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Abstract. A continuum model of a moving transition layer separating two different solid phases of a certain micro-damaged material is proposed. An application of this model to the analysis of the propagation of this layer in a micro-cracked solid is shown.

Key words: microcracks, micro-damaged materials, two-phase solids, interfacial layers

1. Introduction

In the recent literature some interest is paid to the study of propagation of discontinuity surfaces in continuous media. These surfaces are introduced as models of a large variety of phenomena (see for instance [9]) also including various kinds of phase transitions. When these phenomena need more detailed description the modelization is improved by endowing discontinuity surfaces with more sophisticated structure. This is done, for instance, by introducing certain surface fields, [6, 7, 8], describing the passage from one phase to another one. However, this passage may not always be abrupt and hence the interface often has to be regarded as a certain three-dimensional layer (see [8] and the review paper [11] on mushy regions arising in phase transitions) in which we can assume to deal with a certain mixture of both phases. This also occurs in some particular cases of solid-solid phase transitions, [10]. Thus, the question arises how to describe of the interfacial layer properties and the conditions under which this layer can exist and propagate.

In this paper we formulate some necessary existence and propagation conditions for a moving transition layer separating two different solid phases of a micro-damaged medium. These two solid phases correspond to different macroregions in which microcracks are all open or all closed respectively, [15, 16].

Generally speaking, the term *phase* means here a state of a solid specified by a known constitutive relation. Hence the differences in material behaviour for different states of a solid are assumed to arise not only because of the structural phase transition (e.g. martensitic transformations described at the microlevel in [2], [4] and [5]), but because of the existence of different strain energy functions related to different classes of local deformations. Such a situation was considered in [16] where the micro-damaged body behaviour, at the macrolevel, was described in terms of two different strain energy functions related to two different (and varying in time) macroregions (phases) with either all open or all closed microcracks. A phase transition takes place where microcracks are partly closed and partly open and hence in the part of the space in which a strain energy function (at a macrolevel) is not well defined. The considerations carried out in [16] were related to situations in which the aforementioned interfacial state can occur exclusively on certain discontinuity surfaces (which can arise and disappear in the deformation process) separating both phases. In this contribution we investigate the more general situation in which the interfacial state can be present in a certain spatial layer separating the two phases or situated inside one of them and moving across the medium.

The scope of the paper can be outlined as follows:

In Section 2 we introduce certain auxiliary concepts related to the kinematic description of the interfacial layer moving across a solid and separating two different phases. The main hypothesis, formulated in Section 3, is that this interfacial layer may be represented as a certain arrangement of both phases forming a definite banded periodic structure. After scaling down this structure we arrive at the mathematical model of the interfacial layer representing a particular fine mixture of both phases. We adapt here the modelling procedure proposed in [13] and [14] for composite materials. A similarity between a composite material structure and a fine phase mixture was already noted in [2]. The considerations carried out in Sections 2 and 3 are rather general, being related to an arbitrary solid. This solid can show itself in two different phases, each characterized by a separate analytical expression for the strain energy function, and in which the interface is modelled as a shell-like layer moving across the region occupied by the body in its motion. The obtained model is applied in Section 4 to the study of interfacial layers moving across the micro-damaged medium described in [16]. The solution to a special problem, illustrating the general considerations, is given in Section 5.

The conclusions formulated in Section 6 end the paper.

The main results of this contribution are related to a motion of interfacial layers in a micro-cracked solid and can be summarized as follows:

(1) the material behaviour of the interfacial layer depends not only on the material properties of both phases, but also on the propagation speed of the layer,
(2) not every propagation speed of the layer is possible,
(3) if the interfacial layer does not propagate across the medium, then its material properties are similar to those of a micro-layered two-component laminate [1], [3], [13] and [14].

To simplify the treatment only diffusionless phase transitions and exclusively purely mechanical motions are considered throughout the paper; the modelling of more complex phenomena will be presented separately.

2. Preliminaries

Let B_R be a known reference placement of a solid in the euclidean three-space E and $\mathbf{p}(\cdot, t)\colon B_R \to E$ be its configuration at an arbitrary time instant t.

Let us denote by Φ the set of all non-singular 3×3 real matrices interpreted as deformation gradients

$$\mathbf{F}_t(\mathbf{X}) := \nabla \mathbf{p}(\mathbf{X}, t) \in \Phi$$

in an arbitrary configuration of a body. Let W_R^+, W_R^- be two different convex differentiable real-valued functions, both defined in Φ, satisfying for every $\mathbf{F} \in \Phi$ and for an arbitrary rotation matrix $\mathbf{R}$ the conditions

$$W_R^+(\mathbf{R}F) = W_R^+(\mathbf{F}), W_R^-(\mathbf{R}F) = W_R^-(\mathbf{F}),$$

i.e. being invariant under arbitrary rotations. Let there also be known two disjoined non-empty subsets Φ^+ and Φ^- of Φ such that

$$\Phi^0 := \Phi \backslash (\Phi^+ \cup \Phi^-) \neq \emptyset.$$

In the subsequent considerations we shall deal with an elastic homogeneous solid. The main assumption is that to every substantial point of this solid a strain energy function W_R is assigned

$$W_R(\mathbf{F}) := \begin{cases} W_R^+(\mathbf{F}) & \text{if } \mathbf{F} \in \Phi^+, \\ W_R^-(\mathbf{F}) & \text{if } \mathbf{F} \in \Phi^-, \end{cases}$$

while for $\mathbf{F} \in \Phi^0$ the strain energy function is not well defined. This means that for some values of the deformation gradient (i.e. for $\mathbf{F} \in \Phi^0$) the material response of the solid under consideration is not determined.

