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A layer of a saturated binary mixture of soil and water, both of which are true density preserving, is considered. This layer is subjected from above to normal and shear tractions and an inflow of water, and from below to drainage of water and abrasion of till from the rock bed. Sliding processes along the top and bottom interfaces, as well as deformational creep of the sediment and water constituents within the layer generate heat, but here the purely mechanical problem is analysed. We study the steady-state plane flow with vanishing abrasion and balanced inflow and drainage of water. The differential equations governing the horizontal creep flows of the sediment and water decouple from the equations describing the vertical profiles of the vertical water velocity and the solid volume fraction. A stiff second-order ordinary differential equation is shown to describe the distribution of the latter; it genuinely depends on the inflow of water, the fluid viscosity and the thermodynamic pressure, a variable not present in classical formulations and introduced by Svendsen and Hutter in 1995. The singular nature of this equation is resolved by methods of matched asymptotic expansions.

It is shown that in conformity with thermodynamics in the sediment layer, two regions of more dense and less dense solid fraction arise, one of which is a boundary layer. This boundary layer is formed where the fluid enters the sediment layer. Moreover, by using the thickness of this boundary layer as the internal length parameter, we find several constitutive equations for the thermodynamic pressure which adequately describe this creeping flow problem. Viewed as a model for the saturated till layer below ice sheets, the analysis shows that the question of whether soft basal sliding may develop catastrophically is a question of thermodynamics rather than dynamics.

Keywords: saturated binary mixture; soil–water interaction; sheared saturated soil; soft basal sliding; thermodynamic stability; singular perturbations

1. Introduction

In dell' Isola & Hutter (1998), referred to henceforth as I, we presented a saturated binary mixture model of a granular solid and fluid both of which are true density preserving. This model may serve for the description of the dynamics of the thin sediment–water layer below temperate glaciers or ice sheets (see figure 1). In particular, it was shown that the water and till flow within this layer was governed by the

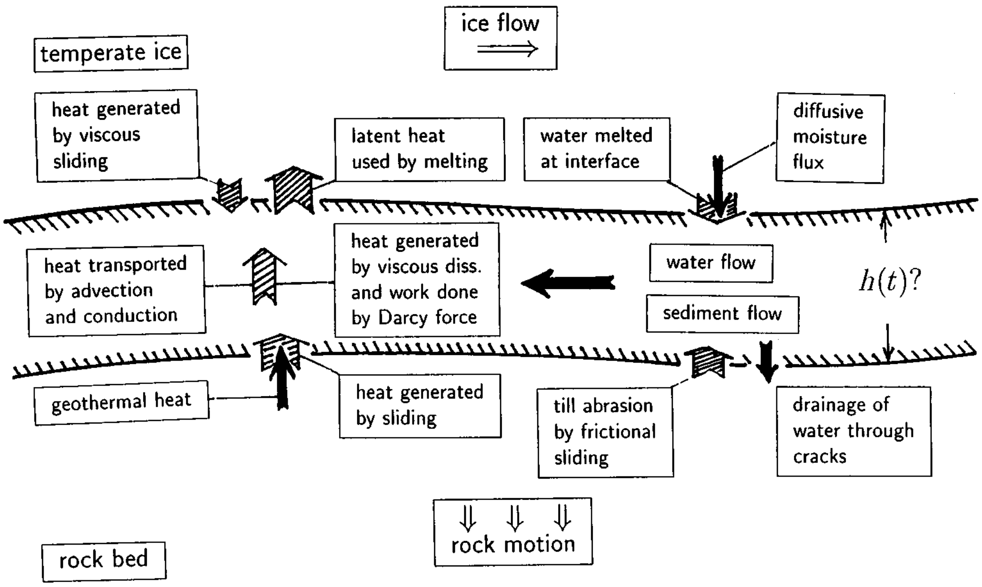

Figure 1. A sketch of the thermodynamic processes affecting the saturated till layer below ice sheets. The arrows on the right indicate the mass flow of water and sediments into and out of the layer, those on the left indicate the flow of heat. The various boxes describe the physical processes that are specified by the mathematical model of dell' Isola & Hutter (1997, 1998).

water inflow from the ice sheet above it, the drainage of water into the rock bed, and the amount of till abraded at the rock–bed interface. The redistribution of mass by the flows of water and sediment within the layer is governed by the force balances of these two constituents; in these balances internal friction and interface friction at the ice–sediment-layer interface and the rock–bed interface are essentially balanced by the pressure and driving shear stresses exerted by the ice sheet from above. The latter are so dominant that gravity forces within the layer can be ignored. The lubrication is provided to the system primarily by the melting processes due to the geothermal heat, the heat generated by viscous dissipation and the work done by the Darcy forces within the layer, and the heat generated by the viscous sliding of the ice over the ice–sediment interface. It was further made clear in I that a binary mixture model of viscous constituents was needed for properly describing the thermodynamically coupled processes, and that the earlier models by MacAyeal (1992), Kamb (1991) and Alley *et al.* (1987*a*, *b*) were likely to be too simple to properly describe the difficult thermomechanical problem.

In § 5 of I, it was made plausible that steady plane shear flow of water and sediment in a layer of constant depth in which the horizontal and vertical velocity components are assumed to be functions only of the vertical coordinate, is likely to form a well-posed boundary-value problem; however, no explicit solution was constructed. In this paper we focus our attention on a very similar problem; our emphasis will now be to construct explicit solutions to a certain boundary-value problem that still contains the essential ingredients of the shear flow that prevails in the till layer below ice sheets. However, we focus attention on the across-layer solid volume fraction profiles, how they form as a result of the across layer water flow, which is an essential ingredient of the subglacial till mechanism. We shall demonstrate that the thermodynamic stability condition and the dynamic equations of balance of mass and momentum constrain the functional behaviour of the thermodynamic pressure of the solid constituent.

The equation from which this result is derived is an ordinary differential equation for the distribution of the solid volume fraction across the layer. Genuine quantities describing this equation are the across-layer fluid-flow rate, the fluid viscosity and the thermodynamic pressure of the solid constituent. In classical theories of saturated soils, the latter two are both ignored (see, for example, Vulliet & Hutter 1988; Drew & Segel 1971; Drew & Lahey 1979; Mackenzie 1984). This equation is second order in the spatial derivative, and the term with the highest derivative is small, giving reason for the use of matched asymptotic expansions. We demonstrate that for Stokes flow, thermodynamic restrictions place this boundary layer at the top of the layer. The solid volume fraction is highest at the rock–bed interface; it is practically constant throughout the layer but quickly decreases as the upper boundary is approached. This demonstrates that the boundary layer develops where the water enters the layer.

Beyond this result, this application of the model also offers guidelines as to the estimation of explicit expressions for the thermodynamic pressure as a function of the solid volume fraction. These functional relations are obtained from requirements of the thickness of the boundary layer. The problem as such does not introduce an internal length scale; the boundary-layer thickness which we choose to be 10^{-3}–10^{-2} m (i.e. of the order of several particle diameters) serves as this length scale. In fact, its thickness is related to the exact value of the flow rate from above, and the above figures correspond to 10^{-3}–10^{0} m a^{-1} as incoming velocity. Together with the thermodynamic stability requirement that the Helmholtz free energy be a convex function of the solid volume fraction, its choice determines the qualitative behaviour of the thermodynamic pressure as a function of the solid volume fraction; it must possess a strong (exponential-like) singularity at a finite value of the solid volume fraction in order that the internal length scale assumes physically meaningful values. We offer explicit proposals for this dependency. Experiments ought to isolate the correct choice.

