The interaction transform for functions on lattices

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Abstract

The paper proposes a general approach of interaction between players or attributes. It generalizes the notion of interaction defined for players modeled by games, by considering functions defined on distributive lattices. A general definition of the interaction transform is provided, as well as the construction of operators establishing transforms between games, their Möbius transforms and their interaction indices.

Key words: lattice function, Möbius transform, interaction transform, group action
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1 Introduction

Set functions or pseudo-Boolean functions are a widely used concept in discrete mathematics, especially in operations research, cooperative game theory, and decision making (see the classical book of Hammer and Rudeanu [17]). In this paper, we are in particular interested by set functions vanishing on the empty set (grounded pseudo-Boolean functions), which we call *games*.

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It is well known that any set function v defined on some finite universe N can be equivalently represented by its Möbius transform m^v (this is the multilinear polynomial form of pseudo-Boolean functions), but other equivalent representations exist as well, in particular the interaction transform I^v [15]. This transform has its origin rooted in cooperative game theory, where it defines a so-called interaction index [16], which is a generalization of the Shapley value to coalitions of players. Apart its application to game theory, the interaction transform has nice mathematical properties by itself: it is expressed through derivatives of the set function, and is closely related to the sequence of Bernoulli numbers. A convenient mathematical framework for expressing these different representations of set functions through an algebra of operators has been done by Denneberg and Grabisch [7].

This paper aims at building a similar construction, where set functions are replaced by more general lattice functions. The motivation for this work stems again from game theory. Indeed, classical games assign to every coalition $S \subseteq N$ a real number, which represents the worth (or cost, power) if all players in S participate to the game, and the others do not. Many generalizations of this elementary setting have been done, e.g., multichoice games of Hsiao and Raghavan [18] where each player is allowed a given number of participation levels, bicooperative games of Bilbao [2], where each player has three options (play in favor, against or not participate), games with restricted cooperation (Faigle and Kern [9], Bilbao et al. [3]), games on antimatroids [1], and other combinatorial structures. Most of the above games can be considered as lattice functions, i.e., real-valued functions defined on a lattice. When the lattice is a product of distributive lattices, Grabisch and Lange have provided a general interpretation for such games [13].

A first step towards an algebra of operators for lattice functions has been done by the authors in [19], where the case of bicooperative games (more generally, bi-set functions) was addressed. The results presented in this paper encompass all previous results in [7] and [19], and gives a general view of the representation of lattice functions through the Möbius and interaction transforms, as well as their inverses. The interaction transform is based on and extends the definition of an interaction index proposed by Grabisch and Labreuche for lattice functions [12].

The paper is organized as follows. Section 2 recalls the necessary material on lattices and games. Section 3 introduces the Möbius transform and derivative of lattice functions. Section 4 gives the definition of the interaction index for lattice functions, which extends the former definition of Grabisch and Labreuche [12] under a form suitable for the definition of an interaction transform. Section 5 gives the algebraic framework for the representation of lattice functions through linear invertible operators, and introduces formally the

Möbius and interaction transform. Section 6 studies a particular subgroup of operators, and gives a simple formula for computing the product and the inverse of such operators. Lastly, Section 7 gives an explicit expression of the interaction transform and its inverse.

$\mathbb{N}$ denotes the set of nonnegative integers. If no ambiguity occurs, we denote by lower case letters $s, t, \ldots$ the cardinal of sets $S, T, \ldots$, and we will often omit braces for singletons.

2 Lattice functions and games

We introduce some basic notions about lattices and distributive lattices. A *lattice* L is any partially ordered set (poset) $(L, \leq)$ in which every pair of elements x, y has a supremum $x \vee y$ and an infimum $x \wedge y$. The greatest element of a lattice (denoted $\top$) and least element $\perp$ always exist. In the sequel, it shall be convenient to lay down the convention $\bigvee \emptyset = \bigwedge \emptyset = \perp$.

A lattice is *distributive* if $\vee, \wedge$ obey distributivity. An element $j \in L$ is *join-irreducible* if it cannot be expressed as a supremum of other elements. Equivalently j is join-irreducible if it covers only one element, where x *covers* y (we also say that y is a *predecessor* of x, and denote $x \succ y$) means that $x > y$ and there is no z such that $x > z > y$. The set of all join-irreducible elements of L is denoted by $\mathcal{J}(L)$.

An important property is that in a distributive lattice, any element x can be written as an irredundant supremum of join-irreducible elements in a unique way (this is called the *minimal decomposition* of x). We denote by $\eta^*(x)$ the set of join-irreducible elements in the minimal decomposition of x, and we denote by $\eta(x)$ the *normal decomposition* of x, defined as the set of join-irreducible elements smaller or equal to x, i.e., $\eta(x) := \{j \in \mathcal{J}(L) \mid j \leq x\}$. Hence $\eta^*(x) \subseteq \eta(x)$, and

$$x = \bigvee_{j \in \eta^*(x)} j = \bigvee_{j \in \eta(x)} j.$$

For any poset $(P, \leq)$, $Q \subseteq P$ is said to be a *downset* of P if $x \in Q$ and $y \leq x$ imply $y \in Q$. We denote by $\mathcal{O}(P)$ the set of all downsets of P. One can associate to any poset $(P, \leq)$ a distributive lattice which is $\mathcal{O}(P)$ endowed with inclusion. As a consequence of the above results, the mapping η is an isomorphism of L onto $\mathcal{O}(\mathcal{J}(L))$ (Birkhoff's theorem, [4]).

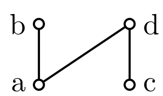
In the whole paper, $N := \{1, \ldots, n\}$ is a finite set which can be thought as the set of players or also voters, criteria, states of nature, depending on the application. We consider finite distributive lattices $(L_1, \leq_1), \ldots, (L_n, \leq_n)$ and their product $L := L_1 \times \cdots \times L_n$ endowed with the product order $\leq$. Elements x of L

can be written in their vector form $(x_1, \dots, x_n)$. L is also a distributive lattice whose join-irreducible elements are of the form $(\bot_1, \dots, \bot_{i-1}, j_i, \bot_{i+1}, \dots, \bot_n)$, for some i and some join-irreducible element j_i of L_i. In the sequel, with some abuse of language, we shall also call j_i this element of L. We denote by $\mathcal{J}(L)$ the set of join-irreducible elements of L (Section 4). A *vertex* of L is any element whose components are either top or bottom. Vertices of L will be denoted by $\top_X$, $X \subseteq N$, whose coordinates are $\top_k$ if $k \in X$, $\bot_k$ else.