Under the aforementioned conditions we deal with what will be called a two-phase solid, each phase being denoted by superscript $+$ or $-$.

For an overview of the problems related to the physical meaning and the interpretation of strain energy functions when dealing with micro-cracked materials see [16] and the references therein.

An example of the above situation was described in [16] where a certain macro-model of a micro-damaged body (with a continuous distribution of micro-cracks) was investigated. In the aforementioned paper, as well as in the present one, Φ^+ and Φ^- are disjoined sets of deformation gradients such that if $\mathbf{F} \in \Phi^+$ then micro-cracks are all open and if $\mathbf{F} \in \Phi^-$ then they are all closed. To these two phases of considered micro-cracked solid two different forms of strain energy function are assigned W_R^+, W_R^- respectively, since in both cases the responses of the material are different. At the same time in [16] the strain energy function was not well defined on certain *surfaces* separating both phases.

In this paper we study those micro-cracked materials for which it is physically meaningful to assume that certain *regions* exist in which not all micro-cracks are all open or all closed and hence a strain energy in these regions is not well defined.

It is easy to see that for an arbitrary configuration $\mathbf{p}(\mathbf{X}, t), \mathbf{X} \in B_R$, of the solid under consideration the set B_R can be partitioned into three mutually disjoined subsets

$$\begin{aligned} B_R^+(t) &:= \{\mathbf{X} \in B_R : \mathbf{F}_t(\mathbf{X}) \in \Phi^+\}, \\ B_R^-(t) &:= \{\mathbf{X} \in B_R : \mathbf{F}_t(\mathbf{X}) \in \Phi^-\}, \\ B_R^0(t) &:= \{\mathbf{X} \in B_R : \mathbf{F}_t(\mathbf{X}) \in \Phi^0\}. \end{aligned}$$

The subsets $B_R^+(t)$ and $B_R^-(t)$ are respectively occupied by the phases with strain energies $W_R^+(\mathbf{F}_t(\mathbf{X}))$ and $W_R^+(\mathbf{F}_t(\mathbf{X}))$. When $\mathbf{X} \in B_R^0$ the strain energy is not well defined. A decomposition similar to that given above can also be introduced in the spatial region $B_t := \mathbf{p}(B_R, t)$ occupied by the solid in the configuration $\mathbf{p}(\cdot, t)$. Those points which belong to $B_R^0(t)$ for every t constitute in B_R (and hence in $B_t := \mathbf{p}(B_R, t)$) the interfacial part of the considered solid. This interfacial part can either separate the parts $B_R^+(t)$ and $B_R^-(t)$ of B_R occupied by the different phases in which the solid can present itself, or can have its boundary in common with only one of the parts $B_R^+(t)$ and $B_R^-(t)$.

It was shown in [16] that these interfaces (occurring there as smooth surface elements) can appear and disappear in short time intervals and hence their motion may be not continuous. This is a main difference between an interfacial surface separating two phases of a solid and a moving discontinuity surface representing a wave in a material continuum.

The aim of this paper is to introduce and apply a new physical description for the moving interfacial region $B_R^0(t)$ under the assumption that it is constituted by a system of *thin layers*. Some of these layers can separate different solid phases or be situated inside one of them. We restrict ourselves to the study of the local properties of the interface so that we consider only one thin shell-like interfacial layer element $J_R(t), J_R(t) \subset B_R^0(t)$ with $t \in T := (t_o, t_f)$. Let $M_R(t)$ stand for the midsurface of this element and let it be parametrized by

$$\mathbf{X} = r(\boldsymbol{\theta}, t), \boldsymbol{\theta} = (\theta_1, \theta_2) \in \Pi,$$

Π being a regular plane region. Let the surface $M_R(t)$ be oriented by the field of unit normals $\mathbf{n}(\boldsymbol{\theta}, t)$. We will also assume that all considered interfacial elements have, for every $t \in T$, a constant thickness $2h(t)$ which is small compared to the curvature radius of the midsurface $M_R(t)$. Under these assumptions the layer $J_R(t)$ can be parametrized by means of the well-known shell-like parametrization

$$\mathbf{X} = \mathbf{X}(\boldsymbol{\theta}, \eta, t) = \mathbf{r}(\boldsymbol{\theta}, t) + \eta \mathbf{n}(\boldsymbol{\theta}, t), \tag{1}$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2) \in \Pi$ parametrize the midsurface $M_R(t)$ and $\eta \in (-h(t), h(t))$. It can be seen that the motion in B_R

$$t \in T \to J_R(t)$$

of the interfacial layer element is uniquely determined by the motion of its midsurface $t \in T \to M_R(t)$ and by the mapping

$$t \in T \to h(t). \tag{2}$$

Let us take into acount the inverse of the relation 1 setting

$$\eta = \zeta(\mathbf{X}, t), \mathbf{X} \in J_R(t), t \in T. \tag{3}$$

We remark that once the motion of the interface layer is known, the function $\zeta(\cdot)$ is also known. Treating now $\mathbf{X}$ and t as independent variables we can define

$$c(\mathbf{X}, t) := \frac{\partial \zeta(\mathbf{X}, t)}{\partial t}, \qquad \mathbf{X} \in J_R, t \in T \tag{4}$$

as the local propagation speed of the interfacial layer at $\mathbf{X} \in J_R, t \in T$.

