7}{4}} \langle \sigma_2\rangle^{-1}| \sim |k^{-\frac{1}{4}} k_1^{-\frac{3}{4}} k_2^{\frac{3}{4}}| \lesssim k^{-\frac{1}{4}} \lesssim 1$$

which yields the result.

With Proposition 7.2 in hand, exactly the same procedure as in Section 5 leads to the uniqueness result in the class $u \in L^\infty_T H^{\frac{1}{4}}(\mathbb{T}) \cap L^4_T W^{\frac{1}{4},4}(\mathbb{T})$ and by Sobolev embedding to the uniqueness in the class $u \in L^\infty_T H^{\frac{1}{2}}(\mathbb{T})$, i.e. unconditional uniqueness in $H^{\frac{1}{2}}(\mathbb{T})$. As in the real line case, it proves the uniqueness of the (energy) weak solutions that belong to $L^\infty(\mathbb{R}; H^{1/2}(\mathbb{T}))$.

Appendix

Proof of Proposition 5.1. We will need that following calculus lemma stated in [10].

Lemma 7.3. *Let $0 < a_- \leq a_+$ such that $a_- + a_+ > \frac{1}{2}$. Then, for all $\mu \in \mathbb{R}$*

$$(7.7)\qquad \int_{\mathbb{R}} \langle y\rangle^{-2a_-}\langle y-\mu\rangle^{-2a_+} dy \lesssim \langle \mu\rangle^{-s},$$

where $s = 2a_-$ if $a_+ > \frac{1}{2}$, $s = 2a_- - \epsilon$, if $a_+ = \frac{1}{2}$, and $s = 2(a_+ + a_-) - 1$, if $a_+ < \frac{1}{2}$ and ϵ denote any small positive number.

The proof of Proposition 5.1 follows closely the one of Proposition 3.5 except in the region σ_2-dominant where we use the approach developed in [16]. Recalling the notation used in (3.24)–(3.25), we need to prove that

$$(7.8)\qquad \big|K\big| \lesssim \|h\|_{L^2_{x,t}}\|f\|_{L^2_{x,t}}\big(\|u\|_{L^2_{x,t}} + \|u\|_{L^4_{x,t}} + \|u\|_{X^{-\theta,\theta}}\big),$$

where

$$(7.9)\qquad K = \int_{\mathcal{D}} \frac{\langle\xi\rangle^{\frac{1}{2}}}{\langle\sigma\rangle^{\frac{1}{2}-2\delta}}\widehat{h}(\xi,\tau)\frac{\langle\xi_1\rangle^{-\frac{1}{2}}}{\langle\sigma_1\rangle^{\frac{1}{2}+\delta}}\widehat{f}(\xi_1,\tau_1)\xi_2\langle\xi_2\rangle^{-s}\widehat{u}(\xi_2,\tau_2)d\nu.$$

For the same reason as in the proof of Proposition 3.5, we can assume that $|\xi_2| \leq 1$. By using a Littlewood-Paley decomposition on h, f and u, K can be rewritten as

$$(7.10)\qquad K = \sum_{N,N_1,N_2} K_{N,N_1,N_2}$$

with

$$K_{N,N_1,N_2} := \int_{\mathcal{D}} \frac{\langle\xi\rangle^{\frac{1}{2}}}{\langle\sigma\rangle^{\frac{1}{2}-2\delta}}\widehat{P_N h}(\xi,\tau)\frac{\langle\xi_1\rangle^{-\frac{1}{2}}}{\langle\sigma_1\rangle^{\frac{1}{2}+\delta}}\widehat{P_{N_1} f}(\xi_1,\tau_1)\xi_2\langle\xi_2\rangle^{-s}\widehat{P_{N_2} u}(\xi_2,\tau_2)d\nu,$$

and the dyadic numbers N, N_1 and N_2 ranging from 1 to $+\infty$. Moreover, we will denote by $K^{\mathcal{A}_{N,N_2}}_{N,N_1,N_2}$, $K^{\mathcal{B}_{N,N_2}}_{N,N_1,N_2}$, $K^{\mathcal{C}_{N,N_2}}_{N,N_1,N_2}$ the restriction of K_{N,N_1,N_2} to the regions $\mathcal{A}_{N,N_2}$, $\mathcal{B}_{N,N_2}$ and $\mathcal{C}_{N,N_2}$ defined in (3.28). Then, it follows that