Lattice functions are real-valued mappings defined over product lattices of the above form. Lattice functions which vanishes at $\bot$ are called *lattice games* (or games) on $(L, \leq)$. We denote by $\mathbb{R}^L$ the set of lattice functions over L, and by $\mathcal{G}(L)$ the subspace of games. Each lattice $(L_i, \leq_i)$ may be different, and represents the (partially) ordered set of actions, choices, levels of participation of player i to the game. A game v is *monotone* if $x \leq y$ implies $v(x) \leq v(y)$ for all $x, y \in L$. Several particular cases of lattice games are of interest.

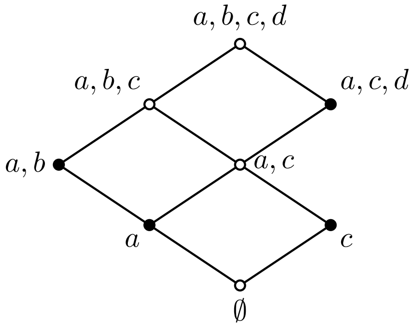
- $L = \{0,1\}^n$. Cooperative games on L are given in the form of *pseudo-Boolean functions* [17]. Indeed, $(L, \leq)$ is isomorphic to the Boolean lattice[1] $(2^N, \subseteq)$ of the subsets of N, also called *coalitions* of N. Monotone games of $\mathcal{G}(2^N)$ are called *capacities* [5], or *non-additive measures* [6], or *fuzzy measures* [22].
- L is the direct product of some linear lattices: $\forall i \in N,\ L_i = \{0, 1, \dots, l_i\}$. This corresponds to multichoice games as introduced by Hsiao and Raghavan [18].
- We propose the following interpretation for games on L in the general case, i.e., L is any direct product of n distributive lattices. We assume that each player $i \in N$ has at her/his disposal a set of *elementary* or *pure* actions $j_1, \dots, j_{n_i}$. These elementary actions are partially ordered (e.g., in the sense of benefit caused by the action), forming a partially ordered set $(\mathcal{J}_i, \leq_i)$. Then by Birkhoff's theorem (see above), the set $(\mathcal{O}(\mathcal{J}_i), \subseteq)$ of downsets of $\mathcal{J}_i$ is a distributive lattice denoted by L_i, whose join-irreducible elements correspond to the elementary actions. The *bottom action* $\bot$ of L_i is the action which amounts to do nothing. Hence, each action in L_i is either a pure action j_k or a combined action $j_k \vee j_{k'} \vee j_{k''} \vee \dots$ consisting of doing all pure actions $j_k, j_{k'}, \dots$ for player i.

For example, let us suppose that for a given player i, elementary actions are a, b, c, d endowed with the order $\leq_i := \{(a,b), (a,d), (c,d)\}$. They form the following poset:



[1] To avoid a heavy notation, we will sometimes denote by 2^m any Boolean lattice isomorphic to 2^M, $|M| = m$.

which in turn form the following lattice L_i of possible actions (black circles represent join-irreducible elements of L_i):



Another interpretation of our framework is borrowed from Faigle and Kern [9]. Let $P := (N, \leq)$ be a partially ordered set of players, where $\leq$ is a relation of *precedence*: $i \leq j$ if the presence of j enforces the presence of i in any coalition $S \subseteq N$. Hence, a (valid) *coalition* of P is a subset S of N such that $i \in S$ and $j \leq i$ entails $j \in S$. Hence, the collection $\mathcal{C}(P)$ of all coalitions of P is the collection of all downsets of P.

It is possible to combine both structures. For each player i in N, let $\mathcal{J}_i := \{j_1, \ldots, j_{n_i}\}$ be the set of elementary actions of player i. Consider the set of all elementary actions $N' := \bigcup_{i \in N} \mathcal{J}_i$ equipped with the partial order $\leq$ induced by the partial orders on each $\mathcal{J}_i$. Then N' might be seen as a set of virtual players whose valid coalitions bijectively correspond to elements of $\prod_{i \in N} \mathcal{O}(\mathcal{J}_i)$.

3 The Möbius transform and derivatives of lattice functions

We introduce in this section some useful material for lattice functions. Let $(P, \leq)$ be any poset. The *Möbius transform* m^f [20] of a mapping $f : P \to \mathbb{R}$ is the unique solution of the equation

$$f(x) = \sum_{y \leq x} m^f(y), \quad x \in P, \tag{1}$$

given by

$$m^f(x) := \sum_{y \leq x} \mu(y, x)\, f(y), \quad x \in P, \tag{2}$$

where $\mu : P \times P \to \mathbb{Z}$ is recursively given by

$$\mu(x, y) = \begin{cases} 1, & \text{if } x = y, \\ -\sum_{x \leq t < y} \mu(x, t), & \text{if } x < y, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

For instance, whenever P is the Boolean lattice 2^N endowed with inclusion, it is well-known that $\mu(A, B) = (-1)^{|B\setminus A|}$, for all subsets A, B such that $A \subseteq B$.

