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Submitted on 27 Oct 2010

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BOLTZMANN EQUATION WITHOUT ANGULAR CUTOFF IN THE WHOLE SPACE: I, GLOBAL EXISTENCE FOR SOFT POTENTIAL

R. ALEXANDRE, Y. MORIMOTO, S. UKAI, C.-J. XU, AND T. YANG

Abstract. It is known that the singularity in the non-cutoff cross-section of the Boltzmann equation leads to the gain of regularity and gain of weight in the velocity variable. By defining and analyzing a non-isotropy norm which precisely captures the dissipation in the linearized collision operator, we first give a new and precise coercivity estimate for the non-cutoff Boltzmann equation for general physical cross sections. Then the Cauchy problem for the Boltzmann equation is considered in the framework of small perturbation of an equilibrium state. In this part, for the soft potential case in the sense that there is no positive power gain of weight in the coercivity estimate on the linearized operator, we derive some new functional estimates on the nonlinear collision operator. Together with the coercivity estimates, we prove the global existence of classical solutions for the Boltzmann equation in weighted Sobolev spaces.

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2000 Mathematics Subject Classification. 35A05, 35B65, 35D10, 35H20, 76P05, 84C40.
Key words and phrases. Boltzmann equation, coercivity estimate, non-cutoff cross sections, global existence, non-isotropic norm, soft potential.
1. Introduction

This is the first part of a series of papers related to the inhomogeneous Boltzmann equation without angular cut-off, in the whole space and for general physical cross-sections. This global project is a natural continuation of our previous study [1] which was specialized to Maxwellian type cross sections.

In this part, we first establish an essential coercivity estimate of the linearized collision operator, in the framework of general cross sections. As shown in [2, 3] for the special Maxwellian case, this estimate will play an important role for the related Cauchy problem.

Based on this estimation, together with Part II [4], we will prove the global existence of classical non-negative solutions to the Boltzmann equation without angular cutoff, for the soft and hard potentials respectively, so that we are able to cover a general physical setting. Finally, in the paper [10], we will study the qualitative properties of solutions, that include full regularity, non-negativity, uniqueness and convergence rates to the equilibrium. This series of works establish a satisfactory theory on the well-posedness and full regularity of classical solutions.

In our presentation, we consider the problem in the physical case with dimension 3. However, our results hold true for any dimension bigger than 2.

Consider

\[ f_t + v \cdot \nabla_x f = Q(f, f), \quad f|_{t=0} = f_0. \]

Here, \( f = f(t, x, v) \) is the density distribution function of particles, having position \( x \in \mathbb{R}^3 \) and velocity \( v \in \mathbb{R}^3 \) at time \( t \). The right hand side of (1.1) is the Boltzmann bilinear collision operator, which is given in the classical \( \sigma \)-representation by

\[ Q(g, f) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} B(v - v_s, \sigma) [g' f' - g f] \, d\sigma dv, \]

where \( f' = f(t, x, v') \), \( f'' = f(t, x, v'') \), \( f_s = f(t, x, v_s) \), \( f = f(t, x, v) \), and for \( \sigma \in \mathbb{S}^2 \),

\[ v' = \frac{v + v_s}{2} + \frac{|v - v_s|}{2} \sigma, \quad v'' = \frac{v + v_s}{2} - \frac{|v - v_s|}{2} \sigma, \]

which gives the relation between the post and pre collisional velocities that follow from the conservation of momentum and kinetic energy.

For monatomic gas, the non-negative cross section \( B(z, \sigma) \) depends only on \( |z| \) and the scalar product \( \frac{v}{|v|} \cdot \sigma \). As in our previous works, we assume that it takes the form

\[ B(v - v_s, \cos \theta) = \Phi(|v - v_s|) b(\cos \theta), \quad \cos \theta = \frac{v - v_s}{|v - v_s|} \cdot \sigma, \quad 0 \leq \theta \leq \frac{\pi}{2}, \]

in which it contains a kinetic factor given by

\[ \Phi(|v - v_s|) = \Phi_s(|v - v_s|) = |v - v_s|^r, \]

and a factor related to the collision angle containing a singularity,

\[ b(\cos \theta) \approx K \theta^{2-2s} \text{ when } \theta \to 0+, \]

for some constant \( K > 0 \).

An important example of this singular cross section is the inverse power law potential \( \rho^{-\gamma} \) with \( r > 1 \), \( \rho \) being the distance between two interacting particles, in which \( s = \frac{\gamma}{r} \in [0, 1] \) and \( \gamma = 1 - 4s \in ]3, 1[ \), cf. [12].

In the theory on the non-cutoff Boltzmann equation, the sign of \( \gamma + 2s \) plays a crucial role. Hence, from now on, the case when \( \gamma + 2s \leq 0 \) is referred to the non-cutoff soft
potential, while the case \( \gamma + 2s > 0 \) to the \textit{non-cutoff hard potential}. Note that this is different from the traditional classification on the index for the inverse power law.

In our present series of works, the well-posedness theory established applies to the general cross-sections with \( \gamma > -3 \) and \( 0 < s < 1 \), that includes the inverse power law as a special example. Note that \( \gamma > -3 \) is needed for the Boltzmann operator to be well-posed, cf. [40].

Being concerned with a close to equilibrium framework, as in [7], the setting of the problem can be formulated as follows. First of all, without loss of generality, consider the perturbation around a normalized Maxwellian distribution

\[
\mu(v) = (2\pi)^{-\frac{d}{2}} e^{-\frac{|v|^2}{2}},
\]

by setting \( f = \mu + \sqrt{\mu} g \). Since \( Q(\mu, \mu) = 0 \), we have

\[
Q(\mu + \sqrt{\mu} g, \mu + \sqrt{\mu} g) = Q(\mu, \sqrt{\mu} g) + Q(\sqrt{\mu} g, \mu) + Q(\sqrt{\mu} g, \sqrt{\mu} g).
\]

Denote

\[
\Gamma(g, h) = \mu^{-1/2} Q(\sqrt{\mu} g, \sqrt{\mu} h).
\]

Then the linearized Boltzmann operator takes the form

\[
Lg = L_1 g + L_2 g = -\Gamma(\sqrt{\mu}, g) - \Gamma(g, \sqrt{\mu}).
\]

Now the original problem (1.1) is reduced to the Cauchy problem for the perturbation \( g \)

\[
\begin{cases}
g_t + v \cdot \nabla_x g + Lg = \Gamma(g, g), & t > 0, \\
g|_{t=0} = g_0.
\end{cases}
\]

This close to equilibrium framework is classical for the Boltzmann equation with angular cutoff, but much less is known for the Boltzmann equation without angular cutoff, though the spectrum of the linearized operator without angular cut-off was analyzed a long time ago by Pao in [3].

However, since the late 1990s, the regularizing effect on the solution, produced by the singularity of the cross-section, has become reachable by rigorous analysis. Let us mention the systematic work on the entropy dissipation method initiated by Alexandre [1] and developed firstly by Lions [26], and then by many others, cf [3, 39, 40] and references therein. Since then, various works have been done on deriving the coercivity estimates in different settings and in different norms for different purposes. In particular, this kind of coercivity estimates has displayed some non-isotropic property in the very loose sense that, on one hand one gets a gain of the regularity in Sobolev norm of fractional order; and on the other hand, one also get a gain the moment to some fractional power in the velocity variable, cf. [2, 3, 5, 6, 7, 16, 21, 22, 24, 31, 32, 38, 39, 40] and references therein. Furthermore, these coercivity estimates have been proven to be very useful in getting the global existence and gain of full regularity in all variables for the Boltzmann equation without angular cutoff, as shown in our previous work [7]. For details about the recent progress in some of the directions mentioned previously, readers are referred to the survey paper by Alexandre, [2].

Since the coercivity estimate plays an important role in the study on the angular non-cutoff Boltzmann equation, such estimate in terms of the indices \( \gamma \) and \( s \), has been pursued by many people. One of the purposes of this paper is to present a precise estimate that gives the essential properties of this singular behavior, that will be stated in the next theorem. Let us note that this result is proved in a general setting and it improves on previous results, such as those obtained in [5, 6, 7, 31, 32]. And this estimate will be used herein and in our
papers \cite{9, 10} on the global existence in the hard potential case, and qualitative study of solutions. To derive the desired coercivity estimate, we generalize the non-isotropic norm introduced in \cite{11} as

\[
\|g\|^2 = \iint \Phi(|v - v_\star|) b(\cos \theta) \mu_\star (g' - g)^2 + \iint \Phi(|v - v_\star|) b(\cos \theta) g^2 \left( \sqrt{\mu'} - \sqrt{\mu} \right)^2,
\]

where the integration is over \(\mathbb{R}^3 \times \mathbb{R}^3 \times S^2\). Note that it is a norm with respect to the velocity variable \(v \in \mathbb{R}^3\) only. We can compare this non-isotropic norm with classical weighted Sobolev norms, see precisely Proposition \(\text{I.1}\).

The introduction of this norm was motivated by the study on the Landau equation which can be viewed as the grazing limit of the Boltzmann equation. It is known that for the Landau equation, see for example \(\cite{19}\), that the essential norm in order to capture the linearized operator. By doing so, a norm can be well defined without loss of any dissipative information in the operator and this can be done directly for the Landau equation mainly because the corresponding Landau operator is a differential operator. However, for the Boltzmann equation without angular cutoff, the collision operator is a singular integral operator so that a direct analog is not obvious or feasible. Therefore, in the first part of this paper, we analyze the properties of the non-isotropic norm and obtain the precise coercivity estimate of the linearized collision operator. At this point, let us mention the different approach undertaken by Gressman-Strain \(\cite{21, 22}\).

We shall use the following standard weighted Sobolev space defined, for \(k, \ell \in \mathbb{R}\), as

\[
H^k_{x,v}(\mathbb{R}^6) = \{ f \in S'(\mathbb{R}^6); \ W_f f \in H^k(\mathbb{R}^3) \}
\]

and

\[
H^\ell_{x,v}(\mathbb{R}^{6x}) = \{ f \in S'(\mathbb{R}^{6x}); \ W_f f \in H^\ell(\mathbb{R}^{6x}) \}
\]

where \(W_f(\nu) = (\nu)^\ell = (1 + |\nu|^2)^{\ell/2}\) is always the weight for \(v\) variables. Herein, \((\cdot, \cdot)_{L^2} = (\cdot, \cdot)_{L^2(\mathbb{R}^3)}\) denotes the usual scalar product in \(L^2 = L^2(\mathbb{R}^3)\) for \(v\) variables. Recall that \(L^2 = H^0_{x,v}\).

We shall use also in the following two different Sobolev spaces, one with \(x\)-derivatives only, another one with \(x, v\) derivatives and weight in the velocity variable \(v\). For \(k \in \mathbb{N}, \ell \in \mathbb{R}\), let

\[
H^k_{x,v}(\mathbb{R}^6) = \{ f \in S'(\mathbb{R}^6); \|f\|_{H^k_{x,v}(\mathbb{R}^6)} = \sum_{|\alpha| + |\beta| \leq N} \|W_f \partial_\mu^{\alpha} \partial_\nu^{\beta} f\|_{L^2(\mathbb{R}^6)} < +\infty \},
\]

\[
\hat{H}^\ell_{x,v}(\mathbb{R}^6) = \{ f \in S'(\mathbb{R}^6); \|f\|_{\hat{H}^\ell_{x,v}(\mathbb{R}^6)} = \sum_{|\alpha| + |\beta| \leq N} \|\hat{W_f} \partial_\mu^{\alpha} \partial_\nu^{\beta} f\|_{L^2(\mathbb{R}^6)} < +\infty \},
\]

where \(\hat{W}_f = (1 + |\nu|^2)^{\ell + \gamma/2\ell/2}\).

We recall that the linearized operator \(L\) has the following null space, which is spanned by the set of collision invariants:

\[
\mathcal{N} = \text{Span} \left\{ \sqrt{\mu}, v_1 \sqrt{\mu}, v_2 \sqrt{\mu}, v_3 \sqrt{\mu}, |\nu|^2 \sqrt{\mu} \right\},
\]

that is, \(\left(Lg, g\right)_{L^2(\mathbb{R}^3)} = 0\) if and only if \(g \in \mathcal{N}\).
Theorem 1.1. Assume that the cross-section satisfies \((1.2)\) with \(0 < s < 1\) and \(\gamma > -3\). Then there exist two generic constants \(C_1, C_2 > 0\) such that for any suitable function \(g\)

\[C_1 \| (1 - P)g \|^2 \leq \left( Lg, g \right)_{L^2} \leq C_2 \| g \|^2,
\]

where \(P\) is the \(L^2\)-orthogonal projection onto the null space \(N\).

This coercivity estimate of the linearized collisional operator will prove to be crucial for the global existence of classical solutions to the Boltzmann equation. For this purpose, the analysis on the nonlinear operator is necessary, and we prove the following upper bound estimate.

Theorem 1.2. For all \(0 < s < 1\), assume that \(\gamma > \max\{-3, -\frac{1}{2} - 2s\}\). Then, one has

\[
\left| (\Gamma(f, g), h)_{L^2} \right| \leq \left\{ \| f \|_{L^2_x \cap Y} \| g \| + \| f \|_{L^2_x \cap Y} \| f \| + \min \| f \|_{L^2_x \cap Y}, \| g \|_{L^2_x \cap Y} \right\} \| h \|,
\]

for suitable functions \(f, g, h\).

We will then concentrate on the global existence of solutions, both weak and strong, for the non-cutoff soft potential case in the framework of small perturbation of an equilibrium state. Even though some estimates hold for the general case and will be used in the forthcoming papers, the condition \(\gamma + 2s \leq 0\) will be imposed in the main existence results. In the Part II [9], we will then present the global existence theory for the hard potential case, that is, the condition \(\gamma + 2s > 0\) imposed. Furthermore, the qualitative behavior of the solutions, such as the uniqueness, non-negativity, regularity and convergence rate to the equilibrium will be investigated in [10]. Note that both the global existence and the qualitative study on the solution behavior were firstly investigated in [7] for the Maxwellian molecule case where a generalized uncertainty principle obtained in [5] was used.

We begin with a local existence of classical solutions that holds true in general case.

Theorem 1.3. Assume that the cross-section satisfies \((1.2)\) with \(\gamma + 2s \leq 0\), \(0 < s < 1\) and \(\gamma > -3\). Let \(N \geq 6\) and \(\ell \geq N\). For a small \(\varepsilon > 0\), if \(\| g_0 \|_{H^\ell (\mathbb{R}^3)} \leq \varepsilon\), then there exists \(T > 0\) such that the Cauchy problem \((1.3)\) admits a solution

\[g \in L^\infty((0, T]; \mathcal{H}_x^\ell (\mathbb{R}^3)).\]

Since we are interested in getting global existence results, the next statement deals with this issue asking only for control of \(x\) derivatives.

Theorem 1.4. Assume that the cross-section satisfies \((1.2)\) with \(\gamma + 2s \leq 0\), \(0 < s < 1\) and \(\gamma > \max\{-3, -\frac{1}{2} - 2s\}\). Let \(N \geq 3\). For a small \(\varepsilon > 0\), if \(\| g_0 \|_{H^N (\mathbb{R}^3; L^2 (\mathbb{R}^3))} \leq \varepsilon\), then the Cauchy problem \((1.3)\) admits a global solution

\[g \in L^\infty([0, +\infty]; H^N (\mathbb{R}^3; L^2 (\mathbb{R}^3))).\]

The above global existence result is in a non-weighted function space without \(v\) derivatives in the framework of weak solutions. On the other hand, we will prove the following global existence result on classical solutions for which the proof is more involved. Note that for the qualitative study on the solution behavior, such as the regularity as will be shown in [10], solutions in a function space with \(x\) and \(v\) derivatives together with weight in \(v\) are needed. Hence, the next theorem also serves for this purpose.
Theorem 1.5. Assume that the cross-section satisfies (1.2) with \( \gamma + 2s \leq 0 \), \( 0 < s < 1 \) and \( \gamma > \max\{ -3, -\frac{1}{2} - 2s \} \). Let \( N \geq 6, \ell \geq N \). For a small \( \varepsilon > 0 \), if \( \|g_0\|_{\tilde{H}^N_\ell(\mathbb{R}^6)} \leq \varepsilon \), then the Cauchy problem (1.3) admits a global solution \( g \in L^\infty([0, +\infty[ ; \tilde{H}^N_\ell(\mathbb{R}^6)) \).

Let us now review some related works on this topic. First of all, the well-posedness theory for the Boltzmann equation has now been well established under the Grad’s angular cutoff assumption. Under this assumption, there exist basically three frameworks of existence of global solutions. The first one was initiated by Grad [18] and firstly completed by Ukai [35, 36, 38] in the framework of weighted \( L^\infty_v \) function space for small perturbation of an equilibrium, where the spectrum analysis was used through a bootstrap argument. An important progress on the existence theory is the introduction of the renormalized solutions for large perturbation in the framework of \( L^1_v \) function space by DiPerna-Lions [17, 25], where the velocity averaging lemma plays a key role. Recently, solutions in \( L^2_v \) framework were established by macro-micro decompositions and energy method for small perturbation of an equilibrium, cf [19, 20, 27, 28].

However, without Grad’s angular cutoff assumption, the established mathematical theories are far less. In this direction, the spectral analysis of the linearized collisional operator was studied by Pao [33]. In 1990’s, some simplified models, such as Kac’s model and the Boltzmann equation in lower dimensions with symmetry, were successfully studied, [13, 14, 15]. In 2000’s, the mathematical theory for the spatially homogeneous Boltzmann equation was satisfactorily solved, [3, 4, 16, 24, 29, 30]. For the original Boltzmann equation in physical space, in the framework of renormalized solutions, the only existing result can be found in [11] where the basic existence result is still lacking. There are some local existence results, [1, 37], see also the reviews [2, 40].

Since 2006, we have been working on the original Boltzmann equation without angular cutoff, cf. [5, 6, 7]. Based on a new generalized uncertainty principle proved in [5], we developed a new approach for the regularity study. In the framework of small perturbation of an equilibrium in the whole space, the first complete global well-posedness theory and regularity were established for the Maxwellian molecule case [7]. As a continuation of these works, we successfully solve, in this series of papers, the fundamental problems, that is, existence, uniqueness, regularity, non-negativity and convergence rates of solutions, so that a complete and satisfactory mathematical theory is now established under some minimal regularity requirement on the initial data. Through this analysis, mathematical tools and techniques from harmonic analysis are used and some new ones are introduced, such as the generalized uncertainty principle and the above non-isotropic norm. Here we would like to mention that recently by using a different method, an existence result on solutions in the torus case was obtained in [21, 22, 23].

Finally, we present the main strategy of analysis in this paper. Based on the essential coercivity estimate on the linearized collisional operator and the non-isotropic norm proven in a first step, what is needed in this paper is then the detailed analysis on the collisional operator in both unweighted and weighted spaces, where its upper bounds, commutators with differentiation in \( v \) and commutators with weights in \( v \) are given. Through this analysis, we can see the role played by the parameters \( \gamma \) and \( s \) in the cross section. With these estimates, the energy method can be applied through the macro-micro decomposition analysis introduced in [19, 20]. Basically, the microscopic component of the solution is controlled by the essential coercivity estimate on the linearized collisional operator in the non-isotropic norm, while the dissipation on the macroscopic component is recovered by the system on the fluid functions through the macro-micro decomposition. Then the nonlinear terms are
essentially of higher order in the non-isotropic norm so that the energy estimate can be closed in the framework of small perturbation.

The rest of the paper is arranged as follows. In the next section, we extend the definition of the non-isotropic norm introduced in [7] and then state the main estimates in this paper. The proof of the upper and lower bound estimates of the non-isotropic norm will be given in Section 3. In Section 4, we will prove the equivalence of the Dirichlet form of the linearized collision operator and the square of the non-isotropic norm. The equivalence of the non-isotropic norms with different kinetic factors and different weights will be shown in Section 5. An upper bound estimate on the nonlinear collision operator which is useful for the well-posedness theory for the Boltzmann equation will be given next. However, because of unnecessary restrictions on the values of the parameter \( \gamma \), we shall amplify such estimations and obtain some new functional estimates in \( v \) variable only in another Section. The functional estimates in both \( v \) and \( x \) variables are given in Section 4. With these estimates, the local and global existence of both weak and classical solutions are given in the last two sections respectively.

**Notations:** Herein, letters \( f, g, \cdots \) stand for various suitable functions, while \( C, c, \cdots \) stand for various numerical constants, independent from functions \( f, g, \cdots \) and which may vary from line to line. Notation \( A \lesssim B \) means that there exists a constant \( C \) such that \( A \leq CB \), and similarly for \( A \gtrsim B \). While \( A \sim B \) means that there exist two generic constants \( C_1, C_2 > 0 \) such that \( C_1 A \leq B \leq C_2 A \).

### 2. Non-isotropic Norm and Estimates of Linearized Collision Operators

For later use, we will need to compare the original cross-section with the situation when its kinetic part is mollified. That is, for the function \( \Phi(z) \) appearing in the cross-section, we denote by \( \tilde{\Phi}(z) = (1 + |z|^2)^\gamma \) its smoothed version. To show the dependence of the estimates on the mollified or non-mollified kinetic factor in the cross-section, we will use the notations \( Q_{\tilde{\Phi}} \) and \( Q_{\Phi} \) to denote the Boltzmann collisional operator when the kinetic part is \( \tilde{\Phi} \) and \( \Phi \) respectively. In particular, \( Q = Q_{\Phi} \). This upper-script will be also used for other operators as well.

First of all, let us recall that
\[
\left( Lg, g \right)_{L^2} = -\left( \Gamma(\sqrt{\mu}, g) + \Gamma(g, \sqrt{\mu}), g \right)_{L^2} \geq 0,
\]
and the definition of the non-isotropic norm
\[
|||g|||^2 = \iiint \Phi(|v - v_*|)b(cos \theta)\mu_* (g' - g)^2 + \iiint \Phi(|v - v_*|)b(cos \theta)\mu (g' - \sqrt{\mu})^2 = J_1 + J_2,
\]
where the integration is over \( \mathbb{R}^3 \times \mathbb{R}^3 \times S^2 \).

The following proposition gives a precise version of Theorem \[1.1\].

**Proposition 2.1.** Assume that the cross-section satisfies (1.2) with \( 0 < s < 1 \) and \( \gamma > -3 \). Then there exist two generic constants \( C_1, C_2 > 0 \) such that
\[
C_1 |||(I - P)g|||^2 \leq \left( Lg, g \right)_{L^2} \leq 2 \left( Lg, g \right)_{L^2} \leq C_2 |||g|||^2
\]
for any suitable function \( g \).
Concerning the lower and upper bounds of the non-isotropic norm we have

**Proposition 2.2.** Assume that the cross-section satisfies (1.2) with $0 < s < 1$ and $\gamma > -3$. Then there exist two generic constants $C_1, C_2 > 0$ such that

$$C_1 \left( \| g \|^2_{H^{\gamma/2}_\rho} + \| \nabla g \|^2_{H^{\gamma/2}_\rho} \right) \leq \| g \|_{\Phi}^2 \leq C_2 \| g \|^2_{H^{\gamma/2}_\rho},$$

for any suitable function $g$.

From this estimate and Theorem 1.1, we can get the following estimate in classical weighted Sobolev spaces

$$C_1 \left( \| (I-P)g \|^2_{H^{\gamma/2}_\rho} + \| (I-P)g \|^2_{H^{\gamma/2}_\rho} \right) \leq \left( Lg, g \right)_{L^2} \leq C_2 \| g \|^2_{H^{\gamma/2}_\rho}.$$

In the following, we will use the lower script $\rho$ on the non-isotropic norm, and so use the notation $\| g \|_{\Phi}$ if we need to specify its dependence on the kinetic factor $\Phi$. Notations $J^0_\rho, J^2_\rho$ will be also used for the same purpose.

Part of the proof on the lower bound of the non-isotropic norm given in Proposition 2.2 is essentially due to the following equivalence relations.

**Proposition 2.3.** Assume that the cross-section satisfies (1.2) with $0 < s < 1$ and $\gamma > -3$. Then we have

$$\| g \|_{\Phi} \sim \| g \|_{\Phi}.$$

Concerning the dependence on the index $\gamma$ in $\Phi = |v - v_1|^{\gamma}$, we have

**Proposition 2.4.** Assume that the cross-section satisfies (1.2) with $0 < s < 1$ and $\gamma > -3$. Then for any $\beta > -3$, we have

$$\| g \|_{\Phi_{\beta}} \sim \| (v - v_1)^{\beta/2} g \|_{\Phi}.$$

2.1. **Bounds on the non-isotropic norm.**

This section is devoted to the proof of Proposition 2.2. We will often use the following elementary estimate stated in velocity dimension $n$, since it will be needed for both cases $n = 2$ and $n = 3$.

**Lemma 2.5.** Let the velocity dimension be $n$, $n \in \mathbb{N}$, $\rho > 0, \delta \in \mathbb{R}$ and let $\mu_{\rho,\delta}(u) = \langle u \rangle^{\rho} e^{-|\delta||u||^2}$ for $u \in \mathbb{R}^n$. If $\alpha > -n$ and $\beta \in \mathbb{R}$, then we have

$$I_{\alpha,\beta}(u) = \int_{\mathbb{R}^n} |w|^\alpha \langle w \rangle^\beta \mu_{\rho,\delta}(w + u) dw \sim \langle u \rangle^{\alpha+\beta}.$$

**Proof.** Since we have

$$\langle u \rangle^\beta (u + w)^{-|\beta|} \leq \langle w \rangle^\beta \leq \langle u \rangle^\beta (u + w)^{|\beta|},$$

it suffices to show (2.3) with $\beta = 0$, by taking $\mu_{\rho,\delta|\beta\rangle}$ instead of $\mu_{\rho,\delta}$. Taking into account the fact that $\alpha > -n$, this estimate is obvious when $|u| \leq 1$. If $|u| \geq 1$, then we have

$$I_{0,0}(u) = 4^{-|\beta|} \langle u \rangle^\alpha \int_{|w| \leq 1/2} \mu_{\rho,\delta}(u + w) dw \geq \langle u \rangle^\alpha,$$

because $|u + w| \leq 1/2$ implies that $4^{-1}(u) \leq |w| \leq 4(u)$. Noticing that $2|w| \geq \langle w \rangle$ if $|w| \geq 1$, we have

$$I_{0,0}(u) \leq \max_{|w| \leq 1} \mu_{\rho,\delta}(u + w) \int_{|w| \leq 1} |w|^\alpha dw + 2^{|\beta|} \int_{|w| \geq 1} \langle w \rangle^\alpha \mu_{\rho,\delta}(u + w) dw$$

$$\leq (\langle u \rangle^\beta e^{-|\delta||u||^2/2} + \langle u \rangle^\alpha) \int_{\mathbb{R}^n} \langle u + w \rangle^\alpha \mu_{\rho,\delta}(u + w) dw \leq \langle u \rangle^\alpha.$$
And this completes the proof of the lemma.

Recall from (2.1) that the non-isotropic norm contains two parts, denoted by \( J_1 \) and \( J_2 \) respectively. The estimation on each part will be given in the following subsections. We start with the estimation on \( J_2 \) because the analysis is easier.

Let us start with the following upper bound on \( J_2 \).

**Lemma 2.6.** Under the same assumptions as in Theorem 1.1, we have

\[
J_2 := \iint b(\cos \theta) \Phi(|v - v_*|) g^2 \left( \sqrt{\mu'} - \sqrt{\mu} \right)^2 d\nu d\sigma \leq \|g\|_{L^{1/2}}^2.
\]

**Proof.** Note that

\[
J_2 \leq 2 \iint b(\cos \theta) \Phi(|v - v_*|) g^2 \left( \mu^{1/4} - \mu^{1/2} \right)^2 \left( \mu^{1/2} + \mu^{1/2} \right) d\nu d\sigma
\]

\[
\leq \iint b|v' - v_*| g^2 \left( \mu^{1/4} - \mu^{1/2} \right)^2 \mu^{1/2} d\nu d\sigma + \iint b|v - v_*| g^2 \left( \mu^{1/4} - \mu^{1/2} \right)^2 \mu^{1/2} d\nu d\sigma
\]

\[= F_1 + F_2.
\]

By the regular change of variables \( v \to v' \), we have

\[
F_1 \leq \iint |v' - v_*| \left( \int b(\cos \theta) \min \left(|v' - v_*|^2 \sigma, 1 \right) d\sigma \right) g^2 \mu^{1/2} d\nu d\nu_*
\]

\[\leq \int \left( \int |v' - v_*|^{3/2} \mu' d\nu' \right) g^2 \nu d\nu_* \leq \|g\|_{L^{1/2}}^2,
\]

where we have used Lemma 2.5 in the case \( n = 3 \) to get the last inequality. A direct estimation show that the same bound holds true for \( F_2 \). And this completes the proof of the lemma.

**Remark 2.7.** Note that the above lemma holds even if \( \Phi \) is replaced by \( \tilde{\Phi} \) by using Lemma 2.4.

We now turn to the lower bound for \( J_2 \).