In § 2 we present the governing equations specialized for the problem at hand. To fully understand equations (2.1), (2.9) and (2.12) the reader may have to consult I; however the remaining text is disjoint from I and self-consistent. Section 3 presents the asymptotic analysis for steady-state conditions and shows that the two-point boundary problem for the solid volume fraction profile is singular; techniques of matched asymptotic expansions are used to solve it. These methods are then also used in the specification of the constitutive relation for the thermodynamic pressure in § 4. In § 5 we summarize our findings and draw inferences.

2. Governing equations

To list the governing equations that form the basis of this study, let $Oxyz$ be a Cartesian coordinate system with origin O, horizontal axes (x, y) and vertical axis z pointing upwards opposite to the direction of gravity. Consider a layer bounded by $z = f_{\mathrm{b}}(x, y, t)$ (bottom surface) and $z = h(x, y, t)$ (top surface) and of infinite horizontal extent. Let this layer be filled by a saturated mixture of sediment and water. The solid volume fraction, the velocity fields of the fluid and solid and the temperature field within this layer are driven by the mass, momentum and energy conservation conditions at the bounding surfaces; they are derived in I and are linked to processes of ice-sheet flow for $z > h(x,y,t)$ and bedrock deformation for $z < f_{\mathrm{b}}(x,y,t)$. Here we treat the input quantities at $z = h(x,y,t)$ and at $z = f_{\mathrm{b}}(x,y,t)$ as prescribed, and we ignore temperature variations, i.e. consider isothermal conditions, since our focus is a qualitative analysis of the till-layer behaviour.

We restrict considerations to plane flow and thus ignore the third, y-coordinate. Moreover, we suppose all process quantities except the saturation pressure to be independent of the horizontal x-coordinate, so that the z-coordinate and the time t will remain as the only surviving independent variables. The analysis will show that with these prerequisites the solid volume fraction profile as well as the horizontal velocity profiles of the till and the fluid within the layer can be determined analytically and/or numerically as functionals of the input quantities.

Time-independent steady conditions prevail, if abrasion of till from the bedrock is ignored, the incumbent pressure is time independent and the drainage of water equals the constant inflow of melt water from above. This steady problem, special as it is, will allow in-depth analysis of the processes and thus provides hints as to the proper parametrization of some of the constitutive equations arising in the model.

From § 2, formula (20), of I, we deduce the following reduced forms of the balances of mass and momentum for the solid and the fluid:

$$\left.\begin{aligned}
\frac{\partial \nu}{\partial t} + (\nu w_{\mathrm{s}})' &= 0,\\
-\frac{\partial \nu}{\partial t} + ((1-\nu) w_{\mathrm{f}})' &= 0,\\
(-\nu\beta_{\mathrm{s}} - \nu p + \mu_{\mathrm{s}} w_{\mathrm{s}}')' + (p + (1-\xi_{\mathrm{s}})\beta_{\mathrm{s}})\nu' - \nu(1-\nu)\tilde{\alpha}(w_{\mathrm{s}} - w_{\mathrm{f}}) &= 0,\\
(-(1-\nu)p + \tfrac{4}{3}\mu_{\mathrm{f}} w_{\mathrm{f}}')' - (p + (1-\xi_{\mathrm{s}})\beta_{\mathrm{s}})\nu' + \nu(1-\nu)\tilde{\alpha}(w_{\mathrm{s}} - w_{\mathrm{f}}) &= 0,\\
-\nu\frac{\partial p}{\partial x} + (\mu_{\mathrm{s}} u_{\mathrm{s}}')' - \nu(1-\nu)\tilde{\alpha}(u_{\mathrm{s}} - u_{\mathrm{f}}) &= 0,\\
-(1-\nu)\frac{\partial p}{\partial x} + (\mu_{\mathrm{f}} u_{\mathrm{f}}')' + \nu(1-\nu)\tilde{\alpha}(u_{\mathrm{s}} - u_{\mathrm{f}}) &= 0,\\
\text{in } z \in (f_{\mathrm{b}}, h), \quad t > 0,&
\end{aligned}\right\} \tag{2.1}$$

in which the prime denotes differentiation with respect to z, and the various symbols have the following meaning: ν is the solid volume fraction; $u_{\mathrm{s}}, u_{\mathrm{f}}$ are the horizontal velocity components of the solid and fluid, respectively; $w_{\mathrm{s}}, w_{\mathrm{f}}$ are the vertical velocity components of the solid and fluid, respectively; p is the saturation pressure; $\beta_{\mathrm{s}}(\nu)$ is the thermodynamic pressure; $\hat{\rho}_{\mathrm{f}}, \hat{\rho}_{\mathrm{s}}$ are the true fluid and sediment mass densities; $\mu_{\mathrm{f}}, \mu_{\mathrm{s}}$ are the (apparent) fluid and solid viscosities; $\tilde{\alpha} =: \hat{\rho}_{\mathrm{f}} g/K$; $a := \hat{\rho}_{\mathrm{f}}/\hat{\rho}_{\mathrm{s}}$;

$$\xi_{\mathrm{s}} := \frac{\rho_{\mathrm{s}}}{\rho} = \frac{\nu\hat{\rho}_{\mathrm{s}}}{\nu\hat{\rho}_{\mathrm{s}} + (1-\nu)\hat{\rho}_{\mathrm{f}}} = \frac{\nu}{\nu + (1-\nu)a};$$

g is the gravity constant; and K is the soil permeability. The thermodynamic pressure $\beta_{\mathrm{s}}(\nu)$ is related to the inner free energy ψ_{I} $(\nu, \vartheta = \text{const.})$ via

$$\beta_{\mathrm{s}}(\nu) = (\nu\hat{\rho}_{\mathrm{s}} + (1-\nu)\hat{\rho}_{\mathrm{f}})\frac{\partial \psi_{\mathrm{I}}}{\partial \nu} \quad (\nu, \vartheta = \text{const.}) \tag{2.2}$$

and will henceforth be written as

$$\beta_{\mathrm{s}} = \beta_0 + \check{\beta}_{\mathrm{s}}(\nu), \quad \check{\beta}_{\mathrm{s}}(0) = 0. \tag{2.3}$$

In the ensuing analysis we shall propose explicit expressions for the function $\beta_{\mathrm{s}}(\nu)$; this must be done in conformity with the thermodynamic stability requirement that ψ_{I} is a convex function of ν. Thus $\partial^2\psi_{\mathrm{I}}/\partial\nu^2$ must always be positive semi-definite; this implies

$$\frac{\mathrm{d}\beta_{\mathrm{s}}}{\mathrm{d}\nu}-\beta_{\mathrm{s}}\frac{1}{\nu+b}\geqslant 0, \quad b=\frac{a}{1-a}, \tag{2.4}$$

a condition which must be satisfied by every constitutive relation for β_{s}.

Equations (2.1) are six partial differential equations for $\nu, p, w_{\mathrm{s}}, w_{\mathrm{f}}, u_{\mathrm{s}}, u_{\mathrm{f}}$, of which the first four are decoupled from the remaining two. Moreover, it follows from $(2.1)_{5,6}$ that $\partial p/\partial x$ does not depend on x, so that p is linear as a function of x.

Equations (2.1) must be complemented by boundary and initial conditions. Before we turn to these, let us transform (2.1) to an analytically more convenient form. Adding $(2.1)_{1,2}$ and integrating the resultant equation over z yields

$$\nu w_{\mathrm{s}}+(1-\nu)w_{\mathrm{f}}=W_C^0(t), \tag{2.5}$$

where $W_C^0(t)$ is an integration constant, a function of t, called the vertical component of the *composite velocity*.