As it will be seen in the next section, *derivatives* of lattice functions are a very useful tool, and have been generalized (in particular) for distributive lattice functions in [12]. Let $(L, \leq)$ be a distributive lattice and $j \in \mathcal{J}(L)$. The *first-order derivative* of the lattice function f w.r.t. j at element $x \in L$ is given by

$$\Delta_j f(x) := f(x \vee j) - f(x).$$

Using the minimal irredundant decomposition $\eta^*(y) = \{j_1, \dots, j_m\}$ of some $y \in L$, we iteratively define the derivative of f w.r.t. y at x by

$$\Delta_y f(x) := \Delta_{j_m}\Big(\dots \Delta_{j_2}\Big(\Delta_{j_1} f(x)\Big) \dots\Big), \quad x \in L.$$

Note that if for some k, $j_k \leq x$, the derivative is null. Also, $\Delta_y f(x)$ does not depend on the order of the j_k's and thus is well defined. Actually, we easily show by induction that

$$\Delta_y f(x) = \sum_{S \subseteq \{1,\dots,m\}} (-1)^{m-s} f\Big(x \vee \bigvee_{k \in S} j_k\Big).$$

We set $\Delta_\perp f(x) := f(x)$, for any $x \in L$.

Note that the derivative w.r.t. y at x takes values at points of $[x, x \vee y]$. If this interval is isomorphic to $2^{\eta^*(y)}$, the derivative is said to be *Boolean*. Equivalently, the derivative is Boolean if $j_k \not\leq x$, $\forall k = 1, \dots, m$, and $[x, x \vee y]$ is Boolean. The reader is invited to refer to [12] for more details about Boolean derivatives. In the same paper, the authors provide a close link between any Boolean derivative and the Möbius transform of a lattice function:

Proposition 1 *Let $x, y \in L$, such that $\Delta_y f(x)$ is Boolean. Then*

$$\Delta_y f(x) = \sum_{z \in [y, x \vee y]} m^f(z).$$

4 The interaction transform for lattice functions

From now on, L is a direct product of n finite distributive lattices. Let $v \in \mathcal{G}(L)$. We propose a general definition of interaction as presented in the introduction. We begin by defining the *importance index*, introduced in [12], as a *power index* of the game defined for elementary actions of every player (that is to say, w.r.t. each join-irreducible element of each lattice L_i). This means that we try to provide an equitable way to share the worth $v(\top)$ between all elementary actions.

For a given elementary action j_i, the importance index is written as a weighted average of the marginal contributions of j_i, taken at vertices of L.

Definition 2 *Let $i \in N$ and j_i any join-irreducible element of L. Let $v \in \mathcal{G}(L)$. The* importance index *w.r.t. j_i of v is defined by*

$$\phi^v(j_i) = \sum_{Y \subseteq N \setminus i} \alpha^1_{|Y|} \, \Delta_{j_i} v(\top_Y),$$

where $\alpha^1_k := \dfrac{k!\,(n-k-1)!}{n!}$*, for all* $k \in \{0, \dots, n-1\}$.

Note that if $L = 2^N$, we obtain the definition of the *Shapley value* [21]. In [14], we proposed an axiomatization of the Shapley value for multichoice games, where the obtained formula is also the one given above (all the L_i's are completely ordered).

As an extension of the importance index for every element of L, and every lattice function $f \in \mathbb{R}^L$, we propose a definition for the *interaction transform*. For any $x \in L$, $I^f(x)$ expresses the interaction in the function among all elementary actions j of the minimal decomposition $x = \bigvee_{j \in \eta^*(x)} j$.

An interaction index has been proposed in [12]. However, the formula was only defined for elements of $\mathcal{J}(L)$. We present here I^f as a mapping defined over L. For that, we give the following generalized definition of $\underline{x}$ for any $x \in L$.

Definition 3 *Let $x \in L$. We call* antecessor *of x the unique element of L defined by $\underline{x} := \bigvee \{ j \in \eta(x) \mid j \notin \eta^*(x)\}$.*

If $x \in \mathcal{J}(L)$, the antecessor of x is obviously its predecessor, in accordance with the notation $\underline{x}$. By the convention of Section 2, the antecessor of $\perp$ is itself. Note also that the definition of $\underline{x} \in L$ is consistent with the structure of direct product of distributive lattices of L. Indeed, we easily check that $\underline{x} = (\underline{x_1}, \dots, \underline{x_n})$.

Lemma 4 *Let $x \in L$. For any $J \subseteq \eta^*(x)$, $\exists!\, y_J \in [\underline{x}, x]$ such that $x = y_J \vee \bigvee_{j \in J} j$. Moreover, the mapping $\mathrm{pred}_x :\ 2^{\eta^*(x)} \to [\underline{x}, x]$ which associates to any J the element y_J, is a bijection.*

PROOF. Let $J \subseteq \eta^*(x)$. Since all j's in J are some maximal elements of $\eta^*(x)$, $\eta(x) \setminus J$ is a downset of $\mathcal{J}(L)$ and thus the normal decomposition of some element $y_J \leq x$. Besides, $y_J \geq \underline{x}$ since $\eta(x) \setminus J \supseteq \eta(x) \setminus \eta^*(x)$, which is the normal decomposition of $\underline{x}$, by definition. This defines the mapping pred_x, which is injective, since $y_J = y_{J'} \Rightarrow \eta(x) \setminus J = \eta(x) \setminus J' \Rightarrow J = J'$. Moreover, surjectivity of pred_x is clear since for any element y of $[\underline{x}, x]$, $\eta(x) \setminus \eta^*(\underline{x}) \subseteq \eta(y) \subseteq \eta(x)$, i.e., there is a subset J of $\eta^*(x)$ such that $\eta(y) = \eta(x) \setminus J$.

The following proposition provides three characterizations and an important property of the antecessor.

Proposition 5 *Let $x \in L$, and $p(x) := \{y \in L \mid y \prec x\}$. Then the following assertions hold.*

(i) $\underline{x} = \bigwedge p(x)$.
(ii) $\underline{x}$ *is the greatest element s.t.* $[\underline{x}, x]$ *contains* $p(x)$.
(iii) $\underline{x}$ *is the least element s.t.* $[\underline{x}, x]$ *is Boolean.*
(iv) $[\underline{x}, x] \cong 2^{\eta^*(x)}$.