In the sequel we will tacitly assume that at least one interfacial layer element (either disjoining two phases or included into one of them) exists and its motion in B_R

$$t \in T \to J_R(t) \subset B_R$$

is described by (1) and (2) which are representing a non-material shell-like element with time-varying thickness moving across B_R.

3. Modelling procedure

The aim of the subsequent analysis is to propose a physical description for the moving interfacial layer whose kinematics was described at the end of the previous section. The idea of the approach proposed is based on the heuristic hypothesis that the interfacial layer can be modelled, at a macrolevel, as a certain fine mixture of both phases. This fine mixture will be obtained performing a limit passage under the modelling assumption that the interface is constituted by a banded heterogeneous structure in which both phases occupy non-intersecting systems of very thin sublayers.

For an example of the limit passage leading from a microdescription to a macrodescription for certain micro-cracked solids the reader is referred to [16].

Let us separate a layer $J_R(t)$ into a large number n of sublayers with equal thickness $\varepsilon = \varepsilon(t) := h(t)/n$ separated by the surfaces

$$\eta = 0, \eta = \pm\varepsilon, \eta = \pm 2\varepsilon, \ldots, \eta = \pm n\varepsilon = \pm h(t).$$

Define by $h^{\varepsilon}(\eta), \eta \in (-h(t), h(t))$ a continuous function such that

$$h^{\varepsilon}(\eta) = +\tfrac{1}{2}\varepsilon \quad \text{for} \quad \eta = \pm\varepsilon, \pm 3\varepsilon, \ldots$$

$$h^{\varepsilon}(\eta) = -\tfrac{1}{2}\varepsilon \quad \text{for} \quad \eta = 0, \pm 2\varepsilon, \pm 4\varepsilon, \ldots$$

and linear in every interval between aforementioned values of η. Setting

$$I^{-} := (0, \varepsilon) \cup (2\varepsilon, 3\varepsilon) \cup \cdots \cup (-2\varepsilon, -\varepsilon) \cup (-4\varepsilon, -3\varepsilon) \cup \cdots$$

$$I^{+} := (\varepsilon, 2\varepsilon) \cup (3\varepsilon, 4\varepsilon) \cup \cdots \cup (-\varepsilon, 0) \cup (-3\varepsilon, -2\varepsilon) \cup \cdots$$

we obtain

$$\frac{\partial h^{\varepsilon}}{\partial \eta}(\eta) = \begin{cases} +1 & \text{if} \quad \eta \in I^{-} \\ -1 & \text{if} \quad \eta \in I^{+} \end{cases}.$$

The above saw-like function was used in modelling periodic-laminates, [13], [14], [15], and will be applied here under the assumption that sublayers for which the gradient of h^{ε} is equal to $+1$ are occupied by – phase and those for which it is equal to -1 by + phase. Hence in the first step of modelling it is assumed that the interfacial element is an aggregate of both phases which occupy thin sublayers of thickness ε. Thus, we deal with a certain banded structure inside every $J_R(t)$ whose motion (in the three-space E) can be assumed to have a similar form to that used for laminated materials [15] and given by

$$\mathbf{p}(\mathbf{X}, t) = \mathbf{P}(\mathbf{X}, t) + h^{\varepsilon}(\eta)\mathbf{Q}(\mathbf{X}, t), \quad \mathbf{X} \in J_R(t), t \in T, \tag{5}$$

where $\mathbf{P}(\mathbf{X}, t), \mathbf{Q}(\mathbf{X}, t)$ are functions which in every ball of radius ε can be treated as constants [15] and η is given by (3). The second term in (5) describes the disturbances in the motion $t \to J_R(t)$ caused by the heterogeneity of the material filling the adjacent sublayers [14].

In the second step of modelling we take into account (3) and (4), treating $\mathbf{X}$ and t as independent variables (i.e. using the Lagrangean description of motion). After passing to the limit for $\varepsilon \to 0$ we obtain

$$\mathbf{F}(\mathbf{X},t) := \nabla\mathbf{p}(\mathbf{X},t) = \nabla\mathbf{P}(\mathbf{X},t) + \begin{Bmatrix} +1 \\ -1 \end{Bmatrix} \nabla\zeta(\mathbf{X},t) \otimes \mathbf{Q}(\mathbf{X},t)$$

$$\dot{p}(\mathbf{X},t) = \dot{P}(\mathbf{X},t) + \begin{Bmatrix} +1 \\ -1 \end{Bmatrix} c(\mathbf{X},t)\mathbf{Q}(\mathbf{X},t) \tag{6}$$

for $\mathbf{X} \in J_R(t), t \in T$, where the signs $+$ and $-$ are related to the situation in which $\mathbf{X}$ belongs to $-$ or $+$ phase respectively.