**Lemma 2.8.** Under the assumptions (1.1) with \( \gamma > -3 \), there exists a constant \( C > 0 \) such that

\[
J_2 := \iint b(\cos \theta) \Phi(|v - v_*|) g^2 \left( \sqrt{\mu'} - \sqrt{\mu} \right)^2 d\nu d\sigma \geq C\|g\|_{L^{1/2}}^2.
\]

**Proof.** We will apply the argument used in [33]. By shifting to the \( \omega \)-representation,

\[
v' = v - ((v - v_*) \cdot \omega) \omega \quad v'_* = v + ((v - v_*) \cdot \omega) \omega \in \Sigma^2,
\]

in view of the change of variables \((v, v_*) \to (v, v_*)\), we get,

\[
J_2 = 4 \iint b(\cos \theta) \sin(\theta/2) \Phi(|v - v_*|) g^2 \left( \sqrt{\mu'} - \sqrt{\mu} \right)^2 d\nu d\omega,
\]

because \( d\sigma = 4 \sin(\theta/2) d\omega \). Then, we use the Carleman representation. The idea of this representation is to replace the set of variables \((v, v_*, \omega)\) by the set \((v, v', v'_*)\). Here, \( v, v' \in \mathbb{R}^3 \) and \( v'_* \in E_{vv'} \), where \( E_{vv'} \) is the hyperplane passing through \( v \) and orthogonal to \( v - v' \). By using the formula

\[
d\nu' d\nu = \frac{dv' dv'}{|v - v'|^2},
\]
cf. page 347 of [39], and by taking the change of variables
\[(v, v', v') \rightarrow (v, v + h, v + y),\]
with \(h \in \mathbb{R}^3\) and \(y \in E_h\), where \(E_h\) is the hyperplane orthogonal to \(h\) passing through the origin in \(\mathbb{R}^3\), we have
\[
J_2 \sim \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \int_{y \in E_h \cap \{h \geq |h|\}} \frac{|y|^{1 + 2s + y} g(v)^2}{|h|^{1 + 2s}} \, dh \, dy \, dv \times (\sqrt{\mu(v + y)} - \sqrt{\mu(v + y + h)})^2 \, dv \frac{dh \, dy}{|h|^2},
\]
because
\[
|h| = |v' - v| = |v'' - v| \tan \frac{\theta}{2} = |v| \tan \frac{\theta}{2}, \quad \theta \in [0, \pi/2],
\]
\[
b(\cos \theta) \sin(\theta/2) \Phi(|v - v'|) = \frac{|v_1 - v'|^{1 + 2s + y}}{|v - v'|^{1 + 2s}} 1_{|v - v'|^{1 + 2s}}.
\]

We decompose \(v = v_1 + v_2\), where \(v_2\) is the orthogonal projection of \(v\) on \(E_h\). Since \(\mu\) is invariant by rotation, we may assume \(v = (0, 0, |v|)\) without loss of generality. By introducing the polar coordinates
\[
h = (\rho \sin \theta \cos \phi, \rho \sin \theta \sin \phi, \rho \cos \theta), \quad \theta \in [0, \pi], \quad \phi \in [0, 2\pi], \quad \rho > 0,
\]
we obtain \(|v_1| = |v| \cos \theta|, |v_1 + h| = |v| \cos \theta + \rho|\) and \(|v_2| = |v| \sin \theta\). Note that if \(0 < \theta \leq \pi/2\), then
\[
(\sqrt{\mu(v + y)} - \sqrt{\mu(v + y + h)})^2 = \mu(v_2 + y) (\sqrt{\mu(v_1)} - \sqrt{\mu(v_1 + h)})^2 \\
\geq \mu(v_2 + y) \mu(v_1) (1 - e^{-\rho^2/4})^2 / (2\pi)^{3/2}.
\]
Therefore, we have for any \(\delta > 0\)
\[
J_2 \geq C \int_{\mathbb{R}^2} g(v)^2 \int_{\mathbb{R}^2} \frac{(\sqrt{\mu(v_1)} - \sqrt{\mu(v_1 + h)})^2}{|h|^{1 + 2s}} \, dh \, dv \\
\times \left( \int_{y \in E_h \cap \{h \geq |h|\}} |y|^{1+2s+\gamma} \mu(v_2 + y) \, dy \right) \, dv \\
\geq C \int_{\mathbb{R}^2} g(v)^2 \int_{\mathbb{R}^2} \frac{\mu(v_1) \left( \int_0^\delta (1 - e^{-\rho^2/4})^2 / \rho^{1+2s} \right.}{\rho^{1+2s}} \\
\times \left( \int_{y \in E_h} |y|^{1+2s+\gamma} \mu(v_2 + y) \, dy \right) \, dv \\
- \int_{y \in E_h \cap \{h \leq |h|\}} |y|^{1+2s+\gamma} \mu(v_2 + y) \, dy \right) \, dv \, \sin \theta \, d\theta \right) \, dv.
\]
Since we have
\[
\int_{y \in E_h \cap \{h \leq |h|\}} |y|^{1+2s+\gamma} \mu(v_2 + y) \, dy \leq \delta^{2s} \int_{y \in E_h} |y|^{1+\gamma} \mu(v_2 + y) \, dy, \quad \text{if} \ \rho \leq \delta,
\]
and it follows from Lemma 2.5 in the case \(n = 2\), that
\[
\int_{y \in E_h} |y|^\beta \mu(v_2 + y) \, dy \sim (v_2)^\beta \quad \text{if} \ \beta > -2,
\]
there exist two constants $C_1, C_2 > 0$ such that if $\rho \leq \delta$, we have
\[
\int_{y \in E_0} |y|^{1+2s+\gamma}\mu(v_2 + y)dy - \int_{y \in E_0 \cap |y| \leq \rho} |y|^{1+2s+\gamma}\mu(v_2 + y)dy \\
\geq C_1(v_2)^{1+2s+\gamma} - C_2\delta^{2s}(v_2)^{1+\gamma}.
\]
Taking a sufficiently small $\delta > 0$ gives
\[
J_2 \geq C \int_{\mathbb{R}^3} g(v)^2J^{\rho} \left\{ \int_{|x|/2-1/(\nu)}^{|x|/2+1/(\nu)} \mu(v_1) \right. \\
\times \left. \left\{ \int_0^\rho \frac{1 - e^{-\rho'/4\theta}}{\rho'^{2s+2}}d\rho' \right\} (v_2)^{1+2s+\gamma} \sin \theta d\theta \right\} dv \\
\geq C_3 \int_{\mathbb{R}^3} (v)^{2s+\gamma}g(v)^2 \left\{ \int_{|x|/2-1/(\nu)}^{|x|/2+1/(\nu)} e^{-|y|^2\cos^2\theta} (v) d\theta \right\} dv \\
\geq C_3 \|g\|_{L^{2s+\gamma/2}}^2.
\]

The proof of the lemma is now completed. \(\square\)

**Remark 2.9.** In the above proof, the factor $|y|^\gamma$ can be replaced by $(\langle \gamma \rangle)^\gamma$, so that Lemma 2.8 is valid even if $\Phi$ is replaced by $\Phi = \langle v - v_*, \rangle^\gamma$. By the above lemma together with Lemma 2.4 and the Remark after it, we can conclude

\[
J_2^\rho \sim \|g\|_{L^{2s+\gamma/2}}^2 \sim J_2^\rho.
\]

(2.4)

We now turn to the estimation of the first term of the non-isotropic norm, that is, $J_1$. We will firstly show that the singular behavior of the cross-section when $v = v_*$ can be smoothed out. This point is given by the following proposition.

**Proposition 2.10.** Under the same assumption as in Theorem 1, we have
\[
J_1^\rho + \|g\|_{L^{2s+\gamma/2}}^2 \sim J_1^\rho + \|g\|_{L^{2s+\gamma/2}}^2.
\]

**Remark 2.11.** This proposition is nothing but Proposition 2.8 by Remark 2.9.

**Proof.** By using similar arguments as in the proof of Lemma 2.8, it follows from the Carleman representation that
\[
J_1^\rho = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{y \in E_0 \cap |y| \leq \rho} \frac{|y|^{1+2s+\gamma}\mu(v)(g(v + y) - g(v + y + \theta))}{|h|^{1+2s}} dv dh dy \\
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{y \in E_0 \cap |y| \leq \rho} \frac{|y|^{1+2s+\gamma}\mu(v)(g(v + y) - g(v + \theta))}{|h|^{1+2s}} dv dh dy,
\]
where the last equality is a direct consequence of the change of variables $(v + y, y) \to (v, -y)$. Similarly, we have
\[
J_1^\rho = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{y \in E_0 \cap |y| \leq \rho} \frac{|y|^{1+2s}(v)^\gamma\mu(v + y)(g(v + y) - g(v))}{|h|^{1+2s}} dv dh dy \\
\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \int_{y \in E_0 \cap |y| \leq \rho} \frac{|y|^{1+2s}(v)^\gamma\mu(v + y)(g(v + h) - g(v))}{|h|^{1+2s}} dv dh dy.
\]
We claim that

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^4} \int_{V \in E_h \cap \{|h| \leq |h|\}} \frac{|v|^{1+2\gamma}}{|h|^{1+s}} \mu(v + y)(g(v + h) - g(v))^2 dh \, dy \\
\leq \|g\|_{L^2}^2 \frac{|h|^{1+s}}{|h|^{2}}.
\]

We decompose \( v = v_1 + v_2 \), where \( v_2 \) is the orthogonal projection of \( v \) on \( E_h \). Then we have \( \mu(v + y) = \mu_v(v_2 + y) \), whence it follows from Lemma 2.5 together with \( 1 + 2s + \gamma > -2 \) that

\[
G(v, h) = \mu(v_1)(v_2)^{1+2s} \sim G(v, h).
\]

It remains to show (2.5) and (2.6). We write

\[
\int_{\mathbb{R}^3} \int_{\mathbb{R}^4} \int_{V \in E_h \cap \{|h| \leq |h|\}} \frac{|v|^{1+2\gamma}}{|h|^{1+s}} \mu(v + y)(g(v + h) - g(v))^2 dh \, dy \\
= \int_{\mathbb{R}^3} \int_{\mathbb{R}^4} \int_{V \in E_h \cap \{|h| \leq |h|\}} + \int_{\mathbb{R}^3} \int_{\mathbb{R}^4} \int_{V \in E_h \cap \{|h| \leq |h|\}} = A_1 + A_2.
\]

Take a small \( \delta > 0 \) such that \( \gamma - \delta > -2 \). Then, in view of \( 1 + \gamma - \delta > -2 \), we have

\[
A_1 \leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^4} \int_{V \in E_h \cap \{|h| \leq |h|\}} \frac{|v|^{1+\gamma-\delta}}{|h|^{1-\delta}} \mu(v + y)(g(v + h) - g(v))^2 dh \, dy \\
= \int_{\mathbb{R}^3} \mu(v_1) \int_{\mathbb{R}^3} \left( \int_{V \in E_h \cap \{|h| \leq |h|\}} |v_1|^{1+\gamma-\delta} \mu(v_2 + y) dy \right) (g(v + h) - g(v))^2 \frac{dh}{|h|^{1-\delta}} dv \\
\leq \int_{\mathbb{R}^3} \mu(v_1)(v_2)^{1+\gamma-\delta} \int_{\mathbb{R}^3} \left( g(v + h) - g(v)^2 \right) \frac{dh}{|h|^{1-\delta}} dv \\
\leq \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left( \mu(v_1 - h) + \mu(v_1) \right)(v_2)^{1+\gamma-\delta} (g(v)^2) \frac{dh}{|h|^{1-\delta}} dv
\]

where we have used the change of variables \( v + h \to v \) for the factor \( g(v + h) \). As in the proof of Lemma 2.8, by assuming \( v = (0, 0, |v|) \), we introduce the polar coordinates

\[
h = (\rho \sin \theta \cos \phi, \rho \sin \theta \sin \phi, \rho \cos \theta), \quad \theta \in [0, \pi], \phi \in [0, 2\pi], \rho > 0.
\]

Since \( |v_1| = |v| \cos \theta, \ |v_1 - h| = ||v| \cos \theta - \rho| \) and \( |v_2| = |v| \sin \theta \), by using the change of variable \( |v| \cos \theta = r \), we obtain

\[
A_1 \leq \int_{\mathbb{R}^3} |g(v)|^2 \int_0^1 \frac{1}{\rho^{1-\delta}} \\
\times \left( \int_{|v|} (1 + |v| - \rho)^{1+\gamma-\delta/2} |v|^{-1/2} e^{-r^2/2} + e^{-r^2/2} \right) dr \, dv.
\]
Similarly, if $1 + 2s - \delta > 1$, then we have
\[
A_2 \leq \int_{\mathbb{R}^2} \left| \int_{|v| \geq 1} \frac{|v|^{1+\gamma+2s-\delta}}{|h|^{1+2s-\delta}} \mu(v + y)(g(v + h) - g(v))^2 \, dv \right| \frac{dhdy}{|h|^2} 
\leq \int_{\mathbb{R}^2} |g(v)|^2 \int_1^\infty \frac{1}{\rho^{1+2s-\delta}} \left( \int_{|v|} (1 + |v|^2 - r^2)^{1+\gamma+2s-\delta}/2 \left( e^{-|r|^2/2} + e^{-r^2/2} \right) dr \right) dp dv.
\]
If $1 + \gamma + 2s - \delta \geq 0$, then
\[
K(v, \rho) = \int_{|v|} (1 + |v|^2 - r^2)^{1+\gamma+2s-\delta}/2 \left( e^{-|r|^2/2} + e^{-r^2/2} \right) dr 
\leq \langle v^{1+\gamma+2s-\delta}/2 \rangle \left( e^{-|r|^2/2} + e^{-r^2/2} \right) dr \leq \langle v^{1+\gamma+2s} \rangle,
\]
which shows
\[
(2.7) \quad A_2 \leq \int_{\mathbb{R}^2} |g(v)|^2 \int_1^\infty K(v, \rho) \frac{1}{\rho^{1+2s-\delta}} \, dp dv \leq \int \langle v^{1+\gamma+2s} \rangle |g(v)|^2 \, dv.
\]
On the other hand, if $1 + \gamma + 2s - \delta < 0$ and $|v| \geq 3$, then
\[
K(v, \rho) \leq \int_0^{|v|} (1 + |v|^2 - r^2)^{1+\gamma+2s-\delta}/2 \left( e^{-|r|^2/2} + 3e^{-r^2/2} \right) dr 
\leq \langle v^{1+\gamma+2s-\delta}/2 \rangle \left( e^{-|r|^2/2} + 3e^{-r^2/2} \right) dr 
\leq \langle v^{1+\gamma+2s} \rangle + \langle v^{1+\gamma+2s-\delta}/2 \rangle \int_{|v|/2}^{|v|/2} (|v| - r)^{1+\gamma+2s-\delta}/2 3e^{-r^2/2} dr,
\]
because
\[
\int_0^{|v|} (|v| - r)^{1+\gamma+2s-\delta}/2 e^{-r^2/2} dr \leq \langle v^{1+\gamma+2s-\delta}/2 \rangle \int_0^{|v|} e^{-r^2/2} dr 
+ e^{-|v|^2/8} \int_{|v|/2}^{|v|/2} (|v| - r)^{1+\gamma+2s-\delta}/2 e^{-r^2/2} dr,
\]
where we have used that $(1 + \gamma + 2s - \delta)/2 > -1$ for small $\delta > 0$ that follows from the assumption $\gamma > -3$. We now consider
\[
\int_1^\infty \frac{dp}{\rho^{1+2s-\delta}} \int_{|v|/2}^{|v|/2} (|v| - r)^{1+\gamma+2s-\delta}/2 e^{-r^2/2} dr \leq \langle v^{1+\gamma+2s-\delta}/2 \rangle \left( \int_{|v|/2}^{|v|/2} (|v|/3)^{1+\gamma+2s-\delta} dp \right) dr 
+ \int_{|v|/2}^{|v|/2} (|v| - r)^{1+\gamma+2s-\delta}/2 \int_{|v|/2}^{|v|/2} (|v|/3)^{1+\gamma+2s-\delta} dp \right) dr 
\leq \langle v^{1+\gamma+2s-\delta}/2 \rangle \left( \frac{1}{2 \log |v|} + |v|^{1+\gamma+2s-\delta}/2 \rangle \right) \leq \langle v^{1+\gamma+2s} \rangle.
\]
Therefore, in the case when $1 + \gamma + 2s - \delta < 0$, we also have (2.7). Similarly, we have
A_1 \leq \int_{\mathbb{R}^2} |g(v)|^2 \int_0^1 \frac{K(v, \rho)}{\rho^{1-\sigma}} d\rho dv \leq \int \langle v \rangle^{\gamma+2} |g(v)|^2 dv,
which shows (2.5). The proof of (2.6) is similar, and thus the proof of the proposition is completed.

Lemma 2.12. There exist constants $C_1, C_2 > 0$ such that
\begin{equation}
J_1 \geq C_1 \|v(v)^{\gamma/2} g\|_{L^p}^2 - C_2 \|g\|_{L^{2+\gamma}}^2.
\end{equation}
The same conclusion holds even with $\mu$ replaced by $\mu^0$ for any fixed $p > 0$.

Proof. It follows from Proposition 2.10 that
\begin{equation}
C \left( J_1^0 + \|g\|^2_{L^{2+\gamma}} \right) \geq 2 J_1^0
\end{equation}
because $\tilde{\Phi}(|v - v'|) - \langle v' - v \rangle \geq \langle v \rangle^2$ and $2(a + b^2) \geq a^2 - 2b^2$. Setting $\tilde{g} = \langle v \rangle^{\gamma/2} g$ and noting $C_\gamma \mu(v)^{1/|v|} \geq \mu(2v)$ for a $C_\gamma > 0$, as in Proposition 1 of [3], we have
\begin{align*}
C_\gamma A_1 & \geq \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} b_\gamma(|v - v'|) \mu(2v) \tilde{g}(v) - \tilde{g}(v') \rangle^2 d\nu d\nu' \\
& = (4\pi)^{-3} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} b_\gamma \langle \mu(0) \tilde{g}(\xi)^2 + \tilde{\mu}(0) |\tilde{g}(\xi)|^2 \rangle d\xi d\xi' \\
& \quad - 2 \Re \langle \tilde{\mu}(\xi') / 2 \tilde{g}(\xi') \tilde{g}(\xi) \rangle d\xi d\xi'.
\end{align*}
Since we have $\tilde{\mu}(0) - \langle \tilde{\mu}(\xi' / 2) \rangle = c(1 - e^{-|\xi'|/\theta}) \geq c|\xi'|$ if $|\xi'| \leq 1$, in view of $|\xi'|^2 = |\xi|^2 \sin^2 \theta/2 \geq |\xi|^2 (\theta/\pi)^2$, we obtain for $|\xi| \geq 1$
\begin{align*}
\int_{\mathbb{R}^2} b_\gamma \langle \mu(0) - \langle \tilde{\mu}(\xi') \rangle \rangle d\xi & \geq \int_{\mathbb{R}^2} \sin \theta b_\gamma(\mu(0) |\tilde{g}(\xi)|^2 (\theta/\pi)^2) d\theta \\
& \geq c'' K |\xi|^2 \int_0^{1/\theta} \theta^{1-2s} d\theta \\
& = c'' K |\xi|^2 \frac{\theta^{2s-2}}{2s}.
\end{align*}
Therefore, we have
\begin{equation}
A_1 \geq C_1 \int_{|\xi| \geq 1} |\xi|^{2s} |\tilde{g}(\xi)|^2 d\xi \geq C_1 2^{-2s} \int_{|\xi| \geq 1} (1 + |\xi|^2)^s |\tilde{g}(\xi)|^2 d\xi
\end{equation}
\begin{align*}
& \geq C_1 2^{-2s} \|v(v)^{\gamma/2} g\|^2_{L^{2+\gamma}} - C_2 \|g\|^2_{L^{2+\gamma}}.
\end{align*}
As for $A_2$, we note that if $\gamma > 0$, then
\begin{align*}
\langle v \rangle & \leq \langle v - v \rangle + \langle v \rangle \leq \sqrt{2} (\gamma - v) + \langle v \rangle \leq (1 + \sqrt{2}) \langle v \rangle.
\end{align*}
and \( \langle v_2 \rangle \leq (1 + \sqrt{2})\langle v_2 \rangle \), which show \( \langle v_2 \rangle^\beta \leq C_\beta \langle v_2 \rangle^\beta \) for any \( \beta \in \mathbb{R} \). It follows that

\[
\left| \langle v \rangle^{\gamma/2} - \langle v' \rangle^{\gamma/2} \right| \leq C_\gamma \int_0^1 \langle v' + \tau(v - v') \rangle^{(\gamma/2) - 1} d\tau |v - v_\theta| \theta
\]

\[
\leq C_\gamma \left( \langle (v^{(\gamma/2-1)}(v_\theta^{(\gamma/2-1)}) \rangle \langle v - v_\theta \rangle \theta,
\right.
\]

and thus we have

\[
A_2 \leq C \int \int_0 \frac{\mu_\gamma}{\langle v \rangle^{\gamma/2}} |g|^2 \left( \langle (v^{(\gamma/2-1)}(v_\theta^{(\gamma/2-1)}) \rangle \langle v - v_\theta \rangle \theta \right)
\]

\[
+ \int_0^\theta \left( \langle v \rangle^{\gamma} + \langle v \rangle^{\gamma} \right) \theta^{1 - 2t} |\langle v - v_\theta \rangle \theta| d\theta dv_\theta.
\]

\[
\leq C \int \int_0 \left( \langle v \rangle^{2m+\gamma} \langle v_\theta \rangle^{2m+\gamma} \right) |g|^2 dv_\theta.
\]

which together with (2.10) yields the desired estimate (2.8). The last estimate of the lemma is obvious by replacing \( \mu \) by \( \mu^\theta \) in each step of the above arguments, so that the proof of the lemma is completed.

Lemma 2.8 together with Lemma 2.12 implies that we have the following lower bound on the non-isotropic norm,

\[
\|g\|_2^2 \geq \|g\|_{L_{\gamma/2}^2}^2 + \|g\|_{L_{\gamma/2}^2}^2.
\]

Therefore, to complete the proof of Proposition 2.2, it remains to show

**Lemma 2.13.** Let \( \gamma > -3 \). Then we have

\[
J_1 \leq \|g\|_{L_{\gamma/2}^2}^2 + \|g\|_{L_{\gamma/2}^2}^2.
\]

The same conclusion holds even if \( \mu \) in \( J_1 \) is replaced by \( \mu^\rho \) for any fixed \( \rho > 0 \).

**Proof.** As for Lemma 2.12, it follows from Proposition 2.10 that, for suitable constants \( C_1, C_2 > 0 \), we have

\[
C_1 J_1^\rho - C_2 \|g\|_{L_{\gamma/2}^2}^2 \leq J_1^\rho
\]

(2.11)

\[
\leq 2 \int \int \int b(\cos \theta) \mu_\gamma(v_\theta) |(\langle v \rangle^{\gamma/2} g' - (\langle v \rangle^{\gamma/2} g) | g|^2 dv_\theta dv_\sigma.
\]

\[
+ 2 \int \int \int b(\cos \theta) \mu_\gamma(v_\theta) |(\langle v \rangle^{\gamma/2} - (\langle v \rangle^{\gamma/2}) | g|^2 dv_\theta dv_\sigma,
\]

\[
= B_1 + B_2,
\]

because \( \Phi(|v - v_\theta|) \sim \langle v - v_\theta \rangle^{\gamma} \) and \( (a + b)^2 \leq 2(a^2 + b^2) \).

By the same argument for \( A_2 \) in the proof of Lemma 2.13, we get \( B_2 \leq \|g\|_{L_{\gamma/2}^2}^2 \).

To estimate \( B_1 \), we apply Theorem 2.1 of [R] about the upper bound on the collision operator in the Maxwellian molecule case. It follows from (2.1.9) of [R] with \( (m, \alpha) = (-\delta, -\delta) \) that

\[
\left( \Theta_{\beta}(F, G), G \right) \leq \|F\|_{L_{\gamma/2}^2} \|G\|_{H_\gamma}^2.
\]
Since $2a(b-a) = -(b-a)^2 + (a^2 - b^2)$, we get
\[
\left( Q^b(F, G), G \right) = \iint b F_s (G' - G) \geq -\frac{1}{2} \iint b F_s (G' - G)^2 + \frac{1}{2} \iint F_s (G'^2 - G^2),
\]
and therefore
\[
\left| \iint b F_s (G' - G)^2 \right| \leq 2 \left( Q^b(F, G), G \right) + \left| \iint F_s (G'^2 - G^2) \right| \leq ||F||_{L^1} ||G||_{L^1} + ||F||_{L^1} ||G||_{L^2},
\]
where we have used the cancellation lemma from [3] for the second term. Choosing $F = \mu \langle \cdot \rangle$ and $G = \langle \cdot \rangle^{1/2} g$, it follows that $B_1 \leq ||g||_{L_{s, y/2}}^2$, completing the proof of the lemma.

2.2. Equivalence to the linearized operator. We will now show that the Dirichlet form of the linearized collision operator is equivalent to the square of the non-isotropic norm, and therefore, the proof of Proposition 2.1 will be given. Let us note that for the bilinear operator $\Gamma(\cdot, \cdot)$, for suitable functions $f, g$, one has
\[
\left( \Gamma(f, g), h \right)_{L^2} = \iint b(\cos \theta) \Phi(|v - v|) \sqrt{\mu_s} (f^g v' - f g) h
\]
and by adding these two lines, it follows that
\[
(2.12) \quad \left( \Gamma(f, g), h \right)_{L^2} = \frac{1}{2} \iint b(\cos \theta) \Phi(|v - v|) (f^g v' - f g) (\sqrt{\mu_s} h - \sqrt{\mu_s} h').
\]

The following lemma shows that $L_1$ dominates $L$.

**Lemma 2.14.** Under the conditions [1.2] on the cross-section with $0 < s < 1$ and $\gamma \in \mathbb{R}$, we have
\[
(2.13) \quad \left( L_1 g, g \right)_{L^2} \geq \frac{1}{2} \left( L g, g \right)_{L^2}.
\]

**Proof.** By standard changes of variables, the following computations hold true
and
\[
(L_2 g, g)_{L^2} = -\left(\Gamma(\sqrt{\mu}, g) + \Gamma(g, \sqrt{\mu})\right)_{L^2(\mathbb{R}^2)}
\]
\[
= \int \int \int B(\mu^{1/2} g - (\mu^{1/2} g' + g(\mu)^{1/2} - g'(\mu)^{1/2})(\mu)^{1/2} g)
\]
\[
= \int \int \int B(\mu^{1/2} g - (\mu^{1/2} g' + g(\mu^{1/2} - g'(\mu)^{1/2})(\mu)^{1/2} g')
\]
\[
= \int \int \int B(\mu^{1/2} g - (\mu^{1/2} g' + g(\mu^{1/2} - g'(\mu)^{1/2})(\mu)^{1/2} g')
\]
\[
= \frac{1}{4} \int \int \int B\left((\mu^{1/2} g - (\mu^{1/2} g' + (\mu^{1/2} g - (\mu^{1/2} g'))^2
\]
Therefore, (2.13) follows from \((\alpha + \beta)^2 \leq 2(\alpha^2 + \beta^2)\) and the proof is completed. \(\square\)

Now for the term \(L_2\), we have

**Lemma 2.15.** One has
\[
\left| (L_2 g, h)_{L^2} \right| \leq \|\mu^{1/10} g\|_{L^2} \|\mu^{1/10} h\|_{L^2}.
\]

**Proof.** It follows from (2.12) that
\[
(L_2 g, h)_{L^2} = -\frac{1}{4} \int \int \int B(g, \sqrt{\mu' - g - \sqrt{\mu', \mu'}}) \left(\sqrt{\mu}, h - \sqrt{\mu}, h'\right)
\]
that is, \(L_2\) is symmetric. Hence it suffices to show the lemma in the case when \(g = h\).
Putting \(G = \sqrt{\mu g}\), we have
\[
-L_2 g = \mu^{-1/2} Q(G, \mu)
\]
\[
= \mu^{-1/2} \int b(\cos \theta) \Phi(|v - \mu|) G_{\mu'}^*(\mu' - \mu) dv_c d\sigma
\]
\[
+ \sqrt{\mu} \int b(\cos \theta) \Phi(|v - \mu|) (G_{\mu'}^* - G_{\mu}) dv_c d\sigma
\]
\[
= I_2(v) + I_2(v).
\]

Thanks to the cancellation lemma, we have \(I_2(v) = \sqrt{\mu(v)}(S \ast G)(v)\) with \(S(v) \sim |v|^\gamma\), whence we have
\[
(I_2, g)_{L^2} \leq \int \int |v - v|^{\gamma} \sqrt{\mu} \sqrt{\mu} \|g\|_g, dv_v dv_c
\]
\[
\leq \int \int |v - v|^{\gamma} (\sqrt{\mu^{1/4}} g^2 \mu^{1/4} g^2 + (\mu^{1/4} g^{1/4})^2) dv_v dv_c
\]
\[
\leq \|v\|_{\mu^{1/4} g}^2 \leq \|\mu^{1/8} g\|_{L^2}^2,
\]
by means of Lemma 2.3.

Writing
\[
\mu' - \mu = \sqrt{\mu'} - \sqrt{\mu} + \sqrt{\mu}(\sqrt{\mu'} - \sqrt{\mu})
\]
and using $\sqrt{\mu'\mu''} = \sqrt{\mu''}$, we have

$$I_1(v) = \int \int b(\cos \theta) \Phi((v-v_\ast))g'(\sqrt{\mu'} + \sqrt{\mu''}) (\sqrt{\mu'} - \sqrt{\mu''}) dv. d\sigma. $$

Hence

$$ (I_1, g)_{L^2} = \int \int b(\cos \theta) \Phi((v-v_\ast))g'(\sqrt{\mu'} - \sqrt{\mu''}) (\sqrt{\mu'} - \sqrt{\mu''}) g' dv. d\sigma$$

$$ + 2 \int \int b(\cos \theta) \Phi((v-v_\ast))G_\ast(\sqrt{\mu'} - \sqrt{\mu''}) g' dv. d\sigma$$

$$ = A_1 + A_2,$$

where we have used the change of variables $(v, v_\ast) \to (v', v_\ast')$ for the second term. We can write

$$ A_1 = \int \int b(\cos \theta) \Phi((v-v_\ast))g(\sqrt{\mu'} + \mu^{1/4} - \mu^{1/4}) g'$$

$$ \times (\mu^{1/4} + \mu^{1/4} + \mu^{1/4}) dv. d\sigma.$$ 

Since we have

$$ |v'|^2 \leq (|v' - v'| + |v'|)^2 \leq (\sqrt{2}|v_\ast - v'| + |v'|)^2$$

$$ \leq (\sqrt{2}|v_\ast| + (\sqrt{2} + 1)|v' - v'|)^2 \leq 4|v_\ast|^2 + 2(\sqrt{2} + 1)^2|v' - v'|^2,$$

and in the same way, $|v|^2 \leq 4|v' - v'|^2 + 2(\sqrt{2} + 1)^2|v_\ast|^2$, we get, by adding the two corresponding inequalities, that $|\mu, \mu' - (\mu, \mu')^{1/(10 + 4\sqrt{2})}$. Moreover, we have $\mu' - \mu, \mu \leq (\mu', \mu)^{1/15}$ because $|v'|^2 \leq (|v' - v| + |v|)^2 \leq (|v_\ast - v| + |v|)^2 \leq 2|v|^2 + 8|v|^2$. Noticing that

$$ \left| (\mu^{1/4} - \mu^{1/4}) (\mu^{1/4} - \mu^{1/4}) \right| \leq |v - v'|^2 \theta^2,$$

we get

$$ (2.15) \quad |A_1| \leq \int \int |v - v'|^{\gamma + 2} \int \int \theta^{1/2} g_{\ast} dv. d\sigma$$

$$ \leq \int \int |v - v'|^{\gamma + 1} (\mu, \mu')^{1/(10 + 4\sqrt{2})} g_{\ast} dv. d\sigma \leq \int \int |v - v'|^{\gamma} (\mu, \mu')^{1/(10 + 4\sqrt{2})} g_{\ast} dv. d\sigma,$$

by an argument similar to the analysis of $I_1$.