PROOF. For any predecessor y of $x \neq \perp$, there is a unique element $j \in \eta^*(x)$ such that $\eta(y) = \eta(x) \setminus j$. Indeed, $y \prec x \Rightarrow \eta(y) \subseteq \eta(x)$, and at least one element of $\eta(x) \setminus \eta(y)$ belongs to $\eta^*(x)$, otherwise $x = y$. If two elements of $\eta^*(x)$ are removed, say j and j', then clearly $y \prec y \vee j \prec x$, which contradicts $y \prec x$. Conversely, for any $j \in \eta^*(x)$, $\eta(x) \setminus j$ is the decomposition into join-irreducible elements of a predecessor of x. Hence $\eta(\bigwedge p(x)) = \bigcap_{j \in \eta^*(x)} \eta(x) \setminus \{j\} = \eta(x) \setminus \eta^*(x)$, which proves (i).

We straightforwardly derive (iv) from Lemma 4. If $[x', x]$ is an interval containing $p(x)$, x' must be a lower bound of any element of $p(x)$, hence by (i), $x' = \underline{x}$ is the greatest possible element, and (ii) is shown. Besides, by Lemma 4, for all $y \in [\underline{x}, x]$, $[y, x]$ is Boolean. At last, for any $z < x$ s.t. $z \notin [\underline{x}, x]$, we have $z < y < x$, where $y \in p(x)$. Hence $[z, x]$ is clearly not Boolean, which proves (iii). As a result, $\underline{x}$ is the sole element such that $[\underline{x}, x]$ is Boolean and contains $p(x)$.

The interaction transform $I^f(x)$ is expressed as a weighted average of the derivatives w.r.t. x, taken at vertices of L.

Definition 6 *Let $f \in \mathbb{R}^L$. Let $x \in L$ and $X := \{i \in N \mid x_i \neq \perp_i\}$. The* interaction transform *w.r.t. x of f is defined by*

$$I^f(x) := \sum_{Y \subseteq N \setminus X} \alpha_{|Y|}^{|X|} \, \Delta_x f(\underline{x} \vee \top_Y), \tag{4}$$

where $\alpha_k^j := \dfrac{k!\,(n-j-k)!}{(n-j+1)!}$*, for all $j = 0, \ldots, n$ and $k = 0, \ldots, n-j$.*

In fact, this extends Definition 2. Besides, the formula overlaps previous definitions of the interaction index introduced and axiomatized in [7,16] for classical cooperative games, and also in [12] for multichoice games whose all L_i's are identical.

We now express the interaction transform in terms of the Möbius transform by means of the following result.

Lemma 7 *For any $x \in L$, $\Delta_x f(y)$ is Boolean for any y such that for all k, $y_k = \bot_k$ or $\top_k$ if $x_k = \bot_k$, and $y_k = \underline{x_k}$ otherwise.*

PROOF. We have to prove that $[y, x \vee y]$ is isomorphic to $2^{\eta^*(x)}$. It suffices to prove that $[y_k, (x \vee y)_k]$ is isomorphic to $2^{\eta^*(x_k)}$ for each coordinate k. If $x_k = \bot_k$, then $[y_k, x_k \vee y_k] = \{y_k\} \cong 2^{\emptyset}$. If $x_k \neq \bot_k$, then $[y_k, x_k \vee y_k] = [\underline{x_k}, x_k] \cong 2^{\eta^*(x_k)}$, by Proposition 5.

The following result generalizes one given in [12].

Theorem 8 *Let $f \in \mathbb{R}^L$ and $x \in L$. Then*

$$I^f(x) = \sum_{z \in [x, \check{x}]} \frac{1}{k(z) - k(x) + 1} m^f(z), \tag{5}$$

where $\check{x}_j := \top_j$ if $x_j = \bot_j$, $\check{x}_j := x_j$ else, and $k(y)$ is the number of coordinates of $y \in L$ not equal to $\bot_j$, $j = 1, \dots, n$.

PROOF. Since the derivative in (4) is Boolean by Lemma 7, we can apply Proposition 1, and we get:

$$I^f(x) = \sum_{Y \subseteq N \setminus X} \alpha_{|Y|}^{|X|} \sum_{z \in [x, x \vee \top_Y]} m^f(z).$$

Consequently, $I^f(x)$ can be linearly expressed in terms of $m^f(z)$, where the z's may belong to any $[x, x \vee \top_Y]$, $Y \subseteq N \setminus X$, i.e., $z \in \bigcup_{Y \subseteq N \setminus X} [x, x \vee \top_Y] = [x, x \vee \top_{N \setminus X}]$, that is to say:

$$I^f(x) = \sum_{z \in [x, \check{x}]} \beta_z m^f(z).$$

To compute β_z for a given $z \in [x, \check{x}]$, we have to examine for which Y's of $N \setminus X$, z belongs to $[x, x \vee \top_Y]$. Note that $z_j = x_j$ for all $j \in X$. If $j \in N \setminus X$, and $z_j \neq \bot_j$, then Y must contain j. As a result:

$$\beta_z = \sum_{Z \subseteq Y \subseteq N \setminus X} \alpha_{|Y|}^{|X|},$$

where $Z := \{j \in N \setminus X \mid z_j \neq \bot_j\}$. Observing that $|X| = k(x)$ and $|Z| = k(z) - k(x)$, we get

$$\begin{aligned}\beta_z &= \sum_{j=k(z)-k(x)}^{n-k(x)} \binom{n-k(z)}{j-k(z)+k(x)} \alpha_j^{k(x)}\\ &= \sum_{j=0}^{n-k(z)} \binom{n-k(z)}{j} \alpha_{j+k(z)-k(x)}^{k(x)}\\ &= \sum_{j=0}^{n-k(z)} \frac{(n-k(z))!\,(j+k(z)-k(x))!}{j!\,(n-k(x)+1)!}.\end{aligned}$$

It has been proved in [11] that

$$\sum_{i=0}^{l} \frac{(m-i-1)!\,l!}{m!\,(l-i)!} = \frac{1}{m-l}, \qquad m \in \mathbb{N} \setminus \{0\}, l \in \{0, \ldots, m-1\}.$$

Applying the above formula with $m = n - k(x) + 1$, $l = n - k(z)$ and $i = n - k(z) - j$, we obtain

$$\beta_z = \frac{1}{k(z) - k(x) + 1},$$

which is the desired result.