Thus, we have arrived at the description of a certain fine mixture: the deformation gradients in $\mathbf{X}$ at the instant t of the phase $+$ and $-$ have to satisfy respectively the conditions

$$\begin{aligned} &\nabla\mathbf{P}(\mathbf{X},t) + \nabla\zeta(\mathbf{X},t) \otimes \mathbf{Q}(\mathbf{X},t) \in \Phi^-, \\ &\nabla\mathbf{P}(\mathbf{X},t) - \nabla\zeta(\mathbf{X},t) \otimes \mathbf{Q}(\mathbf{X},t) \in \Phi^+. \end{aligned} \tag{7}$$

The last conditions have to be verified for every $\mathbf{X} \in J_R(t)$ at every instant t and constitute existence conditions for the proposed model of interfacial layer.

Equation (7) imposes restrictions both on the motion of the interfacial layer $t \in T \to J_R(t)$ across B_R (because $\nabla\zeta(\mathbf{X},t)$ depends on this motion) as well as on the motion of this layer in E, described by the fields $\mathbf{P}$ and $\mathbf{Q}$. The aforementioned vector fields, following [13], will be called macrodeformation and corrector fields respectively, while $\nabla\mathbf{P}$ will be referred to as macrodeformation gradient field.

In order to obtain field equations for $\mathbf{P}(\cdot)$ and $\mathbf{Q}(\cdot)$ we apply a procedure similar to that used in micromodelling some composite materials, [13], [14], [15]. Let us define the mean value $\langle W_R\rangle$ of the strain energy

$$\langle W_R\rangle(\nabla\mathbf{P},\mathbf{Q}) := \tfrac{1}{2}[W_R^+(\nabla\mathbf{p}) + W_R^-(\nabla\mathbf{p})], \tag{8}$$

where

$$\begin{aligned} W_R^+(\nabla\mathbf{p}) &= W_R^+(\nabla\mathbf{P} - \nabla\zeta \otimes \mathbf{Q}), \\ W_R^-(\nabla\mathbf{p}) &= W_R^-(\nabla\mathbf{P} + \nabla\zeta \otimes \mathbf{Q}) \end{aligned} \tag{9}$$

and the mean value of the kinetic energy

$$\tfrac{1}{2}\langle\rho_R\dot{\mathbf{p}}^2\rangle(\dot{\mathbf{P}},\mathbf{Q}) := \tfrac{1}{2}\rho_R(\dot{\mathbf{P}}^2 + c^2\mathbf{Q}^2) \tag{10}$$

where ρ_R is the mass density in the reference configuration (constant in both phases).

If $\mathbf{b}$ is external body force, $\mathbf{T}_R$ is the first Piola–Kirchhof stress tensor in the vicinity of the layer and $\mathbf{n}_R$ is the unit outward normal to $\partial J_R(t)$, we obtain the action functional for the considered system as follows:

$$\mathcal{A}(\nabla\mathbf{P},\dot{P},\mathbf{Q}) = \int_{t_0}^{t_f} (A_1(t) + A_2(t))\,\mathrm{d}t,$$

$$A_1(t) := \int_{J_R(t)} [\tfrac{1}{2}\rho_R(\dot{P}^2 + c^2\mathbf{Q}^2) - \langle W_R\rangle(\nabla\mathbf{P}, \mathbf{Q}) - \rho_R \mathbf{b}\cdot\mathbf{P}]\,\mathrm{d}v(\mathbf{X}),$$

$$A_2(t) := \int_{\partial J_R(t)} (\mathbf{T}_R \cdot \mathbf{n}_R)\cdot\mathbf{P}\,\mathrm{d}a(\mathbf{X}). \tag{11}$$

The principle of stationary action leads to the following system of equations

$$\begin{aligned} &\mathrm{Div}\,\mathbf{S}_R - \rho_R\ddot{P} + \rho\mathbf{b} = \mathbf{0},\\ &\mathbf{H}_R + \rho_R c^2\mathbf{Q} = \mathbf{0}, \end{aligned} \tag{12}$$

where

$$\mathbf{S}_R := \frac{\partial\langle W_R\rangle}{\partial\nabla\mathbf{P}}, \qquad \mathbf{H}_R := \frac{\partial\langle W_R\rangle}{\partial\mathbf{Q}}. \tag{13}$$

Equations (12) and (13) have to be satisfied in $J_R(t)$ for every $t \in T$. The above equations, together with the boundary condition

$$(\mathbf{S}_R - \mathbf{T}_R)\cdot\mathbf{n}_R = \mathbf{0},$$

describe the behaviour of the interfacial layer element when it is possible to model it as a continuous *fine mixture of the phases* $+$ *and* $-$, i.e. when the existence conditions (7) hold and hence $\langle W_R\rangle$ is well defined by means of (8) and (9). Hence (12) and (13) represent *the mathematical model for the interfacial layers under consideration.*

Equation (13) defines a system of internal forces $\mathbf{S}_R$ and $\mathbf{H}_R$ in the interfacial layer; $\mathbf{S}_R$ represents the first Piola–Kirchhoff total stress tensor and $\mathbf{H}_R$ can be called propagational forces, as they are equal to zero if the local propagation speed $c = c(\mathbf{X}, t)$ is zero.

The interfacial layer was modelled above as a two-phase fine mixture with equal phase concentrations. However, physically motivated situations may exist in which such concentrations are different and which can also depend on the variable ζ. In order to describe these situations denote by σ_+ and σ_- (both belonging to $[0,1]$ and such that $\sigma_+ + \sigma_- = 1$) the concentrations respectively of $+$ and $-$ phases inside the interfacial layer $J_R(t)$. We will assume that their distribution

$$\sigma_+ = \sigma_+(\zeta), \qquad \sigma_- = \sigma_-(\zeta), \qquad \zeta \in (-h(t), h(t))$$

is known. For the sake of simplicity we have assumed that both fields σ_+ and σ_- are independent of $\boldsymbol{\theta} \in \Pi$ and $t \in T$.