For $A_2$, we use the regular change of variable $v \to v'$, and denote its inverse transformation by $v' \to \psi_{\gamma}(v') = v$. Then

$$ A_2 = 2 \int \int \sqrt{\mu} g_{\ast} \left\{ \int \int b(\psi_{\gamma}(v') - v_\ast) \cdot \sigma \Phi(\psi_{\gamma}(v') - v_\ast)$$

$$ \times \left( \sqrt{\mu(\psi_{\gamma}(v'))} - \sqrt{\mu(v')} \right) \frac{\partial \psi_{\gamma}(v')}{\partial (v')} d\sigma \right\} g(v') dv. d\sigma'$$

It follows from the Taylor expansion that

$$ \sqrt{\mu(\psi_{\gamma}(v'))} - \sqrt{\mu(v')} = \left( \nabla \sqrt{\mu}(v') \cdot (\psi_{\gamma}(v') - v') \right)$$

$$ + \int_0^1 (1 - \tau) \left( \nabla^2 \sqrt{\mu}(v' + \tau(\psi_{\gamma}(v') - v')) \right) \left( \psi_{\gamma}(v') - v' \right)^2 d\tau.$
Proposition 2.16. Let γ > −3. There exists a constant C > 0 such that
\[ \|g\|^2 \geq \left( L_1 g, g \right)_{L^2} \geq \frac{1}{10}\|g\|^2 - C\|g\|^2_{L^2}. \]

Proof. The equalities
\[ 2\left( L_1 g, g \right)_{L^2} = -2\left( \Gamma(\sqrt{\mu} \cdot g, g) \right)_{L^2} \]
\[ = \iint B \left( \mu_\gamma^{1/2} g' - \mu_\gamma \right) dv \mathrm{d}v \mathrm{d}\sigma \]
\[ = \iint B \left( \mu_\gamma^{1/2} g' - g + g(\mu_\gamma^{1/2} - \mu_\gamma) \right) dv \mathrm{d}v \mathrm{d}\sigma, \]

together with the inequality
\[ 2(a^2 + b^2) \geq (a + b)^2 \geq \frac{1}{2}a^2 - b^2 \]
yields
\[ \|g\|^2 \geq \left( L_1 g, g \right)_{L^2} \geq \frac{1}{4}J_1 - \frac{1}{2}J_2 \geq \frac{1}{4}\|g\|^2 - \frac{3}{4}J_2. \]

It follows from the equality \((a + b)^2 = a^2 + b^2 + 2ab\) that
\[ 2\left( L_1 g, g \right)_{L^2} \geq J_2 - C\|g\|^2_{L^2}, \]
which yields the desired estimate \((2.16)\) together with \((2.17)\).

Indeed, note that
\[ 2\left( L_1 g, g \right)_{L^2} \]
\[ = \iint B \left( \mu_\gamma^{1/2} g' - g + g(\mu_\gamma^{1/2} - \mu_\gamma) \right) dv \mathrm{d}v \mathrm{d}\sigma \]
\[ = J_1 + J_2 + 2\iint B (g' - g) \mu_\gamma^{1/2} (\mu_\gamma^{1/2} - \mu_\gamma) dv \mathrm{d}v \mathrm{d}\sigma. \]
Using the identity $2(\beta - \alpha)\alpha = \beta^2 - \alpha^2 - (\beta - \alpha)^2$, we have
\[
2(g' - g)(\mu'_L)^{1/2}(\mu'_L)^{1/2} - (\mu_L)^{1/2})
= \frac{1}{2}(g'^2 - g^2 - (g' - g)^2)(\mu'_L - \mu_L + (\mu'_L)^{1/2} - (\mu_L)^{1/2})^2
+ \frac{1}{2}g'^2(\mu'_L - \mu_L) + \frac{1}{2}(g'^2 - g^2)(\mu'_L)^{1/2} - (\mu_L)^{1/2})^2
= I_1 + I_2 + I_3 + I_4.
\]

Using the change of variables $(v', v') \rightarrow (v, \nu)$, we see that
\[
\left| \iiint B I_1 dv dv d\sigma \right| = \left| \iiint B \mu_\ast(g^2 - g'^2)dv dv d\sigma \right| \leq C||g||_{L^2}^2,
\]
by means of the cancellation lemma. Furthermore,
\[
\iiint B I_1 dv dv d\sigma = -\frac{1}{2} \iiint B(\mu_\ast + \mu'_L)(g' - g)^2 dv dv d\sigma
+ \iiint B(\mu_\ast)(g' - g)^2 dv dv d\sigma \geq -I_1,
\]
where we have used the change of variables $(v', v') \rightarrow (v, \nu)$. Thus, we obtain (2.18) because the integrals corresponding to the last two terms $I_1$ and $I_4$ vanish, ending the proof of the proposition.

**End of the proof of Proposition 2.1:** It follows from (2.16) and (2.13) that
\[
||g||_{L^2}^2 \geq \left( Lg, g \right)_{L^2} \geq \frac{1}{2} \left( Lg, g \right)_{L^2}.
\]

On the other hand, note that $\left( Lg, g \right)_{L^2} = \left( L(I - P)g, (I - P)g \right)_{L^2}$, from the very definition of the projection operator $P$.

Thus, from Proposition 2.16 and Lemma 2.15, we get
\[
\left( Lg, g \right)_{L^2} = \left( L_1(I - P)g, (I - P)g \right)_{L^2} + \left( L_2(I - P)g, (I - P)g \right)_{L^2}
\geq \frac{1}{10}|||(I - P)g|||^2 - C||(I - P)g||_{L^2}^2.
\]

Since it is known from [31] that we have
\[
\left( Lg, g \right)_{L^2} \geq C||(I - P)g||_{L^2}^2,
\]
we get on the whole
\[
|||(I - P)g|||^2 \leq C\left( Lg, g \right)_{L^2}.
\]

2.3. **Non-isotropic norms with different kinetic factors.** This subsection is devoted to the proof of Proposition 2.4. That is, we will show some equivalence relations between the non-isotropic norms with different kinetic factors and different weights.

For the proof, we introduce some further notations. Let $\rho > 0$, $\mu_\ast(v) = \mu(v)^\rho$, and set
\[
J_{I_1}^\rho(g) = \iiint \Phi_1(|v - v_c|)b(\cos \theta)\mu_\ast(v) (g' - g)^2 dv dv d\sigma.
\]

We simply write $J_{I_1}^\rho(g)$ if $\rho = 1$, and also introduce the notation $J_{I_2}^\rho(g)$ similarly with $\mu$ replaced by $\mu_\ast$. 

Then it follows from (2.4) and the change of variables $v \to v/\sqrt{\rho}$ that

$$J_{x,p}^{\phi}(g) \sim \|g\|_{L^{2 \theta(v)g}}^2 = \|(v)^{\gamma^2/2}g\|_{L^{2 \theta(v)g}}^2 \sim J_{x,p}^{\phi}(v)^{\gamma^2/2}g).$$

By the last assertions of Lemmas 2.12 and 2.13, there exist constants $C_1, C_2 > 0$ such that

$$C_1\|g\|_{H^{2 \theta(v)g}}^2 \leq J_{x,p}^{\phi}(g) + \|g\|_{L^{2 \theta(v)g}}^2 \leq C_2\|g\|_{H^{2 \theta(v)g}}^2.$$  

Furthermore, it follows from (2.9), (2.11) and the proofs of Lemmas 2.12 and 2.13 that

$$J_{1,2}^{\phi}(v)^{\gamma^2/2}g \leq J_{1,2}^{\phi}(g) \leq J_{1,2}^{\phi}(v)^{\gamma^2/2}g, \quad \text{modulo} \quad \|g\|_{L^{2 \theta(v)g}}^2,$$

because we have $C_1\mu_2 \leq \mu(v)^{\gamma^2/2} \leq C_2\mu_1$.

Therefore, to complete the proof of Proposition 2.3, it suffices to show that for any $\rho, \rho' > 0$

$$J_{x,p}^{\phi}(g) \sim J_{x,p}^{\phi}(g), \quad \text{modulo} \quad \|g\|_{L^{2 \theta(v)g}}^2.$$  

In fact, note that

$$J_{x,p}^{\phi}(g) \sim J_{x,p}^{\phi}(v)^{\gamma^2/2}g) \sim J_{x,p}^{\phi}(v)^{\gamma^2/2}g), \quad \text{modulo} \quad \|g\|_{L^{2 \theta(v)g}}^2.$$  

This equivalence looks quite obvious, however, for completeness, we shall give a proof. In fact, (2.20) is a direct consequence of the following lemma, by taking $f = \mu_v$.

**Lemma 2.17.** Assume that (1.3) holds with $0 < s < 1$. Then there exists a constant $C > 0$ such that

$$\iint b f_s^2(g' - g)^2 d\sigma d\nu_s \leq C\|f\|_{L^2_s}^2 \left( J_{x,p}^{\phi}(g) + \|g\|_{L^{2 \theta(v)g}}^2 \right).$$

Once the equivalence (2.20) has been established, we have

**Corollary 2.18.** Assume that (1.2) holds with $0 < s < 1$. Then there exists a constant $C > 0$ such that

$$\iint b f_s^2(g' - g)^2 d\sigma d\nu_s \leq C\|f\|_{L^2_s}^2 \|g\|_{H^s}^2.$$  

**Proof.** It is enough to consider the case $\rho = 1$. As in the proof of Lemma 2.12, it follows from Proposition 2 of [3] that

$$J_{x,p}^{\phi}(g) = \iint b(\cos \theta)\mu_v(g' - g)^2 d\nu_s d\sigma$$

$$= \frac{1}{(2\pi)^3} \int \frac{\xi}{|\xi|} \cdot \sigma \left( \overline{\mu(0)}|\xi| + |\xi| + 1 \right) - 2\Re \overline{\mu} \cdot \overline{\xi} + \overline{\xi} \right) d\xi d\sigma$$

$$= \frac{1}{(2\pi)^3} \int \frac{\xi}{|\xi|} \cdot \sigma \left( \overline{\mu(0)}|\xi| - |\xi| + 1 \right) + 2\Re \overline{\mu} \cdot \overline{\xi} d\xi d\sigma,$$

and

$$A = \iint b(\cos \theta)f_s^2(g' - g)^2 d\nu_s d\sigma$$

$$= \frac{1}{(2\pi)^3} \int \frac{\xi}{|\xi|} \cdot \sigma \left( \overline{f_s(0)}|\xi| - |\xi| + 1 \right) + 2\Re \overline{f_s(0)} \cdot \overline{\xi} d\xi d\sigma.$$
Since $\hat{f}(0) = \|f\|^2_{L^2}$ and $\hat{g}(0) = c_0 > 0$, we obtain

$$c_0 A = c_0 \iint b(\cos \theta) f^2(g - g)^2 dv d\sigma d\rho = \|f\|^2_{L^2} J^0_{1,1}(g)$$

$$- \frac{2}{(2\pi)^3} \|f\|^2_{L^2} \int b \left( \frac{\xi}{|\xi|}, \sigma \right) Re \left( \hat{\mu}(0) - \hat{\mu}(\xi^{-}) \right) \hat{\Phi}(\xi) d\xi d\sigma$$

$$+ \frac{2 c_0}{(2\pi)^3} \int b \left( \frac{\xi}{|\xi|}, \sigma \right) Re \left( \hat{f}^2(0) - \hat{f}^2(\xi^{-}) \right) \hat{\Phi}(\xi) \hat{\Phi}(\xi^-) d\xi d\sigma + \iint b \left( \frac{\xi}{|\xi|}, \sigma \right) Re \left( \hat{f}^2(0) - \hat{f}^2(\xi^{-}) \right) \hat{\Phi}(\xi) \hat{\Phi}(\xi^-) d\xi d\sigma$$

$$= \|f\|^2_{L^2} J^0_{1,1}(g) + A_1 + A_2.$$ 

Write

$$A_2 = \frac{2 c_0}{(2\pi)^3} \int |\hat{\Phi}(\xi)|^2 \left( \int b \left( \frac{\xi}{|\xi|}, \sigma \right) Re \left( \hat{f}^2(0) - \hat{f}^2(\xi^{-}) \right) d\xi \right)$$

$$+ \frac{2 c_0}{(2\pi)^3} \int b \left( \frac{\xi}{|\xi|}, \sigma \right) Re \left( \hat{f}^2(0) - \hat{f}^2(\xi^{-}) \right) \hat{\Phi}(\xi) \hat{\Phi}(\xi^-) d\xi d\sigma$$

$$= \mathcal{A}_{2,1} + \mathcal{A}_{2,2}.$$ 

It follows from Cauchy-Schwarz’s inequality that

$$|A_{2.2}| \leq C \left( \int b \left( \frac{\xi}{|\xi|}, \sigma \right) |\hat{f}^2(0) - \hat{f}^2(\xi^{-})| \hat{\Phi}(\xi) d\xi d\sigma \right)^{1/2}$$

$$\times \left( \int b \left( \frac{\xi}{|\xi|}, \sigma \right) \hat{\Phi}(\xi) \hat{\Phi}(\xi^-) d\xi d\sigma \right)^{1/2}$$

$$= B_1^{1/2} \times B_2^{1/2}.$$ 

Since

$$|\hat{f}^2(0) - \hat{f}^2(\xi^-)| \leq \int f^2(v)|1 - e^{-iv\xi}| dv,$$

we have

$$B_1 \leq C \iint |\hat{\Phi}(\xi)|^2 f^2(v) f^2(w)$$

$$\times \left( \int b \left( \frac{\xi}{|\xi|}, \sigma \right) (|1 - e^{-iv\xi}|^2 + |1 - e^{-iv\xi}|^2) d\sigma \right) dv dwd\xi$$

$$\leq C \|\hat{\Phi}\|^2_{L^2} \|f\|^2_{L^2},$$

because

$$\int b \left( \frac{\xi}{|\xi|}, \sigma \right) (|1 - e^{-iv\xi}|^2) d\sigma$$

$$\leq C \int b \left( \frac{\xi}{|\xi|}, \sigma \right) \sigma^{-1/2} (|v||\xi|^2)^{1/2} d\theta + \int b \left( \frac{\xi}{|\xi|}, \sigma \right) \sigma^{1/2} \sigma^{-1/2} d\theta$$

$$\leq C (v^2)(\xi)^{22}. $$
Then we have $|A_{2,1}| \leq C\|g\|_{H^s}^2\|f\|_{L^2}^2$, because
\[
\int b\left(\frac{\xi}{|\xi|}\right)\Re\left(\tilde{f}^2(0) - \tilde{f}^2(\xi^-)\right)d\sigma = \int f^2(v)\left(\int b\left(\frac{\xi}{|\xi|}\right)(1 - \cos(\nu \cdot \xi^-))d\sigma\right)dv 
\leq C\|\xi\|^2\int f^2(v)(v)^2dv.
\]
Since $\hat{\mu}(\xi)$ is real-valued, it follows that
\[
\Re\left(\hat{\mu}(0) - \hat{\mu}(\xi^-)\right)\Re\left(\tilde{g}(\xi^-)\hat{g}(\xi)\right) = \left(\int (1 - \cos(\nu \cdot \xi^-))\mu(v)dv\right)\Re\left(\tilde{g}(\xi^-)\hat{g}(\xi)\right).
\]
Therefore, by using Cauchy-Schwarz’s inequality and the change of variables $\xi \to \xi^+$ (see the proof of Lemma 2.8 in [2]), we obtain $|A_1| \leq C\|f\|_{L^2}^2\|g\|_{H^s}^2$. Furthermore, it follows from (2.13) that
\[
B_2 = \int b\left(\frac{\xi}{|\xi|}\right)\hat{g}(\xi)\left(\tilde{g}(\xi^-)\hat{g}(\xi)\right)d\xi d\sigma 
\leq C\left(f_{h_\xi}(g) + \|g\|_{H^s}^2\right),
\]
which yields $|A_{2,2}| \leq C\|f\|_{L^2}^2\|f\|_{L^2}^2\|g\|_{H^s}\left(f_{h_\xi}(g) + \|g\|_{H^s}^2\right)^{1/2}$. Hence
\[
|A_2| \leq C\|f\|_{L^2}^2\|g\|_{H^s}\left(f_{h_\xi}(g) + \|g\|_{H^s}^2\right)^{1/2}.
\]
Finally, we have
\[
A \leq C\|f\|_{L^2}^2\|g\|_{H^s}\left(f_{h_\xi}(g) + \|g\|_{H^s}^2\right)^{1/2} \leq C\|f\|_{L^2}^2\left(f_{h_\xi}(g) + \|g\|_{H^s}^2\right),
\]
by means of (2.13) with $\gamma = 0$, completing the proof of the lemma. $\square$

3. Estimates of nonlinear collision operator in velocity space

In this section, we derive various estimates on the nonlinear collision operator. Even though we consider the soft potential case in this paper, some of the following estimates also hold for general case so that they will be used in Part II.

3.1. Upper bounds in general case. In this sub-section, we will establish various functional estimates which hold true under the more general assumption $0 < s < 1$ and $\gamma > -3$. In particular, all the results in this part are independent of the sign of $\gamma + 2s$.

Proposition 3.1. For all $0 < s < 1$ and $\gamma > -3$, we have

\[
\left\|G(f, g), h\right\|_{L^2} \leq \|h\|_{H^s} \left\|\|f\|_{L^\infty}\|g||\varphi_h + \left\|\nabla f\|_{L^\infty} + \|f\|_{L^\infty}\right\|\|g\|_{H^s}^2\right\|_{H^s}.\n\]

Proof. Direct calculation gives
\[
\left\langle G(f, g), h\right\rangle_{L^2} = \iiint b\Phi_y \mu_+^{1/2}(f_y' - f_y) dv dv d\sigma 
\leq \frac{1}{2} \iiint b\Phi_y \mu_+^{1/2}(f_y' - f_y)(b\Phi_y)' \mu_{-1/4} h - \mu_{-1/4} h + \frac{1}{2} \iiint b\Phi_y \mu_+^{1/2}(f_y' - f_y)(b\Phi_y)' \mu_{-1/4} h.\n\]
Noticing that
\[ \mu_*^{1/4} h - \mu_*^{1/4} h' = \mu_*^{1/4} (h - h') + \left( \mu_*^{1/4} - \mu_*^{1/4} \right) n, \]
by using the Cauchy-Schwarz inequality, we have
\[
\left\| (f, g), h \right\| \leq \left( \iiint b \Phi_j \frac{1}{\mu_1^{1/2}} \left( f'_j g' - f_j g \right)^2 \, dr dv \right)^{1/2} \| h \|_{\Phi_j} \cdot
\]
where we have used the fact that the non-isotropic norm is invariant by replacing \( \mu \) by \( \mu^\rho \) for any fixed \( \rho > 0 \) (see the previous section). We then estimate
\[
A \leq 3 \left( \iiint b \Phi_j \frac{1}{\mu_1^{1/2}} \left( (\mu_1^{1/8} f'_j) - (\mu_1^{1/8} f_j) \right)^2 \, dr dv \right)^{1/2} \| g \|_{\Phi_j}^2 + \left( \iiint b \Phi_j \frac{1}{\mu_1^{1/2}} \left( (\mu_1^{1/8} f'_j) - (\mu_1^{1/8} f_j) \right)^2 \, dr dv \right) \| h \|_{\Phi_j}^2 + \left( \iiint b \Phi_j \frac{1}{\mu_1^{1/2}} \left( (\mu_1^{1/8} f'_j) - (\mu_1^{1/8} f_j) \right)^2 \, dr dv \right) \| g \|_{\Phi_j}^2 \cdot
\]
where we have used the fact that \( \gamma + 2 \sigma > -3 \) and the fact that
\[
\int b(\cos \theta) \min(\theta^2 |v - v_s|^2, 1) \, d\sigma \leq |v - v_s|^2 \int_{0}^{\min(\pi/2, |v - v_s|^{-1})} \theta^{1-2\sigma} \, d\theta + \int_{\min(\pi/2, |v - v_s|^{-1})}^\pi \theta^{1-2\sigma} \, d\theta \leq |v - v_s|^2 
\]
and this completes the proof of the proposition. \( \square \)

**Lemma 3.2.** Let \( \gamma \geq 0 \). Assume that (1.2) holds with \( 0 < s < 1 \). Then
\[
\left\| \iiint \Phi_j (|v - v_s|) b f'_j (g' - g)^2 \, dr dv \right\| \leq \| f \|_{r^{2\gamma/2}}^2 \| g \|_{\Phi_j}^2 .
\]
Proof. Since $\Phi_y(|v-v_c|) \leq (v')^y + (v_c)^y$, we have
\[
\int_0^1 \int b(\cos \theta) \Phi(|v-v_c|) f^2_z (g' - g)^2 \, d\tau \, dv.
\]
Noticing that
\[
\left| (v')^{y/2} - (v_c)^{y/2} \right| \leq C_y \int_0^1 (v' + \tau(v - v_c))^{(y/2-1)v} 
\]
we have
\[
A_3 \leq \int f^2_z (v^{y/2} + (v_c)^{y/2})^2 \, dv.
\]
Applying Corollary 2.18 to $A_1$ and $A_2$, it follows that
\[
A_1 + A_2 \leq \|f\|^2_{L^2_z} \|\gamma/2\|^2_{\eta_{tv}} + \|v^{y/2} f\|^2_{L^2_z} \|\eta_{tv}\|_{\eta_{tv}}^2
\]
where we have used Proposition 2.4 in the last inequality. 

**Proposition 3.3.** For all $0 < s < 1$ and $\gamma > -3$, one has
\[
|\Gamma(f, g, h)|_L^2 \leq \|f\|_{L^2_z} \|g\|_{L^2_v} \|h\|_{L^2_v} + \|f\|_{L^2_z} \|g\|_{L^2_v} \|h\|_{L^2_v}
\]

\[
+ \min \left( \|f\|_{L^2_z} \|g\|_{L^2_v} \|h\|_{L^2_v}, \|f\|_{L^2_z} \|g\|_{L^2_v} \|h\|_{L^2_v} \right)
\]

\[
+ \mu^{1/40} f_{L^2_z} \|\mu^{1/40} g_{H_1} \|_{H_1} + \mu^{1/40} \|\mu^{1/40} g_{H_1} \|_{H_1}^2.
\]

Proof. We will use the decomposition
\[
\Phi_y(z) = |z|^{1/2} I_{|z| \\leq 1} + |z| I_{|z| > 1} = \Phi_A(z) + \Phi_B(z).
\]
We denote by $\Gamma_A(\cdot, \cdot)$, $\Gamma_B(\cdot, \cdot)$ the collision operators with the kinetic factor in the cross section given by $\Phi_A$ and $\Phi_B$ respectively. Similarly to the proof of Proposition 3.1, we
have
\[
\left| \left[ \Gamma_\Phi(f, g), h \right] \right| \leq \left( \iint b \Phi_\mu \mu^{1/2} (f', g')^2 \, dv \right)^{1/2} \left\| A \right\| \Phi_y.
\]
Since \( \Phi_y \leq 2^8 \Phi_y \), we have
\[
A \leq \iint b \Phi_\mu \mu^{1/4} (\mu^{1/8} f')^2 g^2 \, dv \, dv \, dv
+ \iint b \Phi_\mu \mu^{1} (\mu^{1/8} f')^2 (g'-g)^2 \, dv \, dv
+ \iint b \Phi_\mu \mu^{1/4} (\mu^{1/8} - \mu^{1/8} f')^2 (f', g')^2 \, dv \, dv
= A_1 + A_2 + A_3.
\]
In order to estimate \( A_1 \), we make use of Lemma 3.2. Since \( \Phi_y \mu^{1/4} \leq \langle v \rangle \gamma \), we have by putting \( f = \mu^{1/8} f \) and \( g = \langle v \rangle \gamma^{1/2} g \),
\[
A_1 \leq \iint b \langle \langle v \rangle \gamma^{1/2} g \rangle^2 (\mu^{1/8} f')^2 \, dv \, dv \, dv
\leq \| \langle \langle v \rangle \gamma^{1/2} g \rangle^2 \|_{L^2} \| \mu^{1/8} f \|_{L^2} \| g \|_{L^2} \| g \|_{L^2}.
\]
We decompose the estimation on \( A_2 \) as
\[
A_2 \leq \iint b \langle \langle v \rangle \gamma^{1/2} g \rangle^2 (\mu^{1/8} f')^2 \, dv \, dv \, dv
\leq \| \langle \langle v \rangle \gamma^{1/2} g \rangle^2 \|_{L^2} \| \mu^{1/8} f \|_{L^2} \| g \|_{L^2} \| g \|_{L^2}.
\]
For \( A_{2.2} \), we note that if \( v_\tau = v' + \tau (v - v') \) for \( \tau \in [0, 1] \), then
\[
\langle v \rangle \gamma^{1/2} - \langle v' \rangle \gamma^{1/2} \leq C_\tau \int_0^1 \langle v' + \tau (v - v') \rangle \gamma^{(y-1)} \, d\tau \| v - v_\tau \| \theta.
\]
Thus, \( \langle v \rangle \gamma^{1/2} \leq C_\tau \langle \langle v' \rangle \gamma^{(y-1)} \rangle \| v - v_\tau \| \theta \),
\[
A_{2.2} \leq \iint \frac{(\mu^{1/8} f)^2}{\langle v \rangle^{1/2}} \| g \|_{L^2}^2 (\langle v \rangle \gamma^{(y-2)} \langle v \rangle \gamma^{(y-2)}) \int_0^{(v-v_\tau)^2} \theta^{-1/2} (v - v_\tau \theta)^2 d\theta
+ \iint_0^{(v-v_\tau)^2} (v') \gamma^{(y-2)} \langle v' \rangle \gamma^{(y-2)} \, d\theta \, dv \, dv
\leq \iint \langle v \rangle^{2+y} \langle v \rangle^{2+y} \| g \|_{L^2}^2 \, dv \, dv \leq \| \mu^{1/10} f \|_{L^2} \| g \|_{L^2}^2.
Noticing that \( (\mu_{1/8} - \mu_{1/8})^2 \leq \min(|v - v_s|^2 \theta^2, 1) \), we have
\[
A_3 \leq \iint b(\cos \theta) \min(|v - v_s|^2 \theta^2, 1) d\sigma f^2_g g^2 dv dv_s.
\]
\[
\leq \iint (v - v_s)^{2s} f^2_g g^2 dv dv_s.
\]
\[
\leq \iint (v_v)^{2s} f^2_g (v_v)^{2s} g^2 dv dv_s \leq \|f\|^2_{L^{\infty/2}} \|g\|^2_{L^{\infty/2}},
\]
when \( \gamma + 2s \geq 0 \) because \( (v - v_s)^{2s} \leq (v_v)^{2s} (v_v)^{2s} \).

To consider the case \( \gamma + 2s < 0 \), we divide the space \( \mathbb{R}^3 \times \mathbb{R}_v \) into three parts
\[
U_1 = \{ |v - v_s| \leq |v_s|/8 \}, \quad U_2 = \{ |v - v_s| > |v_s|/8 \} \cap \{ |v_s| \leq 1 \},
\]
U_3 = \{ |v - v_s| > |v_s|/8 \} \cap \{ |v_s| > 1 \}.

Then we have
\[
\frac{1}{3} A_3 = \iiint \Phi_r (\mu_{1/8}^r - \mu_1^r)^2 (f_g g)^2 dv dv_s.
\]
\[
= \int_{U_1} d\sigma dv dv_s + \int_{U_2} d\sigma dv dv_s + \int_{U_3} d\sigma dv dv_s.
\]
\[
= A_{3,1} + A_{3,2} + A_{3,3}.
\]

Since \( |v' - v_s| \leq |v - v_s| \leq |v_s|/8 \), we have \( 7|v_s|/8 \leq |v|, |v| \leq 9|v_s|/8 \) and \( |v'|^2 = |v|^2 + |v_s|^2 - |v|^2 \geq |v_s|^2/2 \) in \( U_1 \). Hence, in this region, we have \( \mu_{1/4} \leq C \mu_{1/8} \leq C(\mu_{1/4})^{1/20} \), and this to
\[
A_{3,1} \leq \iiint (\mu_{1/4})^{1/20} (v - v_s)^{2s} f^2 g^2 dv dv_s \leq \|f\|^2_{L^{\infty/2}} \|g\|^2_{L^{\infty/2}}.
\]

Furthermore, we have
\[
A_{3,2} \leq \iiint (v - v_s)^{2s} f^2 g^2 dv dv_s \leq \|f\|^2_{L^{\infty/2}} \|g\|^2_{L^{\infty/2}},
\]
because \( (v - v_s)^{2s} \leq (v_v)^{2s} v_v^{-1} v_v^{-1} (v_v)^2 \leq 2(v_v)^{-1} (v_v)^{-1} \) in \( U_2 \). Since \( (v_v)^{-1} \leq 8|v_v|^{-1} \leq 16(v_v)^{-1} \) in \( U_3 \), we get
\[
A_{3,3} \leq \|f\|^2_{L^{\infty/2}} \|g\|^2_{L^{\infty/2}}.
\]

Therefore, we have, when \( \gamma + 2s \leq 0 \)
\[
A_3 \leq \|f\|^2_{L^{\infty/2}} \|g\|^2_{L^{\infty/2}}.
\]

By considering another partition in \( \mathbb{R}^6_{\mu,v} \) with \( \mu \) and \( v \) exchanged, the estimate \( A_3 \leq \|f\|^2_{L^{\infty/2}} |g|_{L^2}^2 \) holds, because \( |v'_v - v_v| \leq |v - v_s| \leq |v_s|/8 \) implies \( 7|v_s|/8 \leq |v_v|, |v_v| \leq 9|v_s|/8 \).