5 Linear transformations on sets of lattice functions

In [7], Denneberg and Grabisch laid down a general framework of transformations of set functions by introducing an algebraic structure on set functions and operators (set functions of two variables), which enable the writing of the formulae given in the previous section under a simplified algebraic form. Then in [19], the authors did the same for bi-set functions, by introducing *incidence algebras* [8]. Although this tool may be useful in combinatorics of order theory, we do not now proceed in the same way for lattice functions, making the choice to use a more suitable algebraic structure, namely the group actions.

We call *operator* on L a real-valued function on $L \times L$. A binary operation $\star$ (multiplication or convolution) between operators is introduced as follows:

$$(\Phi \star \Psi)(x, y) := \sum_{t \in L} \Phi(x, t)\, \Psi(t, y).$$

Endowed with $\star$, the set of operators contains the identity element

$$\Delta(x, y) := \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{otherwise,} \end{cases} \qquad x, y \in L,$$

and also satisfies associativity, which makes it a monoid. When it exists, we denote by Φ^{-1} the inverse of an operator Φ, that is to say the operator satisfying $\Phi \star \Phi^{-1} = \Phi^{-1} \star \Phi = \Delta$. Consequently, the set of all inversible operators is a group. We denote it by $\mathbb{G}$. We denote by ${}^t\Phi$ the *transpose* of the operator Φ, i.e., ${}^t\Phi(x, y) := \Phi(y, x)$ for all $x, y \in L$.

Let $\leqq$ be any partial order on L included in the usual order $\leq$, and $\lneqq$ the associated strict order. We denote by $I(L, \leqq)$ the set of intervals of L w.r.t. the order $\leqq$, i.e., the family of subsets $[x, y]_{\leqq} := \{t \in L \mid x \leqq t \leqq y\}$, with $x \leqq y$. An operator Φ is said to be *unit upper-triangular* (resp. *unit lower-triangular*) relatively to $\leqq$, or shortly UUT$_{\leqq}$ (resp. ULT$_{\leqq}$), if it equals 1 on the diagonal of L^2, and vanishes at all pairs (x, y) s.t. $[x, y]_{\leqq} = \emptyset$ (resp. $[y, x]_{\leqq} = \emptyset$):

$$\Phi(x, y) = \begin{cases} 1, & \text{if } x = y, \\ 0, & \text{if } x \not\leqq y, \end{cases} \qquad x, y \in L.$$

Note that the transpose of any UUT$_{\leqq}$ operator is ULT$_{\leqq}$.

Proposition 9 *The subset $G(\leqq)$ of all UUT$_{\leqq}$ operators endowed with $\star$, is a subgroup of $\mathbb{G}$. The inverse Φ^{-1} of $\Phi \in G(\leqq)$ computes recursively through*

$$\begin{aligned} &\Phi^{-1}(x, x) = 1, \\ &\Phi^{-1}(x, y) = -\sum_{x \leqq t \lneqq y} \Phi^{-1}(x, t)\, \Phi(t, y), \quad x \lneqq y. \end{aligned}$$

PROOF. $G(\leqq)$ being nonempty, it suffices to check that it is closed under multiplication and inversion. For any $\Phi, \Psi \in G(\leqq)$, $\Psi \star \Phi$ clearly belongs to $G(\leqq)$. Besides, let us examine $\Phi^{-1}(x, y)$ for $x \lneqq y$.

$$\Phi^{-1} \star \Phi(x, y) = \sum_{t \leqq y} \Phi^{-1}(x, t)\, \Phi(t, y).$$

Then

$$\Phi^{-1}(x, y)\, \Phi(y, y) + \sum_{t \lneqq y} \Phi^{-1}(x, t)\, \Phi(t, y) = \Delta(x, y) = 0.$$

Thus:

$$\Phi^{-1}(x, y) = -\sum_{t \lneqq y} \Phi^{-1}(x, t)\, \Phi(t, y).$$

In addition, we easily verify that the unit upper-triangular operator satisfying the above formula (which implies that the sum is over $x \leqq t \lneqq y$), suits as the inverse of Φ.

Applying this result for the *Zeta operator* $Z \in G(\leq)$:

$$Z(x,y) := \begin{cases} 1, & \text{if } x \leq y, \\ 0, & \text{otherwise}, \end{cases} \qquad x, y \in L, \tag{6}$$

we recognize the recursive formula (3) (Section 3) of the Möbius operator, i.e., $Z^{-1} = \mu$.

In order to rewrite formulae (1), (2) and also (5) in a reduced form, we introduce some group actions of $\mathbb{G}$ on the set of lattice functions: a *left* (resp. *right*) *group action* of a group $(\mathscr{G}, *)$ on a set $\mathcal{S}$ is a binary function

$$\mathscr{G} \times \mathcal{S} \to \mathcal{S} \quad (\text{resp. } \mathcal{S} \times \mathscr{G} \to \mathcal{S})$$

denoted by

$$(\Phi, f) \mapsto \Phi * f \quad (\text{resp. } (f, \Phi) \mapsto f * \Phi),$$

satisfying the following axioms:

GA1. $(\Psi * \Phi) * f = \Psi * (\Phi * f) \quad (\text{resp. } f * (\Phi * \Psi) = (f * \Phi) * \Psi)$, for all Φ, Ψ in $\mathscr{G}$ and $f \in \mathcal{S}$.

GA2. $E * f = f \quad (\text{resp. } f * E = f)$, for every $f \in \mathcal{S}$, where E is the identity element of $(\mathscr{G}, *)$.

Let $\Phi \in \mathbb{G}$, and f be a lattice function over L. For x belonging to L, we define:

$$(\Phi \star f)(x) := \sum_{t \in L} \Phi(x,t)\, f(t), \tag{7}$$

$$(f \star \Phi)(x) := \sum_{t \in L} f(t)\, \Phi(t,x). \tag{8}$$

It is easy to verify that (7) and (8) respectively define a left and a right group action of $\mathbb{G}$ on $\mathbb{R}^L$. Note that the subgroup $G(\leq)$ is not stable under the transpose operation.