Following the approach used above we define

$$\eta_i := 0, \pm 2\varepsilon, \pm 4\varepsilon, \ldots, -h(t) \leqslant \eta_i \leqslant h(t)$$

and partition every part of the layer $J_R(t)$, bounded by the coordinate surfaces $\eta = \eta_i - \varepsilon$, $\eta = \eta_i + \varepsilon$ (ε being previously determinated), into the two sublayers

$$(\eta_i - \varepsilon, \eta_i - \varepsilon + 2\varepsilon\sigma_-(\eta_i)), \qquad (\eta_i - \varepsilon + 2\varepsilon\sigma_-(\eta_i), \eta_i + \varepsilon)$$

occupied respectively by the $+$ and $-$ phases. Let $h^\varepsilon(\eta)$, $\eta \in (-h(t) + 2\varepsilon, h(t))$ be a continuous function, linear inside each of the aforementioned sublayers, which on the surfaces separating sublayers attains the values

$$h^\varepsilon(\eta_i + \varepsilon) = h^\varepsilon(\eta_i - \varepsilon) = \varepsilon \ \min\{\sigma_+(\eta_i), \sigma_-(\eta_i)\},$$
$$h^\varepsilon(\eta_i - \varepsilon + 2\varepsilon\sigma_-(\eta_i)) = -\varepsilon \ \min\{\sigma_+(\eta_i), \sigma_-(\eta_i)\}.$$

Denoting

$$\varphi_+(\eta) := \frac{\min\{\sigma_+(\eta), \sigma_-(\eta)\}}{\sigma_+(\eta)},$$

$$\varphi_-(\eta) := -\frac{\min\{\sigma_+(\eta), \sigma_-(\eta)\}}{\sigma_-(\eta)},$$

$$\eta \in (-h(t), h(t))$$

we see that

$$\frac{\partial h^\varepsilon}{\partial \eta}(\eta) = \begin{cases} \varphi_+(\eta) & \text{if} \quad \eta \in (\eta_i - \varepsilon + 2\varepsilon\sigma_-(\eta_i), \eta_i + \varepsilon), \\ \varphi_-(\eta) & \text{if} \quad \eta \in (\eta_i - \varepsilon, \eta_i - \varepsilon + 2\varepsilon\sigma_-(\eta_i)). \end{cases}$$

Passing to the limit $\varepsilon \to 0$, instead of (6) we now obtain

$$\mathbf{F}(\mathbf{X}, t) := \nabla\mathbf{p}(\mathbf{X}, t) = \nabla\mathbf{P}(\mathbf{X}, t) + \left\{ \begin{matrix} \varphi_+(\eta) \\ \varphi_-(\eta) \end{matrix} \right\} \nabla\zeta(\mathbf{X}, t) \otimes \mathbf{Q}(\mathbf{X}, t),$$

$$\dot{p}(\mathbf{X}, t) = \dot{P}(\mathbf{X}, t) + \left\{ \begin{matrix} \varphi_+(\eta) \\ \varphi_-(\eta) \end{matrix} \right\} c(\mathbf{X}, t)\mathbf{Q}(\mathbf{X}, t) \tag{14}$$

where $\eta = \zeta(\mathbf{X}, t)$.

The interfacial layer existence conditions become

$$\begin{aligned} &\nabla\mathbf{P}(\mathbf{X}, t) + \varphi_+(\zeta(\mathbf{X}, t)) \ \nabla\zeta(\mathbf{X}, t) \otimes \mathbf{Q}(\mathbf{X}, t) \in \Phi^-, \\ &\nabla\mathbf{P}(\mathbf{X}, t) + \varphi_-(\zeta(\mathbf{X}, t))\nabla\zeta(\mathbf{X}, t) \otimes \mathbf{Q}(\mathbf{X}, t) \in \Phi^+ \end{aligned} \tag{15}$$

and must hold for every $\mathbf{X} \in B_R, t \in T$. Under (15) we obtain

$$\begin{aligned} W_R^+(\nabla\mathbf{p}) &= W_R^+(\nabla\mathbf{P} + \varphi_-(\zeta(\mathbf{X}, t))\nabla\zeta \otimes \mathbf{Q}), \\ W_R^-(\nabla\mathbf{p}) &= W_R^-(\nabla\mathbf{P} + \varphi_+(\zeta(\mathbf{X}, t))\nabla\zeta \otimes \mathbf{Q}) \end{aligned} \tag{16}$$

and the mean strain energy will be given by

$$\langle W_R \rangle(\nabla\mathbf{P}, \mathbf{Q}) := \sigma_+(\zeta(\mathbf{X}, t))W_R^+(\nabla\mathbf{p}) + \sigma_-(\zeta(\mathbf{X}, t))W_R^-(\nabla\mathbf{p}). \tag{17}$$