$$\big|K\big| \le \big|K_{\mathcal{A}}\big| + \big|K_{\mathcal{B}}\big| + \big|K_{\mathcal{C}}\big|, \tag{7.11}$$

where

$$K_{\mathcal{A}} := \sum_{N,N_1,N_2} J^{\mathcal{A}_{N,N_2}}_{N,N_1,N_2},\ K_{\mathcal{B}} := \sum_{N,N_1,N_2} K^{\mathcal{B}_{N,N_2}}_{N,N_1,N_2} \text{ and } K_{\mathcal{C}} := \sum_{N,N_1,N_2} J^{\mathcal{C}_{N,N_2}}_{N,N_1,N_2}\ ,$$

so that it suffices to estimate $\big|K_{\mathcal{A}}\big|$, $\big|K_{\mathcal{B}}\big|$ and $\big|K_{\mathcal{C}}\big|$. Recall that, due to the structure of $\mathcal{D}$, one of the following case must hold:

(1) high-low interaction: $N_1 \sim N$ and $N_2 \le N_1$
(2) high-high interaction: $N_1 \sim N_2$ and $N \le N_1$.

Estimate for $\big|K_{\mathcal{A}}\big|$. In the first case, it follows from the triangular inequality, Plancherel's identity and Hölder's inequality that

$$\begin{aligned}\big|K_{\mathcal{A}}\big| &\lesssim \|h\|_{L^2_{x,t}} \sum_{N_1}\sum_{N_2\le N_1} \frac{N_1^{\frac12}}{(N_1N_2)^{\frac12-2\delta}} \Big\| P_{N_1}\Big(J_x^{-\frac12} P_{N_1}\Big(\frac{\widehat{f}}{\langle\sigma_1\rangle^{\frac12+\delta}}\Big)^{\vee} P_-\partial_x J_x^{-s} P_{N_2} u\Big)\Big\|_{L^2_{x,t}} \\ &\lesssim \|h\|_{L^2_{x,t}} \sum_{N_1}\sum_{N_2\le N_1} \frac{N_2^{\frac12-s+2\delta}}{(N_1)^{\frac12-2\delta}} \Big\|P_{N_1}\Big(\frac{\widehat{f}}{\langle\sigma_1\rangle^{\frac12+\delta}}\Big)^{\vee}\Big\|_{L^4_{x,t}} \|P_{N_2}u\|_{L^4_{x,t}} \\ &\lesssim \|h\|_{L^2_{x,t}} \|u\|_{L^4_{x,t}} \sum_{N_1} N_1^{4\delta-s} \Big\|P_{N_1}\Big(\frac{\widehat{f}}{\langle\sigma_1\rangle^{\frac12+\delta}}\Big)^{\vee}\Big\|_{L^4_{x,t}}.\end{aligned}$$

Then, it is deduced from the Cauchy-Schwarz inequality in N_1 that

$$\big|K_{\mathcal{A}}\big| \lesssim \|h\|_{L^2_{x,t}} \Big(\sum_{N_1} \Big\|P_{N_1}\Big(\frac{\widehat{f}}{\langle\sigma_1\rangle^{\frac12+\delta}}\Big)^{\vee}\Big\|^2_{L^4_{x,t}}\Big)^{\frac12} \|u\|_{L^4_{x,t}}, \tag{7.12}$$

since $s > 10\delta$. On the other, estimate (7.12) also holds in the case of high-high interaction by arguing exactly as in (3.32), so that estimate (2.9) yields

$$\big|K_{\mathcal{A}}\big| \lesssim \|h\|_{L^2_{x,t}} \|f\|_{L^2_{x,t}} \|u\|_{L^4_{x,t}}. \tag{7.13}$$

Estimate for $\big|K_{\mathcal{B}}\big|$. The estimate

$$\big|K_{\mathcal{B}}\big| \lesssim \|h\|_{L^2_{x,t}} \|f\|_{L^2_{x,t}} \|u\|_{L^4_{x,t}}, \tag{7.14}$$

follows arguing as in (7.12).