Now, (1) and (2) respectively rewrites as

$$f = m^f \star Z, \quad \text{and} \quad m^f = f \star Z^{-1}, \quad f \in \mathbb{R}^L.$$

Similarly, if we set down:

$$\Gamma(x,y) := \begin{cases} \dfrac{1}{k(y) - k(x) + 1}, & \text{if } \forall i \in N, x_i = \perp_i \text{ or } y_i = x_i, \\ 0, & \text{otherwise}, \end{cases} \qquad x, y \in L, \tag{9}$$

we notice that $\Gamma \in G(\leq)$, and we can write from (5) the relation:

$$I^f = \Gamma \star m^f, \quad f \in \mathbb{R}^L,$$

which in turns gives by inversion

$$m^f = \Gamma^{-1} \star I^f, \quad f \in \mathbb{R}^L. \tag{10}$$

It is also possible to do without left group actions. Indeed, we easily show that the left action $\mathbb{G} \times \mathbb{R}^L \to \mathbb{R}^L$ can be converted into the right action $\mathbb{R}^L \times \mathbb{G} \to \mathbb{R}^L$ by $(\Phi, f) \mapsto (f, {}^t\Phi)$. Consequently,

$$I^f = m^f \star {}^t\Gamma, \quad f \in \mathbb{R}^L.$$

Note that ${}^t\Gamma$ and ${}^t\Gamma^{-1}$ are unit lower-triangular.

As a conclusion of these results, any lattice function may be seen as the interaction transform or the Möbius transform of some lattice function. This actually generalizes a result *(equivalent representations)* of [10] by the result below (see Figure 1).

Theorem 10 *Operators Z and Γ generate a commutative diagram in $\mathbb{R}^L$.*

PROOF. From axioms **GA1** and **GA2**, it follows that for every Φ in $G(\leq)$, the function which maps f in $\mathbb{R}^L$ to $f \star \Phi$ (or $\Phi \star f$) is a bijective map from $\mathbb{R}^L$ to $\mathbb{R}^L$. Applying the result for $\Phi = Z$ and $\Phi = \Gamma$, the result follows.

We call *interaction operator*, the operator $\mathbb{I} := Z^{-1} \star {}^t\Gamma$. Hence, the interaction transform of $f \in \mathbb{R}^L$ is given by $I^f = f \star \mathbb{I}$. Note that $\mathbb{I}$ is neither UUT nor ULT.

$$\begin{tikzcd}
f \arrow[rr, shift left, "\mathbb{I}"] \arrow[dr, shift right, "Z^{-1}"'] & & I^f \arrow[ll, shift left, "\mathbb{I}^{-1}"] \arrow[dl, shift right, "{}^t\Gamma^{-1}"'] \\
& m^f \arrow[ul, shift right, "Z"'] \arrow[ur, shift right, "{}^t\Gamma"'] &
\end{tikzcd}$$

Fig. 1. Lattice functions and their representations (operators act on the right)

6 The Möbius and Bernoulli operators

We now aim at giving an explicit formula for the Möbius operator and the Bernoulli operator[2] Γ^{-1}. Let $\sim$ be an equivalence relation on the set $I(L, \leqq)$. We denote by $\overline{[x,y]_{\leqq}}$ the class of any interval $[x,y]_{\leqq}$ by the relation $\sim$. We consider the following property for operators of $G(\leqq)$ relatively to this relation:

$$\begin{aligned}&\Phi \text{ is constant on each equivalence class of } \sim, \text{ i.e.,}\\ &\forall [x,y],[x',y'] \in I(L,\leqq), \quad \overline{[x,y]_{\leqq}} = \overline{[x',y']_{\leqq}}, \text{ then } \Phi(x,y) = \Phi(x',y').\end{aligned}$$

Such operators are said to be *compatible* (relatively to relation $\sim$). In the same way, the relation $\sim$ is said to be *compatible*, if the set of compatible operators is stable under multiplication.

We now consider the particular equivalence relation $\cong$ (order isomorphism) on $I(L, \leqq)$. Then it is a compatible equivalence relation (see [8]). One can notice that relatively to $\cong$ and the usual order, Z is compatible. However, it is not the case of Γ in the general case; for instance, if $L := L_1 = \{0, 1, 2\}$, $\frac{1}{2} = \Gamma(0, 1) \neq \Gamma(1, 2) = 0$ although $[0, 1] \cong [1, 2]$.

We denote by $\tilde{G}(\leqq)$ the subset of $G(\leqq)$ of compatible operators relatively to the relation $\cong$. It is possible to reduce the algebra structure of operators when dealing with the elements of $\tilde{G}(\leqq)$: to any $\Phi \in \tilde{G}(\leqq)$, we associate the following function φ defined on $\tilde{I}(L, \leqq)$, quotient set of $I(L, \leqq)$ by $\cong$:

$$\varphi(\overline{[x,y]_{\leqq}}) := \Phi(x,y), \quad [x,y]_{\leqq} \in I(L,\leqq). \tag{11}$$

The identity operator Δ clearly belongs to $\tilde{G}(\leqq)$, and has for associated function

$$\delta(\overline{[x,y]_{\leqq}}) := \begin{cases}1, & \text{if } x = y,\\ 0, & \text{otherwise,}\end{cases} \quad [x,y]_{\leqq} \in I(L,\leqq).$$

Let $\tilde{g}(\leqq) := \{\varphi : \tilde{I}(L, \leqq) \to \mathbb{R} \mid \forall x \in L, \varphi(\overline{\{x\}}) = 1\}$. Clearly, (11) being reversible, we see that any real-valued mapping φ on $\tilde{I}(L, \leqq)$ such that $\varphi(\overline{\{x\}}) = 1$, $x \in L$, determines uniquely an operator of $\tilde{G}(\leqq)$. For $\varphi, \psi \in \tilde{g}(\leqq)$, we define

$$\varphi \star \psi(\overline{[x,y]_{\leqq}}) := \Phi \star \Psi(x,y), \quad [x,y]_{\leqq} \in I(L,\leqq), \tag{12}$$

where Φ and Ψ are the operators of $\tilde{G}(\leqq)$ respectively induced by φ and ψ.