For the mean value of the kinetic energy we obtain

$$\tfrac{1}{2}\langle \rho_R \, \dot{p}^2 \rangle(\dot{\mathbf{P}}, \mathbf{Q}) := \tfrac{1}{2}\rho_R[\sigma_+(\dot{\mathbf{P}} + \varphi_+ \ c\mathbf{Q})^2 + \sigma_-(\dot{\mathbf{P}} + \varphi_- \ c\mathbf{Q})^2]. \tag{18}$$

From the principle of stationary action we obtain again (12_1) in which the mean strain energy $\langle W_R \rangle$ is now defined by (16) and (17). The layer existence conditions are now given by (15) while (12_2) is replaced by

$$\mathbf{H}_R + \rho_R \, \psi \, c^2 \mathbf{Q} = 0, \qquad \psi(\eta) := \frac{\min\{\sigma_+(\eta), \sigma_-(\eta)\}}{\sigma_-(\eta) \; \sigma_+(\eta)},$$

where $\mathbf{H}_R$ is given by (13_2) in which the means strain energy is given by (16) and (17).

In order to explain the above generalization of the proposed modelling procedure we shall consider two special cases. First, assume that it is physically motivated the supposition that the passage across the interface from one phase to another is continuous. In this case we can set

$$\sigma_+ = \frac{1}{2}\left[1 + \frac{\eta}{h(t)}\right], \qquad \sigma_- = \frac{1}{2}\left[1 - \frac{\eta}{h(t)}\right],$$

provided that the passage from $-$ phase to $+$ phase takes place in the ζ-axis direction.

Second, let the interfacial layer be situated inside, for instance, the $+$ phase. In some cases it may be reasonable to require that this interface is constituted by a mixture of $+$ and $-$ phases in which concentration varies smoothly. Such a situation will be described by setting:

$$\sigma_+ = \left[\frac{\eta}{h(t)}\right]^2, \qquad \sigma_- = 1 - \left[\frac{\eta}{h(t)}\right]^2.$$

This case constitutes a continuum model of the physical situation investigated in [16], where the existence of such interfacial layers, but with a discrete structure, was shown.

It should be emphasized that if there is no physical information about the internal structure of the interfacial layer, then the assumption $\sigma_+ = \sigma_- = 0.5$ and hence the model introduced before seem to be the most reasonable.

4. Applications

In this section we want to apply the model developed up to now to the study of the propagation of a particular interfacial layer. More precisely, we consider a certain variant of the micro-cracked material of which the macromodel was introduced and investigated in [15] and [16]: the considered solid body is assumed damaged because of the presence of planar stochastically distributed microcracks (parallel to the laminae interfaces) whose planes are oriented in B_R by a regular field **n** of unit vectors. It will be assumed that the region B_R can be partitioned into three regions: i.e. the region in which the microcracks are open (phase $+$), the region in which the microcracks are closed (phase $-$) and the interfacial layer in which the microcracks are partly closed and partly open. In the last case we assume that the strain energy is not defined.

We will limit our study to the case

(1) of linear elastic micro-cracked materials in which the vector field **n** is constant,
(2) of interfacial layer propagation in the direction of **n**.

Therefore, we will restrict our consideration to uniaxial strain states. In this case the strain energy functions W_R^+ and W_R^- (the strain energy in the medium with all open and with all closed cracks respectively) and the sets of admissible local deformations are assumed to be given, in terms of the infinitesimal strain tensor **e**, by

$$\Phi^+ := \{\mathbf{e} : \mathbf{n}\cdot\mathbf{en} > \alpha_+\}; \qquad W_R^+(\mathbf{e}) = \tfrac{1}{2}E^+(\mathbf{n}\cdot\mathbf{en})^2,$$
$$\Phi^- := \{\mathbf{e} : \mathbf{n}\cdot\mathbf{en} < \alpha_-\}; \qquad W_R^-(\mathbf{e}) = \tfrac{1}{2}E^-(\mathbf{n}\cdot\mathbf{en})^2,$$

where the constants $\alpha_+, \alpha_-, \alpha_+ > \alpha_-$, which we can call *strain thresholds*, play the role of the phase-defining thresholds. Moreover, the constants E^+ and E^- are longitudinal elastic moduli relative to the uniaxial strain and to the phase with all open and all closed microcracks respectively. They are such that $E^- > E^+$: when the cracks are open the stiffness of the material is lower than in the case of closed cracks. In the model discussed in [15] and [16] it is assumed that $\alpha_+ = \alpha_- = 0$. In the sequel we will study the propagation of a longitudinal plane displacement wave, opening (or closing) microcracks, in which the phase change occurs inside a layer. We will use the following denotations:

(1) U is the component of displacement vector field along **n** and $U_{,n}$ its derivative in the direction of **n**,
(2) $\xi(t)$ is the position of the midsurface of the interfacial layer and $2h(t)$ its thickness at any instant t,
(3) S is the normal stress component, $S = \mathbf{n}\cdot\mathbf{S}_R\mathbf{n}$,
(4) we define the longitudinal wave propagation speeds in both phases by

$$c_+ := \sqrt{\frac{E^+}{\rho_R}}, \qquad c_- := \sqrt{\frac{E^-}{\rho_R}}.$$

The average strain energy for the considered continuum and displacement fields is easily evaluated once (8) is recalled

$$\langle W_R\rangle = \tfrac{1}{2}\tilde{E}(U_{,n})^2 + \Delta E\mathbf{Q}\,U_{,n} + \tfrac{1}{2}\tilde{E}(\mathbf{Q})^2$$