As a conclusion, we have in summary that
\[
\left\| (\Gamma_A(f,g), h) \right\| \leq \left\| f \right\|_{L^{\infty/2}} \|g\|_{L^2} \|h\|_{\Phi_v} + \|g\|_{L^2} \left\| f \right\|_{L^{\infty/2}} \|h\|_{\Phi_v} + \min \left( \|f\|_{L^2} \|g\|_{L^2} \cdot \left\| f \right\|_{L^{\infty/2}} \|g\|_{L^2} \right) \|h\|_{\Phi_v}.
\]

We now turn to \( \Gamma_B \). For this, firstly, it holds that
\[
\left\| (\Gamma_B(f,g), h) \right\| \leq \left( \iint b(\cos \theta) \min(|v - v_s|^2 \theta^2, 1) d\sigma f^2 g^2 dv dv_s \right)^{1/2} \|h\|_{\Phi_v},
\]
\[
=B^{1/2} \|h\|_{\Phi_v}.
\]
Since $|v - v_*| \leq 1$ implies $|v|^2 \leq 2 + 2|v_*|^2$ and then $\mu_* \leq \varepsilon_0^{1/2}$, we have
\[
B \leq \int_{|v - v_*| \leq 1} b(\Phi_{\gamma} \mu_*^{1/10} \mu^{1/10}(\mu^{1/8} f)' - (\mu^{1/8} f'))^2 g^2 dv dv_*
\]
\[
+ \int_{|v - v_*| \leq 1} b (\Phi_{\gamma} \mu_*^{1/10} \mu^{1/10}(\mu^{1/8} f)'(g - g)^2 dv dv_*
\]
\[
+ \int_{|v - v_*| \leq 1} b(\Phi_{\gamma} \mu_*^{1/10} \mu^{1/10}(\mu^{1/8} - \mu_*^{1/8})^2 (f_*)^2 g^2 dv dv_*
\]
\[= B_1 + B_2 + B_3.
\]
Obviously,
\[B_1 \leq \|\mu^{1/20} g\|_{L^\infty}^2 \|\mu^{1/8} f\|_{L^1} \leq \|\mu^{1/20} g\|_{L^\infty}^2 \|f\|_{L^1},
\]
Since $|\mu^{1/8} - \mu_*^{1/8}| \leq |v - v_*|\theta$, we see that by the change of variables $(v', v'_*) \to (v, v_*)$
\[B_3 \leq \int_{|v - v_*| \leq 1} b(\Phi_{\gamma} \mu_*^{1/10} \mu^{1/10}(\mu^{1/8} - \mu_*^{1/8})^2 (f_*)^2 g^2 dv dv_*
\]
\[\leq \int_{|v - v_*| \leq 1} (\mu^{1/20}_* f_*)^2 (\mu^{1/8}_* - \mu^{1/8}_*)^2 |v - v_*|^{1+2/\gamma} (\int b(\cos \theta \theta^2/\sigma) dv dv_*
\]
\[\leq \int (\mu^{1/20} f_*)^2 (\sup_{v_*} \int \|\mu^{1/8}_* - \mu^{1/8}_*\|_{L^1}^2 dvdv_*
\]
\[\leq \|\mu^{1/20} f\|_{L^2}^2 \|\mu^{1/8}_* - \mu^{1/8}_*\|_{L^2}^2,
\]
where we have used the Hardy inequality if $\gamma + 2 < 0$, cf. [54]. If one writes
\[B_2 \leq \int_{|v - v_*| \leq 1} b(\Phi_{\gamma} \mu_*^{1/10} \mu^{1/10}(\mu^{1/8} f)' - (\mu^{1/8} f'))^2 g^2 dv dv_*
\]
\[+ \int_{|v - v_*| \leq 1} b (\Phi_{\gamma} \mu_*^{1/10} \mu^{1/10}(\mu^{1/8} f)'(g - g)^2 dv dv_*
\]
\[= B_{2.1} + B_{2.2},
\]
then the second term $B_{2.2}$ has a similar upper bound as $B_3$. It remains to estimate
\[B_{2.1} = \int_{|v - v_*| \leq 1} b(\Phi_{\gamma} \mu_*^{1/10} \mu^{1/10}(\mu^{1/8} f)' - (\mu^{1/8} f'))^2 g^2 dv dv_*
\]
by the change of variables $(v', v'_*) \to (v, v_*)$. By firstly putting $F = \mu^{1/8} f$ and $G = \mu^{1/40} g$,
and denoting by $v_\tau = v' + \tau (v - v_*)$ for $\tau \in [0, 1]$, then by using
\[|G(v) - G(v')|^2 = \left| \int_0^1 \nabla G(v_\tau) \cdot (v - v') d\tau \right|^2 \leq |v - v_*|^2 (\sin^2 \theta/2) \left( \int_0^1 |\nabla G(v_\tau)|^2 d\tau \right),
\]
we have
\[B_{2.1} \leq \int_0^1 \{ \int_{|v - v_*| \leq 1} b (|v - v_*|^{1+2/\gamma} \sin^2 \theta/2 F^2 |\nabla G(v_\tau)|^2 dv dv_* d\tau \} d\tau.
\]
To estimate this term, we need the change of variables
\[v \to v_\tau = \frac{1 + \tau}{2} v + \frac{1 - \tau}{2} (v - v_*) |v + v_*|.
\]
The Jacobian of this transform is bounded from below uniformly in $\nu_s$, $\sigma$ and $\tau$, because

$$
|\partial(v_\nu, \sigma, \tau) / \partial(\nu_s, \sigma)\rangle = \left| \text{det} \left( \frac{1 + \tau}{2} I + \frac{1 - \tau}{2} \sigma \otimes k \right) \right| (k = \nu_s / |\nu_s|)
= (1 + \tau)^3 \left| 1 + \tau \sigma \cdot k \right| = (1 + \tau)^3 \left| \frac{2\tau}{1 + \tau} + 2 \frac{1 - \tau}{1 + \tau} \cos^2 \frac{\theta}{2} \right|
\geq (1 + \tau)^3 \left| 2\tau / (1 + \tau) + 1 - \tau \right| \geq (1 + \tau)^3 \left| \frac{1}{2} \right|.
$$

If we set $\tilde{b} = b(k \cdot \sigma)(1 - k \cdot \sigma)$, then we have $\int_{\mathbb{S}^2} \tilde{b} d\sigma < \infty$. Therefore,

$$
B_{2,1} \leq \int_0^1 \int_{\mathbb{S}^2} \tilde{b} \left\| \frac{\nabla G(\nu)}{|\nu - \nu_s|} \right\|^2 \int_{\mathbb{S}^2} \left\| \frac{\nabla G(\nu)}{|\nu - \nu_s|} \right\|^2 dv \, d\tau
\leq \int_0^1 \int_{\mathbb{S}^2} \left| \nu \right| d\sigma \int_{\mathbb{S}^2} \left| \nu \right| d\sigma
\leq \left\| F \right\|_{L^2(t, \mathbb{S}^2)} \left\| |D|^{-\gamma/2 - 1} V G \right\|_{L^2(t, \mathbb{S}^2)} \leq \left\| |F| \right\|_{L^2(t, \mathbb{S}^2)} \left\| |G| \right\|_{H^{\gamma/2 - 1}(t, \mathbb{S}^2)},
$$

where we have used $|\nu - \nu_s| = |\nu - \nu_s|$. Finally we obtain

$$
B \leq \left\| |F| \right\|_{L^2(t, \mathbb{S}^2)} \left\| |G| \right\|_{L^2(t, \mathbb{S}^2)} = \left\| \mu^{1/40} |f| \right\|_{L^2(t, \mathbb{S}^2)} + \left\| \mu^{1/40} |g| \right\|_{L^2(t, \mathbb{S}^2)} = \left\| |f| \right\|_{L^2(t, \mathbb{S}^2)} + \left\| |g| \right\|_{L^2(t, \mathbb{S}^2)} = \left\| \mu^{1/40} |f| \right\|_{H^{\gamma/2 - 1/2}}.
$$

This concludes the proof of Proposition 3.3. \(\square\)

Note that the above estimation is good enough for proving the local existence for the general case. However, the above upper bound related to $B$ is given in Sobolev space with positive index and this cannot be controlled by the non-isotropic norm. Hence, it is not sufficient for the proof of global existence.

### 3.2. A simple proof of Theorem 1.2 for $\gamma > -3/2$.

We first give a simple proof of upper bound estimates on the Boltzmann nonlinear operator when $\gamma > -\frac{3}{2}$. We state it as

**Proposition 3.4.** Assume that $0 < s < 1$ and $\gamma > -3/2$. Then

$$
\left| \left\langle \Gamma(f, g), h \right\rangle \right| \leq \left( \left\| f \right\|_{L^2} \left\| g \right\|_{L^2} \right) \left\| \phi_s \right\|_\mu \leq \left( \left\| f \right\|_{L^2} \left\| g \right\|_{L^2} \right) \left\| \phi_s \right\|_\mu,
$$

Furthermore, together with $\gamma \geq -3s$, one has

$$
\left| \left\langle \Gamma(f, g), h \right\rangle \right| \leq \left( \left\| f \right\|_{L^2} \left\| g \right\|_{L^2} \right) \left\| \phi_s \right\|_\mu.
$$

Let us note that the first statement deals with general values of $\gamma > -3/2$, that is not necessarily linked with the value of $s$. For the second statement, note that the condition $\gamma \geq -3s$ is always true in the physical cases mentioned above. Indeed recall here that $\gamma = 1 - 4s$, and that $0 < s < 1$. Therefore, we can conclude that together with the constraint $\gamma > -3/2$, the physical range $0 < s < 5/8$ is allowed.
The case when $\gamma \geq 0$. Note that
\begin{equation}
(3.1) \quad (\Gamma(f, g), \tilde{h})_{L^2} = \left(\mu^{-1/2} \mathcal{O}(\mu^{1/2} f, \mu^{1/2} g), \tilde{h}\right)_{L^2}.
\end{equation}
\begin{align*}
&= \iint \int \Phi_r b(\cos \theta) \mu^{1/2} (f^{*}_r g' - f g) h \\
&= \frac{1}{2} \iint \int \Phi_r b(\cos \theta) (f^{*}_r g' - f g)(\mu^{1/2} h - \mu^{1/2} h') \\
&\leq \frac{1}{2} \left( \iint \Phi_r b(\cos \theta) (f^{*}_r g' - f g)^2 \right)^{1/2} \\
&\times \left( \iint \Phi_r b(\cos \theta) (\mu^{1/2} h - (\mu')^{1/2} h')^2 \right)^{1/2} \\
&\leq \frac{1}{2} A^{1/2} \times B^{1/2}.
\end{align*}

For $B$, we have
\begin{align*}
B &= \iint \int \Phi_r b(\cos \theta) \left((\mu')^{1/2} (h' - h) + h((\mu')^{1/2} - (\mu)_r^{1/2})\right)^2 \\
&\leq 2 \iint \int \Phi_r b(\cos \theta) \left((\mu')^{1/2} (h' - h)^2 + h^2((\mu')^{1/2} - (\mu)_r^{1/2})\right)^2 \\
&\leq 2 \iint \int \Phi_r b(\cos \theta) h^2 + 2 \iint \int \Phi_r b(\cos \theta) h^2((\mu')^{1/2} - (\mu)_r^{1/2})^2 \\
&= 2 ||h||^2_{\mu^{1/2} r}.
\end{align*}

where we have used the change of variables $(v, v_*) \to (v', v'_*)$ for the first term and $(v, v_*) \to (v_*, v)$ for the second term. Similarly,
\begin{align*}
A &= \iint \int \Phi_r b(\cos \theta) \left(f^{*}_r (g' - g) + g (f^{*}_r - f_0)\right)^2 \\
&\leq 2 \iint \int \Phi_r b(\cos \theta) \left(f^{*}_r (g' - g)^2 + g^2 (f^{*}_r - f_0)^2\right) \\
&\leq 2 \iint \int \Phi_r b(\cos \theta) f^{*}_r (g' - g)^2 + 2 \iint \int \Phi_r b(\cos \theta) g^2 (f^{*}_r - f_0)^2.
\end{align*}

Then (3.2) implies that
\begin{equation}
A \leq \frac{B}{||f||^2_{L^{2,1/2}, \mu^{1/2} r} \times \frac{||g||^2_{L^{2,1/2}, \mu^{1/2} r}}{||f||^2_{L^{2,1/2}, \mu^{1/2} r}}},
\end{equation}
which completes the proof in the case when $\gamma \geq 0$.

The case when $-3/2 < \gamma < 0$. As in Subsection 2.3, it is easy to check that for any fixed $\rho > 0$,
\begin{equation}
(3.2) \quad ||g||^2_{\mu^{1/2} r, \phi} - J_{1, \rho}^{(g)} + J_{2, \rho}^{(g)} \sim J_{1, \rho}^{(g)} + ||g||^2_{L^{2,1/2}, \mu^{1/2} r} \\
- \iint \int \Phi_r b_{\mu^{1/2} r} (g' - g)^2 + 2 \iint \int \Phi_r b_{\mu^{1/2} r} \left(\sqrt{\mu'} - \sqrt{\mu_0}\right)^2.
\end{equation}

where the assumption $2\gamma > -3$ is required for the existence of the above integral, and more precisely for
\begin{equation}
\int |v|^{2\gamma}(v_*)^{2\gamma}(v + v_*) d\nu_* \sim (v)^{\gamma+2s}.
\end{equation}
Instead of (3.1), we write
\[
(\Gamma(f, g), h) = \iiint b\Phi_\gamma \mu_1^{1/8}(f_g' - f_g) hdvdv, \quad \ldots
\]
\[
= \frac{1}{2} \iiint (b\Phi_\gamma)^{1/2}(f_g' - f_g)(b\Phi_\gamma)^{1/2}(\mu_1^{1/4}h - \mu_1^{1/4}h')
+ \frac{1}{2} \iiint (b\Phi_\gamma)^{1/2}(f_g' - f_g)(b\Phi_\gamma)^{1/2}(\mu_1^{1/4}h - \mu_1^{1/4}h')h.
\]
Noticing that
\[
\mu_1^{1/4}h - \mu_1^{1/4}h' = \mu_1^{1/4}(h - h') + (\mu_1^{1/4} - \mu_1^{1/4})h,
\]
by Cauchy-Schwarz’s inequality and (3.2), we have
\[
\|(\Gamma(f, g), h)\| \lesssim A^{1/2} \|h\|_0, \quad \ldots
\]
We estimate
\[
A \lesssim 3 \iiint b\Phi_\gamma \mu_1^{1/4}(\mu_1^{1/8}f)_g' - (\mu_1^{1/8}f)_g' \|g\|^2 d\nu \|h\|_0,
+ \iiint b\Phi_\gamma \mu_1^{1/8}(\mu_1^{1/8}f)_g' (g' - g) \|g\|^2 d\nu \|h\|_0,
+ \iiint b\Phi_\gamma \mu_1^{1/4}(\mu_1^{1/8} - \mu_1^{1/8})(f_g' - f_g) \|g\|^2 d\nu \|h\|_0,
\]
\[
= A_1 + A_2 + A_3.
\]
Since \(\Phi_\gamma(\nu - \nu_0)\mu_1^{1/4} \lesssim \langle \nu \rangle', \) we have by means of Corollary 2.18
\[
A_1 \lesssim \iiint b((\langle \nu \rangle)^{1/2}g)' \|g\|^2 d\nu \|h\|_0 \|g\|_0^2,
\]
\[
\lesssim \|\langle \nu \rangle^{1/2}g\|_0^2 \|\mu_1^{1/8}f\|_0^2 \|g\|_0^2 \lesssim \|\langle \nu \rangle^{1/2}g\|_0^2 \|f\|_0^2,
\]
where we have used Propositions 2.2 and 2.4 in the last inequality. As for \(A_2,\) we decompose it as follows :
\[
A_2 \lesssim \iiint b(\mu_1^{1/8}f)_g' \|g\|^2 d\nu \|h\|_0 \|g\|_0^2,
\]
\[
+ \iiint b((\langle \nu \rangle)^{1/2} - (\langle \nu \rangle)^{1/2})' \|g\|^2 d\nu \|h\|_0.
\]
Apply Corollary 2.18 again to \(A_2.\) Then
\[
A_2 \lesssim \|\mu_1^{1/8}f\|_0^2 \|\langle \nu \rangle^{1/2}g\|_0^2 \lesssim \|f\|_0^2 \|g\|_0^2,
\]
The estimation for \(A_2.\) is the same as the one for \(A_2\) in the proof of Lemma 2.13. By using the change of variables \((\nu', \nu') \rightarrow (\nu, \nu),\) we obtain
\[
A_{2.2} \lesssim \iiint (\langle \nu \rangle^{2+\gamma} \nu'_0)^{2\gamma} \mu_1^{1/8}f_0^2 d\nu \|h\|_0 \|g\|_0^2 \lesssim \|\mu_1^{1/10}f\|_0^2 \|g\|_0^2 \lesssim \|\mu_1^{1/10}f\|_0^2 \|g\|_0^2 \lesssim \|\mu_1^{1/10}f\|_0^2 \|g\|_0^2 \lesssim \|\mu_1^{1/10}f\|_0^2 \|g\|_0^2 \lesssim \|\mu_1^{1/10}f\|_0^2 \|g\|_0^2.$$
Noticing that $\left(\mu^{1/8} - \mu^{1/8}_s\right)^2 \leq \min(|v - v_s|^2, 1)$, we have

$$A_3 \leq \int \int b \Phi_v(\int S \delta v \min(|v - v_s|^2, 1), \int d\sigma) f^2 g^2 dv dv,$$

$$\leq \int \int (v - v_s)^{\gamma + 2s} f^2 g^2 dv dv,$$

$$\leq \int \int (v_s)^{\gamma + 2s} f^2 (v)^{\gamma + 2s} g^2 dv dv \leq ||f||_{L_{\gamma+2s}^2} ||g||_{L_{\gamma+2s}^2}.$$

if $\gamma + 2s \geq 0$ because of $(v - v_s)^{\gamma + 2s} \leq (v_s)^{\gamma + 2s} (v)^{\gamma + 2s}$.

To consider the case $\gamma + 2s < 0$, we divide $\mathbb{R}^3 \times \mathbb{R}^3$ into three parts

$$U_1 = \{|v - v_s| \leq |v_s|/8\}, \quad U_2 = \{|v - v_s| > |v_s|/8 \cap |v_s| \leq 1\}, \quad U_3 = \{|v - v_s| > |v_s|/8 \cap |v_s| > 1\}.$$

Then we have

$$\frac{1}{3} A_3 = \int \int b \Phi_v \mu^{1/4} \left(\mu^{1/8} - \mu^{1/8}_s\right)^2 \left(f \frac{1}{3} g\right) dv dv,$$

$$= \int \int U_1 \int d\sigma dv dv + \int \int U_2 \int d\sigma dv dv + \int \int U_3 \int d\sigma dv dv,$$

$$= A_{3,1} + A_{3,2} + A_{3,3}.$$

Since $|v' - v_s| \leq |v - v_s| \leq |v_s|/8$ implies $7|v_s|/8 \leq |v'|$, $|v| \leq 9|v_s|/8$ and $|v'|^2 = |v|^2 + |v_s|^2 - |v|^2 \geq |v_s|^2/2$. Hence, we have $\mu^{1/4} \leq C \mu^{1/8} \leq C(\mu, \mu_s)^{1/20}$ on $U_1$, which leads to

$$A_{3,1} \leq \int \int \left(\mu^{1/4}_s\right)^{1/20} (v - v_s)^{\gamma + 2s} f^2 g^2 dv dv \leq C ||f||_{L_{\gamma+2s}^2} ||g||_{L_{\gamma+2s}^2}.$$

Furthermore, we have

$$A_{3,2} \leq \int U_2 \int (v - v_s)^{\gamma + 2s} f^2 g^2 dv dv \leq ||f||_{L_{\gamma+2s}^2} ||g||_{L_{\gamma+2s}^2},$$

because $(v - v_s)^{-1} \leq (v)^{-1} (v_s)^{-1} \leq 2 (v)^{-1} (v_s)^{-1}$ on $U_2$. Since $(v - v_s)^{-1} \leq 8 |v_s|^{-1} \leq 16 (v_s)^{-1}$ on $U_3$, we get

$$A_{3,3} \leq ||f||_{L_{\gamma+2s}^2} ||g||_{L_{\gamma+2s}^2}.$$

Therefore, we have in the case when $\gamma + 2s < 0$

$$A_3 \leq ||f||_{L_{\gamma+2s}^2} ||g||_{L_{\gamma+2s}^2}.$$

If one considers another partition in $R^6_{\gamma v}$ with $v$ and $v_s$ exchanged, then the estimate

$$A_3 \leq ||f||_{L_{\gamma+2s}^2} ||g||_{L_{\gamma+2s}^2}$$

holds, because $|v' - v| \leq |v_s - v| \leq |v_s|/8$ implies $7|v_s|/8 \leq |v'|$, $|v_s| \leq 9|v|/8$.

As a conclusion, when $\gamma > -3/2$ and $\gamma + 2s \leq 0$ we have

$$\left(\langle \Gamma(f, g), h \rangle \right) \leq \left( ||f||_{L_{\gamma+2s}^2} ||g||_{L_{\gamma+2s}^2} \right),$$

$$+ \min \left( ||f||_{L_{\gamma+2s}^2} ||g||_{L_{\gamma+2s}^2} \right) ,$$

which concludes the proof of the first statement of Proposition 3.4.
The case $\gamma + 2s < 0$, $\gamma \geq -3s$. We go back to the definition of $A_3$, that is (we have performed the usual change of variables)

$$A_3 \sim \iint b\Phi, \mu_+^{1/2}(\mu_+^{1/8} - \mu_+^{1/8})^2 f_+^2 g^2 d\tau dv, \quad \leq \iint b\Phi, (\mu_+^{1/8} - \mu_+^{1/8})^2 f_+^2 g^2 d\tau dv.$$

We estimate the spherical integral as usual, that is over the sets

$$|v' - v| \leq \frac{1}{2} < v > \quad \text{and} \quad |v' - v| \geq \frac{1}{2} < v >$$

It follows by Taylor formula that, on the first set (which is the singular part), one has, for another non important and non negative constant $c$

$$(\mu_+^{1/8} - \mu_+^{1/8})^2 \leq \theta^2 |v - v|^2 \mu'.$$

On the other set, we just estimate the square by 1. Note that on the second set we have, $|v' - v| \geq < v >$.

Then we find, by now standard computations, that

$$A_3 \leq \int \int < v - v' >^\gamma \mu_+^{1/2} |v' - v|^2 < v >^2 f_+^2 g^2 + \int \int < v - v' >^\gamma |v' - v|^2 < v >^2 f_+^2 g^2 = A_{3,1} + A_{3,2}$$

Now, for $A_{3,1}$, we write $< v - v' >^{1+2s} < v >^{\gamma + 2s} < v >^2$ and we see that we may absorb all the powers of $< v >$ with the maxwellian, to get, for another non negative constant $d$

$$A_{3,1} \leq \|\mu f\|^2_{L_{2}^{1/2,2}} \|g\|^2_{L_{2}^{1/2,2}}.$$

For $A_{3,2}$, we write

$$< v - v' >^\gamma < v >^{-2s} < v >^{\gamma + 2s} < v >^2 < v >^{\gamma - 2s} < v >^2 = < v >^\gamma < v >^2 < v >^{-2s}.$$

Note that the power $-\gamma - 4s$ which enters the power over $< v >$ can be written $-\gamma - 4s = -(\gamma + 2s) - 2s$

the first term being positive. Of course $-\gamma - 4s \leq 0$ iff $\gamma \geq -4s$, and this is true since we have assumed that $\gamma \geq -3s$. Furthermore $\gamma + 4s \geq -\gamma - 2s$ again because $\gamma \geq -3s$. Therefore we obtained

$$A_{3,2} \leq ||f||^2_{L_{2}^{1/2,2}} \|g\|^2_{L_{2}^{1/2,2}},$$

concluding the proof of the second statement.

Let us note that the proof of Proposition 3.4 gives the following corollary.

**Corollary 3.5.** With the regularized potential, together with assumptions (1.2), $0 < s < 1$ and $\gamma \geq -3s$, one has

$$\left\| \left( \Gamma_{\Phi} (f, g), h \right) \right\| \leq \|f\|^2_{L_{2}^{1/2,2}} \|g\|^2_{L_{2}^{1/2,2}} \|h\|. \|h\|_{\Phi}.$$

Note carefully that in this last result, the constraint $\gamma > -3/2$ is removed, and we have retained the constraint $\gamma \geq -3s$, which is always true for physical cases, as we saw above.
3.3. **Proof of Theorem 1.2** The general case is long and will be divided into several steps. One of the key ingredients in the proof is to split the term \((\Gamma(f, g); h)\) into two parts. In this way, one part can be dealt with the method introduced in the previous subsection, while the other one will be analyzed by direct Fourier transform.

**Lemma 3.6.** For any integer \(k \geq 2\) we can write

\[
\mu^{1/2} = (\mu^{\alpha} - \mu^{\beta})^{k} \sum_{i=1}^{k+2} \alpha_i^2 \mu_{\alpha} \mu_{\beta}^{n} + \sum_{i=1}^{k} \alpha_i^3 \mu_{\alpha} \mu_{\beta}^{n}
\]

\[
\mu_{0} \frac{d}{dx} \mu_{0} = \mu(x, v_{0}) + \sum_{i=1}^{k} \alpha_i^3 \mu_{0} \mu_{0}^{n}.
\]

In the above, \(\alpha_i^j\) are real numbers for all \(i\) and \(j\), and the other exponents are strictly positive, at the exception of \(b_i^2 = 0\), with \(b_i^j > 0\).

**Proof.** Differentiating \(k - 1\) times the identity \(\sum_{j=0}^{2k} x^j = \frac{1 - x^{2k+1}}{1 - x}\) we have

\[
\sum_{j=k+1}^{2k} \frac{j!}{(j - k + 1)!} x^{j-k+1} = \left( \frac{1}{1 - x} \right)^{(k-1)} - \sum_{j=0}^{k-1} \frac{(k - 1)}{j} \left( \frac{1}{1 - x} \right)^{(j)} \left( x^{2k+1} \right)^{(k-1-j)}
\]

\[
= \frac{(k - 1)!}{(1 - x)^k} \left[ 1 - \sum_{j=0}^{k-1} \frac{(2k + 1)!}{j!} \left( \sum_{n=0}^{k-j-1} \frac{(-1)^n}{n!(2k - j - n + 1)!} \right) x^{2k-j-1} \right].
\]

By setting \(x = B/A\) and multiplying the above identity by \(A^{2k+2} \left( 1 - B/A \right)^k / (k - 1)!\), we obtain

\[
A^{2k+2} = (A - B)^{k} \sum_{j=k+1}^{2k} \frac{j!}{(j - k + 1)!} A^{2k-j+1} B^{j-k+1} + \sum_{j=0}^{k-1} \frac{(2k + 1)!}{j!} \left( \sum_{n=0}^{k-j-1} \frac{(-1)^n}{n!(2k - j - n + 1)!} \right) A^{2k-j+1} B^{2k-j+1},
\]

which gives the desired formula. \(\square\)

With the help of Lemma 3.6, we can analyze \((\Gamma(f, g); h)\) as follows. Write

\[
(\Gamma(f, g), h) = (\Gamma_{\mu}(f, g), h) + (\Gamma_{rest}(f, g), h),
\]

with

\[
(\Gamma_{\mu}(f, g), h) = \iiint b\left( \frac{v - v_{0}}{|v - v_{0}|}, \sigma \right) \Phi_{\mu}(v - v_{0}) \mu(v, v_{0})(f_{g'}' - f,g) hdv_{0}, d\sigma,
\]

and \((\Gamma_{rest}(f, g); h)\) is a finite linear combination of terms in the form of

\[
(\Gamma_{mod}(f, g), h) = \iiint b\left( \frac{v - v_{0}}{|v - v_{0}|}, \sigma \right) \Phi_{\mu}(v - v_{0})(f_{g'}' - f,g) \mu_{\alpha} \mu_{\beta}^{n} hdv_{0}, d\sigma
\]

\[
= (Q(\mu_{\alpha} f, \mu_{\beta} g), \mu_{\beta} h),
\]

with \(d_{i} > 0, c_{i} > 0\), for \(1 \leq i \leq 3\).

The following two propositions give estimates on each of these scalar products, and all together imply Theorem 1.2.
Proposition 3.7. For all $0 < s < 1$ and $\gamma > -3$, one has
\[
\| (\Gamma_\mu(f, g), h) \| \leq \left\| \| f \|_{L^2_{s+\gamma}} \| \| g \|_{L^2_{s+\gamma}} \| \| h \|_{L^2_{s+\gamma}} \right\| + \min\left(\| f \|_{L^2_{s+\gamma}}\| g \|_{L^2_{s+\gamma}}\| h \|_{L^2_{s+\gamma}}\right) \cdot \| h \|_{L^2_{s+\gamma}}.
\]

Proof. Since $\mu(v)$ is a finite sum of $(\mu^a - \mu^b)^2\mu^c\mu^d$ with $a, b > 0$ and $c \geq 0$, by setting $H = \mu^e h$, we can write
\[
(\Gamma_\mu(f, g), h) = \int b \Phi(f, g) d\sigma dv d\mu,
\]
and then correspondingly we can write
\[
F = f h^c, \quad G = g h^c, \quad H = h h^c,
\]
for some positive constants $c_1, c_2$ and $c_3$.

Let $0 \leq \phi(v) \leq 1$ be a smooth radial function with value 1 for $v$ close to 0, and 0 for large values of $v$. Set
\[
\Phi_\gamma(v) = \Phi_\gamma(v)\phi(v) + \Phi_\gamma(v)(1 - \phi(v)) = \Phi_\gamma(v) + \Phi_\gamma(v).
\]
And then correspondingly we can write
\[
Q(F, G) = Q_\gamma(F, G) + Q_\gamma(F, G).
\]
where the kinetic factor in the collision operator is defined according to the decomposition respectively. To prove Proposition 3.8, it suffices to prove the following two lemmas, by taking \( m = -s \) in the statements. The general form below for any real \( m \) will be needed in Part II.

**Lemma 3.9.** For all \( 0 < s < 1 \) and \( \gamma > -3 \), one has
\[
\left| \left( Q_s(F,G), H \right) \right| \leq C \| F \|_{L_{1/2}^{1/2}} \| G \|_{\phi_s} + \| G \|_{L_{1/2}^{1/2}} \| F \|_{\phi_s} \| H \|_{\phi_s}.
\]

*Proof.* One has for some positive constant \( \beta \)
\[
\left| \left( Q_{\beta}(F,G), H \right) \right| = \int \int b \Phi_{\beta}(v - v_\gamma) \mu_{\beta}^{\gamma} [F^G - F, G] Hdvdv, d\sigma
\]
\[
= \frac{1}{2!} \int \int b \Phi_{\beta}(v - v_\gamma) [F^G F - F^G] \mu_{\beta}^{\gamma} [H^H - H] Hdvdv, d\sigma
\]
\[
\leq A^{1/2} B^{1/2},
\]
where
\[
A = \int \int b \Phi_{\beta}(v - v_\gamma) [F^G F - F^G] \mu_{\beta}^{\gamma} dvvdv, d\sigma,
\]
and
\[
B = \int \int b \Phi_{\beta}(v - v_\gamma) \mu_{\beta}^{\gamma} [H^H - H] \mu_{\beta}^{\gamma} dvvdv, d\sigma.
\]

\( B \) is clearly estimated from above by the dissipative norm of \( H \), while for \( A \), we note that \( \Phi_{\beta} \leq \Phi_{\gamma} \). The proof of Proposition 3.8 in Subsection 3.4 can then be applied to \( A \). And this gives the desired estimate and then completes the proof of the lemma. \( \square \)

Next, let us note that, from the Appendix, \( |\hat{\Phi}_{\gamma}(\xi)| \leq |\xi|^{-3 - \gamma} \). We shall prove

**Lemma 3.10.** Let \( m \in \mathbb{R} \). For all \( 0 < s < 1 \) and \( \gamma > \max\{-3, -\frac{1}{2} - 2s\} \), one has
\[
\| Q_s(F,G), H \| \leq \left( \| F \|_{L_{1/2}^{1/2}} \| G \|_{\phi_s} + \| F \|_{L_{1/2}^{1/2}} \| G \|_{\phi_s} + \| F \|_{H^{\infty}} \| G \|_{H^{\infty}} \right) \| H \|_{H^{\infty}}.
\]

It is important to note that even though the statement of Lemma 3.10 is not as sharp as the one of Lemma 3.9 by recalling (3.3), we have all the needed weights because we are dealing with functions of the form \( F, G \) and \( H \) that contain Gaussians. Hence, these two lemmas together imply Proposition 3.8.