Next, by summing $(2.1)_{3,4}$ and integrating the resulting equation over z one obtains the saturation pressure in the form

$$p=-\nu\beta_{\mathrm{s}}+\mu_{\mathrm{s}}w_{\mathrm{s}}'+\tfrac{4}{3}\mu_{\mathrm{f}}w_{\mathrm{f}}'+p_0(t). \tag{2.6}$$

Evaluating p' and substituting the resulting expression into $(2.1)_3$ or $(2.1)_4$ leads to the following equation:

$$\nu' F(\nu)-\nu(\tfrac{4}{3}\mu_{\mathrm{f}}w_{\mathrm{f}}')'+(1-\nu)(\mu_{\mathrm{s}}w_{\mathrm{s}}')'-\nu(1-\nu)\tilde{\alpha}(w_{\mathrm{s}}-w_{\mathrm{f}})=0, \tag{2.7}$$

where

$$F(\nu):=(\nu-\xi_{\mathrm{s}})\beta_{\mathrm{s}}-\nu(1-\nu)\frac{\mathrm{d}\beta_{\mathrm{s}}}{\mathrm{d}\nu}. \tag{2.8}$$

Equations $(2.1)_1$, (2.5) and (2.7) form a well-posed initial boundary-value problem (IBVP) for ν, w_{s} and w_{f}, provided the inflow of the fluid (melting rate) at the top boundary, $(1-\nu(h,t))w_{\mathrm{f}}(h,t)$, and abrasion rate at the bottom boundary, $\nu w_{\mathrm{s}}(f_{\mathrm{b}},t)$, are *a priori* known, the incumbent pressure, $p_{\mathrm{i}}(t)$, at the top surface is given and the drainage function at the bottom interface, $(1-\nu(f_{\mathrm{b}},t))w_{\mathrm{f}}(f_{\mathrm{b}},t)$, is prescribed. In addition, an initial distribution for the volume fraction, $\nu(z,0)$, must be prescribed, and the kinematic conditions be obeyed that (i) the top boundary moves with the solid particles, whereas (ii) the bottom interface velocity coincides with the bottom abrasion rate. These requirements can be expressed by the following statements.

(1) At the top interface $z=h(t)$:

$$\left.\begin{aligned} w_{\mathrm{f}}(1-\nu)&=V_{\mathrm{f0}},\\ \frac{\partial h}{\partial t}&=w_{\mathrm{s}},\\ -\alpha\nu^{1}p_{\mathrm{i}}&=-\nu(\beta_{\mathrm{s}}+p)+\mu_{\mathrm{s}}w_{\mathrm{s}}',\\ -(1-\alpha\nu^{1})p_{\mathrm{i}}&=-(1-\nu)p+\tfrac{4}{3}\mu_{\mathrm{f}}w_{\mathrm{f}}',\end{aligned}\right\} \tag{2.9}$$

in which V_{f0} is the volume flow per unit area of water from the top into the layer (see I, formula (61)). The first of (2.9) is a mass balance, the second a kinematic statement, and the third and forth combine the continuity of the traction condition across the top surface with the postulate of how this traction is divided amongst the constituents (see I, formulae (32), (33), in which $l = \frac{2}{3}$). Adding $(2.9)_{3,4}$ and using (2.6) implies that

$$p_{\mathrm{i}}(t) = p_0(t). \tag{2.10}$$

With this result and (2.6), relations (2.9) may be rewritten as follows:

$$\left.\begin{aligned} w_{\mathrm{f}}(1-\nu) &= V_{\mathrm{f}0}, \\ \frac{\partial h}{\partial t} &= w_{\mathrm{s}}, \\ C_1 := (\nu - \alpha\nu^{\mathrm{l}})p_{\mathrm{i}} + \nu(1-\nu)\beta_{\mathrm{s}} - (1-\nu)\mu_{\mathrm{s}}w_{\mathrm{s}}' + \tfrac{4}{3}\nu\mu_{\mathrm{f}}w_{\mathrm{f}}' &= 0. \end{aligned}\right\} \tag{2.11}$$

(2) At the bottom interface $z = f_{\mathrm{b}}(t)$:

$$\left.\begin{aligned} \frac{\partial f_{\mathrm{b}}}{\partial t} &= -w_{\mathrm{b}} = -\frac{\mathcal{M}_{\mathrm{s}}^{\mathrm{b}}}{\rho_{\mathrm{r}}}, \\ (1-\nu)\,\mathrm{sgn}(w_{\mathrm{f}} - w_{\mathrm{b}})(w_{\mathrm{f}} - w_{\mathrm{b}}) &= \frac{\mathcal{M}_{\mathrm{f}}^{\mathrm{b}}}{\hat{\rho}_{\mathrm{f}}} = \frac{\mathfrak{m}_{\mathrm{f}}^{\mathrm{b}}}{\hat{\rho}_{\mathrm{f}}}\sigma_{\mathrm{f}}^4, \\ \nu(w_{\mathrm{s}} - w_{\mathrm{b}}) &= -\frac{\mathcal{M}_{\mathrm{s}}^{\mathrm{b}}}{\hat{\rho}_{\mathrm{s}}} \end{aligned}\right\} \tag{2.12}$$

and

$$\sigma_{\mathrm{f}} := (1-\nu)p - \tfrac{4}{3}\mu_{\mathrm{f}}w_{\mathrm{f}}'. \tag{2.13}$$

Here ρ_{r} is the rock density and $\mathcal{M}_{\mathrm{s}}^{\mathrm{b}}/\rho_{\mathrm{r}}$ the abrasion rate, while $\mathcal{M}_{\mathrm{f}}^{\mathrm{b}}/\hat{\rho}_{\mathrm{f}}$ is the drainage rate of water into the rock bed. $(2.12)_1$ is the kinematic equation and $(2.12)_{2,3}$ are jump conditions of mass, and a constitutive relation for the drainage function proposed in I, formula (52), has been used. Note, that requesting vanishing abrasion rate makes the bottom-interface location steady, so that $z = 0$ may be identified with it. In this case (2.12) reduce to

$$w_{\mathrm{b}} = w_{\mathrm{s}} = 0, \quad (1-\nu)w_{\mathrm{f}} = \frac{\mathfrak{m}_{\mathrm{f}}^{\mathrm{b}}}{\hat{\rho}_{\mathrm{f}}}\sigma_{\mathrm{f}}^4, \quad \text{at } z = 0. \tag{2.14}$$

If at the same time $\nu, V_{\mathrm{f}0}$ and p_{i} are also time independent then $(2.1)_1$ and $(2.11)_1$ imply

$$(1-\nu)w_{\mathrm{f}} = V_{\mathrm{f}0} \quad \forall z \in [0, h], \tag{2.15}$$

and thus $\mathcal{M}_{\mathrm{f}}^{\mathrm{b}}/\hat{\rho}_{\mathrm{f}} = V_{\mathrm{f}0}$, which in our application is negative.

(3) Initial conditions $t = 0$. As $\partial\nu/\partial t$, $\partial f_{\mathrm{b}}/\partial t$ and $\partial h/\partial t$ are the only field variables of which the time derivatives appear in the field equations, initial conditions must be imposed for these. We shall denote them by $\nu_0(z), h_0$ and f_{b0}.