[2] This name will be justified in Corollary 14.

Proposition 11 *$(\tilde{G}(\leqq), \star)$ and $(\tilde{g}(\leqq), \star)$ are isomorphic groups. δ is the identity element of $(\tilde{g}(\leqq), \star)$.*

PROOF. We successively show that $(\tilde{G}(\leqq), \star)$ is a subgroup of $(G(\leqq), \star)$, then $(\tilde{G}(\leqq), \star)$ and $\cong (\tilde{g}(\leqq), \star)$ are isomorphic. By definition of a compatible equivalence relation, the closure of $\tilde{G}(\leqq)$ under convolution follows, and the closure under inversion is straightforwardly derived from Proposition 9. Considering the bijection given by (11), for $\Phi, \Psi \in \tilde{G}(\leqq)$, we denote by φ, ψ and $f_{\Phi\star\Psi}$ the respective images of Φ, Ψ and $\Phi \star \Psi$. Applying (11) for $f_{\Phi\star\Psi}$ and (12) for $\varphi \star \psi$, we get $f_{\Phi\star\Psi} = \varphi \star \psi$.

We now address the particular order relation $\leqq$ that enables the writing of operation $\star$ in $\tilde{g}(\leqq)$ in terms of binomial coefficients, which makes clear the terminology "convolution". From the description of (9) of Γ, we define the following binary relation in L:

$$x \trianglelefteq y \quad \text{iff} \quad \forall i \in N, x_i = \perp_i \text{ or } x_i = y_i.$$

One can easily check that $\trianglelefteq$ in an order relation. Besides, for all x, y s.t. $x \trianglelefteq y$, we naturally define the element $y - x$ of L by

$$(y-x)_i := \begin{cases} y_i, & \text{if } x_i = \perp_i, \\ \perp_i, & \text{if } x_i = y_i, \end{cases} \quad i \in N.$$

Note that if $x \trianglelefteq y$, $k(y-x) = k(y) - k(x)$.

By the following result, one can easily check that $\Gamma \in \tilde{G}(\trianglelefteq)$.

Lemma 12 *Let $x, y \in L$ such that $x \trianglelefteq y$. Then*

$$[x, y]_{\trianglelefteq} \cong 2^{k(y-x)}.$$

As a consequence, the elements of $\tilde{I}(L, \trianglelefteq)$ are given by the classes of intervals of $I(L, \trianglelefteq)$ which are isomorphic to some Boolean lattice.

PROOF. Let $z \in [x, y]_{\trianglelefteq}$. For any $i \in N$, either $x_i = \perp_i$ and $y_i \neq x_i$, or $x_i = y_i$. The first case implies $z_i = \perp_i$ or $z_i = y_i$ (with $y_i \neq \perp_i$), and the second case implies $z_i = x_i = y_i$. As a result,

$$[x, y]_{\trianglelefteq} = \prod_{i \in N | x_i \neq y_i} \{\perp_i, y_i\} \times \prod_{i \in N | x_i = y_i} \{y_i\} \cong 2^{k(y-x)}.$$

Let $w(\mathcal{J}(L))$ be the *width* of $\mathcal{J}(L)$, that is to say the cardinal of a maximal antichain of $\mathcal{J}(L)$, that is also the sum of the cardinals of maximal antichains of the $\mathcal{J}(L_i)$'s. As a result, the greatest intervals of L isomorphic to a Boolean lattice, are isomorphic to $2^{w(\mathcal{J}(L))}$. Note that $n \leq w(\mathcal{J}(L)) \leq |\mathcal{J}(L)|$.

Considering the elements of $\tilde{I}(L, \leq)$, we denote by $\overline{m}$ the class of all Boolean intervals isomorphic to 2^m, $m = 0, \ldots, w(\mathcal{J}(L))$. In the same way, $\overline{\overline{m}}$ denotes the element of $\tilde{I}(L, \trianglelefteq)$ representing all intervals $[x, y]_\trianglelefteq$ s.t. $k(y - x) = m$, $m = 0, \ldots, n$. Clearly, all these classes are nonempty. In particular, $\overline{0}$ and $\overline{\overline{0}}$ are the unique elements of $\tilde{g}(\leq)$ and $\tilde{g}(\trianglelefteq)$ containing singletons of L: $\overline{0} = \overline{\overline{0}} = \{\{x\} \mid x \in L\}$. Consequently, the identity element of $\tilde{g}(\leq)$ (resp. $\tilde{g}(\trianglelefteq)$) simply writes as the function which is 1 onto $\overline{0}$ (resp. $\overline{\overline{0}}$), and 0 elsewhere. One can note that in the general case, $\tilde{I}(L, \trianglelefteq) = \{\overline{\overline{0}}, \ldots, \overline{\overline{n}}\}$, but $\tilde{I}(L, \leq) \supsetneq \{\overline{0}, \ldots, \overline{w(\mathcal{J}(L))}\}$ (there are some classes having not a "Boolean type").

By (6) and (9), the associated functions $\zeta \in \tilde{g}(\leq)$ of Z and $\gamma \in \tilde{g}(\trianglelefteq)$ of Γ respectively write

$$\begin{aligned} \zeta(\alpha) &= 1, \quad \alpha \in \tilde{I}(L, \leq), \\ \text{and} \quad \gamma(\overline{\overline{m}}) &= \frac{1}{m+1}, \quad m = 0, \ldots, n. \end{aligned}$$

Theorem 13 *For all $\varphi, \psi \in \tilde{g}(\leq)$, and any $m \in \{0, \ldots, w(\mathcal{J}(L))\}$,*

$$\varphi \star \psi(\overline{m}) = \sum_{j=0}^{m} \binom{m}{j} \varphi(\overline{j})\, \psi(\overline{m-j}).$$

Besides, the inverse of φ computes recursively through

$$\begin{aligned} \varphi^{-1}(\overline{0}) &= 1, \\ \varphi^{-1}(\overline{m}) &= -\sum_{j=0}^{m-1} \binom{m}{j} \varphi^{-1}(\overline{j})\, \varphi(\overline{m-j}). \end{aligned}$$

The same formulae hold for $\varphi \star \psi(\overline{\overline{m}})$ and $\varphi^{-1}(\overline{\overline{m}})$, $\varphi, \psi \in \tilde{g}(\trianglelefteq)$ and $m \in \{0, \ldots, n\}$.