For the proof of Lemma 3.10, first of all, by using the formula from the Appendix of [3], we have
\[
\mathcal{F}(Q_s(F,G))(\xi) = \int \int_{\mathbb{R}^2} b\left( \frac{\xi}{|\xi|} \cdot \sigma \right) \Phi_s(\xi) \mathcal{F}(\xi^- + \xi_s) \hat{G}(\xi^- - \xi_s)
\]
\[
- \int \int_{\mathbb{R}^2} b\left( \frac{\xi}{|\xi|} \cdot \sigma \right) \Phi_s(\xi) \mathcal{F}(\xi^-) \hat{G}(\xi - \xi_s).
\]

We change variables in \( \xi_s \), in the first integral to obtain
\[
(Q_s(F,G), H) = \int \int b\left( \frac{\xi}{|\xi|} \cdot \sigma \right) \Phi_s(\xi_s) \mathcal{F}(\xi_s) \hat{G}(\xi_s) \hat{H}(\xi) d\xi d\xi_s, d\sigma.
\]
\[
= \int \int \int_{|\xi| \leq |\xi_s|} \cdots d\xi d\xi_s, d\sigma + \int \int \int_{|\xi| \leq |\xi_s|} \cdots d\xi d\xi_s, d\sigma
\]
\[
= A_1(F,G,H) + A_2(F,G,H).
\]
$A_2(F, G, H)$ can be naturally decomposed into

$$
A_2 = \iiint b(\frac{\xi}{|\xi|}, \sigma)1_{|\xi| \geq \frac{1}{2}|\dot{\Phi}_\tau|}(\xi, \sigma)\dot{F}(\xi, \sigma)\dot{G}(\xi, \sigma)\dot{H}(\xi) d\xi d\sigma.
$$

For $A_1$, we use the Taylor expansion of $\Phi_\tau$ at order 2 to have

$$
A_1 = A_{1,1}(F, G, H) + A_{1,2}(F, G, H)
$$

where

$$
A_{1,1} = \iiint b(\xi, \sigma)1_{|\xi| \geq \frac{1}{2}|\dot{\Phi}_\tau|}(\xi, \sigma)\dot{F}(\xi, \sigma)\dot{G}(\xi, \sigma)\dot{H}(\xi) d\xi d\sigma,
$$

and $A_{1,2}(F, G, H)$ is the remaining term corresponding to the second order term in the Taylor expansion of $\Phi_\tau$. The $A_{i,j}$ with $i, j = 1, 2$ are estimated by the following lemmas.

**Lemma 3.11.** Let $m \in \mathbb{R}$. For all $\gamma > \max(-3, -\frac{3}{2} - 2s)$, one has

$$
|A_{i,j}| \leq \left(\|F\|_{L^2} + \|G\|_{L^2} + \|F\|_{L^\infty} \|G\|_{L^2} + \|F\|_{H^s} \|G\|_{H^s}\right)\|H\|_{H^m}, \quad j = 1, 2.
$$

**Proof.** We first consider $A_{1,1}$. By writing

$$
\xi^- = \frac{|\xi|}{2}\left(\frac{\xi}{|\xi|}, \sigma\right) - \frac{1}{2}\left(\frac{\xi}{|\xi|}, \sigma\right) + \frac{1}{2}\left(\frac{\xi}{|\xi|}, \sigma\right),
$$

we see that the integral corresponding to the first term on the right hand side vanishes because of the symmetry on $S^2$. Hence, we have

$$
A_{1,1} = \iiint K(\xi, \xi)\dot{F}(\xi, \sigma)\dot{G}(\xi, \sigma)\dot{H}(\xi) d\xi d\sigma,
$$

where

$$
K(\xi, \xi) = \iiint b(\xi, \sigma)1_{|\xi| \geq \frac{1}{2}|\dot{\Phi}_\tau|}(\xi, \sigma)\dot{F}(\xi, \sigma)\dot{G}(\xi, \sigma)\dot{H}(\xi) d\xi d\sigma.
$$

Note that $|\nabla \dot{\Phi}_\tau(\xi)| \leq \frac{1}{|\xi|},$ from the Appendix. If $\sqrt{\gamma} \leq |\xi|$, then $|\xi^-| \leq |\xi|/2$ and this implies that for $0 \leq \theta \leq \pi/2,$

$$
|K(\xi, \xi)| \leq \int_0^{\pi/2} d\theta \int_0^{\theta} \frac{\langle \xi \rangle}{(\langle \xi \rangle)^{3\gamma+1}} \leq \left(\frac{\langle \xi \rangle}{\langle \xi \rangle^{3\gamma+2}}\right)\frac{\langle \xi \rangle}{(\langle \xi \rangle)^{3\gamma+1}} \langle \xi \rangle^{1-x}.
$$

On the other hand, if $\sqrt{\gamma} \geq |\xi|,$ then

$$
|K(\xi, \xi)| \leq \int_0^{|\xi|} d\theta \int_0^{\theta} \frac{\langle \xi \rangle}{(\langle \xi \rangle)^{3\gamma+1}} \leq \left(\frac{\langle \xi \rangle}{\langle \xi \rangle^{3\gamma+2}}\right)\frac{\langle \xi \rangle}{(\langle \xi \rangle)^{3\gamma+1}} \langle \xi \rangle^{1-x}.
$$

Since $\langle \xi \rangle^{-(3\gamma+2s)} \in L^2$ when $\gamma > -3/2 - 2s$, we obtain the desired estimate for $A_{1,1}$. 
Now we turn to $A_{1,2}(F,G,H)$, which comes from the second order term of the Taylor expansion. Note that

$$A_{1,2} = \iint \left| b\left(\frac{\xi}{|\xi|}\right)\right| |\tau|^{\frac{1}{2}}(\nabla^2 \hat{\Phi})(\xi - \tau \xi) \cdot \hat{\xi} - \tau \hat{\xi} - \hat{\xi} \cdot \hat{\xi} \cdot \hat{\xi} d\tau d\xi d\xi_\ast.$$  

From the Appendix, we have

$$|\nabla^2 \hat{\Phi}(\xi - \tau \xi)| \leq \frac{1}{\langle \xi - \tau \xi \rangle^{3+\gamma/2}} \leq \frac{1}{\langle \xi \rangle^{3+\gamma/2}},$$

because $|\xi^-| \leq \langle \xi \rangle/2$. Similar to $A_{1,1}$, we can obtain

$$|A_{1,2}| \leq \int_{\mathbb{R}} \tilde{K}(\xi,\xi) \hat{F}(\xi) \hat{G}(\xi - \xi) \tilde{H}(\xi) d\xi d\xi_d \xi_\ast,$$

where $\tilde{K}(\xi,\xi)$ has the following upper bound

$$\tilde{K}(\xi,\xi) \leq \int_{\mathbb{R}} \min(|\xi|, \langle \xi \rangle^{-2}) \theta^{1-2s} d\theta \frac{\langle \xi \rangle^2}{\langle \xi \rangle^{3+\gamma/2}} \leq \frac{\langle \xi \rangle^{2s}}{\langle \xi \rangle^{3+\gamma/2}},$$

which yields the desired estimate for $A_{1,2}$. And this completes the proof of the lemma. □

**Lemma 3.12.** Let $m \in \mathbb{R}$. For all $0 < s < 1$ and $\gamma > \max(-3,-\frac{3}{2} - 2s)$, one has

$$|A_{2,1}| + |A_{2,2}| \leq \|F\|_{1/2} \|G\|_{1/m+2s} + \|F\|_{1/m+2s} \|G\|_{1/2} \|H\|_s.$$  

**Proof.** In view of the definition of $A_{2,2}$, since we assume that $\theta \geq 1/|\xi| |\xi|^{-1}$, we also have $1/|\xi| |\xi|^{-1} \leq \tilde{\xi}$, that is, $\langle \xi \rangle \leq |\xi|$. We can then directly compute the spherical integral appearing inside $A_{2,2}$ together with $\Phi$ by using the inequality

$$\frac{1}{\langle \xi \rangle^{3+\gamma} \langle \xi \rangle^{2s}} \leq \frac{1}{\langle \xi \rangle^{3+\gamma + 2s}} \langle \xi \rangle^{-m} \langle (\xi)^{(m+2)s} + (\xi - \xi)^{(m+2)s} \rangle,$$

to obtain the estimate for $A_{2,2}$.

We now turn to

$$A_{2,1} = \iint b \left| 1_{|\xi| \leq \|\xi\|} \hat{\Phi}(\xi - \xi) \hat{F}(\xi) \hat{G}(\xi - \xi) \tilde{H}(\xi) d\xi d\xi_d \xi_\ast.$$  

Firstly, note that we can work on the set $|\xi - \xi - \xi | \geq \frac{1}{2} |\xi^-|^3$. In fact, on the complementary of this set, we have $|\xi - \xi - \xi | \leq \frac{1}{2} |\xi^-|^3$ so that $|\xi - \xi - \xi | \geq |\xi|$, and in this case, we can proceed in the same way as for $A_{2,2}$. Therefore, it suffices to estimate

$$A_{2,1,p} = \iint b \left( 1_{|\xi| \leq \|\xi\|} \left| 1_{|\xi| \leq \|\xi|} \hat{\Phi}(\xi - \xi) \hat{F}(\xi) \hat{G}(\xi - \xi) \tilde{H}(\xi) d\xi d\xi_d \xi_\ast \right| \right.$$  

$$= \iint b \left( \frac{1}{\langle \xi \rangle^{3+\gamma + 2s}} \langle \xi \rangle^{-m} \langle (\xi)^{(m+2)s} + (\xi - \xi)^{(m+2)s} \rangle \hat{\Phi}(\xi - \xi) \hat{F}(\xi) \hat{G}(\xi - \xi) \tilde{H}(\xi) d\xi d\xi_d \xi_\ast \right.$$  

$$\times \tilde{H}(\xi) \sum_{j=1}^{3} \hat{F}_j(\xi) \hat{G}_j(\xi - \xi) d\xi_d \xi_\ast,$$

where $\hat{F}_1 = (\xi)^{(m+2)s} \hat{F}$, $\hat{G}_1 = \hat{G}$ and $\hat{F}_2 = \hat{F}$, $\hat{G}_2 = (\xi)^{(m+2)s} \hat{G}$. On the set for the above integral, we have $|\xi - \xi - \xi | \leq \langle \xi \rangle^{2s}$, because $|\xi^-| \leq |\xi|$ that follows from $|\xi^-|^2 \leq 2|\xi - \xi| \leq |\xi^-| |\xi|$. By the Cauchy-Schwarz inequality, we have

$$|A_{2,1,p}| \leq \sum_{j=1}^{3} D_j^{1/2} D_j^{1/2},$$
Proposition 3.13. Assume that

\[ D = \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} b(\xi) \cdot \sigma \right) \mathbf{1}_{\mathcal{F} = \mathcal{I} / (\xi)} \| \hat{\phi}_s(\xi - \xi') \|^2 (\xi - \xi')^2 d\sigma d\xi, \]

and

\[ D_j = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} b(\xi) \cdot \sigma \frac{1}{(\xi - \xi')^{2x}} (\xi + \xi')^{2x} \| \hat{f}_j(\xi - \xi') \|^2 d\sigma d\xi. \]

Since \( \int_{\mathbb{R}^3} b(\xi) \cdot \sigma \mathbf{1}_{\mathcal{F} = \mathcal{I} / (\xi)} d\sigma \leq \| \xi \|^2 \), we obtain

\[ D_j \leq \| F \|_{L^2}^2 \| G \|_{L^2}^2. \]

For \( D \), we use the change of variables in \( \xi' \), \( u = \xi - \xi' \) to get

\[ D = \int_{\mathbb{R}^3} \left( \int_{\mathbb{R}^3} b(\xi) \cdot \sigma \right) \mathbf{1}_{\mathcal{F} = \mathcal{I} / (\xi)} \| \hat{\phi}_s(\xi - \xi') \|^2 (\xi - \xi')^2 d\sigma d\xi. \]

By noting that \( |\xi'| \geq \frac{1}{2} |u| + |\xi'| \) implies \( |\xi'| \geq |u|/\sqrt{10} \), we have

\[ D \leq \int_{\mathbb{R}^3} \left( \frac{|\xi|}{|u|} \right)^{2x} \| \hat{\phi}_s(\xi') \|^2 \| \xi \|^2 d\xi \leq \| H \|_{L^2}^2. \]

This completes the proof of the lemma. \( \square \)

3.4. Estimation of commutators. By using the arguments similar to those used in previous subsections, we now prove the following estimation on commutators.

Proposition 3.13. Assume that \( 0 < s < 1 \) and \( \gamma > -3 \). Then, for any \( l \geq 1 \), one has

\[ \left| \left( W_l \Gamma(f, g) - \Gamma(f, W_l g), \ h \right) \right|_{L^2} \]

\[ \leq \left( \| f \|_{L^2}^2 \| W_l g \|_{L^2}^2 + \| g \|_{L^2}^2 \| W_l f \|_{L^2}^2 \right) + \min \left( \| f \|_{L^2} \| W_l g \|_{L^2}, \| f \|_{L^2} \| W_l f \|_{L^2} \right) \]

\[ + \min \left( \| g \|_{L^2} \| W_l f \|_{L^2}, \| g \|_{L^2} \| W_l f \|_{L^2} \right) \| h \|_{\Phi_*}. \]

And for any \( 0 < l < 1 \), one has

\[ \left| \left( W_l \Gamma(f, g) - \Gamma(f, W_l g), \ h \right) \right|_{L^2} \]

\[ \leq \left( \| f \|_{L^2}^2 \| W_l g \|_{L^2}^2 + \| g \|_{L^2}^2 \| W_l f \|_{L^2}^2 \right) + \min \left( \| f \|_{L^2} \| W_l g \|_{L^2}, \| f \|_{L^2} \| W_l f \|_{L^2} \right) \]

\[ + \min \left( \| g \|_{L^2} \| W_l f \|_{L^2}, \| g \|_{L^2} \| W_l f \|_{L^2} \right) \| h \|_{\Phi_*}. \]

Remark 3.14. Assume that \( 0 < s < 1 \) and \( \gamma > -3 \). Then, for any \( l \geq 0 \), one has

\[ \left| \left( \hat{W} \Gamma(f, g) - \Gamma(f, \hat{W} g), \ h \right) \right|_{L^2} \]

\[ \leq \left( \| f \|_{L^2}^2 \| W_l g \|_{L^2}^2 + \| g \|_{L^2}^2 \| W_l f \|_{L^2}^2 \right) + \min \left( \| f \|_{L^2} \| W_l g \|_{L^2}, \| f \|_{L^2} \| W_l f \|_{L^2} \right) \]

\[ + \min \left( \| g \|_{L^2} \| W_l f \|_{L^2}, \| g \|_{L^2} \| W_l f \|_{L^2} \right) \| h \|_{\Phi_*}. \]

where we use (5.3) if \( l |\gamma/2 + s| \geq 1 \) and (5.7) if \( l |\gamma/2 + s| < 1 \).
Proof. In view of the decomposition given for \( \Gamma \), it is enough to consider (3.3) with
\[
\mu(v, v_*) = (\mu^c - \mu_*^c)^2 \mu_*^c,
\]
for some constants \( a, c > 0 \). Indeed, all the other terms have compensation by some Gaussian function so that any algebraic weight is not a problem. For this term, the commutator is then given by
\[
\left[ W\Gamma_{\mu}(f, g) - \Gamma_{\mu}(f, W_{g}\mu), \ h \right]_{L^2} = \iiint\Phi_\mu(v - v_*)\mu(v, v_*)f_v'g'(W_{1} - W_{1}')hdvdv_{\sigma}.
\]
which can be written as
\[
\iiint b\Phi_\mu(v - v_*)\mu(v, v_*)f_v'g'(W_{1} - W_{1}')hdvdv_{\sigma}
\]
\[
= \iiint b\Phi_\mu(v - v_*)\mu(v, v_*)f_v'g'(W_{1} - W_{1}')dvdv_{\sigma}
\]
\[
+ \iiint b\Phi_\mu(v - v_*)(\mu(v', v_') - \mu(v, v_*)\mu_v g(W_{1} - W_{1}')hdvdv_{\sigma}
\]
\[
+ \iiint b\Phi_\mu(v - v_*)f_v g(W_{1} - W_{1}')hdvdv_{\sigma}
\]
\[
= A + B + C,
\]
by the usual change of variables. For \( A \), we use the Cauchy-Schwarz inequality to get
\[
A = \iiint b\Phi_\mu(v - v_*)\mu(v, v_*)f_v'g'(W_{1} - W_{1}')dvdv_{\sigma}
\]
\[
\leq \left( \iiint b\Phi_\mu(v - v_*)(\mu^c - \mu_*^c)^4 \mu_*^c |f_v'|^2 |g'|^2 |W_{1} - W_{1}'|^2 dvdv_{\sigma} \right)^{1/2} \|h\|_{\Phi_\mu},
\]
where
\[
U = \iiint b < v - v_* >^\gamma |f_v|^2 |g'| |W_{1} - W_{1}'|^2 dvdv_{\sigma}.
\]
If \( l \geq 1 \), by using the Taylor’s formula, we have
\[
|W_{1} - W_{1}'|^2 \leq \min[\theta^2 |v - v_*|^2, (v > + < v_* >)^2] (v > l - 1 + < v >^{2l - 1}),
\]
and then
\[
\iiint b |W_{1} - W_{1}'|^2 dvdv_{\sigma} \leq (\varepsilon < v >^{2l - 2 +} + < v_* >^{2l - 2l}.
\]
Then we note immediately that \( U \) is similar to the term \( A_3 \) in the proof of Proposition 3.3, because we have a Gaussian inside the definition of \( U \). Taking into account the weights here gives
\[
U \leq \|f\|_{L^2} \|W_{r - s}g\|_{L^2} + \min[\|f\|_{L^2} |W_{r - s}g|_{L^2}, |W_{r - s}f|_{L^2} |W_{r - s}g|_{L^2}]
\]
\[
+ \|g|_{L^2} |W_{r - s}f|_{L^2} + \min[|g|_{L^2} |W_{r - s}f|_{L^2}, |g|_{L^2} |W_{r - s}f|_{L^2}]
\]
If \( 0 < l \leq 1 \), then we note that
\[
|W_{1} - W_{1}'|^2 \leq \varepsilon < v >^{2l} + < v_* >^{2l} \text{ and } |W_{1} - W_{1}'|^2 \leq \theta^2 |v - v_*|^2,
\]
so that
\[
|W_{1} - W_{1}'|^2 \leq \min[\theta^2 |v - v_*|^2, < v >^{2l} + < v_* >^{2l}].
\]
Then we obtain
\[
(3.8) \quad \iiint b |W_{1} - W_{1}'|^2 dvdv_{\sigma} \leq |v - v_*|^2 (\varepsilon < v >^{2l - 2 +} + < v_* >^{2l - 2l}).
\]
and therefore the same argument gives

\[ U \leq \|f\|_{L^2_x} \|W_{\eta^\gamma}g\|_{L^2_x} + \min\{\|f\|_{L^2_x}, \|W_{\eta^\gamma}f\|_{L^2_x}, \|W_{\eta^\gamma}g\|_{L^2_x}\} \]

which gives the final conclusion, for \( \gamma > -3 \). Terms \( B \) and \( C \) can be dealt with similarly so that we omit the details for brevity. And this completes the proof of the proposition. \( \square \)

4. Functional estimates in full space

In this section, we prove the estimations on the collision operators in some weighted function space of variables \((x, v) \in \mathbb{R}^6\). Together with the essential coercivity estimates proved in Section 2, we give coercivity results for the linear operator in some weighted spaces. These tools are crucial for the proofs of the existence results, both in the local and global cases. Recall the assumption \( \gamma + 2s \leq 0 \).

Let \( N \in \mathbb{N}, l \in \mathbb{R} \), we define the weighted function spaces

\[ B^N_l(\mathbb{R}^6) = \left\{ g \in \mathcal{S}'(\mathbb{R}^6); \|g\|_{B^N_l(\mathbb{R}^6)}^2 = \sum_{|\nu| + |\beta| \leq N} \int_{\mathbb{R}^6} \|W_{\eta^\gamma} \partial_x^\nu \partial_v^\beta g(x, \cdot)\|_0^2 \, dx < +\infty \right\}, \]

\[ \tilde{B}^N_l(\mathbb{R}^6) = \left\{ g \in \mathcal{S}'(\mathbb{R}^6); \|g\|_{\tilde{B}^N_l(\mathbb{R}^6)}^2 = \sum_{|\nu| + |\beta| \leq N} \int_{\mathbb{R}^6} \|W_{\eta^\gamma} \partial_x^\nu \partial_v^\beta g(x, \cdot)\|_0^2 \, dx < +\infty \right\}, \]

and also

\[ X^N(\mathbb{R}^6) = \left\{ g \in \mathcal{S}'(\mathbb{R}^6); \|g\|_{X^N(\mathbb{R}^6)}^2 = \sum_{|\nu| + |\beta| \leq N} \int_{\mathbb{R}^6} \|\partial_x^\nu \partial_v^\beta g\|_0^2 \, dx < +\infty \right\}. \]

4.1. Estimations without weight. First of all, one has

**Lemma 4.1.** For all \( 0 < s < 1, \gamma > -3 \), and for any \( \alpha, \beta \in \mathbb{N}^3 \),

\begin{align*}
&\|\partial_\beta^\sigma \mathbf{P} g\|_{X^0(\mathbb{R}^6)} + \|\mathbf{P}(\partial_\beta^\sigma g)\|_{X^0(\mathbb{R}^6)} \leq C_{\eta} \|\partial_\beta^\sigma g\|_{L^2(\mathbb{R}^6)}, \\
&\frac{\eta_0}{2} \|g\|_{L^2(\mathbb{R}^6)}^2 - C \|g\|_{L^2(\mathbb{R}^6)}^2 \leq (\mathcal{L}_{g, g})_{L^2(\mathbb{R}^6)} \leq 2(\mathcal{L}_{1, g, g})_{L^2(\mathbb{R}^6)} \leq \|g\|_{X^0(\mathbb{R}^6)}^2 \leq \|g\|_{X^1(\mathbb{R}^6)}^2, \quad \text{and} \quad \frac{\eta_0}{2} \|g\|_{L^2(\mathbb{R}^6)}^2 + \|g\|_{L^2(\mathbb{R}^6)}^2 \leq \|g\|_{X^0(\mathbb{R}^6)}^2 \leq \|g\|_{L^2(\mathbb{R}^6)}^2 \leq \|g\|_{L^2(\mathbb{R}^6)}^2.
\end{align*}

**Proof.** From [20], one has

\[ \mathbf{P} g = (a_g(t, x) + v.b_g(t, x) + |v|^2 c_g(t, x)) \mu^{1/2}, \]

where

\[ a_g(t, x) = \int_{\mathbb{R}^3} (2 - \frac{|v|^2}{2})g((t, x), v) \mu^{1/2}(v) \, dv, \]

\[ b_g(t, x) = \int_{\mathbb{R}^3} g((t, x), v) \mu^{1/2}(v) \, dv, \]

and

\[ c_g(t, x) = \int_{\mathbb{R}^3} \left( \frac{|v|^2}{6} - \frac{1}{2} \right)g((t, x), v) \mu^{1/2}(v) \, dv. \]
Thus, \( (4.3) \) can be obtained by using integration by parts. To get \( (4.3) \), we use the results from Section 3 to obtain

\[
\|g\|_{X^N(R^3)} \geq (Lg, g)_{L^2(R^3)} \geq \eta_0 \| (1 - P) g \|_{X^N(R^3)}^2
\]

\[
\geq \frac{\eta_0}{2} \| g \|_{X^N(R^3)}^2 - C \| P g \|_{X^N(R^3)}^2 \geq \frac{\eta_0}{2} \| g \|_{X^N(R^3)}^2 - C \| g \|_{L^2(R^3)}^2.
\]

Finally, \( (4.3) \) follows directly from Section 3. \( \square \)

The following Lemma is an application of the Sobolev embedding for functions with values in a Hilbert space.

**Lemma 4.2.**

\[
\sup_{x \in \mathbb{R}^3} \| f(x, \cdot) \|_{\mathcal{B}_1} \leq \| f \|_{X^2(R^3)}.
\]

**Proof.** It follows from the definition that

\[
\left( \sup_{x \in \mathbb{R}^3} \| f(x, \cdot) \|_{\mathcal{B}_1} \right)^2 \leq \int \int \int B \mu \left( \sup_{x \in \mathbb{R}^3} (f(x, \nu') - f(x, \nu))^2 \right) dv d\nu d\sigma
\]

\[
+ \int \int \int B \left( \sup_{x \in \mathbb{R}^3} f(x, \nu)^2 \right) (|\sqrt{\mu'} - \sqrt{\mu}|)^2 dv d\nu d\sigma
\]

\[
\leq \int \int \int B \mu \left( \sum_{|\alpha| \leq 2} \int (\partial^\alpha_x f(x, \nu') - \partial^\alpha_x f(x, \nu))^2 dx \right) dv d\nu d\sigma
\]

\[
+ \int \int \int B \left( \sum_{|\alpha| \leq 2} \int f(x, \nu)^2 dx \right) (|\sqrt{\mu'} - \sqrt{\mu}|)^2 dv d\nu d\sigma
\]

\[
\leq \sum_{|\alpha| \leq 2} \int \| \partial^\alpha_x f(x, \cdot) \|_{\mathcal{B}_1}^2 dx.
\]

**Proposition 4.3.** Under the assumption of Theorem 3.3, for any \( N \geq 3 \), we have, for all \( \alpha \in \mathbb{N}^3, |\alpha| \leq N, \)

\[
|\partial^\alpha x \Gamma(f, g), \mathbf{h}|_{L^2(R^3)} \leq \left( \| f \|_{H^N(R^3; L^2(R^3))} \| g \|_{X^N(R^3)} + \| f \|_{X^N(R^3)} \| g \|_{H^N(R^3; \mathbf{h})} \right) \| \mathbf{h} \|_{X^N(R^3)}.
\]

**Proof.** Firstly, if \( |\alpha_1| \leq N - 2 \), we get from Theorem 3.3, Lemma 4.2 and usual Sobolev embedding, replacing the "min" term by the corresponding terms without the weights that

\[
|\left( \partial^\alpha_x \Gamma(f, g), \mathbf{h} \right)|_{L^2(R^3)} \leq \left( \left( \| \partial^\alpha_x f \|_{L^2(R^3)} \| \partial^\alpha_x g \|_{L^2(R^3)}^2 + \| \partial^\alpha_x f \|_{L^2(R^3)} \| \partial^\alpha_x g \|_{L^2(R^3)}^2 + \| \partial^\alpha_x f \|_{L^2(R^3)} \| \partial^\alpha_x g \|_{L^2(R^3)}^2 \right) dx \right)^{1/2} \| \mathbf{h} \|_{X^N(R^3)}
\]

\[
\leq \left( \| f \|_{H^{N-1; \alpha_3} L^2(R^3)} \| g \|_{X^{N-2} \mathbb{R}^3} \right) \| \mathbf{h} \|_{X^N(R^3)}.
\]

If \( |\alpha_1| = N - 1, N \), then \( |\alpha_2| + 2 \leq N \), we get in a similar way, again from Theorem 3.2 that

\[
|\left( \partial^\alpha_x \Gamma(f, g), \mathbf{h} \right)|_{L^2(R^3)} \leq \left( \left( \| \partial^\alpha_x f \|_{L^2(R^3)} \| \partial^\alpha_x g \|_{L^2(R^3)}^2 + \| \partial^\alpha_x f \|_{L^2(R^3)} \| \partial^\alpha_x g \|_{L^2(R^3)}^2 + \| \partial^\alpha_x f \|_{L^2(R^3)} \| \partial^\alpha_x g \|_{L^2(R^3)}^2 \right) dx \right)^{1/2} \| \mathbf{h} \|_{X^N(R^3)}
\]

\[
\leq \left( \| f \|_{H^{N-1; \alpha_3} L^2(R^3)} \| g \|_{X^{N-2} \mathbb{R}^3} \right) \| \mathbf{h} \|_{X^N(R^3)}.
\]
Remark 4.4. The above proof shows that, for $|\alpha| < N$,

$$
\left|\left(\partial_{\alpha}^1 f, \partial_{\beta}^3 g, h\right)_{L^2(\mathbb{R}^3)}\right| \leq \|f\|_{H^\gamma(\mathbb{R}^3)} \|g\|_{V^\gamma(\mathbb{R}^3)} \left(\|f\|_{L^3(\mathbb{R}^3)} + \|g\|_{H^\gamma(\mathbb{R}^3)}\right).
$$

Finally, the estimate on the linear operator $L_2$ can be given as follows, which in fact can be deduced from Section 2.

Proposition 4.5. For all $0 < s < 1$, $\gamma > -3$ and any $\alpha \in \mathbb{N}^3$, we have

$$
\left|\left(\partial_{\alpha}^1 L_2(f), h\right)_{L^2(\mathbb{R}^3)}\right| \leq C_{\alpha} \|f\|_{H^\gamma(\mathbb{R}^3)} \|g\|_{H^\gamma(\mathbb{R}^3)} \|f\|_{L^2(\mathbb{R}^3)}.
$$

4.2. Estimation with weight. We now prove the following upper bound with weights.

Proposition 4.6. For all $0 < s < 1$, $\gamma > -3$, and for any $N \geq 6$, $\ell \geq N$, $|\alpha| + |\beta| \leq N$, we have

$$
\left|\left(\partial_{\alpha}^1 L_2(f), h\right)_{L^2(\mathbb{R}^3)}\right| \leq \|f\|_{H^\gamma(\mathbb{R}^3)} \|\eta_3^\gamma(f)\|_{L^\beta(\mathbb{R}^3)}.
$$

Proof. By using Leibniz formula, we have

$$
\left(\partial_{\alpha}^1 L_2(f), h\right)_{L^2(\mathbb{R}^3)} = \sum_{\alpha_1, \alpha_2, \alpha_3} C_{\alpha_1, \alpha_2, \alpha_3} \left(\partial_{\alpha_1}^1 f, \partial_{\alpha_2}^3 g, \partial_{\alpha_3}^3 g, h\right)_{L^2(\mathbb{R}^3)}
$$

Note that $\mathcal{T}$ shares the same upper bound properties as $\Gamma$ given in the previous propositions. In fact, by using Proposition 3.13

$$
A := \left(\mathcal{T}(\partial_{\alpha}^1 f, \partial_{\beta}^3 g, \partial_{\beta}^3 g, h)\right)_{L^2(\mathbb{R}^3)}
$$

For this, we divide the discussion into two cases.