We conclude this formulation of the problem by making plausible that the above IBVP is well posed. To this end let $\nu(z, t_0)$ be known and evaluate (2.5) at the bottom and top interfaces at $t = t_0$; this yields

$$\left.\begin{aligned} \nu(f_{\mathrm{b}})w_{\mathrm{s}}(f_{\mathrm{b}}) + (1-\nu(f_{\mathrm{b}}))w_{\mathrm{f}}(f_{\mathrm{b}}) &= W_C^0(t_0), \\ \nu(h)w_{\mathrm{s}}(h) + (1-\nu(h))w_{\mathrm{f}}(h) &= W_C^0(t_0). \end{aligned}\right\} \tag{2.16}$$

Note that from (2.9) and (2.12), $w_{\mathrm{s}}(f_{\mathrm{b}})$ and $w_{\mathrm{f}}(h)$ are known because the water inflow at the top and the abrasion rate at the bottom are prescribed. Thus, (2.16) constitutes two equations for $w_{\mathrm{f}}(f_{\mathrm{b}})$ and $w_{\mathrm{s}}(h)$ as functions of $W_C^0(t_0)$. Next, from $(2.11)_3$ we see that C_1 depends on $w_{\mathrm{s}}'(h)$ and $w_{\mathrm{f}}'(h)$:

$$C_1 = \hat{C}_1(w_{\mathrm{s}}'(h), w_{\mathrm{f}}'(h)) = 0. \tag{2.17}$$

However, recalling from (2.5) that

$$w_{\mathrm{s}}' = \frac{\nu'}{\nu^2}(w_{\mathrm{f}} - W_C^0) + w_{\mathrm{f}}'\left(1 - \frac{1}{\nu}\right), \quad \forall z \in [f_{\mathrm{b}}, h], \tag{2.18}$$

we alternatively write $C_1 = \tilde{C}_1(w_{\mathrm{f}}(h), w_{\mathrm{f}}'(h), W_C^0(t_0)) = 0$ and as $w_{\mathrm{f}}(h)$ is known,

$$C_1 = \check{C}_1(W_C^0(t_0), w_{\mathrm{f}}'(h)) = 0. \tag{2.19}$$

Moreover, from $(2.12)_2$ we may derive

$$C_2 := (1 - \nu(f_{\mathrm{b}}))(w_{\mathrm{f}}(f_{\mathrm{b}}) - w_{\mathrm{b}}) - \frac{\mathfrak{m}_{\mathrm{f}}^{\mathrm{b}}}{\hat{\rho}_{\mathrm{f}}}\sigma_{\mathrm{f}}^4 = 0, \tag{2.20}$$

in which σ_{f} is defined in (2.13). This shows that C_2 is a function of $w_{\mathrm{f}}(f_{\mathrm{b}})$, $w_{\mathrm{f}}'(f_{\mathrm{b}})$ and $w_{\mathrm{s}}'(f_{\mathrm{b}})$, namely

$$C_2 = \hat{C}_2(w_{\mathrm{f}}(f_{\mathrm{b}}), w_{\mathrm{f}}'(f_{\mathrm{b}}), w_{\mathrm{s}}'(f_{\mathrm{b}})) = 0. \tag{2.21}$$

Using (2.18) at $z = f_{\mathrm{b}}$ and $(2.16)_1$, we may write instead of (2.21)

$$C_2 = \check{C}_2(w_{\mathrm{f}}'(f_{\mathrm{b}}), W_C^0(t_0)) = 0. \tag{2.22}$$

Next, consider the equation (2.7) in which w_{s}' is replaced by use of (2.5). What emerges is a second-order ODE for $w_{\mathrm{f}}(z)$. Two initial conditions for this ODE are obtained from (2.22) and $(2.16)_1$. Via integration we obtain

$$w_{\mathrm{f}} = \tilde{w}_{\mathrm{f}}(z; W_C^0(t_0)). \tag{2.23}$$

Evaluating the function $\partial\tilde{w}_{\mathrm{f}}(\cdot)/\partial z$ for $z = h$ we obtain

$$w_{\mathrm{f}}'|_{z=h} = \frac{\partial}{\partial z}\tilde{w}_{\mathrm{f}}(h; W_C^0(t_0)) =: \tilde{w}_{\mathrm{f}}'(h; W_C^0(t_0)). \tag{2.24}$$

With this expression substituted into (2.19), this latter equation becomes an equation determining $W_C^0(t_0)$, and so $w_{\mathrm{s}}(h)$ is now known from $(2.16)_2$. Since the vertical velocity profiles of the solid and fluid are thus fully determined we may evaluate

$$\begin{gathered}\frac{\partial \nu}{\partial t} \text{ with the aid of } (2.1)_1,\\ \frac{\partial h}{\partial t} \text{ with the aid of } (2.9)_2,\\ \left[\frac{\partial f_{\mathrm{b}}}{\partial t} \text{ with the aid of } (2.12)_1\right]\end{gathered}$$

at time $t = t_0$. A forward step in time now determines v, h, f_{b} at time $t = t_0 + \Delta t$. Thus the problem is well posed.

From this point onwards we now suppose steady conditions with vanishing abrasion rate and time-independent incumbent pressure.

Table 1. *Characteristic values of physical quantities*

variable	value	dimension
$[\mu_{\rm f}]$	2×10^{-3}	kg m^{-1} s^{-1}
$[\lvert V_{\rm f0}\rvert]$	10^{-3}–10^{0}	ma^{-1}
$[H]$	1–10	m
$[\beta_1]$	10^{5}–10^{7}	Pa
$\tilde{\alpha}$	10^{9}–10^{11}	kg m^{-3} s^{-1}
$\mathfrak{m}_{\rm f}^{\rm b}$	10^{-28}–10^{-31}	kg^{-3} m^{2} s^{7}

3. Asymptotic analysis for steady-state drainage and vanishing abrasion rate

Consider equation (2.7) with the specializations (2.14) and (2.15):

$$\left.\begin{aligned}\nu' F(\nu) - \nu(\tfrac{4}{3}\mu_{\rm f} w_{\rm f}')' + \nu(1-\nu)\tilde{\alpha} w_{\rm f} = 0,\\ (1-\nu) w_{\rm f} = V_{\rm f0},\end{aligned}\right\} \tag{3.1}$$

in which $\tilde{\alpha}$ may, but need not, be treated as constant. Introducing the scalings

$$\left.\begin{aligned} z = [H]\tilde{z}, \qquad w_{\rm f} = [V_{\rm f0}]\tilde{w}_{\rm f},\\ \mu_{\rm f} = [\mu_{\rm f}]\tilde{\mu}_{\rm f}, \quad F(\nu) = [\beta_1]\tilde{F}(\nu),\end{aligned}\right\} \tag{3.2}$$

in which the bracketed terms have fixed values of a typical order of magnitude for the variable which they are scaling, and the variables $\tilde{(\cdot)}$ are dimensionless, equations (3.1) can be transformed to the following dimensionless second-order ODE:

$$\nu' \frac{\tilde{F}(\nu)}{\nu} + \delta_{\rm f}\left(\tilde{\mu}_{\rm f}\left(\frac{1}{1-\nu}\right)'\right)' - \eta_{\rm f} = 0, \tag{3.3}$$

in which primes are redefined as $(\cdot)' = {\rm d}/{\rm d}\tilde{z}$ and

$$\left.\begin{aligned}\delta_{\rm f} &:= \frac{4}{3}\frac{[\mu_{\rm f}][\lvert V_{\rm f0}\rvert]}{[H][\beta_1]} = O(10^{-20}\text{–}10^{-17}),\\ \eta_{\rm f} &:= \tilde{\alpha}\frac{[\lvert V_{\rm f0}\rvert][H]}{[\beta_1]} = O(10^{-7}\text{–}10^{-4}).\end{aligned}\right\} \tag{3.4}$$

The coefficients $\delta_{\rm f}$ and $\eta_{\rm f}$ are dimensionless quantities determined exclusively by the scales and by $\tilde{\alpha}$ of which the above estimates were obtained using the values in table 1 as suggested in I. Both coefficients are small and the differential equation (3.3) defines a singular perturbation problem. A more convenient form of (3.3) is obtained by the transformation

$$y = \frac{1}{1-\nu}, \quad h(y) := -\frac{\tilde{F}((y-1)/y)}{(1-y)y}, \quad y \in [1, \infty); \tag{3.5}$$

then (3.3) becomes

$$(\tilde{\mu}_{\rm f}\delta_{\rm f})y'' + (h(y) + \tilde{\mu}_{\rm f}'\delta_{\rm f})y' - \eta_{\rm f} = 0. \tag{3.6}$$