where

$$\tilde{E} := \tfrac{1}{2}(E^+ + E^-); \qquad \Delta E := E^- - E^+.$$

Therefore, we obtain the following expression for the normal stress S in terms of **Q** and $U_{,n}$

$$S = \tilde{E}U_{,n} + \Delta E\mathbf{Q}$$

while the second from (12) becomes

$$\tilde{E}\mathbf{Q} + \Delta E U_{,n} - \rho_R c^2\mathbf{Q} = 0.$$

This last equation allows us to express **Q** in terms of $U_{,n}$ and c^2

$$\mathbf{Q} = -\frac{\Delta E}{\tilde{E} - \rho_R c^2}U_{,n}$$

so that S can be expressed in terms of $U_{,n}$ and c^2 as follows

$$S = \left(\tilde{E} - \frac{\Delta E^2}{\tilde{E} - \rho_R c^2}\right) U_{,n}. \tag{19}$$

We will call *interfacial longitudinal modulus* the quantity

$$E(c^2) := \tilde{E} - \frac{\Delta E^2}{\tilde{E} - \rho_R c^2}.$$

The interfacial longitudinal modulus must be positive everywhere inside the interfacial layer. This physical condition implies the following restrictions on the propagation speed c

$$c < c^+ \quad \text{or} \quad c > c^-. \tag{20}$$

The field U has to satisfy the hyperbolic equation implied by the first from (12),

$$S_{,n} - \rho_R \frac{\mathrm{d}^2 U}{\mathrm{d}t^2} + b = 0 \quad \text{in} \quad B_R^0(t), \tag{21}$$

in which the expression (19) for S has to be substituted.

Equation (21) has to be considered together with the continuity conditions for S and U on both surfaces separating the interfacial layer from the remaining part of the solid. If this layer is included between $+$ and $-$ phase, then the position of these surfaces is given by

$$U_{,n}(\xi(t) + h(t), t) = \alpha_+; \qquad U_{,n}(\xi(t) - h(t), t) = \alpha_-.$$

It has to be emphasized that the above analysis has a physical sense only if condition (7) holds inside the interfacial layer. In the framework of the linearized theory the aforementioned condition yields

$$U_{,n} + Q = \left(1 - \frac{\Delta E}{\tilde{E} - \rho_R c^2}\right) U_{,n} < \alpha_-,$$

$$U_{,n} - Q = \left(1 + \frac{\Delta E}{\tilde{E} - \rho_R c^2}\right) U_{,n} > \alpha_+. \tag{22}$$

These inequalities must be satisfied at every point of the interfacial layer if it is to be modelled as an ideal mixture of phases $+$ and $-$, as done in the present paper.

Outside the interfacial layer the standard equations of linear elasticity with strain energy, equal respectively to W_R^+ and W_R^-, are assumed to hold.

We remark that:

(1) the interfacial longitudinal modulus reduces to the effective modulus found in [13] when c vanishes;
(2) when c tends to c^+ or to c^- respectively from lower or higher values then at $\xi(t) + h(t)$ or $\xi(t) - h(t)$ respectively the interfacial longitudinal modulus vanishes.

5. Illustrative example

The analysis carried out in Section 2–4 was quite formal and one can ask whether the model of the interfacial layer presented here satisfies the existence conditions (7) or (22) and hence has physical sense. In order to give an answer to the above question, using the results found in Section 4, we will consider below a simple problem illustrating the behaviour of micro-damaged solids under consideration.

Let the micro-cracked solid occupy a thick layer bounded by coordinate planes $x = 0, x = L$ and the vector **n**, determining the orientation of microcracks, coincide with the versor of x-axis. The solid is subject to a body force acting along the x-axis and having constant value b and to the external pressure $p = p(t)$ applied at the plane $x = 0$.

Moreover, let the displacement of the solid be equal to zero at $x = L$.

Hence, we deal with an uniaxial strain state corresponding to a displacement field with only one non-vanishing component $U(x,t)$ in which the material properties of the solid are determined by the elasticity moduli E^+ and E^-. We must also recall that we assume as known the two strain thresholds α_+ and α_- with $\alpha_+ > \alpha_- > 0$. It means that if $U_{,x} > \alpha_+$ or $U_{,x} < \alpha_-$ then the microcracks are all open or all closed and the corresponding stress strain relations are given respectively by

$$S = E^+ U_{,x} \quad \text{or} \quad S = E^- U_{,x}.$$

Our aim is to find the displacement field $U(x,t), x \in [0,L], t \in T$. For the sake of simplicity its evolution will be treated only in the quasi-stationary case.