Case 1: $|\alpha_1| + |\beta_1| \leq N - 2$. We have, by using $H^2(\mathbb{R}^3) \subset L^\infty(\mathbb{R}^3)$, and $\gamma + 2s \leq 0$ that

$$
A \leq \|f\|_{H^\gamma(\mathbb{R}^3)} \left(\|W_{\ell=\gamma} f\|_{L^2(\mathbb{R}^3)} + \|\partial_{\beta}^3 g\|_{L^2(\mathbb{R}^3)}\right).
$$
Next, we deal with the following term and the discussion is also divided into several steps:

\[
\sum_{\alpha_2, \beta, \ell} c_{\alpha_2, \beta, \ell} \left( T(\partial_{\beta}^\ell f, W_{\ell-\gamma} \partial_{\beta}^\ell g, \mu_{\beta}), h \right)
\]

Next, we deal with the following term and the discussion is also divided into several steps:

**Case I:** \( |\alpha_1| + |\beta| \leq 1 \). From Proposition 3.1, we get

\[
B = \left| T(\partial_{\beta}^\ell f, W_{\ell-\gamma} \partial_{\beta}^\ell g, \mu_{\beta}), h \right|_{L^2(\mathbb{R}^3)}
\]

\[
\leq ||h||_{L^2_{\beta,\ell}} \left( ||f||_{L^4(\mathbb{R}^3)} ||W_{\ell-\gamma} \partial_{\beta}^\ell g||_{L^2(\mathbb{R}^3)} + ||f||_{L^4(\mathbb{R}^3)} ||W_{\ell-\gamma} \partial_{\beta}^\ell g||_{L^2(\mathbb{R}^3)} \right)
\]

**Case II:** \( 2 \leq |\alpha_1| + |\beta| \leq 3 \). Again from Proposition 3.1, one has

\[
B \leq ||h||_{L^2_{\beta,\ell}} \left( ||f||_{L^4(\mathbb{R}^3)} ||W_{\ell-\gamma} \partial_{\beta}^\ell g||_{L^2(\mathbb{R}^3)} + ||f||_{L^4(\mathbb{R}^3)} ||W_{\ell-\gamma} \partial_{\beta}^\ell g||_{L^2(\mathbb{R}^3)} \right)
\]

**Case III:** \( 4 \leq |\alpha_1| + |\beta| \leq N-1 \). Then \( |\alpha_2| + |\beta| \leq N-4 \), and from Proposition 3.3, we get

\[
B \leq ||h||_{L^2_{\beta,\ell}} \left( ||\partial_{\beta}^\ell f||_{L^2_{\gamma,\ell}(\mathbb{R}^3)} ||W_{\ell-\gamma} \partial_{\beta}^\ell g||_{L^2(\mathbb{R}^3)} + ||\partial_{\beta}^\ell f||_{L^2_{\gamma,\ell}(\mathbb{R}^3)} ||W_{\ell-\gamma} \partial_{\beta}^\ell g||_{L^2(\mathbb{R}^3)} \right)
\]

Finally, the proof of the proposition is completed.

\[
4.3. \text{Estimation with modified weight}
\]

In the sequel, we shall often use the inequalities

\[
||f||_{L^2_{\gamma,\ell}} \leq ||f||_{L^2} \quad \|f\|_{L^2_{\gamma,\ell}} \leq \|f\|_{L^2}
\]

The first inequality is valid since we are assuming that \( \gamma + 2s \leq 0 \), while the second one comes from Section 3.

To obtain the global existence, the upper bound given in Proposition 4.4 for the general case is not enough because this bound cannot be controlled by the coercivity estimate of the linearized operator that contains weight in the case of soft potential. To
overcome this, we now prove the following upper bound under the conditions of Theorem 1.5.

**Proposition 4.7.** Under the assumptions of Theorem 1.5 on the parameters $\gamma$ and $s$, for any $N \geq 6$, $\ell \geq N$, $|\alpha| + |\beta| \leq N$, one has

\[
|\langle \check{W}_y, \check{W}_g \rangle_{L^2(\mathbb{R}^N)}| \leq \left(\|f\|_{H^s(\mathbb{R}^N)} \|g\|_{H^s(\mathbb{R}^N)} \right) + \|h\|_{H^s(\mathbb{R}^N)}.
\]

**Proof.** First, notice that from Remark 3.14 and (4.5), we have for $\gamma > -3$,

\[
|\langle \check{W} \Gamma, g \rangle_{L^2(\mathbb{R}^N)}| \leq \left(\|f\|_{L^2(\mathbb{R}^N)} \|g\|_{L^2(\mathbb{R}^N)} \right) \|h\|_{L^2(\mathbb{R}^N)}.
\]

Recall the definition of $\mathcal{T}$ to deduce that

\[
|\langle \check{W}_y \partial_\alpha f, \check{W}_g \partial_\beta h \rangle_{L^2(\mathbb{R}^N)}| \leq \sum_{\beta_1 + \beta_2 + \beta_3 = \beta} |\langle \check{W}_y \mathcal{T}(\partial_{\beta_1} f, \partial_{\beta_2} g, \partial_{\beta_3} h) \rangle_{L^2(\mathbb{R}^N)}|
\]

\[
= \sum_{\beta_1 + \beta_2 + \beta_3 = \beta} \left(\langle \mathcal{T}(\partial_{\beta_1} f, \partial_{\beta_2} g, \partial_{\beta_3} h) \rangle_{L^2(\mathbb{R}^N)} \right)
\]

\[
+ |\langle \check{W}_y \mathcal{T}(\partial_{\beta_1} f, \partial_{\beta_2} g, \partial_{\beta_3} h) \rangle_{L^2(\mathbb{R}^N)}|.
\]

By using Theorem 1.2, we obtain

\[
|\langle \mathcal{T}(\partial_{\beta_1} f, \partial_{\beta_2} g, \partial_{\beta_3} h) \rangle_{L^2(\mathbb{R}^N)}| \leq \left(\|\partial_{\beta_1} f\|_{L^2(\mathbb{R}^N)} \|\partial_{\beta_2} g\|_{L^2(\mathbb{R}^N)} \|\partial_{\beta_3} h\|_{L^2(\mathbb{R}^N)} \right) \|\check{W}_y \partial_\beta h\|_{L^2(\mathbb{R}^N)}.
\]

Moreover, (4.2) implies that

\[
|\langle \check{W}_y \partial_\beta f, \check{W}_g \partial_\beta h \rangle_{L^2(\mathbb{R}^N)}| \leq \left(\|\partial_\beta f\|_{L^2(\mathbb{R}^N)} \|\partial_\beta g\|_{L^2(\mathbb{R}^N)} \|\partial_\beta h\|_{L^2(\mathbb{R}^N)} \right) \|\check{W}_y \partial_\beta h\|_{L^2(\mathbb{R}^N)}.
\]

As a consequence,

\[
|\langle \check{W}_y \partial_\beta f, \check{W}_g \partial_\beta h \rangle_{L^2(\mathbb{R}^N)}| \leq \sum_{\beta_1 + \beta_2 + \beta_3 = \beta} \left(\|\partial_{\beta_1} f\|_{L^2(\mathbb{R}^N)} \|\partial_{\beta_2} g\|_{L^2(\mathbb{R}^N)} \|\partial_{\beta_3} h\|_{L^2(\mathbb{R}^N)} \right) \|\check{W}_y \partial_\beta h\|_{L^2(\mathbb{R}^N)}
\]

\[
+ \|\check{W}_y \partial_\beta f\|_{L^2(\mathbb{R}^N)} \|\partial_{\beta_2} g\|_{L^2(\mathbb{R}^N)} \|\check{W}_y \partial_\beta h\|_{L^2(\mathbb{R}^N)} dx
\]

\[
+ \|\check{W}_y \partial_{\beta_1} f\|_{L^2(\mathbb{R}^N)} \|\partial_{\beta_2} g\|_{L^2(\mathbb{R}^N)} \|\check{W}_y \partial_\beta h\|_{L^2(\mathbb{R}^N)} dx
\]

\[
+ \|\check{W}_y \partial_{\beta_1} f\|_{L^2(\mathbb{R}^N)} \|\partial_{\beta_2} g\|_{L^2(\mathbb{R}^N)} \|\check{W}_y \partial_\beta h\|_{L^2(\mathbb{R}^N)} dx
\]

\[
= \sum_{\alpha, \beta_1, \beta_2} (G^1_{\alpha, \beta_1, \beta_2} + G^2_{\alpha, \beta_1, \beta_2}).
\]
Now these terms are discussed in the following two cases.

- When $|\alpha_1| + |\beta_1| \leq N/2$, we have

$$G^{1}_{\alpha_1,\beta_1} \leq \left\| \partial^{\infty}_{\alpha_1} f \right\|_{L^2(\mathbb{R}^3)} \left\| \partial^{\infty}_{\beta_1} g \right\|_{\mathcal{L}_1(\mathbb{R}^3)}$$

$$G^{2}_{\alpha_1,\beta_1} \leq \left\| \partial^{\infty}_{\alpha_1} f \right\|_{L^2(\mathbb{R}^3)} \left\| \partial^{\infty}_{\beta_1} g \right\|_{\mathcal{L}_1(\mathbb{R}^3)}$$

- When $|\alpha_1| + |\beta_1| \geq N/2$, we have

$$G^{1}_{\alpha_1,\beta_1} \leq \left\| \partial^{\infty}_{\alpha_1} f \right\|_{L^2(\mathbb{R}^3)} \left\| \partial^{\infty}_{\beta_1} g \right\|_{\mathcal{L}_1(\mathbb{R}^3)}$$

$$G^{2}_{\alpha_1,\beta_1} \leq \left\| \partial^{\infty}_{\alpha_1} f \right\|_{L^2(\mathbb{R}^3)} \left\| \partial^{\infty}_{\beta_1} g \right\|_{\mathcal{L}_1(\mathbb{R}^3)}$$

Here, we have used Lemma 4.3 to get

$$\left\| \partial^{\infty}_{\alpha_1} f \right\|_{L^2(\mathbb{R}^3)} \leq \left\| \partial^{\infty}_{\alpha_1} g \right\|_{L^2(\mathbb{R}^3)}$$

for $|\alpha_1| + |\beta_1| \leq N/2$. Therefore, we complete the proof of the proposition.

4.4. Weighted coercivity of the linearized operator. We turn to the weighted lower estimates, more precisely, the lower bound for $(\mathcal{L}_1 g, W_i g)_{L^2(\mathbb{R}^3)}$. Let us recall that in Section 3 it was shown that, if $\gamma > -3$, then there exists a constant $C > 0$ such that

$$(4.9) \quad \left\| g \right\|^2_{\mathcal{L}_1(\mathbb{R}^3)} \geq \left( \mathcal{L}_1 g, g \right)_{L^2(\mathbb{R}^3)} \geq \eta_0 \left\| g \right\|^2_{\mathcal{L}_1(\mathbb{R}^3)} - C \left\| g \right\|^2_{L^2(\mathbb{R}^3)}.$$ 

In the estimation on the weighted linearized collisional operator $\mathcal{L}_1$, we need to consider the commutator estimate of the weight and the linearized operator. However, we cannot apply the Proposition 3.13 directly because the error term then will have the weight of the order of $s + \gamma/2$. The purpose of the following proposition is to show that the error term coming from this commutator has the weight of order $\gamma/2$ only.

Proposition 4.8. For all $0 < s < 1$, $\gamma > -3$, and for any $l \geq 0$, there exists a positive constant $C$ such that

$$(4.10) \quad (\mathcal{L}_1 g, W_i g)_{L^2(\mathbb{R}^3)} \geq \frac{\eta_0}{2} \left\| W_i g \right\|^2_{\mathcal{L}_1(\mathbb{R}^3)} - C \left\| W_i g \right\|^2_{L^2(\mathbb{R}^3)}.$$ 

Moreover, for any $\beta \in \mathbb{N}^3 \setminus \{0\}$, one has

$$(W_i \partial_{\beta_1} \mathcal{L}_1 g, W_i \partial_{\beta_2} g)_{L^2(\mathbb{R}^3)} \geq \frac{\eta_0}{2} \left| \left\| W_i \partial_{\beta_1} g \right\|_{\mathcal{L}_1(\mathbb{R}^3)}^2 - C \left\| W_i \partial_{\beta_1} g \right\|^2_{L^2(\mathbb{R}^3)} \right.$$ 

$$- C \left( \sum_{\beta_1 < \beta} \left\| W_i \partial_{\beta_1} g \right\|^2_{\mathcal{L}_1(\mathbb{R}^3)} \right) \left\| W_i \partial_{\beta_2} g \right\|^2_{\mathcal{L}_1(\mathbb{R}^3)}.$$ 

Proof. According to (4.2), it is enough to show that

$$\left| (\mathcal{L}_1 g, W_i g)_{L^2(\mathbb{R}^3)} \right| \leq C_0 \left\| W_i g \right\|^2_{L^2(\mathbb{R}^3)} + \delta \left\| W_i g \right\|^2_{L_{2x_1}^2(\mathbb{R}^3)}.$$
where $\delta > 0$ can be arbitrarily small. By using the above expression, one has

\[
(W_l, L_l g, W_l g)_{L^2} = - \iiint b \Phi_l (\mu_l')^{1/2} g' - (\mu_l)^{1/2} g) \mu_l^{1/2} W_l W_l g dv d\sigma dv
\]

\[
= - \iiint b \Phi_l (\mu_l')^{1/2} W_l g' - (\mu_l)^{1/2} W_l g \mu_l^{1/2} W_l g dv d\sigma dv
\]

\[
= - \iiint b \Phi_l \left( (\mu_l')^{1/2} W_l g' - (\mu_l)^{1/2} W_l g + (\mu_l')^{1/2} g'(W_l - W_l') \right) \mu_l^{1/2} W_l g dv d\sigma dv
\]

\[
= 1/2 \iiint b \Phi_l \left( (\mu_l')^{1/2} W_l g' - (\mu_l)^{1/2} W_l g \right)^2 dv d\sigma dv
\]

\[
- \iiint b \Phi_l \left( (\mu_l')^{1/2} g'(W_l - W_l') \right) \mu_l^{1/2} W_l g dv d\sigma dv
\]

\[
= (L_l (W_l g), (W_l g))_{L^2} + I,
\]

where

\[
I = (W_l, L_l g, W_l g)_{L^2} = - \iiint b \Phi_l (\mu_l')^{1/2} g' \mu_l^{1/2} W_l g (W_l - W_l') dv d\sigma dv.
\]

Changing variables yields

\[
I = - \iiint b \Phi_l (\mu_l')^{1/2} g' \mu_l^{1/2} W_l g' (W_l' - W_l) dv d\sigma dv.
\]

Adding the above two equations gives

\[
2I = - \iiint b \Phi_l (\mu_l')^{1/2} g' \mu_l^{1/2} W_l g' (W_l' - W_l) dv d\sigma dv.
\]

Then, by using the Cauchy-Schwarz inequality with respect to full variables, we find

\[
|I| \leq \iiint b \Phi_l (\mu_l')^{1/2} g' (W_l' - W_l) dv d\sigma dv.
\]

When $l \geq 1$, since

\[
\int_{S^2} b (W_l' - W_l)^2 dv d\sigma \leq |v - v_*|^{2(l-1)} [v - v_*]^{2l-2x}
\]

\[
\leq v^{2l-2x} < v_*^{2l-2x} (v^{2l-2x} + v_*^{2l-2x})
\]

\[
\leq v_*^{2l-2x} < v^{2l-2x} + v^{2l-2x} + v_*^{2l-2x} \leq v^{2l-2x} + v^{2l-2x} < 2l < v > 2l,
\]

by using Section 2, we have

\[
|I| \leq \|W_l g\|_{L^2}^2.
\]

When $0 < l \leq 1$, by (3.8), we have

\[
\int_{S^2} b |W_l' - W_l|^2 dv d\sigma \leq |v|^{2(1-s)} < v_*^{2(1-s)} |v - v_*|^{2l}.
\]

Consider

\[
I = \iint_{|v| \leq R} + \iint_{|v| \geq R} = I_1 + I_2.
\]

It is obvious that for any fixed $R$,

\[
|I_1| \leq \|W_l g\|_{L^2}^2.
\]
Therefore, we have

\[
I_2 = \int_{|v| \leq R} \int_{|\theta| \leq \frac{1}{2} |\gamma|} b(\cos \theta) |W'_{\beta} - W_{\beta}|^2 \, d\tau + \int_{|v| \leq R} \int_{|\theta| \leq \frac{1}{2} |\gamma|} b(\cos \theta) |W'_{\beta} - W_{\beta}|^2 \, d\tau = I_{2,1} + I_{2,2},
\]

which are the singular part and the non-singular part respectively. For the singular part \(I_{2,1}\), note that

\[
\int_{|\theta| \leq \frac{1}{2} |\gamma|} b(\cos \theta) |W'_{\beta} - W_{\beta}|^2 \, d\tau \leq |v - v_*|^2 < |v|^{2l-1} \frac{|v|^2 - |v_*|^2}{|v|^{2l-2}} \leq |v - v_*|^{2s} < |v|^{2l-2s} \leq |v|^{2l} < v_* >^{2l};
\]

while for the non-singular part \(I_{2,2}\), one has

\[
\int_{|\theta| \leq \frac{1}{2} |\gamma|} b(\cos \theta) |W'_{\beta} - W_{\beta}|^2 \, d\tau \leq |v - v_*|^{2s} < |v|^{2l} < v_* >^{2l} |v|^{-2s}.
\]

Therefore, we have

\[
I_{2,1} \leq \|W_g\|_{L^2}^2,
\]

and

\[
I_{2,2} \leq R^{-2s} \|W_g\|_{L^2}^2.
\]

By taking \(R\) large enough, we complete the proof of (4.10).

Now we turn to the derivatives in \(v\) variable. For \(\beta \in \mathbb{N}^3 \setminus \{0\}\), we have

\[
\partial_\beta \mathcal{L}_1(g) = \mathcal{L}_1(\partial_\beta g) + \sum_{\beta_1 < \beta} C_{\beta_1, \beta_2} \mathcal{T}(\partial_{\beta_1} \mu, \partial_{\beta_2} g, \partial_{\beta_2} \mu).
\]

By (4.10), we have

\[
(W_\beta \partial_\beta \mathcal{L}_1(g), W_\beta \partial_\beta g) = (W_\beta \mathcal{L}_1(\partial_\beta g), W_\beta \partial_\beta g) + \sum_{\beta_1 < \beta} C_{\beta_1, \beta_2} (W_\beta \mathcal{T}(\partial_{\beta_1} \mu, \partial_{\beta_2} g, \partial_{\beta_2} \mu), W_\beta \partial_\beta g) \geq \frac{\gamma_0}{2} \|W_\beta \partial_\beta g\|_{\Phi}^2 - C \|W_\beta \partial_\beta g\|_{L^2}^2 + II,
\]

where

\[
II = \sum_{\beta_1 < \beta} C_{\beta_1, \beta_2} (W_\beta \mathcal{T}(\partial_{\beta_1} \mu, \partial_{\beta_2} g, \partial_{\beta_2} \mu); W_\beta \partial_\beta g).
\]

Recall that the operator \(\mathcal{T}\) shares the same commutator properties as \(\Gamma\). As in the proofs given in Section 2, the linearized operator \(\mathcal{T}(\partial_{\beta_1} \mu, \partial_{\beta_2} g, \partial_{\beta_2} \mu)\) satisfies

\[
|\mathcal{T}(\partial_{\beta_1} \mu, \partial_{\beta_2} g, \partial_{\beta_2} \mu)| \leq \|W_\beta \partial_\beta g\|_{\Phi} \|W_\beta \partial_\beta g\|_{\Phi}.
\]

Hence

\[
|II| \leq \left( \sum_{\beta_1 < \beta} \|W_\beta \partial_{\beta_1} g\|_{\Phi} \right) \|W_\beta \partial_\beta g\|_{\Phi}.
\]

This completes the proof of the proposition.

\[\square\]

5. Local existence

In the following two subsections, we prove Theorem 1.3 and the local existence of solutions in the function space considered in Theorem 1.4.
5.1. Classical solutions. We now proceed to the proof of Theorem 3. The restriction of soft potential $γ + 2x ≤ 0$ will play an important role.

Consider the following Cauchy problem for a linear Boltzmann equation with a given function $f$,

$$\partial_t g + v \cdot \nabla_x g + L_1 g = \Gamma(f, g) - \mathcal{L}_2 f, \quad g|_{t=0} = g_0,$$

which is equivalent to the problem:

$$\partial_t G + v \cdot \nabla_x G = Q(F, G), \quad G|_{t=0} = G_0,$$

with $F = \mu + \sqrt{\mu} f$ and $G = \mu + \sqrt{\mu} g$. The proof is based on energy estimates in the functional space $\mathcal{H}^α_γ(\mathbb{R}^d)$.

For $N ≥ 6$, $ℓ ≥ N$ and $α, β ∈ \mathbb{N}^d$, $|α| + |β| ≤ N$, taking

$$\varphi(t, x, v) = (-1)^{|α|+|β|} β^2 γ^2 W^2_0 γ^2 γ^2 γ^2 t(t, x, v),$$

as a test function for equation (5.3), we get

$$\frac{1}{2} \frac{d}{dt} \left[ W - y β γ^2 γ^2 γ^2 g(t) \right]_{L^2(\mathbb{R}^d)}^2 + \left( W - y β γ^2 γ^2 γ^2 L_1(g), W - y β γ^2 γ^2 γ^2 \right)_{L^2(\mathbb{R}^d)} + \left( W - y β γ^2 γ^2 γ^2 β γ^2 γ^2 γ^2, \nabla_x g, W - y β γ^2 γ^2 γ^2 \right)_{L^2(\mathbb{R}^d)} = \left( W - y β γ^2 γ^2 γ^2 β γ^2 γ^2 γ^2, \right)_{L^2(\mathbb{R}^d)} - \left( W - y β γ^2 γ^2 γ^2 β γ^2 γ^2 γ^2, W - y β γ^2 γ^2 γ^2 \right)_{L^2(\mathbb{R}^d)},$$

where we have used the fact that

$$\left( v \cdot \nabla_x (W - y β γ^2 γ^2 γ^2), W - y β γ^2 γ^2 γ^2 \right)_{L^2(\mathbb{R}^d)} = 0.$$

We have immediately

$$\left| \left( W - y β γ^2 γ^2 γ^2, \nabla_x g, W - y β γ^2 γ^2 γ^2 \right)_{L^2(\mathbb{R}^d)} \right| \leq ||f||_{2γ^2 γ^2 γ^2} ||g||_{2γ^2 γ^2 γ^2}.$$

By Proposition 4.3, one has

$$\left| \left( W - y β γ^2 γ^2 γ^2 L_1(g), W - y β γ^2 γ^2 γ^2 \right)_{L^2(\mathbb{R}^d)} \right| \leq ||f||_{2γ^2 γ^2 γ^2} ||g||_{2γ^2 γ^2 γ^2}.$$

Now by using (4.4), with $f = \mu$, we have

$$\left| \left( W - y β γ^2 γ^2 γ^2 L_1(g), W - y β γ^2 γ^2 γ^2 \right)_{L^2(\mathbb{R}^d)} \right| \leq C ||\varphi||_{2γ^2 γ^2 γ^2} + \delta ||\varphi||_{2γ^2 γ^2 γ^2}.$$

Applying Proposition 4.6, we get for any $|α| + |β| ≤ N$,

$$\frac{1}{2} \frac{d}{dt} \left[ ||W - y β γ^2 γ^2 γ^2 g(t)||_{L^2(\mathbb{R}^d)}^2 \right] + \left( L_1(W - y β γ^2 γ^2 γ^2 g(t)), W - y β γ^2 γ^2 γ^2 \right)_{L^2(\mathbb{R}^d)} \leq ||f||_{2γ^2 γ^2 γ^2} ||\varphi||_{2γ^2 γ^2 γ^2} + ||f||_{2γ^2 γ^2 γ^2} ||\varphi||_{2γ^2 γ^2 γ^2} ||\varphi||_{2γ^2 γ^2 γ^2} + \delta ||\varphi||_{2γ^2 γ^2 γ^2}.$$

By using now the coercivity estimate from Section 2 and Lemma 4.1, and by taking summation over $|β| ≤ N$, the Cauchy-Schwarz inequality and soft potential assumption imply that

$$\frac{d}{dt} \left[ ||\varphi||_{2γ^2 γ^2 γ^2}^2 \right] + \frac{n_0}{2} ||\varphi||_{2γ^2 γ^2 γ^2}^2 ≤ C \left[ ||f||_{2γ^2 γ^2 γ^2} ||\varphi||_{2γ^2 γ^2 γ^2} + ||f||_{2γ^2 γ^2 γ^2} ||\varphi||_{2γ^2 γ^2 γ^2} + \delta ||\varphi||_{2γ^2 γ^2 γ^2} \right].$$

In conclusion, we are ready to prove the following proposition.
Proposition 5.1. Assume that $0 < s < 1$, $\gamma > -3$ and let $N \geq 6, \ell \geq N$. Suppose that $g_0 \in \mathcal{H}^s_\ell (\mathbb{R}^6)$ and

$$f \in L^\infty([0, T]; \mathcal{H}^s_\ell (\mathbb{R}^6)) \bigcap L^2([0, T]; \mathcal{B}^s_\ell (\mathbb{R}^6)).$$

If $g \in L^\infty([0, T]; \mathcal{H}^s_\ell (\mathbb{R}^6)) \bigcap L^2([0, T]; \mathcal{B}^s_\ell (\mathbb{R}^6))$ is a solution of the Cauchy problem (5.1), then there exists $\epsilon_0 > 0$ such that if

$$\|f\|_{L^\infty([0, T]; \mathcal{H}^s_\ell (\mathbb{R}^6))} + \|f\|_{L^2([0, T]; \mathcal{B}^s_\ell (\mathbb{R}^6))} \leq \epsilon_0^2,$$

we have

$$\|g\|_{L^\infty([0, T]; \mathcal{H}^s_\ell (\mathbb{R}^6))}^2 + \|g\|_{L^2([0, T]; \mathcal{B}^s_\ell (\mathbb{R}^6))}^2 \leq C e^{\gamma T} (\|g_0\|_{\mathcal{H}^s_\ell (\mathbb{R}^6)}^2 + \epsilon_0^2 T),$$

for a constant $C > 0$ depending only on $N$ and $\ell$.

Proof. From (5.3), we have, for $t \in [0, T]$,

$$\|g(t)\|_{\mathcal{H}^s_\ell (\mathbb{R}^6)}^2 + \frac{\eta_0}{2} e^{C T} \int_0^T e^{-C s} \|g(s)\|_{\mathcal{B}^s_\ell (\mathbb{R}^6)}^2 ds \leq e^{C T} \|g_0\|_{\mathcal{H}^s_\ell (\mathbb{R}^6)}^2 + \int_0^T e^{-C s} \|g(s)\|_{\mathcal{B}^s_\ell (\mathbb{R}^6)}^2 ds.$$  

Then

$$\|g\|_{L^\infty([0, T]; \mathcal{H}^s_\ell (\mathbb{R}^6))}^2 + \frac{\eta_0}{2} \|g\|_{L^2([0, T]; \mathcal{B}^s_\ell (\mathbb{R}^6))}^2 \leq e^{C T} \|g_0\|_{\mathcal{H}^s_\ell (\mathbb{R}^6)}^2 + \int_0^T e^{-C s} \|g(s)\|_{\mathcal{B}^s_\ell (\mathbb{R}^6)}^2 ds + \int_0^T e^{-C s} \|f(s)\|_{\mathcal{B}^s_\ell (\mathbb{R}^6)}^2 ds.$$  

Hence, if we choose $C e^{\gamma T} \epsilon_0 \leq \frac{\eta_0}{2}, C e^{\gamma T} \epsilon_0^2 \leq \frac{1}{2},$ then (5.3) implies that

$$\frac{1}{2} \|g\|_{L^\infty([0, T]; \mathcal{H}^s_\ell (\mathbb{R}^6))}^2 + \frac{\eta_0}{4} \|g\|_{L^2([0, T]; \mathcal{B}^s_\ell (\mathbb{R}^6))}^2 \leq e^{C T} \|g_0\|_{\mathcal{H}^s_\ell (\mathbb{R}^6)}^2 + C e^{\gamma T} \epsilon_0^2.$$

And this completes the proof of the proposition. \qed

From the energy estimate (5.4), one can deduce the local existence as in [7], and we have proved the following precise version of Theorem 1.3.

Theorem 5.2. Under the assumptions of Theorem 1.3, let $N \geq 6, \ell \geq N$. There exist $\epsilon_1, T > 0$ such that if $g_0 \in \mathcal{H}^s_\ell (\mathbb{R}^6)$ and

$$\|g_0\|_{\mathcal{H}^s_\ell (\mathbb{R}^6)} \leq \epsilon_1,$$

then the Cauchy problem (1.3) admits a solution

$$g \in L^\infty([0, T]; \mathcal{H}^s_\ell (\mathbb{R}^6)) \cap L^2([0, T]; \mathcal{B}^s_\ell (\mathbb{R}^6)).$$

Remark 5.3. By using the Proposition 4.7, we can get the same results as Theorem 5.2 if we replace $\mathcal{H}^s_\ell (\mathbb{R}^6), \mathcal{B}^s_\ell (\mathbb{R}^6)$ by $\mathcal{H}^s_\ell (\mathbb{R}^6), \mathcal{B}^s_\ell (\mathbb{R}^6)$ respectively. In other words, Theorem 1.3 holds also in the function space $\mathcal{H}^s_\ell (\mathbb{R}^6)$. 
5.2. $L^2$-solutions. Under some more restrictive conditions on the parameters $\gamma$ and $s$, we can prove local existence of solutions with only differentiation in the $x$ variable. That is, we will deduce the energy estimate for the equation (5.3) in the function space $H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3))$. For $N \geq 3$ and $\beta \in \mathbb{R}^3$, $|\beta| \leq N$, by taking
\[
\varphi(t, x, v) = (-1)^m \partial_t^m \partial_x^\beta g(t, x, v),
\]
as a test function on $\mathbb{R}^3_+ \times \mathbb{R}^3_+$, we get
\[
\frac{1}{2} \frac{d}{dt} \|\partial_t^\beta g\|_{L^2(\mathbb{R}^3)}^2 + \left(\partial_t^\beta \mathcal{L}_1(g), \partial_t^\beta g\right)_{L^2(\mathbb{R}^3)} = \left(\partial_t^\beta \mathcal{L}_2(f), \partial_t^\beta g\right)_{L^2(\mathbb{R}^3)},
\]
where we have used the fact that $\left(\nu \cdot \nabla_x (\partial_t^\beta g), \partial_t^\beta g\right)_{L^2(\mathbb{R}^3)} = 0$.

Applying now Proposition 4.3, Proposition 4.5, we get for any $N \geq 3$ and $|\beta| \leq N$,
\[
\frac{1}{2} \frac{d}{dt} \|\partial_t^\beta g\|_{L^2(\mathbb{R}^3)}^2 + \left(\mathcal{L}_1(\partial_t^\beta g), \partial_t^\beta g\right)_{L^2(\mathbb{R}^3)} \leq \left\|\mathcal{L}_2(\mathbf{f})\right\|_{L^2(\mathbb{R}^3)} \left\|\partial_t^\beta g\right\|_{L^2(\mathbb{R}^3)} + \left\|\mathcal{L}_2(f)\right\|_{L^2(\mathbb{R}^3)} \left\|\partial_t^\beta g\right\|_{L^2(\mathbb{R}^3)} + \left\|\mathcal{L}_2(f)\right\|_{L^2(\mathbb{R}^3)} \left\|\partial_t^\beta g\right\|_{L^2(\mathbb{R}^3)}.
\]

By using now the coercivity estimate (4.2), and by taking summation over $|\beta| \leq N$, the Cauchy-Schwarz inequality leads to
\[
\frac{d}{dt} \|g\|_{L^2(\mathbb{R}^3)}^2 + \frac{\nu_0}{2} \|g\|_{X^{N}(\mathbb{R}^3)}^2 \leq C \left\|\mathcal{L}_2(\mathbf{f})\right\|_{L^2(\mathbb{R}^3)} \left\|g\right\|_{L^2(\mathbb{R}^3)} + \left\|\mathcal{L}_2(f)\right\|_{L^2(\mathbb{R}^3)} \left\|g\right\|_{L^2(\mathbb{R}^3)} + \left\|\mathcal{L}_2(f)\right\|_{L^2(\mathbb{R}^3)} \left\|g\right\|_{L^2(\mathbb{R}^3)}.
\]

We are now ready to prove the following

**Proposition 5.4.** Under the assumptions of Theorem 4.4, let $N \geq 3$, $g_0 \in H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3))$ and
\[
f \in L^\infty([0, T]; H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3)) \cap L^2([0, T]; X^{N}(\mathbb{R}^6))).
\]
If $g \in L^\infty([0, T]; H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3)) \cap L^2([0, T]; X^{N}(\mathbb{R}^6)))$ is a solution of the Cauchy problem (5.3), then there exists $\epsilon_0 > 0$ such that if
\[
|\mathbf{f}(t)|_{L^2([0, T]; X^{N}(\mathbb{R}^3))} + |\mathbf{f}(t)|_{L^2([0, T]; X^{N}(\mathbb{R}^6))} \leq \epsilon_0 T,
\]
we have
\[
|g(t)|_{L^2([0, T]; H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3)))} + \|g\|_{L^2([0, T]; X^{N}(\mathbb{R}^3))} \leq Ce^{CT} \left\{\|g_0\|_{H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3))} + \epsilon_0 T\right\},
\]
for a constant $C$ depending only on $N$.