It follows also from (2.8) that

$$h(y(\nu)) = -(1-\nu)^3\left(\tilde{\beta}_{\rm s}\frac{1}{\nu+b} + \frac{\rm d}{{\rm d}\nu}\tilde{\beta}_{\rm s}\right), \tag{3.7}$$

which is an explicit dependence on ν. The differential equation (3.6), even though it is determining the solid mass distribution in the layer, is governed by the viscosity of the fluid. If the latter is constant, $\tilde{\mu}_{\mathrm{f}}' = 0$, and $\tilde{\mu}_{\mathrm{f}} = 1$, so that

$$\delta_{\mathrm{f}} y'' + h(y) y' = \eta_{\mathrm{f}} \tag{3.8}$$

describes the constant fluid-viscosity case. Imposing inviscidity of the fluid implies $\delta_{\mathrm{f}} = 0$ and reduces (3.8) to first order. Likewise, assuming no inflow (and no outflow) through the layer, $V_{\mathrm{f0}} = 0$, implies $\delta_{\mathrm{f}} = 0$ and $\eta_{\mathrm{f}} = 0$, whence $y' = 0$ or $\nu(\tilde{z}) =$ const. Apart from this the equation is chiefly governed by thermodynamics of the solid, as h is given by (3.7) in terms of $\tilde{\beta}_{\mathrm{s}}$ and $\mathrm{d}\tilde{\beta}_{\mathrm{s}}/\mathrm{d}\nu$. In the geophysical and geotechnical literature (see, for instance, Drew & Segel 1971; Ehlers 1993), $\tilde{\beta}_{\mathrm{s}}$ is not even introduced and the fluid is in most cases assumed to be inviscid, so that η_{f} would have to vanish. This then would mean that V_{f0} had to vanish in this case: no water flow across the layer would be permissible.

Returning to the full equation (3.6) or (3.8), we now impose the boundary conditions. At the top interface $(2.11)_3$ must hold; eliminating w_{f} and w_{s} in the third of these equations and non-dimensionalizing the resulting equation yields

$$(\nu - \alpha \nu^{l}) \frac{p_{\mathrm{i}}}{[\beta_1]} + \tilde{\beta}_{\mathrm{s}} \nu (1-\nu) + \delta_{\mathrm{f}} \tilde{\mu}_{\mathrm{f}} \frac{\nu \nu'}{(1-\nu)^2} = 0, \quad \text{at } \tilde{z} = 1. \tag{3.9}$$

At the bottom boundary (2.12) must hold, of which the steady state reduction takes the form

$$(1-\nu) w_{\mathrm{f}} = V_{\mathrm{f0}} = -\frac{\mathfrak{m}_{\mathrm{f}}^{\mathrm{b}}}{\hat{\rho}_{\mathrm{f}}} \sigma_{\mathrm{f}}^4, \quad \text{at } \tilde{z} = 0. \tag{3.10}$$

Substituting in turn relation (2.13) for σ_{f} and equation (2.6) for p, and non-dimensionalizing the resulting expression as before, yields

$$(1-\nu) \frac{p_{\mathrm{i}}}{[\beta_1]} - \tilde{\beta}_{\mathrm{s}} \nu (1-\nu) + \delta_{\mathrm{f}} \tilde{\mu}_{\mathrm{f}} \frac{\nu \nu'}{(1-\nu)^2} - \mathcal{F} = 0, \quad \text{at } \tilde{z} = 0, \tag{3.11}$$

where

$$\mathcal{F} := \left(\frac{V_{\mathrm{f0}} \hat{\rho}_{\mathrm{f}}}{\mathfrak{m}_{\mathrm{f}}^{\mathrm{b}}} \right)^{1/4} \frac{1}{[\beta_1]} \simeq O(10^{-1}\text{–}10^{0}), \tag{3.12}$$

and an estimate for $\mathcal{F}$ has been determined with the aid of table 1. Introducing the transformation (3.5), (3.9) and (3.12) take the forms

$$\left. \begin{aligned} -\delta_{\mathrm{f}} \tilde{\mu}_{\mathrm{f}} y' + \frac{1}{y} \tilde{\beta}_{\mathrm{s}} - \frac{1}{y-1} \frac{p_{\mathrm{i}}}{[\beta_1]} + \frac{y}{y-1} \mathcal{F} = 0, \quad \text{at } \tilde{z} = 0, \\ -\delta_{\mathrm{f}} \tilde{\mu}_{\mathrm{f}} y' + \frac{1}{y} \tilde{\beta}_{\mathrm{s}} + \left(1 - \alpha \left(\frac{y-1}{y} \right)^{l-1} \right) \frac{p_{\mathrm{i}}}{[\beta_1]} = 0, \quad \text{at } \tilde{z} = 1, \end{aligned} \right\} \tag{3.13}$$

in which $\tilde{\beta}_{\mathrm{s}}$ is now regarded as a function of y, and p_{i} and $\mathcal{F}$ are prescribed.

The differential equation (3.6) and the boundary conditions together define a nonlinear two-point boundary-value problem (TPBVP) which we will solve as $\delta_{\mathrm{f}} \to 0$, in which case it is prone to develop boundary layers either at the top or at the bottom interfaces, and in principle such layers are possible at the interior of the interval $\tilde{z} \in [0,1]$. The question of where the boundary layer forms is one of thermodynamics, as it will depend on the sign and form of the function $h(y)$ in $y \in [1, \infty)$ and, consequently, can be traced back to the functional form of the Helmholtz free energy. The question is non-trivial because $h(y)$ is a first-order (linear) differential operator in $\tilde{\beta}_{\mathrm{s}}$ through (2.8). In the linear case, when $\mu_{\mathrm{f}}' = 0$ and the coefficient $h = h(\tilde{z})$ of the first-order derivative in the considered ODE is a function of $\tilde{z}$ only, then $h > 0$ ($h < 0$) locates the boundary layer at $\tilde{z} = 0$ ($\tilde{z} = 1$), and when h changes sign at $\tilde{z} = \tilde{z}_{\mathrm{T}} \in [0, 1]$ such that $h(\tilde{z}_{\mathrm{T}}) = 0$, there is a transition layer near $\tilde{z}_{\mathrm{T}}$ (see Bush 1992, p. 155), where this computation is done explicitly.

To corroborate this qualitative behaviour of the above *nonlinear* singular perturbation problem we solve (3.8) subject to the boundary conditions:

$$y(0) = y_0, \quad y(1) = y_{\mathrm{h}}, \quad \text{with } y_0 > y_{\mathrm{h}}. \tag{3.14}$$

These boundary conditions imply $\nu(0) > \nu(1)$ which is the physically expected behaviour. The inequality can be qualitatively inferred from (3.13) (see Appendix A).

The outer solution of (3.8) takes the form

$$\mathcal{H}(y, y^*) + C_0 = \eta_{\mathrm{f}}(\tilde{z} - \tilde{z}^*), \quad \text{with } \mathcal{H}(y, y^*) := \int_{y^*}^{y} h(\bar{y})\,\mathrm{d}\bar{y}, \tag{3.15}$$

where C_0 is a constant of integration.