Estimate for $\big|K_{\mathcal{C}}\big|$. First observe that

$$\big|K_{\mathcal{C}}\big| \lesssim \int_{\widetilde{\mathcal{C}}} \frac{|\xi|^{\frac12}}{\langle\sigma\rangle^{\frac12-2\delta}} |\widehat{h}(\xi,\tau)| \frac{|\xi_1|^{-\frac12}}{\langle\sigma_1\rangle^{\frac12+\delta}} |\widehat{f}(\xi_1,\tau_1)| \frac{|\xi_2|^{(1+\theta-s)}}{\langle\sigma_2\rangle^{\theta}} \frac{\langle\sigma_2\rangle^{\theta}}{|\xi_2|^{\theta}} |\widehat{u}(\xi_2,\tau_2)| d\nu, \tag{7.15}$$

where

$$\widetilde{\mathcal{C}} = \big\{(\xi,\xi_1,\tau,\tau_1) \in \mathcal{D} \mid (\xi,\xi_1,\tau,\tau_1) \in \bigcup_{N,N_2} \mathcal{C}_{N,N_2}\big\}.$$

Since $|\sigma_2| > |\sigma|$ and $|\sigma_2| > |\sigma_1|$ in $\widetilde{\mathcal{C}}$, (3.28) implies that $|\sigma_2| \gtrsim |\xi\xi_2|$. Applying twice the Cauchy-Schwarz inequality, it is deduced that

$$\big|K_{\mathcal{C}}\big| \lesssim \sup_{\xi_2,\tau_2} \big(L_{\widetilde{\mathcal{C}}}(\xi_2,\tau_2)\big)^{\frac12} \|f\|_{L^2_{\xi,\tau}} \|g\|_{L^2_{\xi,\tau}} \|h\|_{L^2_{\xi,\tau}},$$

where

$$L_{\widetilde{\mathcal{C}}}(\xi_2,\tau_2)=\frac{|\xi_2|^{2+2(\theta-s)}}{\langle\sigma_2\rangle^{2\theta}}\int_{\mathcal{C}(\xi_2,\tau_2)}\frac{|\xi||\xi_1|^{-1}}{\langle\sigma\rangle^{1-4\delta}\langle\sigma_1\rangle^{1+2\delta}}d\xi_1 d\tau_1,$$

and

$$\widetilde{\mathcal{C}}(\xi_2,\tau_2)=\big\{(\xi_1,\tau_1)\in\mathbb{R}^2 \mid (\xi,\xi_1,\tau,\tau_1)\in\mathcal{C}\big\}.$$

Thus, to prove that

$$\big|K_{\mathcal{C}}\big|\lesssim\|h\|_{L^2_{x,t}}\|f\|_{L^2_{x,t}}\|u\|_{X^{-\theta,\theta}}, \tag{7.16}$$

it is enough to prove that $L_{\widetilde{\mathcal{C}}}(\xi_2,\tau_2)\lesssim 1$ for all $(\xi_2,\tau_2)\in\mathbb{R}^2$. We deduce from (7.7) and (3.28) that

$$L_{\widetilde{\mathcal{C}}}(\xi_2,\tau_2)\lesssim\frac{|\xi_2|^{2+2(\theta-s)}}{\langle\sigma_2\rangle^{1+2\delta}}\int_{\xi_1}\frac{|\xi||\xi_1|^{-1}}{\langle\sigma_2+2\xi\xi_2\rangle^{1-4\delta}}d\xi_1,$$

since $\theta=1+\delta$. To integrate with respect to ξ_1, we change variables

$$\mu_2=\sigma_2+2\xi\xi_2 \quad\text{so that}\quad d\mu_2=2\xi_2 d\xi_1 \quad\text{and}\quad |\mu_2|\le 4|\sigma_2|.$$

Moreover, (3.26) and (3.28) imply that

$$\frac{|\xi||\xi_1|^{-1}|\xi_2|^{1+2(\theta-s)}}{|\xi_1|^2}\le|\xi\xi_2|^{\frac12+\theta-s}\lesssim|\sigma_1|^{\frac12+\theta-s}$$

in $\widetilde{\mathcal{C}}$. Then,

$$\begin{aligned}L_{\widetilde{\mathcal{C}}}(\xi_2,\tau_2)&\lesssim\frac{|\xi_2|^{1+2(\theta-s)}}{\langle\sigma_2\rangle^{1+2\delta}}\int_0^{4|\sigma_2|}\frac{|\xi||\xi_1|^{-1}}{\langle\mu_2\rangle^{1-4\delta}}d\mu_2\\&\lesssim\frac{\langle\sigma_2\rangle^{\frac12+\theta-s+4\delta}}{\langle\sigma_2\rangle^{1+2\delta}}\lesssim\langle\sigma_2\rangle^{3\delta-s}\lesssim 1,\end{aligned}$$

since $s-3\delta>0$.

Finally, we conclude the proof of Proposition 5.1 gathering (7.8), (7.11), (7.13), (7.14) and (7.16).

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