It can be easily shown that the evolution of the midsurface and of the thickness of the interfacial layer, provided that it exists, is given by

$$\xi(t) = \frac{1}{\rho_R b}\left(p(t) - \frac{\alpha_+ E^+ + \alpha_- E^-}{2}\right);$$

$$h(t) = \frac{\alpha_+ E^+ + \alpha_- E^-}{2\rho_R b} =: h = \text{const} \tag{23}$$

respectively. Hence the local propagation speed is given by

$$c(t) = \frac{\dot{p}(t)}{\rho_R b}.$$

The above formulas are valid if both the conditions

$$\xi(t) - h > 0 \quad \text{and} \quad \xi(t) + h < L \tag{24}$$

hold for every $t \in T$ and $c(t)$ satisfies (20). The inequalities (24) combined with conditions (23) have to be interpreted as necessary conditions imposed on the pressure $p(t)$ under which the interfacial layer may exist. For the parts of the solid situated outside this layer we have

$$\begin{aligned} &E^- U_{,xx}(x,t) + \rho_R b = 0 \quad \text{if} \quad x \in (\xi(t) + h, L) \\ &E^+ U_{,xx}(x,t) + \rho_R b = 0 \quad \text{if} \quad x \in (0, \xi(t) - h) \end{aligned}. \tag{25}$$

In the region occupied by the interfacial layer the stress-strain relation cannot be specified in the framework of standard elasticity theory. Thus, we model it using the results found in Sections 3 and 4. Combining (19) and (21) we obtain

$$\left(\tilde{E} - \frac{\Delta E^2}{\tilde{E} - \rho_R c^2(t)}\right) U_{,xx}(x,t) + \rho_R b = 0$$
$$\text{if} \quad x \in (\xi(t) - h, \xi(t) + h), \tag{26}$$

where the conditions (20) are assumed to be verified.

Since $\xi(t), c(t)$ and h are given by (23), (25) and (26) together with the continuity conditions for the displacement U and the stress S for $x = \xi(t) - h$ and $\xi(t) + h$ as well as with the boundary conditions

$$U(0,t) = 0, \quad E^+ U_{,1}(0,t) = p(t) \quad \forall t \in T$$

allow, with simple algebra, the analytical expression for the unique solution of the considered evolution problem, provided that for every $t \in T$ the conditions (24) are verified.

However, the model applied may have no physical sense if this solution does not verify the existence condition (22) which now reads

$$\left(1 - \frac{\Delta E}{\tilde{E} - \rho_R c^2(t)}\right) U_{,x}(x,t) < \alpha_-,$$
$$\left(1 - \frac{\Delta E}{\tilde{E} - \rho_R c^2(t)}\right) U_{,x}(x,t) > \alpha_+, \quad \forall x \in (\xi(t) - h, \xi(t) + h).$$

The last inequalities imply that

$$\left(1 - \frac{\Delta E}{\tilde{E} - \rho_R c^2(t)}\right) < \frac{\alpha_-}{\alpha_+}, \quad \left(1 - \frac{\Delta E}{\tilde{E} - \rho_R c^2(t)}\right) > \frac{\alpha_+}{\alpha_-} \tag{27}$$

which represent the conditions imposed on the material parameters $\Delta E, \tilde{E}, \rho_R, \alpha_-$ and α_+ under which the interfacial layer, propagating at the instant $t \in T$ with the speed $c(t) \in (-\infty, c^+) \cup (c^-, \infty)$, can be modelled as an ideal mixture of the two considered solid phases.

A necessary condition to (27), as $\alpha_+ > \alpha_-$, is

$$c < c^+, \forall t \in T \tag{28}$$

which excludes the higher propagation speeds.

In concluding this Section we consider two special cases of the conditions (27):

(1) Let us assume that $\alpha_- = \alpha_+$. In this case (as $\Delta E > 0$) the conditions (27) are equivalent to condition (28), if one recalls (20). If we assume that $\alpha_- = \alpha_+ = 0$, (as done in [16]) then (23) implies that the interfacial layer becomes an interfacial surface.

(2) If $\alpha_+ > 0 > \alpha_-$ then the conditions (27) are never satisfied. In this case the proposed model of the interfacial layer as fine mixture of both phases does not make sense from the physical viewpoint. Indeed our reasoning proves that two phases with the phase-defining thresholds verifying $\alpha_+ > 0 > \alpha_-$ cannot coexist as constituents of the fine mixture introduced in this paper.

6. Conclusions

Summarizing the general results obtained in Section 3 we can conclude that:

(1) in the proposed model the properties of the medium constituting the interfacial layer can be easily determined once the constitutive properties of phases $+$ and $-$ and the motion of the layer are known,
(2) since c is the relative velocity in the interfacial layer with respect to the referencial space B_R then the equations (12) and (13) are Galilean invariant,
(3) the second from (13) is an algebraic relation $\mathbf{Q}$ and $\nabla\mathbf{P}$. The definition (8) of average strain energy implies that using it one can express $\mathbf{Q}$ in terms of $\nabla\mathbf{P}$ and c^2 and therefore the strain energy, for the introduced interfacial fine mixture can be regarded as a function of the macrodeformation gradient $\nabla\mathbf{P}$ and the speed c,
(4) if the interfacial layer coincides with a material one (i.e. if it is not moving across B_R) then the speed c vanishes and the equations (12) and (13) reduce to those found in [14] where some macromodels for micro-laminated materials are considered.

Moreover, in Section 4 of this paper we have proven that:

(1) the interfacial layers can be modelled as fine mixtures of both phases only if the propagation speed satisfies the inequalities (20),
(2) the material behaviour of the medium filling the interfacial layer depends not only on the material properties of both phases but also on the propagation speed of the layer,
(3) since the interfacial longitudinal elastic modulus has to be always positive not every propagation speed of the interfacial layer is possible.

Finally, in Section 5 we have shown that the class of systems described by the model proposed in this paper is not empty.

The aforementioned results lead to the conclusion that the proposed approach to the modelling of interfacial layers in the micro-damaged two-phase solids seems to deserve both mathematical and physical interest.

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