Suppose now that $h(y) > 0\ \forall y \in [1, \infty)$ and assume that the boundary layer is at $\tilde{z} = 0$. Then writing $\tilde{z} = \delta_{\mathrm{f}}^{p}\zeta$, accordingly transforming the differential equation (3.8), balancing the terms on the left-hand side and applying the principle of least degeneracy (see Bush 1992, p. 159) yields $p = 1$ and

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\zeta^2} + h(y)\frac{\mathrm{d}y}{\mathrm{d}\zeta} = \delta_{\mathrm{f}}\eta_{\mathrm{f}} \tag{3.16}$$

as the transformed ODE in the stretched coordinates. Its lowest-order solution in the limit $\eta_{\mathrm{f}}\delta_{\mathrm{f}} \to 0$ can be obtained by dropping its right-hand side, implying the first integral

$$\frac{\mathrm{d}y}{\mathrm{d}\zeta} + \mathcal{H}(y, y^*) = C_1, \tag{3.17}$$

and the second

$$\zeta = \int_{1}^{y} \frac{\mathrm{d}\bar{y}}{C_1 - \mathcal{H}(\bar{y}, y^*)} + C_2. \tag{3.18}$$

At $\tilde{z} = 0$, i.e. $\zeta = 0$, $y = y_0$, which fixes C_2 so that

$$\zeta = \int_{y_0}^{y} \frac{\mathrm{d}\bar{y}}{C_1 - \mathcal{H}(\bar{y}, y^*)}. \tag{3.19}$$

Matching this solution with the outer solution requires that, as $\zeta \to \infty$, $y \to y_{\mathrm{h}}$. Thus

$$+\infty = \int_{y_0}^{y_{\mathrm{h}}} \frac{\mathrm{d}\bar{y}}{C_1 - \mathcal{H}(\bar{y}, y^*)}. \tag{3.20}$$

The integrand must therefore be non-integrable at the upper limit $\bar{y} = y_{\mathrm{h}}$; the necessary condition for this to hold is

$$C_1 = \mathcal{H}(y_{\mathrm{h}}, y^*), \tag{3.21}$$

so that

$$-\infty = \int_{y_\mathrm{h}}^{y_0} \frac{\mathrm{d}\bar{y}}{\mathcal{H}(y_\mathrm{h}, y^*) - \mathcal{H}(\bar{y}, y^*)}. \tag{3.22}$$

Moreover, to have this equal to $-\infty$, $\mathcal{H}(y_\mathrm{h}, y^*) < \mathcal{H}(y_\mathrm{h}^+, y^*)$. So, as y_h is the smallest value which y can attain, $\mathcal{H}(y, y^*)$ is monotonically increasing in y, and thus $h(y)$ is indeed positive. Repeating this analysis for $h \leqslant 0$ and $y_\mathrm{h} \leqslant y_0$ we find that, irrespective of whether $y_\mathrm{h} > y_0$ or $y_\mathrm{h} < y_0$: (1) the boundary layer is at $\tilde{z} = 0$, if $h > 0$; (2) the boundary layer is at $\tilde{z} = 1$, if $h < 0$.

This demonstration is not a rigorous proof that the same properties also apply when (3.8) is being solved subject to the two boundary conditions (3.13), but we take the position that alternatives are unlikely; indeed in Appendix A we parametrize the solution of the ODE (3.8) in terms of the variables y_0 and y_h and prove that no further singularity arises when mixed boundary conditions like (3.13) are considered. Here it suffices to state that with (3.17) and (3.21) we have

$$\frac{\mathrm{d}y}{\mathrm{d}\tilde{z}}(y) = -\frac{1}{\delta_\mathrm{f}} \int_{y_\mathrm{h}}^{y} h(\bar{y})\,\mathrm{d}\bar{y} < 0. \tag{3.23}$$

In absolute value, the largest derivative arises at $y = y_0$, the edge of the boundary layer. Since h is monotonically increasing with y, (3.23) implies the bounds

$$-\frac{1}{\delta_\mathrm{f}}\left(\max_{y\in[y_\mathrm{h}, y_0]} h(y)\right)(y_0 - y_\mathrm{h}) \leqslant \frac{\mathrm{d}y}{\mathrm{d}\tilde{z}}(y) \leqslant -\frac{1}{\delta_\mathrm{f}}\left(\min_{y\in[y_\mathrm{h}, y_0]} h(y)\right)(y_0 - y_\mathrm{h}). \tag{3.24}$$

This allows us to find estimates for y' in (3.13) that support the computation in Appendix A.

It is now necessary to use the thermodynamic inequality (2.4) together with (3.7) to see whether h obeys one or both of the above inequalities. It is easy to see that (2.4) and the thermodynamic pressure $\tilde{\beta}_\mathrm{s} > 0$ imply $h < 0$, thus excluding any boundary layer at the bottom interface.

4. The specification of $\tilde{\boldsymbol{\beta}}_\mathrm{s}$

The inner solution of (3.8) has the form

$$\zeta = \frac{\tilde{z} - 1}{\delta_\mathrm{f}} = \int_{y}^{y_0} \frac{\mathrm{d}\bar{y}}{\mathcal{H}(y_0, y_0^*) - \mathcal{H}(\bar{y}, y^*)}. \tag{4.1}$$

Assume now that $h(y)$ is bounded $\forall y \in [y_\mathrm{h}, y_0]$; then according to the mean value theorem of integral calculus

$$\mathcal{H}(y_0, y^*) - \mathcal{H}(y, y^*) = -h(y_0)(y - y_0) + o(|y - y_0|), \tag{4.2}$$

which upon substitution in (4.1) implies

$$(\tilde{z} - 1)\frac{h(y_\mathrm{h})}{\delta_\mathrm{f}} = \ln\left(\frac{y_\mathrm{h}/y_0 - 1}{y/y_0 - 1}\right) + O(|y - y_0|). \tag{4.3}$$

This equation allows us to estimate $h(y_0)$. To this end we note that the right-hand side is negative for $y \in [y_\mathrm{h}, y_0]$ for $y_\mathrm{h} < y_0$, and varies from $-\infty$ to 0 with a logarithmic singularity at y_h. The thickness of the boundary layer is obviously $\delta_\mathrm{f}/|h(y_0)|$ and not

δ_{f}. To have it of reasonable thickness, i.e. to have it of the dimension of several particle diameters we choose $\varepsilon = 10^{-p}$, $p \in [2, 4]$, where

$$\varepsilon := \frac{\delta_{\mathrm{f}}}{|h(y_0)|} \Rightarrow |h(y_0)| = \frac{\delta_{\mathrm{f}}}{\varepsilon}, \tag{4.4}$$

implying $h(y_0) \simeq 10^{-18}$–10^{-13} with a mean of, perhaps, 10^{-15}.

To model this boundary layer we pick two trial functions as examples

$$\left.\begin{aligned} &(1) \quad h(y) = -\exp(-k_1/\nu(y)), \\ &(2) \quad h(y) = -1 + \tanh^2(k_2/\nu(y)). \end{aligned}\right\} \tag{4.5}$$

These trial functions exhibit the fortunate behaviour that explicit analytical expressions for $\tilde{\beta}_{\mathrm{s}}$ can be found (e.g. by MAPLE) which satisfy the thermodynamic stability inequality. These choices are very delicate as the trial functions for $\tilde{\beta}_{\mathrm{s}}$,

$$\nu^{2m}, \quad m > 0; \quad \ln(\nu_* - \nu); \quad \left(\frac{\nu}{\nu - \nu_*}\right)^{2n}, \quad n > 0; \quad \tanh^{-1}(\nu/\nu_*), \tag{4.6}$$

where $\nu_* \in [0, 1]$, lead to values of $\varepsilon = 10^{-p}$ with $p > 10$. This is physically meaningless, because in physical units the boundary layer would turn out to be of the order of magnitude of 10^{-8} m! In other words logarithmic and algebraic singularities of $\tilde{\beta}_{\mathrm{s}}$ at ν_* are not sufficiently strong to generate a boundary layer of the order of 10^{-3} to 10^{-2} m (which would physically be expected). Exponential singularities are needed for $\tilde{\beta}_{\mathrm{s}}$ at the point $\nu = \nu_*$, and such singularities are exactly obtained with the choices (4.5). We cannot give a systematic rule of how these functions were determined. It was essentially trial and error by selecting either $h(y)$ or $\tilde{\beta}_s(y)$ and determining the other function by using (3.7). The choices needed to be such that the thermodynamic stability requirement was satisfied *and* ε in (4.4) assumed reasonable values.