**Proof.** From (5.5), we have, for $t \in [0, T]$,
\[
|g(t)|_{H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3))} + \|g\|_{L^2(\mathbb{R}^3)} + \int_0^t e^{-CT} \|g(s)\|_{X^{N}(\mathbb{R}^6)} ds \leq e^{CT} \|g_0\|_{H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3))} + Ce^{CT} \int_0^t e^{-CT} |\mathbf{f}(s)|_{H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3))} \|g(s)\|_{X^{N}(\mathbb{R}^6)} ds
\]
\[
+ Ce^{CT} \int_0^t e^{-CT} |\mathbf{f}(s)|_{H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3))} \|g(s)\|_{X^{N}(\mathbb{R}^6)} ds + \int_0^t e^{-CT} |\mathbf{f}(s)|_{H^N(\mathbb{R}^3_+; L^2(\mathbb{R}^3))} ds \right\}.
\]
Then
\[
\|g\|_{L^2([0,T];H^N(\mathbb{R}^3;L^2(\mathbb{R}^3)))}^2 + \frac{\eta_0}{2}\|g\|_{L^2([0,T];X^N(\mathbb{R}^3))}^2 \leq e^{CT}\|g_0\|_{H^N(\mathbb{R}^3;L^2(\mathbb{R}^3))}^2
\]
\[+ C e^{CT}\left\{\|f\|_{L^2([0,T];H^N(\mathbb{R}^3;L^2(\mathbb{R}^3)))} \|g\|_{L^2([0,T];X^N(\mathbb{R}^3))} + \|f\|_{L^2([0,T];H^N(\mathbb{R}^3;L^2(\mathbb{R}^3)))}^2 + T\|f\|_{L^2([0,T];H^N(\mathbb{R}^3;L^2(\mathbb{R}^3)))}^2\right\}.\]

By choosing \( T \) small as in Proposition 5.3, we complete the proof. \( \square \)

As in [3], the energy estimate (5.7) yields

**Theorem 5.5.** Under the assumptions of Theorem 1.4 for \( N \geq 3 \), there exist \( \epsilon_1 \), \( T > 0 \) such that if \( g_0 \in H^N(\mathbb{R}^3;L^2(\mathbb{R}^3)) \) and
\[
\|g_0\|_{H^N(\mathbb{R}^3;L^2(\mathbb{R}^3))} \leq \epsilon_1,
\]
then the Cauchy problem (1.3) admits a solution
\[
g \in L^\infty([0,T];H^N(\mathbb{R}^3;L^2(\mathbb{R}^3))) \cap L^2([0,T];X^N(\mathbb{R}^3)).
\]

6. **Global solutions**

We are now ready to prove the global existence of weak and classical solutions in the following two subsections.

6.1. **\( L^2 \)-solutions.** We now conclude for the global existence issue in Theorem 1.4. We already gave the macro-micro decomposition of solutions introduced in [23]:
\[
g = Pg + (I-P)g = g_1 + g_2,
g_1 = (a + v \cdot b + |v|^2) \sqrt{\mu}, \quad \mathcal{A} = (a, b, c).
\]

Notice that
\[
\|g\|_{H^N(\mathbb{R}^3;L^2(\mathbb{R}^3))}^2 \sim \|\mathcal{A}\|_{H^N(\mathbb{R}^3)}^2 + \|g_2\|_{H^N(\mathbb{R}^3;L^2(\mathbb{R}^3))}^2,
\]
\[
\|g_2\|_{X^N(\mathbb{R}^3)}^2 \sim \|\mathcal{A}\|_{H^N(\mathbb{R}^3)}^2 + \|g_2\|_{X^N(\mathbb{R}^3)}^2.
\]

The temporal energy functional and dissipation integral of solutions are defined by
\[
E_N = \|g\|_{H^N(\mathbb{R}^3;L^2(\mathbb{R}^3))}^2 = \|g_1\|_{H^N(\mathbb{R}^3;L^2(\mathbb{R}^3))}^2 + \|g_2\|_{H^N(\mathbb{R}^3;L^2(\mathbb{R}^3))}^2
\]
\[
\sim \|\mathcal{A}\|_{H^N(\mathbb{R}^3)}^2 + \|g_2\|_{H^N(\mathbb{R}^3;L^2(\mathbb{R}^3))}^2,
\]
\[
D_N = \|\nabla_a g\|_{X^{N-1}(\mathbb{R}^3;L^2(\mathbb{R}^3))}^2 + \|g_2\|_{X^N(\mathbb{R}^3)}^2 \sim \|\nabla_a \mathcal{A}\|_{H^{N-1}(\mathbb{R}^3)}^2 + \|g_2\|_{X^N(\mathbb{R}^3)}^2,
\]
respectively. Let \( g = g(t, x, v) \) be a solution to
\[
g_t + v \cdot \nabla_x g + Lg = \Gamma(g, g), \quad g_{t=0} = g_0.
\]

We start with the macroscopic energy estimate. It is well-known that the macroscopic component \( g_1 = Pg \sim \mathcal{A} = (a, b, c) \), satisfies the following set of equations
\[
\begin{equation}
\begin{aligned}
\nu_1 v_1^\mu_1^{1/2} : & & \nabla_x c &= -\partial_t r + l_c + h_c, \\
\nu_1 v_2^\mu_1^{1/2} : & & \partial_t c + \partial_i b_i &= -\partial_t r + l_c + h_i, \\
\nu_2 v_2^\mu_1^{1/2} : & & \partial_t b_i + \partial_j b_j &= -\partial_t r_i + l_{ij} + h_{ij}, \quad i \neq j, \\
\mu_1^{1/2} : & & \partial_t a &= -\partial_t r_a + l_a + h_a,
\end{aligned}
\end{equation}
\]
where
\[
r = (g_2, e)_{L^2(\mathbb{R}^3)}, \quad l = -(v \cdot \nabla_x g_2 + Lg_2, e)_{L^2(\mathbb{R}^3)}, \quad h = (\Gamma(g, g), e)_{L^2(\mathbb{R}^3)},
\]
stand for \( r, \cdots, h \), while

\[ e \in \text{span}\{v[v\mu^{1/2}, v^2\mu^{1/2}, v\mu^{1/2}, \mu^{1/2}]\}. \]

**Lemma 6.1.** Assume \( N \geq 3 \) and let \( \partial^\alpha = \partial_{x_1}^\alpha, \alpha \in \mathbb{N}^3, |\alpha| \leq N \). Then,

\[ \|\partial^\alpha \mathcal{A}^2\|_{L^2(\mathbb{R}^4)} \leq \|\mathcal{A}\|_{L^2(\mathbb{R}^4)}\|\mathcal{A}\|_{L^2(\mathbb{R}^4)} \]

Proof. Firstly, one has, for \( |\alpha| = 0 \)

\[ \|\mathcal{A}^2\|_{L^2(\mathbb{R}^4)} \leq \|\mathcal{A}\|_{L^2(\mathbb{R}^4)}\|\mathcal{A}\|_{L^2(\mathbb{R}^4)} \]

Also for \( |\alpha| = 1 \), we have

\[ \|\partial \mathcal{A}^2\|_{L^2(\mathbb{R}^4)} \leq \|\mathcal{A}\|_{L^2(\mathbb{R}^4)}\|\mathcal{A}\|_{L^2(\mathbb{R}^4)}, \]

and for \( 2 \leq |\alpha| \leq N \),

\[ \|\partial^\alpha \mathcal{A}^2\|_{L^2(\mathbb{R}^4)} \leq \sum\limits_{k \leq |\alpha|} \|\partial^\alpha \mathcal{A}^{\partial^{-k} \mathcal{A}}\|_{L^2(\mathbb{R}^4)} \]

\[ \leq \sum\limits_{k \leq |\alpha|} \|\partial^\alpha \mathcal{A}_{\partial^{-k} \mathcal{A}}\|_{L^2(\mathbb{R}^4)} \leq \|\mathcal{A}\|_{H^{N-1}(\mathbb{R}^4)}\|\mathcal{A}\|_{H^{N-1}(\mathbb{R}^4)}. \]

This completes the proof of the lemma. \( \square \)

**Lemma 6.2.** Assume \( \gamma > -3, N \geq 3 \). Let \( \partial^\alpha = \partial_{x_i}^\alpha, \partial_i = \partial_{x_i}, |\alpha| \leq N - 1 \). Then, one has

\[ \|\partial \partial^\alpha r\|_{L^2(\mathbb{R}^4)} + \|\partial^\alpha f\|_{L^2(\mathbb{R}^4)} \leq \|g_2\|_{L^2(\mathbb{R}^4)} \equiv A_1, \]

\[ \|\partial \partial^\alpha h\|_{L^2(\mathbb{R}^4)} + \|\partial^\alpha f\|_{L^2(\mathbb{R}^4)} \leq \|g_2\|_{L^2(\mathbb{R}^4)} \equiv A_2. \]

Proof. (6.3) follows from

\[ \|\partial \partial^\alpha r\|_{L^2(\mathbb{R}^4)} = \|\partial_{\partial^\alpha g_2, e}\|_{L^2(\mathbb{R}^4)} = \|(\tilde{\mathcal{R}}^{-1} \partial \partial^\alpha g_2, \tilde{\mathcal{R}} e)\|_{L^2(\mathbb{R}^4)} \leq \|\partial \partial^\alpha g_2\|_{L^2(\mathbb{R}^4)} \leq \|g_2\|_{L^2(\mathbb{R}^4)}, \]

and

\[ \|\partial \partial^\alpha f\|_{L^2(\mathbb{R}^4)} \leq \|\tilde{\mathcal{R}}^{-1} \partial \partial^\alpha g_2, \tilde{\mathcal{R}} e\|_{L^2(\mathbb{R}^4)} + \|(\tilde{\mathcal{R}}^{-1} \partial \partial^\alpha g_2, \tilde{\mathcal{R}} e)\|_{L^2(\mathbb{R}^4)} \leq \|\partial \partial^\alpha g_2\|_{L^2(\mathbb{R}^4)} \leq \|g_2\|_{L^2(\mathbb{R}^4)}. \]

Here, we have used (4.15). We prove (6.4) as follows. Firstly, set

\[ H(f, g) = (\Gamma(f, g), e)_{L^2(\mathbb{R}^4)} \]

\[ = \iint b(\cos \theta)|v - \nu| f g (\mu^{1/2} \nu, \mu^{1/2} \nu) dv d\nu d\sigma \]

\[ = \iint b(\cos \theta)|v - \nu| f g (\mu^{1/2} \nu, \mu^{1/2} \nu) (q(v) - q(\nu)) dv d\nu d\sigma, \]

where \( q(v) \) is a polynomial. Of course, \( h = H(g, g) \). We now write the Taylor expansion of the second order of the function \( q(v) \) as

\[ q(v) - q(v') = (\nabla q)(v') \cdot (v' - v) + \frac{1}{2} \int_0^1 \nabla^2 q(v + \tau(v' - v)) d\tau (v' - v)^2. \]
Firstly, we have
\[ v' - v = \frac{1}{2} [v - v_*] (\sigma - (\sigma \cdot k)k) + \frac{1}{2} (\sigma \cdot k) (v - v_*), \]
and notice that by virtue of the symmetry
\[ \int_{\mathbb{R}^2} b(\sigma \cdot k)(\sigma - (\sigma \cdot k)k) d\sigma = 0. \]
Therefore, we have
\[
H(f, g) = \frac{1}{2} \iint (\mu^{1/2} f)(\mu^{1/2} g), |v - v_*| \left[ \int_{\mathbb{R}} b(\cos \theta)(\cos \theta - 1) d\sigma \right] (\nabla q)(v) \cdot (v - v_*) dv dv,
\]
and notice that by virtue of Lemma 6.1 and for \( s \leq 1 \), we get
\[
|H_1(f, g)| \leq \int \frac{1}{\mu^{1/8}(\mu^{1/2} f)(\mu^{1/2} g)}, |v - v_*|^{s+1} dv dv,
\]
\[
\leq \left( \int \frac{1}{\mu^{1/4} [g^2]^{1/2}, \mu^{1/4} |v - v_*|^{s+1} dv, dv} \right)^{1/2} \left( \int \frac{1}{\mu^{1/4} [f^2]^{1/2}, \mu^{1/4} |v - v_*|^{s+1} dv, dv} \right)^{1/2}
\]
\[
\leq \|f\|_{L_2} \|g\|_{L_2} [\|v\|^{s+1/2} \mu^{1/8} f \|_{L_2} ] [\|v\|^{s+1/2} \mu^{1/8} g \|_{L_2} ] \leq \|f\|_{L_2} \|g\|_{L_2},
\]
for any \( l, m \in \mathbb{R} \). Similarly, \( \gamma + 2 > -3 \) implies,
\[
|H_2(f, g)| \leq \iint b(\cos \theta)(\cos \theta - 1) f(\mu^{1/2} f), |v - v_*|^{s+2} (\nabla q(v) + \nabla q(v')) d\sigma dv dv,
\]
\[
\leq \|v\|^{s+2} \mu^{1/8} f \|_{L_2} \|v\|^{s+2} \mu^{1/8} g \|_{L_2} \leq \|f\|_{L_2} \|g\|_{L_2}.
\]
Combining these two estimates yields
\[
|H(f, g)| \leq \|f\|_{L_2} \|g\|_{L_2}.
\]
Now \( h \) is computed as follows.
\[
h = H(g, g) = \sum_{i,j=1,2} H(g_i, g_j) = \sum_{i,j=1,2} H^{(ij)}.
\]
Firstly, we have
\[
H^{(11)} \sim \mathcal{A}^2 H(\varphi_1, \varphi_1),
\]
where \( \varphi_1 \in N \). Applying (6.5) for \( l = m = 0 \), we get by virtue of Lemma 6.1 and for \( |\alpha| \leq N - 1 \) that
\[
||\partial^\alpha H^{(11)}||_{L_2(\mathbb{R}^2)} \leq ||\partial^\alpha \mathcal{A}^2||_{L_2(\mathbb{R}^2)} \leq ||\nabla \mathcal{A}||_{H^{N-1}(\mathbb{R}^2)} ||\mathcal{A}||_{H^{N-1}(\mathbb{R}^2)},
\]
while taking \( l, m \) to be 0 or \( s + \gamma/2 \) in \( \tilde{\mathcal{F}} \) and by the Leibniz rule and by \( \tilde{\mathcal{F}} \),

\[
\|\partial^s H^{(1)}\|_{L^2(\mathbb{R}^d)} \leq \sum_{\alpha_1 + \alpha_2 = \alpha} \|\partial_{\alpha_1} \mathcal{A}\|_{L^2(\mathbb{R}^d)} \|\partial_{\alpha_2}^2 g_2\|_{L^2(\mathbb{R}^d)} \leq \|\mathcal{A}\|_{L^2(\mathbb{R}^d)} \|g_2\|_{L^2(\mathbb{R}^d)},
\]

\[
\|\partial^s H^{(2)}\|_{L^2(\mathbb{R}^d)} \leq \sum_{\alpha_1 + \alpha_2 = \alpha} \|\partial_{\alpha_1}^2 g_2\|_{L^2(\mathbb{R}^d)} \|\partial_{\alpha_2}^2 \mathcal{A}\|_{L^2(\mathbb{R}^d)} \leq \|g_2\|_{L^2(\mathbb{R}^d)} \|\mathcal{A}\|_{H^{s-1}(\mathbb{R}^d)},
\]

\[
\|\partial^s H^{(3)}\|_{L^2(\mathbb{R}^d)} \leq \sum_{\alpha_1 + \alpha_2 = \alpha} \|\partial_{\alpha_1} g_2\|_{L^2(\mathbb{R}^d)} \|\partial_{\alpha_2}^3 \mathcal{A}\|_{L^2(\mathbb{R}^d)} \leq \|g_2\|_{H^s(\mathbb{R}^d)} \|\mathcal{A}\|_{L^2(\mathbb{R}^d)} \|g_2\|_{L^2(\mathbb{R}^d)}.
\]

Now the proof of Lemma 6.2 is completed. \( \square \)

Next, we shall prove

**Lemma 6.3.** Assume \( \gamma > -3 \). Let \( |a| \leq N - 1 \). Then

\[
(6.6) \quad \|\nabla \partial^s \mathcal{A}\|_{L^2(\mathbb{R}^d)}^2 \leq \frac{d}{dt}\left( (\partial^s r, \nabla \partial^s \mathcal{A})_{L^2(\mathbb{R}^d)} + (\partial^s b, \nabla \partial^s \mathcal{A})_{L^2(\mathbb{R}^d)} \right) + \|\nabla \partial^s \mathcal{A}\|_{L^2(\mathbb{R}^d)}^2 + \|\nabla b\|_{H^s(\mathbb{R}^d)}^2 + \|\nabla b\|_{L^2(\mathbb{R}^d)}^2 \|\mathcal{A}\|_{L^2(\mathbb{R}^d)},
\]

**Proof.** (a) Estimate of \( \nabla \partial^s a \). Let \( A_1, A_2 \) be as in Lemma 6.2. From (5.3),

\[
\|\nabla \partial^s a\|_{L^2(\mathbb{R}^d)}^2 = (\nabla \partial^s a, \nabla \partial^s a)_{L^2(\mathbb{R}^d)}
\]

\[
= (\partial^s (-\partial_b - \partial_r + \gamma r + h), \nabla \partial^s a)_{L^2(\mathbb{R}^d)}
\]

\[
\leq R_1 + C_\eta (A_1^2 + A_2^2) + \eta \|\nabla \partial^s a\|_{L^2(\mathbb{R}^d)}^2,
\]

Here,

\[
R_1 = - (\partial^s \partial_b + \partial^s \partial_r, \nabla \partial^s a)_{L^2(\mathbb{R}^d)}
\]

\[
= - \frac{d}{dt} (\partial^s (b + r), \nabla \partial^s a)_{L^2(\mathbb{R}^d)} + (\nabla \partial^s (b + r), \partial_r \partial^s a)_{L^2(\mathbb{R}^d)}
\]

\[
\leq - \frac{d}{dt} (\partial^s (b + r), \nabla \partial^s a)_{L^2(\mathbb{R}^d)} + C_\eta (\|\nabla \partial^s b\|_{L^2(\mathbb{R}^d)}^2 + A_1^2) + \eta \|\partial_r \partial^s a\|_{L^2(\mathbb{R}^d)}^2.
\]

(b) Estimate of \( \nabla \partial^s b \). From (6.2),

\[
\Delta \partial^s b_i + \partial^2 \partial^s b_i = \sum_{j \neq i} \partial_i \partial^s (\partial_j b_i + \partial_j b_j) + \partial_i \partial^s (2\partial_j b_j - \sum_{j \neq i} \partial_j b_j)
\]

\[
= \partial_i \partial^s (-\partial_r + \gamma r + h),
\]

\[
\|\nabla \partial^s b\|_{L^2(\mathbb{R}^d)}^2 + \|\partial_r \partial^s b\|_{L^2(\mathbb{R}^d)}^2 = - (\Delta \partial^s b_i + \partial^2 \partial^s b_i, \partial^s b_i)_{L^2(\mathbb{R}^d)} = R_2 + R_3 + R_4,
\]

where

\[
R_2 = (\partial^s r, \partial^s b)_{L^2(\mathbb{R}^d)} + \frac{d}{dt} (\partial^s r, \partial^s b)_{L^2(\mathbb{R}^d)} + (\partial^s \partial_r, \partial^s b)_{L^2(\mathbb{R}^d)}
\]

\[
\leq \frac{d}{dt} (\partial^s r, \partial^s b)_{L^2(\mathbb{R}^d)} + C_\eta A_1^2 + \eta \|\partial_r \partial^s b\|_{L^2(\mathbb{R}^d)}^2,
\]

\[
R_3 = - (\partial^s l, \partial^s b)_{L^2(\mathbb{R}^d)} \leq C_\eta A_1^2 + \eta \|\partial_r \partial^s b\|_{L^2(\mathbb{R}^d)}^2,
\]

\[
R_4 = - (\partial^s h, \partial^s b)_{L^2(\mathbb{R}^d)} \leq C_\eta A_2^2 + \eta \|\partial_r \partial^s b\|_{L^2(\mathbb{R}^d)}^2.
\]

(c) Estimate of \( \nabla \partial^s c \). From (6.3),

\[
\|\nabla \partial^s c\|_{L^2(\mathbb{R}^d)}^2 = (\nabla \partial^s c, \nabla \partial^s c)_{L^2(\mathbb{R}^d)} = (\partial^s (-\partial_r + \gamma r + h), \nabla \partial^s c)_{L^2(\mathbb{R}^d)}
\]

\[
\leq R_5 + C_\eta (A_1^2 + A_2^2) + \eta \|\nabla \partial^s c\|_{L^2(\mathbb{R}^d)}^2,
\]
Under the assumptions of Theorem 1.4, for $N$

Proof. We apply

which completes the proof of Lemma 6.3. □

In view of Section 2, we have, for all

Proof of Lemma 6.4 can then be concluded by plugging these two estimates into (6.8).

Proof of (6.7): Write

Estimation of $J^{11}$. We shall estimate

Finally, by choosing $|a| \leq N - 1$, and using Lemma 5.2, we obtain

which completes the proof of Lemma 6.3. □

We now turn to the estimation on the microscopic component $g_2$ in the function space $H^N(\mathbb{R}^N; L^2(\mathbb{R}^N))$. Actually, we shall establish

**Lemma 6.4.** Under the assumptions of Theorem 1.4, for $N \geq 3$,

(6.7) $\frac{d}{dt} E_N + \|g_2\|_{\mathcal{H}^N(\mathbb{R}^N)} \leq C_N^1 |E_N|.$

Proof. We apply $\partial^*_{11}$ to (5.1) and take the $L^2(\mathbb{R}^N)$ inner product with $\partial^*_{11} g$. Since the inner product including $v \cdot \nabla x g$ vanishes by integration by parts, we get

(6.8) $\frac{1}{2} \frac{d}{dt} \|\partial^*_{11} g\|_{\mathcal{H}^N(\mathbb{R}^N)}^2 + (\mathcal{L} \partial^*_{11} g, \partial^*_{11} g)_{L^2(\mathbb{R}^N)} = (\partial^*_{11} \Gamma(g, g), \partial^*_{11} g)_{L^2(\mathbb{R}^N)}.$

In view of Section 2, we have, for all $\gamma > -3$,

\[ \sum_{|a| \leq N} (\mathcal{L} \partial^*_{11} g, \partial^*_{11} g)_{L^2(\mathbb{R}^N)} \geq \eta_0 \sum_{|a| \leq N} \sum_{|b| \leq N} \int_{\mathbb{R}^N} \|\partial^*_{11} (I - P) g\|^2 dx = \eta_0 \|g_2\|^2_{\mathcal{H}^N(\mathbb{R}^N)}, \]

while we shall show below that for $|a| \leq N, N \geq 3$,

(6.9) $\|\partial^*_{11} \Gamma(g, g), \partial^*_{11} g)_{L^2(\mathbb{R}^N)} \| \leq C_N |E_N|.$

Lemma 6.4 can then be concluded by plugging these two estimates into (6.8).

Proof of (6.7): Write

Estimation of $J^{11}$. We shall estimate
Firstly, for any function $\phi, \psi \in \mathcal{N}$, set
\[ \Psi_1 = (\Gamma(\phi, \psi), h)_{L^2(\mathbb{R}^3)}. \]
We shall prove that for $\gamma > -3$,
\begin{equation}
|\Psi_1| \leq ||h||_{L^2(\mathbb{R}^3)},
\end{equation}
holds for any $m \in \mathbb{R}$.

**Proof of (6.10).** Notice that
\[
\Psi_1 = \iint \int b(\cos \theta)|v - \nu_i|^\gamma (\mu_r)^{1/2}(\phi, \psi') d\nu dr d\sigma,
\]
where $q = q(v)$ and $r = r(v)$ are some polynomials. First, write
\[
q_r' - q_r = (q_r' - q_r)(r' - r) + (q_r' - q_r)r + q_r(r' - r) = S_1 + S_2 + S_3,
\]
and make a decomposition
\[
\Psi_1 = \sum_{i=1}^3 \int \int \int b(\cos \theta)|v - \nu_i|^\gamma (\mu_r)^{1/2}(\mu_r)^{1/2} S_i d\nu dr d\sigma.
\]
Since $|S_1| \leq R_1(v, v_*)|v - v_*|^2 \theta^2$ where $R_1(v, v_*)$ is a polynomial of $v, v_*$, it holds that
\[
|\Psi_1^1| \leq \int |v - v_*|^\gamma \left[ \int b(\cos \theta)|v - v_*|^\gamma (\mu_r)^{1/2}(\mu_r)^{1/2} |R_1(v, v_*)| |h| d\nu dr d\sigma \right. \leq \int \int \int |v - v_*|^\gamma |h| d\nu d\sigma.
\]
for any $m \in \mathbb{R}$.

On the other hand, the Taylor expansion of the second order gives
\[
g(q_r') - g(q_r) = (\nabla g)(v_*) \cdot (v_r' - v_r) + \frac{1}{2} \int_0^1 \nabla^2 g(v_* + \tau(v_r' - v_r)) d\tau (v_r' - v_r).^2.
\]
Since
\[
\left| \int_0^1 \nabla^2 g(v_* + \tau(v_r' - v_r)) d\tau (v_r' - v_r)^2 \right| \leq |R_2(v, v_*)| |v - v_*|^2 \theta^2,
\]
where $R_2$ is a polynomial of $v, v_*$, by symmetry, we have
\[
|\Psi_1^2| \leq \int |v - v_*|^{\gamma + 1}(\mu_r)^{1/2}(\mu_r)^{1/2} \left( |\nabla g(v_*)| + |R_2(v, v_*)| |v - v_*| \right) |r| h d\nu dr d\sigma \leq \int \int \int ((v)^{\gamma + 1} + (v)^{\gamma + 2} \mu^{1/4}) r |h| d\nu d\sigma \leq ||h||_{L^2(\mathbb{R}^3)},
\]
for any $m \in \mathbb{R}$.

The estimation on $\Psi_1^3$ can be carried out exactly in the same way to have
\[
|\Psi_1^3| \leq \int \int \int ((v)^{\gamma + 1} + (v)^{\gamma + 2} \mu^{1/4}) |h| d\nu d\sigma \leq ||h||_{L^2(\mathbb{R}^3)},
\]
for any $m \in \mathbb{R}$. This completes the proof of (6.10).
Take \( m = s + \gamma/2 \) in (4.11) and use (4.5) to obtain \(|\Psi_1| \leq \|h\|\). Set \( h = \partial_\gamma g_2 \). Now by the Sobolev embedding theorem, for \( \alpha_1 + \alpha_2 = \alpha, 1 \leq |\alpha| \leq N \), we have

\[
|J^{11}| \leq \int_{\mathbb{R}^3} |\partial^{\alpha_1}_x \mathcal{A} x | |\partial^{\alpha_2} \mathcal{A} x | |\partial^{\alpha}_x g_2|\,dx
\]

\[
\leq \left\{ \begin{array}{ll}
\|\partial^\alpha_x \mathcal{A} x \|_{L^2} \|\partial^{\alpha_2} \mathcal{A} x \|_{L^2} \|\partial_\gamma g_2\|_{L^2(\mathbb{R}^3)} \leq \|\mathcal{A}\|_{H^N} \|\nabla \mathcal{A}\|_{H^{N-1}} \|g_2\|_{L^2(\mathbb{R}^3)}, & |\alpha_1| \leq N - 2, \\
\|\partial^\alpha_x \mathcal{A} x \|_{L^2} \|\partial^{\alpha_2} \mathcal{A} x \|_{L^2} \|\partial_\gamma g_2\|_{L^2(\mathbb{R}^3)} \leq \|\nabla \mathcal{A}\|_{L^2} \|\mathcal{A}\|_{H^{N-1}} \|g_2\|_{L^2(\mathbb{R}^3)}, & |\alpha_2| < 1,
\end{array} \right.
\]

and for \( |\alpha_1 + \alpha_2| = 0 \), we have

\[
\int_{\mathbb{R}^3} |\mathcal{A}\|_{L^2} \|g_2\|_{L^2} \,dx \leq \|\mathcal{A}\|_{L^2} \|\mathcal{A}\|_{L^2} \|g_2\|_{L^2(\mathbb{R}^3)} \leq \|\nabla \mathcal{A}\|_{L^2} \|\mathcal{A}\|_{L^2} \|g_2\|_{L^2(\mathbb{R}^3)} \leq E_N^{1/2} \mathcal{D}_N.
\]

**Estimation of \( J^{12} \):** First, notice

\[
J^{12} \sim \int_{\mathbb{R}^3} (\partial^{\alpha_1}_x \mathcal{A} (\Gamma(\phi, \partial_\gamma g_2), \partial_\gamma g_2))_{L^2(\mathbb{R}^3)} \,dx.
\]

For some \( \phi \in \mathcal{N} \), set

\[
\Psi_2 = (\Gamma(\phi, g), h)_{L^2(\mathbb{R}^3)}.
\]

By virtue of Theorem 1.3, we get

\[
|\Psi_2| \leq \left( \|\phi\|_{L^2_x(\mathbb{R}^3)} \|g_2\|_{L^2} \|\phi\|_{L^2_x(\mathbb{R}^3)} + \|\phi\|_{L^2_x(\mathbb{R}^3)} \|g_2\|_{L^2} \right) |h| \|\phi\|_{L^2_x(\mathbb{R}^3)}
\]

\[
\leq \left( \|\phi\|_{L^2_x(\mathbb{R}^3)} \|g_2\|_{L^2} \|\phi\|_{L^2_x(\mathbb{R}^3)} + \|\phi\|_{L^2_x(\mathbb{R}^3)} \|g_2\|_{L^2} \right) |h| \|\phi\|_{L^2_x(\mathbb{R}^3)}
\]

where we have chosen the first factor in the min term.