Using (4.5) in (4.4) yields

$$\left.\begin{aligned} &(1) \quad h(y_0) = \exp\left(-\frac{k_1}{\nu(y_0)}\right) = 10^{-15} \Rightarrow k_1 \simeq 13.6, \\ &(2) \quad h(y_0) = 1 - \tanh^2\left(\frac{k_2}{\nu(y_0)}\right) = 10^{-15} \Rightarrow k_2 \simeq 7.2, \end{aligned}\right\} \tag{4.7}$$

with the functions (4.5) being given and definition (3.7) of $h(y)$ in terms of $\tilde{\beta}_{\mathrm{s}}$ and its derivative integration determining $\tilde{\beta}_{\mathrm{s}}$. The constant of integration β_0 must then be chosen such that the resulting $\tilde{\beta}_{\mathrm{s}}$ function is in conformity with the thermodynamic stability condition. $\beta_0 = 0$ in both cases is sufficient for this. We refrain from presenting these results explicitly, because they can easily be obtained by using e.g. MAPLE.

The formulae for $\tilde{\beta}_{\mathrm{s}}$ that are obtained by using (4.5) are very lengthy and inconvenient expressions. Thus it may be advantageous to try to select $\tilde{\beta}_{\mathrm{s}}$ in simple form such that the thermodynamic stability condition (2.4) is satisfied and to determine the function $h(y)$ by differentiation. This we have done in several cases and table 2 gives the selection of five different choices of which one does not conform with the stability condition. For the first choice (a), $\tilde{\beta}_{\mathrm{s}}$ is exponentially decaying as $\nu \to 0$ for all $k > 0$ and assumes a finite value at $\nu = 1$. The corresponding function $-h(y(\nu))$ is zero at $\nu = 0$, very small in its neighbourhood, forms a hump with a maximum at

Table 2. *Properties of functions $\tilde{\beta}_s$ and corresponding functions h which do (OK) or do not (NO) satisfy the thermodynamic stability criterion*

(*a*)	$\tilde{\beta}_s(\nu) = \exp\left(-\dfrac{k}{\nu}\right)$	OK
(*b*)	$\tilde{\beta}_s(\nu) = (\nu + b)^2 \exp\left(-\dfrac{k}{\nu}\right)$	OK
(*c*)	$\tilde{\beta}_s(\nu) = \dfrac{1}{(1-\nu)^3} \exp\left(-\dfrac{k}{\nu}\right)$	OK
(*d*)	$\tilde{\beta}_s(\nu) = \exp\left(\dfrac{1}{1-\nu}\right) \exp\left(-\dfrac{k}{\nu}\right)$	NO
(*e*)	$\tilde{\beta}_s(\nu) = \dfrac{1}{(\nu_L-\nu)^\gamma} \exp\left(-\dfrac{k}{\nu}\right), \quad \gamma \geqslant 3$	OK
(*a*)	$h(y(\nu)) = -(1-\nu)^3 e^{-k/\nu}\left(\dfrac{1}{\nu+b} + \dfrac{k}{\nu^2}\right)$	OK
(*b*)	$h(y(\nu)) = -(1-\nu)^3 (\nu+b) e^{-k/\nu}\left(3 + k\dfrac{b}{\nu^2} + \dfrac{k}{\nu}\right)$	OK
(*c*)	$h(y(\nu)) = -e^{-k/\nu}\left(\dfrac{k}{\nu^2} + \dfrac{1}{\nu+b} + \dfrac{3}{1-\nu}\right)$	OK
(*d*)	$h(y(\nu)) = -(1-\nu)^3 e^{1/(1-\nu)} e^{-k/\nu}\left(\dfrac{1}{\nu+b} + \dfrac{1}{(1-\nu)^2} + \dfrac{k}{\nu^2}\right)$	NO
(*e*)	$h(y(\nu)) = -\dfrac{(1-\nu)^3}{(\nu_L-\nu)^\gamma} e^{-k/\nu}\left(\dfrac{1}{\nu+b} + \dfrac{\gamma}{\nu_L-\nu} + \dfrac{k}{\nu^2}\right)$	OK

about ν_0 and finally reaches zero at $\nu = 1$. This qualitative behaviour also persists for the choice (*b*) of table 2; in fact it can be shown that

$$\tilde{\beta}_s(\nu) = (1-\nu)^\gamma \exp(-k/\nu), \quad -3 < \gamma < \infty$$

generates such a hump for $-h(y(\nu))$, whereas the choice (*c*) ($\gamma \leqslant -3$) makes $-h(y(\nu))$ monotonically increasing with increasing ν with an algebraic singularity at $\nu = 1$. If $\tilde{\beta}_s(\nu)$ is not finite at $\nu = 1$ but has an exponential singularity, the thermodynamic stability condition can no longer be satisfied (case (*d*) of table 2). Now it can be shown (see Appendix A) that with the hump-like $h(y(\nu))$ the original boundary condition (3.13) at $\tilde{z} = 0$ cannot be satisfied (as can be easily inferred with the aid of (3.24)); by contrast, however, this is possible for a function $h(y(\nu))$, which becomes singular as $\nu \to \nu_L$, where $\nu_L = 1$ in case (*d*) of table 2. As physically $\nu = 1$ can never be achieved, functions $\tilde{\beta}_s(v)$, and consequently $h(y(\nu))$, must be modified as shown in item (*e*) of table 2, in which ν_L is a constitutive quantity (a constant) with a value of approximately 0.8–0.85.

5. Concluding remarks

In this paper a layer of saturated binary-mixture of soil and water, of which both are true density preserving, is considered. This layer is subjected from above to the overburden pressure of the overlying ice and the shear traction exerted by its horizontal movement. Melt water from the ice above is pressed into the layer. At the lower boundary, the rock bed, abrasion of till may add sediment mass to the layer and drainage of water contributes to a true filtering process from above through the layer, thereby affecting the sediment concentration profile and influencing the effective resistance of the layer to horizontal shearing. It turned out that the determination of the vertical flow plus the vertical profile of the solid volume fraction uncouple from those of the horizontal flow (except for the abrasion rate), and ignoring the abrasion rate led to a steady problem if also the bordering tractions were assumed to be time independent.

The analysis of this steady problem is interesting because it shows that the distribution of the solid volume fraction across the layer is determined by the vertical water transport through the layer, the viscosity of the water and the constitutive equation for the thermodynamic pressure of the solid. All three are lumped together in one differential equation for the solid volume fraction profile (see (3.6) or (3.8)). In a theory ignoring the thermodynamic pressure, i.e. assuming the Helmholtz free energy to be independent of the solid volume fraction—this is so for most von Terzaghi type theories (see de Boer 1996; Mackenzie 1984; Ehlers 1993 and others)—this equation collapses to a physically non-acceptable statement, implying that the effect of the filter flow on the distribution of the solid volume fraction cannot be studied†.

The problem is also interesting because the differential equation exhibits boundary-layer structure; the boundary layer grows linearly both with the fluid viscosity and the water transport across the layer. However, it is also influenced by the behaviour of the thermodynamic pressure as a function of the solid volume fraction. This function must satisfy the thermodynamic stability criterion that the Helmholtz free energy is a convex function of the solid volume fraction, and this condition constrains the boundary layer to be at the top rather than at the bottom surface (when the granules are rock material and the fluid is water). This property has important glaciological implications, because high solid volume fraction concentrations are to be expected at the (abrading) bed. This high value is reduced through the boundary layer, so that under Stokes flow conditions the larger part of the layer has relatively large values of the solid volume fraction, conditions which favour low shear velocities within the layer. Conversely, if the boundary layer were at the bottom surface, where the solid volume fraction is minimal, the larger part of the layer would have low solid volume fraction, making the layer weak to shear deformations.

This physical mechanism has never been envisaged by geologists and glaciologists who deal with the destabilization of an ice sheet such as West Antarctica (see, for example, Alley *et al.* 1987*a*, *b*; MacAyeal 1992; Kamb 1991; and others). In I, the previous formulations might have been too simple and might have missed some destabilizing mechanisms that enhance the potential for the disintegration of the West Antarctic Ice Sheet. We proved in I by applying a systematic mixture model (Svendsen & Hutter 1995) that additional processes contribute to this destabilization, but did not give a direct demonstration. However, we proved in the present paper that a classical mixture theory which ignores the dependence of the Helmholtz free energy on the solid volume fraction is too simple to discover the behaviour, and

† Indeed, (3.10) reduces in this case to $y'' = \tilde{\alpha}H^2/\mu_{\mathrm{f}} = 1/\varepsilon$ ($\simeq 10^{15}$), of which the solution subject to $y(0) = y_0$, $y(1) = y_{\mathrm{h}}$ produces a solid volume fraction profile which at $\tilde{z} = \frac{1}{2}$ is larger than unity, $\nu(\frac{1}{2}) \simeq 1 + 8\varepsilon$, which is physically meaningless.

we show how thermodynamic completion of such a model does allow this expected behaviour.

Future analyses will have to include acceleration terms and the abrasion process and thus focus on time-dependent behaviour.

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Appendix A.

In this appendix we demonstrate that the boundary conditions (3.14) are in conformity with the boundary conditions (3.13), provided $\tilde{\beta}_{\mathrm{s}}$ is chosen accordingly. The required properties $\tilde{\beta}_{\mathrm{s}}$ must fulfil are

$$\begin{aligned}&(1)\quad \tilde{\beta}_{\mathrm{s}}(\nu)\to 0,\ \text{exponentially as } \nu\to 0,\\ &(2)\quad \tilde{\beta}_{\mathrm{s}}(\nu)\to\infty,\ \text{algebraically,}\\ &\qquad\qquad \text{with a power } \gamma\geqslant 3 \text{ for } \nu\to\nu_{\mathrm{L}}, \text{where } \nu_{\mathrm{L}}\in(\alpha^{-1/(l-1)},1).\end{aligned}$$

To see this, consider the boundary condition (3.13):

$$\left.\begin{aligned}-\delta_{\mathrm{f}}\tilde{\mu}_{\mathrm{f}}y'+\frac{1}{y}\tilde{\beta}_{\mathrm{s}}-\frac{1}{y-1}\frac{p_{\mathrm{i}}}{[\beta_1]}+\frac{y}{y-1}\mathcal{F}=0,\quad \text{at } \tilde{z}=0,\\ -\delta_{\mathrm{f}}\tilde{\mu}_{\mathrm{f}}y'+\frac{1}{y}\tilde{\beta}_{\mathrm{s}}+\left(1-\alpha\left(\frac{y-1}{y}\right)^{l-1}\right)\frac{p_{\mathrm{i}}}{[\beta_1]}=0,\quad \text{at } \tilde{z}=1.\end{aligned}\right\}\tag{A 1}$$

The boundary layer is located at $\tilde{z}=1$, and $|y'|$ is very large and needs to be estimated. Such an estimate follows from (3.23) and takes the form

$$y'(y)=\frac{1}{\delta_{\mathrm{f}}}(\mathcal{H}(y_0,y^*)-\mathcal{H}(y,y^*))+O(\eta_{\mathrm{f}}).\tag{A 2}$$

At $\tilde{z}=0$, i.e. $y=y_0$, the term y' in $(\mathrm{A}\,1)_1$ is given by (3.8) with $\delta_{\mathrm{f}}=0$, while at $\tilde{z}=1$ (A 2) must be substituted in $(\mathrm{A}\,1)_2$, implying

$$\left.\begin{aligned}-\frac{\delta_{\mathrm{f}}\tilde{\mu}_{\mathrm{f}}\eta_{\mathrm{f}}}{h(y(0))}+\frac{1}{y}\tilde{\beta}_{\mathrm{s}}-\frac{1}{y-1}\frac{p_{\mathrm{i}}}{[\beta_1]}+\frac{y}{y-1}\mathcal{F}=0,\quad \text{at } \tilde{z}=0,\\ \tilde{\mu}_{\mathrm{f}}(-\mathcal{H}(y_0,y^*)+\mathcal{H}(y,y^*))+\frac{1}{y}\tilde{\beta}_{\mathrm{s}}+\left(1-\alpha\left(\frac{y-1}{y}\right)^{l-1}\right)\frac{p_{\mathrm{i}}}{[\beta_1]}=0,\quad \text{at } \tilde{z}=1,\end{aligned}\right\}\tag{A 3}$$

or, after some trivial manipulations,

$$\left.\begin{aligned}\frac{\tilde{\beta}_{\mathrm{s}}}{y}&=-\frac{y}{y-1}\mathcal{F}+\frac{1}{y-1}\frac{p_{\mathrm{i}}}{[\beta_1]}+\frac{\delta_{\mathrm{f}}\eta_{\mathrm{f}}\tilde{\mu}_{\mathrm{f}}(y)}{h(y)},\quad \text{at } \tilde{z}=0,\\ \frac{\tilde{\beta}_{\mathrm{s}}}{y}&=\tilde{\mu}_{\mathrm{f}}(\mathcal{H}(y_0,y^*)-\mathcal{H}(y,y^*))-\left(1-\alpha\left(\frac{y-1}{y}\right)^{l-1}\right)\frac{p_{\mathrm{i}}}{[\beta_1]},\quad \text{at } \tilde{z}=1.\end{aligned}\right\}\tag{A 4}$$

However, in the case of constant viscosity, $\tilde{\mu}_{\mathrm{f}}=1$, and as in the physically admissible range for y we have

$$\mathcal{F}\frac{1}{y}\frac{p_{\mathrm{i}}}{[\beta_1]}$$

(see (3.12)), these relations become (approximately)

$$\left.\begin{aligned}\tilde{\beta}_{\mathrm{s}} &\simeq \frac{\delta_{\mathrm{f}}\eta_{\mathrm{f}}\tilde{\mu}_{\mathrm{f}}(y)}{h(y)}y + \frac{y}{y-1}\frac{p_{\mathrm{i}}}{[\beta_1]}, \quad \text{at } \tilde{z}=0,\\ \tilde{\beta}_{\mathrm{s}} &= -y\left(-\tilde{\mu}_{\mathrm{f}}(\mathcal{H}(y_0,y^*)-\mathcal{H}(y,y^*)) + \left(1-\alpha\left(\frac{y-1}{y}\right)^{l-1}\right)\frac{p_{\mathrm{i}}}{[\beta_1]}\right), \quad \text{at } \tilde{z}=1,\end{aligned}\right\} \tag{A 5}$$

which imply (under the above-stated general properties for $\tilde{\beta}_{\mathrm{s}}$, which are, for instance verified by the example given in case (*e*) of table 2) that

$$y_0 \simeq \frac{1}{1-\nu_{\mathrm{L}}}, \quad \nu_{\mathrm{h}} \approx (1/\alpha)^{1/(l-1)}. \tag{A 6}$$

Indeed, the factor of $p_{\mathrm{i}}/[\beta_1]$ on the right-hand side of $(\mathrm{A}\,5)_2$ must be non-positive, implying $(\mathrm{A}\,6)_2$. Moreover,

$$\mathcal{H}(y_0,y^*)-\mathcal{H}(y,y^*) = -\int_y^{y_0} h(\bar{y})\,\mathrm{d}\bar{y},$$

appearing in $(\mathrm{A}\,5)_2$, must be sufficiently negative to counterbalance the last addend on the right-hand side. Because of the second itemized property of $\tilde{\beta}_{\mathrm{s}}$, this is possible for all $y \approx y_{\mathrm{L}}$ ($\nu \approx \nu_{\mathrm{L}}$).

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