Now we have

\[
|J^{12}| \leq \int_{\mathbb{R}^3} |(\partial^{\alpha_1}_x \mathcal{A} x | |\partial^{\alpha_2} \mathcal{A} x | |\partial_\gamma g_2|\,dx
\]

\[
\leq \left\{ \begin{array}{ll}
\|\partial^\alpha_x \mathcal{A} x \|_{L^2} \int_{\mathbb{R}^3} |\partial^{\alpha_2} \mathcal{A} x | |\partial_\gamma g_2|\,dx \leq \|\mathcal{A}\|_{H^N(\mathbb{R}^3)} \|g_2\|_{L^2(\mathbb{R}^3)}, & |\alpha_1| \leq N - 2, \\
\|\partial^\alpha_x \mathcal{A} x \|_{L^2} \int_{\mathbb{R}^3} |\partial^{\alpha_2} \mathcal{A} x | |\partial_\gamma g_2|\,dx \leq \|\mathcal{A}\|_{H^N(\mathbb{R}^3)} \|g_2\|_{L^2(\mathbb{R}^3)}, & |\alpha_2| \leq 1.
\end{array} \right.
\]

**Estimation of \( J^{21} \):** A similar argument applies to

\[
J^{21} \sim \int_{\mathbb{R}^3} (\partial^{\alpha_2}_x \mathcal{A} (\Gamma(\phi, g_2), \partial_\gamma g_2)_{L^2(\mathbb{R}^3)} \,dx.
\]

In fact, for \( \phi \in \mathcal{N} \), set

\[
\Psi_3 = (\Gamma(f, \phi), h).
\]

Again by virtue of Theorem 1.3 and taking the second factor in the min term, (4.5) gives

\[
|\Psi_3| \leq \|f\|_{L^2} \|h\|_{L^2_x(\mathbb{R}^3)}.
\]

Thus, proceeding as for \( J^{12} \) yields

\[
|J^{21}| \leq \|\mathcal{A}\|_{H^N(\mathbb{R}^3)} \|g_2\|_{L^2(\mathbb{R}^3)}^2.
\]
Estimation on $J^{22}$: It follows from the Leibniz rule that
\[ |J^{22}| \leq \int |\Gamma(\partial_t \alpha g_2, \partial_{x_1} \alpha g_2) + \partial_{x_2} \alpha g_2)|dx. \]

Different from the above, we now use Theorem 1.2 and (8.5) in the following way.
\[ |\Gamma(f, g)| \leq \left( ||f||_{L^2(\mathbb{R})} \right) \left( ||g||_{L^2(\mathbb{R})} \right) \left( ||\alpha||_{L^2(\mathbb{R})} \right), \]

\[ + \min \left( ||f||_{L^2(\mathbb{R})} ||g||_{L^2(\mathbb{R})} \right) \left( ||\alpha||_{L^2(\mathbb{R})} \right) \left( ||\gamma||_{L^2(\mathbb{R})} \right) \]
\[ \leq \left( ||f||_{L^2(\mathbb{R})} ||g||_{L^2(\mathbb{R})} \right) \left( ||\alpha||_{L^2(\mathbb{R})} \right) \left( ||\gamma||_{L^2(\mathbb{R})} \right). \]

This is valid for the assumptions imposed on $\gamma$ and $s$ from Theorem 1.4. Then,
\[ |J^{22}| \leq \int \left( ||\partial_t \alpha g_2||_{L^2(\mathbb{R})} \right) \left( ||\partial_{x_1} \alpha g_2||_{L^2(\mathbb{R})} \right) \left( ||\partial_{x_2} \alpha g_2||_{L^2(\mathbb{R})} \right) \left( ||\partial_{x_2} \alpha g_2||_{L^2(\mathbb{R})} \right) \left( ||\alpha||_{L^2(\mathbb{R})} \right) \left( ||\gamma||_{L^2(\mathbb{R})} \right) \left( ||\gamma||_{L^2(\mathbb{R})} \right) \left( ||\gamma||_{L^2(\mathbb{R})} \right) \left( ||\gamma||_{L^2(\mathbb{R})} \right). \]

Suppose $|\alpha_1| \leq N - 2$. Then, by the Sobolev embedding theorem,
\[ |J^{22}| \leq ||\partial_t \alpha g_2||_{L^2(\mathbb{R})} ||\partial_{x_1} \alpha g_2||_{L^2(\mathbb{R})} \left( ||\partial_t \alpha \alpha g_2||_{L^2(\mathbb{R})} \right) \left( ||\partial_{x_1} \alpha g_2||_{L^2(\mathbb{R})} \right) \left( ||\partial_{x_2} \alpha g_2||_{L^2(\mathbb{R})} \right) \left( ||\alpha||_{L^2(\mathbb{R})} \right) \left( ||\gamma||_{L^2(\mathbb{R})} \right) \left( ||\gamma||_{L^2(\mathbb{R})} \right) \left( ||\gamma||_{L^2(\mathbb{R})} \right) \left( ||\gamma||_{L^2(\mathbb{R})} \right). \]

Similarly, when $|\alpha_1| > N - 2$ then $|\alpha_2| \leq 1$, we get
\[ |J^{22}| \leq ||\partial_t \alpha g_2||_{L^2(\mathbb{R})} ||\partial_{x_1} \alpha g_2||_{L^2(\mathbb{R})} \left( ||\partial_t \alpha \alpha g_2||_{L^2(\mathbb{R})} \right) \left( ||\partial_{x_1} \alpha g_2||_{L^2(\mathbb{R})} \right) \left( ||\partial_{x_2} \alpha g_2||_{L^2(\mathbb{R})} \right) \left( ||\alpha||_{L^2(\mathbb{R})} \right) \left( ||\gamma||_{L^2(\mathbb{R})} \right) \left( ||\gamma||_{L^2(\mathbb{R})} \right) \left( ||\gamma||_{L^2(\mathbb{R})} \right) \left( ||\gamma||_{L^2(\mathbb{R})} \right). \]

Now, combining the above estimates yields the estimate (8.5) and this completes the proof of the Lemma 6.4.

Taking a suitable linear combination of (6.3) and (6.7) gives the

**Proposition 6.5. (Global Energy Estimate without Weight)** Under the assumptions of Theorem 1.4 for $N \geq 3$, there exists a constant $C > 0$ such that
\[ \frac{d}{dt} \mathcal{E}_N + D_N \leq C\mathcal{E}_N^{1/2} D_N \]
holds as far as $g$ exists.

This proposition assures that a usual continuation argument of local solutions can be carried out under the smallness assumption of initial data. Thus, we established the existence of global solutions in the space $H^N(\mathbb{R}_+^3; L^2(\mathbb{R}_+^3))$.

### 6.2. Classical solutions

We now turn to the energy estimates involving also $v$ derivatives of solutions. To close this type of energy estimate, we then need to use the weighted norms in the $v$ variable, cf. also Guo [19], with the weight function $W$. Recall that we assume $s + \gamma/2 \leq 0$. Set
\[ \mathcal{E}_{N,t} = \mathcal{E}_N + ||g||_{H^N(\mathbb{R})}^2 + ||v||_{H^N(\mathbb{R})}^2 \sim ||\mathcal{A}||_{H^N(\mathbb{R})}^2 + ||g||_{H^N(\mathbb{R})}^2, \]
\[ D_{N,t} = D_N + ||g||_{H^N(\mathbb{R})}^2 \sim ||\nabla^s v||_{H^{N-1}(\mathbb{R})} + ||g||_{H^N(\mathbb{R})}^2. \]
Recall
\[ \partial_t^\sigma = \partial_t \partial_t^\sigma, \quad |\sigma| + |\beta| \leq N, \quad \beta \neq 0, \quad N \geq 6, \]
and apply \( \tilde{W}_{t-\gamma} \partial_t^\sigma g_2 \) to (6.1) to get
\[
\partial_t (\tilde{W}_{t-\gamma} \partial_t^\sigma g_2) + v \cdot \nabla (\tilde{W}_{t-\gamma} \partial_t^\sigma g_2) + \mathcal{L}_1 (\tilde{W}_{t-\gamma} \partial_t^\sigma g_2) \\
= \tilde{W}_{t-\gamma} \partial_t^\sigma \Gamma(g, g) + [v \cdot \nabla, \tilde{W}_{t-\gamma} \partial_t^\sigma] g_2 - \tilde{W}_{t-\gamma} \partial_t^\sigma [P, v \cdot \nabla] g \\
+ \{ \mathcal{L}_1, \tilde{W}_{t-\gamma} \partial_t^\sigma g_2 \} - \tilde{W}_{t-\gamma} \partial_t^\sigma \mathcal{L}_2 (g_2).
\]
Then take the \( L^2(\mathbb{R}^3) \) inner product with \( \tilde{W}_{t-\gamma} \partial_t^\sigma g_2 \) to get
\[
(6.11) \quad \frac{1}{2} \frac{d}{dt} \| \tilde{W}_{t-\gamma} \partial_t^\sigma g_2 \|^2_{L^2(\mathbb{R}^3)} + D \leq K.
\]
Here, \( D \) is a dissipation rate given by
\[
D = (\mathcal{L}_1 (\tilde{W}_{t-\gamma} \partial_t^\sigma g_2), \tilde{W}_{t-\gamma} \partial_t^\sigma g_2)_{L^2(\mathbb{R}^3)}.
\]
Due to the coercivity inequality from Section 3, which holds true for \( \gamma > -3 \), we get
\[
D \geq D_0 \int_{\mathbb{R}^3} \| (1 - P) \tilde{W}_{t-\gamma} \partial_t^\sigma g_2 \|^2 \, dx \\
\geq D_0 \| \tilde{W}_{t-\gamma} \partial_t^\sigma g_2 \|^2_{L^2(\mathbb{R}^3)} - C \| \partial_t \partial_t^\sigma \|^2_{L^2(\mathbb{R}^3)}.
\]
where we have used, with \( \psi \in N \) and \( m \in \mathbb{N} \),
\[
\| P \tilde{W}_{t-\gamma} \partial_t^\sigma g_2 \|^2 = \| \partial_t^m (\psi, \tilde{W}_{t-\gamma} \partial_t^\sigma g_2)_{L^2(\mathbb{R}^3)} \|_{L^2(\mathbb{R}^3)}^2
\]
\[
= \| (\Psi, \tilde{W}_{-\gamma} \partial_t^\sigma g_2)_{L^2(\mathbb{R}^3)} \|_{L^2(\mathbb{R}^3)}^2
\]
\[
\leq \| \partial_t \partial_t^\sigma \|^2_{L^2(\mathbb{R}^3)}.
\]
Note that we will use the above estimate later by choosing \( m = -|s + \gamma/2| \). On the other hand, \( K \) is given by
\[
K = (\tilde{W}_{t-\gamma} \partial_t^\sigma \Gamma(g, g), \tilde{W}_{t-\gamma} \partial_t^\sigma g_2)_{L^2(\mathbb{R}^3)} + \| [v \cdot \nabla, \tilde{W}_{t-\gamma} \partial_t^\sigma] g_2 + \| \mathcal{L}_1, \tilde{W}_{t-\gamma} \partial_t^\sigma g_2 \|^2_{L^2(\mathbb{R}^3)} \\
- \| \tilde{W}_{t-\gamma} \partial_t^\sigma \|_{L^2(\mathbb{R}^3)}^2 + \| \tilde{W}_{t-\gamma} \partial_t^\sigma \|_{L^2(\mathbb{R}^3)}^2 \\
= K_1 + K_2 + K_3 + K_4 + K_5.
\]
(1) **Estimation of \( K_1 \):** First, we show that
\[
(6.12) \quad |K_1| \leq E_{10}^{1/2} D_{N.E}.
\]
For the proof, write
\[
K_1 = \sum_{i,j=0}^2 \| \tilde{W}_{t-\gamma} \partial_t^\sigma \|_{L^2(\mathbb{R}^3)}^2 \\
= K_{111} + K_{112} + K_{121} + K_{122}.
\]
(1) **Estimation on \( K_{111} \):** Proceeding as in the computation for \( \Psi_1 \) in (6.10), we get for
\( \gamma > -3 \),
\[
\| (\tilde{W}_{t-\gamma} \partial_t^\sigma \Gamma(\varphi_2, \varphi_m), \tilde{W}_{t-\gamma} \partial_t^\sigma g_2)_{L^2(\mathbb{R}^3)} \| \leq \| \tilde{W}_{t-\gamma} \partial_t^\sigma g_2 \|_{L^2(\mathbb{R}^3)}.
\]
which leads to

\[
K_{111} \sim \sum_{\alpha_1 + \alpha_2 = \alpha} \int_{\mathbb{R}^3} ||(\partial_x^\alpha \mathcal{A})(\partial_x^\alpha \mathcal{A})|| \tilde{W}_{t-\bar{y}B_\mu} g_2(x) dx \leq \sum_{\alpha_1 + \alpha_2 = \alpha} ||(\partial_x^\alpha \mathcal{A})(\partial_x^\alpha \mathcal{A})||_{L^2(\mathbb{R}^3)} ||g_2||_{L^\infty(\mathbb{R}^3)} \\
\leq ||\mathcal{A}||_{H^\infty(\mathbb{R}^3)}^2 (||\nabla \mathcal{A}||_{L^2(\mathbb{R}^3)}^2 + ||g_2||_{L^\infty(\mathbb{R}^3)}^2) \leq \mathcal{E}_{N/2}^{1/2} \mathcal{D}_{N,t}.
\]

Here, we used that for \(\alpha_1 = \alpha_2 = 0\),

\[
||\mathcal{A}^2||_{L^2(\mathbb{R}^3)} \leq ||\mathcal{A}||_{L^\infty(\mathbb{R}^3)} ||\mathcal{A}||_{L^\infty(\mathbb{R}^3)} \leq ||\mathcal{A}||_{H^\infty(\mathbb{R}^3)} ||\nabla \mathcal{A}||_{L^2(\mathbb{R}^3)}.
\]

**(2) Estimation on \(K_{112}\):** Since \(g_1 \sim \mathcal{A} \Phi\), Proposition 4.7 implies

\[
|K_{112}| \leq \sum_{\mu_1 + \mu_2 = \mu} \int_{\mathbb{R}^3} ||\tilde{W}_{t-\bar{y}B_{\mu_1}^2} g_1||_{L^\infty(\mathbb{R}^3)} ||\tilde{W}_{t-\bar{y}B_{\mu_2}^2} g_2||_{L^\infty(\mathbb{R}^3)} dx \leq \sum_{\mu_1 + \mu_2 = \mu} L_{\mu_1, \mu_2}.
\]

We have, for \(|\alpha_1| < N/2\)

\[
L_{\mu_1, \mu_2} \leq ||\mathcal{A}||_{L^\infty(\mathbb{R}^3)} \int_{\mathbb{R}^3} ||\tilde{W}_{t-\bar{y}B_{\mu_1}^2} g_1||_{L^\infty(\mathbb{R}^3)} ||\tilde{W}_{t-\bar{y}B_{\mu_2}^2} g_2||_{L^\infty(\mathbb{R}^3)} dx \\
\leq ||\mathcal{A}||_{H^\infty(\mathbb{R}^3)} ||g_1||_{L^\infty(\mathbb{R}^3)}^2 \leq ||\mathcal{A}||_{H^\infty(\mathbb{R}^3)} ||g_2||_{L^\infty(\mathbb{R}^3)}^2.
\]

while for \(|\alpha_2| \leq N/2\)

\[
L_{\mu_1, \mu_2} \leq ||\tilde{W}_{t-\bar{y}B_{\mu_1}^2} g_2||_{L^\infty(\mathbb{R}^3)} \int_{\mathbb{R}^3} ||\mathcal{A}||_{L^\infty(\mathbb{R}^3)} ||\tilde{W}_{t-\bar{y}B_{\mu_2}^2} g_2||_{L^\infty(\mathbb{R}^3)} dx \\
\leq ||\tilde{W}_{t-\bar{y}B_{\mu_1}^2} g_2||_{L^\infty(\mathbb{R}^3)} ||\mathcal{A}||_{L^\infty(\mathbb{R}^3)} ||g_2||_{L^\infty(\mathbb{R}^3)}^2 \\
\leq ||\mathcal{A}||_{H^\infty(\mathbb{R}^3)} ||g_2||_{L^\infty(\mathbb{R}^3)}^2.
\]

Consequently,

\[
|K_{112}| \leq \mathcal{E}_{N/2}^{1/2} \mathcal{D}_{N,t}.
\]

**(3) Estimation on \(K_{121}\):** As for \(K_{112}\), we get

\[
|K_{121}| \leq \sum_{\mu_1 + \mu_2 = \mu} \int_{\mathbb{R}^3} ||\tilde{W}_{t-\bar{y}B_{\mu_1}^2} g_1||_{L^\infty(\mathbb{R}^3)} ||\tilde{W}_{t-\bar{y}B_{\mu_2}^2} g_2||_{L^\infty(\mathbb{R}^3)} dx \leq \mathcal{E}_{N/2}^{1/2} \mathcal{D}_{N,t}.
\]
(4) Estimation on $K_{122}$: We shall re-use (1.6) in the form, 

$$|K_{122}| = |(\langle \tilde{W}_{\xi,-\eta} \partial_{\mu}^2 \Gamma \rangle (g_2), g_2, \tilde{W}_{\xi,-\eta} \partial_{\mu}^2 g_2)_{L^2(\mathbb{R})}|$$

$$\leq \sum_{a_1} \sum_{a_2} \sum_{\mu_1, \mu_2, \mu_3} \int_{\mathbb{R}^3} \left( |\langle \tilde{W}_{\xi,-\eta} \partial_{\mu_1}^2 \partial_{\mu_2}^2 g_2 \rangle | \right) |\tilde{W}_{\xi,-\eta} \partial_{\mu_3}^2 g_2| \Phi_{a_1} \Phi_{a_2} \Phi_{a_3} \right| dx$$

$$\leq \sum_{a_1} \sum_{a_2} \sum_{\mu_1, \mu_2, \mu_3} \int_{\mathbb{R}^3} \left( |\langle \tilde{W}_{\xi,-\eta} \partial_{\mu_1}^2 \partial_{\mu_2}^2 g_2 \rangle | \right) |\tilde{W}_{\xi,-\eta} \partial_{\mu_3}^2 g_2| \Phi_{a_1} \Phi_{a_2} \Phi_{a_3} \right| dx$$

The case $|\alpha_1| \leq 1$ can be computed similarly, we finally conclude

$$|K_{122}| \leq \|g_2\|_{L^p(\mathbb{R})} \|g_2\|_{L^q(\mathbb{R})} \leq E_{N, \ell}^{1/2} D_{N, \ell},$$

and therefore the estimate (1.12) holds.

(II) Estimation of $K_2$: On the other hand, we have, for $|\alpha| + |\beta| = N, \beta \neq 0,$

$$|K_2| = |(v \cdot \nabla, \tilde{W}_{\xi,-\eta} \partial_{\mu}^2 g_2)_{L^2(\mathbb{R})}|$$

$$\leq \|\tilde{W}_{\xi,-\eta} \partial_{\mu_1}^2 \partial_{\mu_2}^2 g_2\|_{L^2(\mathbb{R})}$$

$$\leq C_8 \|\tilde{W}_{\xi,-\eta} \partial_{\mu_1}^2 \partial_{\mu_2}^2 g_2\|_{L^2(\mathbb{R})} + \|\tilde{W}_{\xi,-\eta} \partial_{\mu_1}^2 \partial_{\mu_2}^2 g_2\|_{L^2(\mathbb{R})}$$

(III) Estimation of $K_3$: Again we assume $\beta \neq 0$ or $|\alpha| \leq N - 1.$

$$|K_3| = |(\tilde{W}_{\xi,-\eta} \partial_{\mu}^2 (P, v \cdot \nabla) g_2)_{L^2(\mathbb{R})}|$$

where $D_{\gamma}$ is the dissipation integral with only $x$ derivatives.

(IV) Estimation of $K_4$: The main ingredients of the estimation are the commutator estimates $I$ and $II$ established in the proof of Proposition 4.8 that are valid for $\gamma > -3.$ We
re-produce them here.

\[ |I| = ||W_t, L_1 g, W_t g)_{L^2( \mathbb{R})}|| \leq ||W_t g||_{L^2( \mathbb{R})}^2. \]

\[ |II| = ||W_t[\partial_{x_t} L_1 g, W_t \partial_t g)_{L^2( \mathbb{R})}|| \leq \sum_{\beta + \beta' + \gamma = 0} \|W_t T(\partial_{x_t} \mu^{1/2}, \partial_{x_t} g, \partial_{x_t} \mu^{1/2}; W_t \partial_x g)_{L^2( \mathbb{R})}\| \|W_t \partial_t g \|_{\Phi_y}. \]

We also need the interpolation inequality

\[ (6.13) \|\hat{W}^i \partial_x h\|_{L^2} \leq C_0 \|\hat{W}^i \partial_x h\|_{L^2} + \|\hat{W}^i \partial_x h\|_{H^1} \leq C_0 \|\hat{W}^i \partial_x h\|_{L^2} + \|\hat{W}^i \partial_x h\|_{W^1}. \]

We shall prove

\[ (6.14) \|K_4\| \leq C_0 \|g_x\|_{L^2}^2 + \|g_x\|_{L^2}^2. \]

To this end, first, notice that

\[ K_4 = (W_t L_2 g, \partial_x g, W_t \partial_x g)_{L^2( \mathbb{R})}^2 = (W_t L_2 g, g, W_t \partial_x g)_{L^2( \mathbb{R})}^2 \]

Then, by virtue of the estimate for |I| and the interpolation inequality (6.13),

\[ |K_{41}| \leq \|W_t \partial_x g\|_{L^2( \mathbb{R})}^2 \]

\[ \leq C_0 \|W_t \partial_x g\|_{L^2( \mathbb{R})}^2 + \|W_t \partial_x g\|_{L^2( \mathbb{R})}^2 \]

\[ \leq C_0 \|W_t \partial_x g\|_{L^2( \mathbb{R})}^2 + \|W_t \partial_x g\|_{L^2( \mathbb{R})}^2 \]

On the other hand, the estimate for |II| leads to

\[ |K_{42}| \leq \sum_{\beta + \gamma = 0} \int_{\mathbb{R}} \|W_t \partial_x g\|_{L^2( \mathbb{R})}^2 \|W_t \partial_x g\|_{L^2( \mathbb{R})}^2 dx \]

This completes the proof of (6.14).

(V) Estimation of $K_5$: Further, by Proposition 4.3 that holds for $\gamma > -3$, we can proceed as in the computation for $K_{41}$ to obtain

\[ |K_5| \leq \|W_t \partial_x g\|_{L^2( \mathbb{R})}^2 \leq \|W_t \partial_x g\|_{L^2( \mathbb{R})}^2 \]

\[ \leq C_0 \|W_t \partial_x g\|_{L^2( \mathbb{R})}^2 + \|W_t \partial_x g\|_{L^2( \mathbb{R})}^2 \]

\[ \leq C_0 \|g_x\|_{L^2( \mathbb{R})}^2 + \|g_x\|_{L^2( \mathbb{R})}^2. \]
Conclusion: Plug the above estimates into (6.11) to deduce, for $|\alpha + \beta| \leq N, |\beta| \geq 1$,

\begin{equation}
\frac{d}{dt} \left( \|\bar{\partial}_\beta^m g_2\|_{L^2_\nu(\mathbb{R}^3)}^2 \right) + \left( \frac{1}{2} \|\hat{W}^{(\beta_1 - 1)\beta_2 + 1}\hat{g}_2\|_{L^2_\nu(\mathbb{R}^3)}^2,\|\bar{\partial}_\beta^m a\|_{L^2_\nu(\mathbb{R}^3)}^2 \right) \\
+ \frac{1}{2} \|\bar{\partial}_\beta^m g_2\|_{L^2_\nu(\mathbb{R}^3)}^2 \\
\leq \|\bar{\partial}_\beta^m g_2\|_{L^2_\nu(\mathbb{R}^3)}^2 + \mathcal{E}_{N,\ell}^{1/2} \mathcal{D}_{N,\ell} \\
+ \|\bar{\partial}_\beta^m g_2\|_{L^2_\nu(\mathbb{R}^3)}^2 + |\delta| \|\hat{g}_2\|_{L^2_{\beta}(\mathbb{R}^3)}^2 + \mathcal{D}_{N,\ell} \\
+ \|\hat{g}_2\|_{L^2_{\beta}(\mathbb{R}^3)}^2 + \|\hat{g}_2\|_{L^2_{\beta}(\mathbb{R}^3)}^2.
\end{equation}

We can then make a suitable linear combination of (5.3), (5.7), and (6.15) to deduce the following energy estimate.

**Proposition 6.6. (Global Energy Estimate with Weight)** Under the assumptions of Theorem 1.5, for $N \geq 6, \ell \geq N$,

\[
\frac{d}{dt} \mathcal{E}_{N,\ell} + \mathcal{D}_{N,\ell} \leq \mathcal{E}_{N,\ell}^{1/2} \mathcal{D}_{N,\ell}
\]

holds as far as $g$ exists.

We can now conclude in a standard way that the global classical solutions exist for small initial data in the weighted space $\tilde{\mathcal{H}}^{1/2}$, and this completes the proof of Theorem 1.5.

7. Appendix

Let us recall that $\Phi_{\nu} = |\nu|^\gamma$. Let $\phi$ be a smooth, positive radial function that takes value $1$ for small value and $0$ for large value of $|\nu|$. Set $\Phi_{\nu}(\nu) = |\nu|^\gamma \phi(\nu)$. We shall show the following

**Lemma 7.1. Assume $\gamma > -3$. Then, for all integer $k$, one has**

\[ |D^k \hat{\Phi}_{\nu}(\xi)| \leq \xi^{> -3 - \gamma}, \text{ for all } \xi \in \mathbb{R}^3. \]

**Proof.** Since $\Phi_{\nu}$ is bounded and compactly supported, clearly, for any integer $k, |D^k \hat{\Phi}_{\nu}(\xi)| \leq 1$ so we can only consider the case when $|\xi| >> 1$.

We first consider the case: $-3 < \gamma < 0$. We use the fact that the Fourier transform of $|\nu|^\gamma$ is (up to constant) $|\xi|^{-3 - \gamma}$, see Page 243 of [84].

Let $\psi = \phi(\xi)$ a smooth positive function supported on $|\xi| \leq 1$, and is equal to $1$ for on $|\xi| \leq 1/2$. Write

\[
\hat{\Phi}_{\nu}(\xi) = \int_{\eta} \frac{1}{|\xi - \eta|^{3+\gamma}} \hat{\phi}(\eta) d\eta = J_1 + J_2,
\]

where

\[
J_1 = \int_{\eta} \frac{1}{|\xi - \eta|^{3+\gamma}} \psi(\xi - \eta) \hat{\phi}(\eta) d\eta, \quad \text{and} \quad J_2 = \int_{\eta} \frac{1}{|\xi - \eta|^{3+\gamma}} [1 - \psi(\xi - \eta)] \hat{\phi}(\eta) d\eta.
\]

For $J_1$, the support is on $|\xi - \eta| \leq 1$. This means that $|\eta| \geq |\xi| - 1 \geq c|\xi|$, for some constant $c$ and because we have assumed $|\xi| >> 1$. Then, we can use the decay of $\hat{\phi}$ to get, for any $m$ positive

\[
J_1 \leq \int_{|\xi - \eta| \leq 1} \frac{1}{|\xi - \eta|^{3+\gamma}} < \eta >^{-m} d\eta \leq |\xi >^{-m}.
\]
For $J_2$, the integration is over $|\xi - \eta| \geq 1/2$. So we can replace $|\xi - \eta|$ by $<\xi - \eta>$ to get

$$J_2 = \int_{|\xi - \eta| \geq 1/2} \frac{1}{<\xi - \eta>^{3+\gamma}} <\eta>^{-m} d\eta.$$  

Choose $m = M + 3 + \gamma$ with $M$ large enough. Then

$$J_2 = \int_{|\xi - \eta| \geq 1/2} \frac{1}{<\xi - \eta>^{3+\gamma}} <\eta>^{-M} <\eta>^{-3-\gamma} d\eta$$

$$\leq <\xi>^{-3-\gamma} \int_{|\xi - \eta| \geq 1/2} <\eta>^{-M} d\eta.$$  

Thus, we have shown that $|\Phi_\gamma(\xi)| \leq <\xi>^{-3+\gamma}$, which proves the Lemma in the case when $k = 0$.

This proof works well for derivatives. For example, consider the case when $k = 1$. First note that

$$\nabla \Phi_\gamma(\xi) = \int_{\eta} \frac{1}{|\xi - \eta|^{3+\gamma}} \nabla \phi(\eta) d\eta = K_1 + K_2,$$

where

$$K_1 = \int_{\eta} \frac{1}{|\xi - \eta|^{3+\gamma}} \psi(\xi - \eta) \nabla \phi(\eta) d\eta \quad \text{and} \quad K_2 = \int_{\eta} \frac{1}{|\xi - \eta|^{3+\gamma}} [1 - \psi(\xi - \eta)] \nabla \phi(\eta) d\eta.$$  

$K_1$ is estimated directly as for $J_1$, with all the decay.

For $K_2$, integration by parts gives

$$K_2 = -\int_{\eta} \left[ \frac{1}{|\xi - \eta|^{3+\gamma}} [1 - \psi(\xi - \eta)] \nabla \phi(\eta) \right] d\eta$$

$$+ \int_{\eta} \frac{1}{|\xi - \eta|^{3+\gamma}} \nabla \psi(\xi - \eta) \nabla \phi(\eta) d\eta.$$  

Here, the first term has the good decay in $-3 - \gamma - 1$, while the second one has all the decay.

We now consider the case $\gamma \geq 0$. Of course, for $\gamma = 0$, the result is clear, because then $\Phi_\gamma$ is in $S$.

For $2 > \gamma > 0$ we have

$$|v|^\gamma \varphi(|v|) = \int \left( -\Delta_v e^{iv\xi} \right) F_v (v^\gamma \varphi(v)) (\xi) d\xi / (2\pi^2)$$

$$= -\int e^{iv\xi} \Delta_v F_v (v^\gamma \varphi(v)) (\xi) d\xi / (2\pi^2),$$  

which gives

$$\left| \partial_\xi^a F_v (v^\gamma \varphi(|v|)) (\xi) \right| = \left| \partial_\xi^a \Delta_v F_v (v^\gamma \varphi(|v|)) (\xi) \right| \leq C_a <\xi>^{-3-\gamma-|v|},$$  

by using the previous negative case since $\gamma - 2 < 0$. The remaining cases are similar and this completes the proof of the lemma.

$$\square$$

Acknowledgements: The research of the first author was supported in part by the Zhiyuan foundation and Shanghai Jiao Tong University. The research of the second author was supported by Grant-in-Aid for Scientific Research No.22540187, Japan Society of the Promotion of Science. The last author’s research was supported by the General Research Fund of Hong Kong, CityU No.103109, and the Lou Jia Shan Scholarship programme of Wuhan University. The authors would like to thank the financial supports from City University of Hong Kong, Kyoto University, Rouen University and Wuhan University for their visits.
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