PROOF. Let $\varphi, \psi \in \tilde{g}(\leq)$, $m \in \{0, \ldots, w(\mathcal{J}(L))\}$, and any interval $[x, y]$ of L such that $[x, y] \cong 2^m$. Note that $\forall t \in [x, y]$, $[x, t]$ and $[t, y]$ are also Boolean, with $[x,t] \cong 2^j$, $[t,y] \cong 2^{j'}$ s.t. $j + j' = m$. Then by (12),

$$\begin{aligned}
\varphi \star \psi(\overline{m}) &= \Phi \star \Psi(x,y) \\
&= \sum_{x \le t \le y} \Phi(x,t)\, \Psi(t,y) \\
&= \sum_{j=0}^{m} \sum_{t \in [x,y] \mid \overline{[x,t]} = \overline{j}} \varphi(\overline{j})\, \psi(\overline{m-j}) \\
&= \sum_{j=0}^{m} \binom{m}{j} \varphi(\overline{j})\, \psi(\overline{m-j}).
\end{aligned}$$

By definition of φ^{-1}, then by Proposition 9, we have also

$$\begin{aligned}
\varphi^{-1}(\overline{0}) &= \Phi^{-1}(x,x) = 1, \\
\text{and for } m \neq 0,\ \varphi^{-1}(\overline{m}) &= \Phi^{-1}(x,y) \\
&= -\sum_{x \le t < y} \Phi^{-1}(x,t)\, \Phi(t,y) \\
&= -\sum_{j=0}^{m-1} \sum_{t \in [x,y] \mid \overline{[x,t]} = \overline{j}} \varphi^{-1}(\overline{j})\, \varphi(\overline{m-j}) \\
&= -\sum_{j=0}^{m-1} \binom{m}{j} \varphi^{-1}(\overline{j})\, \varphi(\overline{m-j}).
\end{aligned}$$

Now, by Lemma 12, any interval of $I(L, \trianglelefteq)$ is Boolean. Consequently, for $\varphi, \psi \in \tilde{g}(\trianglelefteq)$, and $m \in \{0, \ldots, n\}$, we obtain the same formulae for $\varphi \star \psi(\overline{\overline{m}})$ and $\varphi^{-1}(\overline{\overline{m}})$.

Note that the above result is not general and does not apply for any $\tilde{g}(\leqq)$. Actually, $\tilde{G}(\leq)$ and $\tilde{G}(\trianglelefteq)$ are very particular subgroups of $G(\leq)$, which refer to particular algebras, namely of *binomial type* in the framework of incidence algebras.

Let $(B_m)_{m \in \mathbb{N}}$ be the *sequence of Bernoulli numbers*, computed recursively through

$$\begin{aligned}
&B_0 = 1, \\
&B_m = -\frac{1}{m+1} \sum_{j=0}^{m-1} \binom{m+1}{j} B_j, \quad m \in \mathbb{N} \setminus \{0\}.
\end{aligned}$$

$(B_m)_m$ starts with $1, -1/2, 1/6, 0, -1/30, 0, 1/42 \ldots$, and it is well-known that $B_m = 0$ for $m \geq 3$ odd. From Theorem 13, we derive the following result.

Corollary 14 *The inverses of* ζ *in* $\tilde{g}(\leq)$ *and* γ *in* $\tilde{g}(\trianglelefteq)$ *are given by*

$$\zeta^{-1}(\alpha) = \begin{cases} (-1)^m, & \alpha = \overline{m} \quad (m = 0, \ldots, w(\mathcal{J}(L))), \\ 0, & \textit{otherwise}, \end{cases}$$
$$\textit{and} \;\; \gamma^{-1}(\overline{\overline{m}}) = B_m, \quad m = 0, \ldots, n.$$

PROOF. By applying Theorem 13, we get $\zeta^{-1}(\overline{0}) = 1 = (-1)^0$, and for any $m \in \{1, \ldots, w(\mathcal{J}(L))\}$, by induction on m, we get

$$\begin{aligned}
\zeta^{-1}(\overline{m}) &= -\sum_{j=0}^{m-1} \zeta^{-1}(\overline{j})\, \zeta(\overline{m-j}), \\
&= -\Big((-1+1)^m - (-1)^m\Big) \\
&= (-1)^m.
\end{aligned}$$

Then we check that $\zeta^{-1}(\alpha) = 0$ for all $\alpha \in \tilde{I}(L, \leq) \setminus \{\overline{j} \mid j = 0, \ldots, w(\mathcal{J}(L))\}$, suits as the inverse of ζ. Indeed, let α be such an element, and $[x, y]$ be any interval s.t. $\overline{[x, y]} = \alpha$ ($[x, y]$ is not Boolean). Note that $\underline{y} \in [x, y]$. Then

$$\begin{aligned}
\zeta \star \zeta^{-1}(\alpha) &= Z \star Z^{-1}(x, y) \\
&= \sum_{x \leq t \leq y} Z(x, t)\, Z^{-1}(t, y) \\
&= \sum_{\underline{y} \leq t \leq y} Z(x, t)\, Z^{-1}(t, y) + \sum_{x \leq | t \ngeq \underline{y}} Z(x, t)\,.
\end{aligned}$$

By Proposition 5–(iv), the first sum is $(1+(-1))^{\eta^*(y)}$. Besides, by Proposition 5–(iii), all the $[t, y]$'s in the second sum are not Boolean, and thus the $Z^{-1}(t, y)$'s vanish.

Now, $\gamma^{-1}(\overline{\overline{0}}) = 1 = B_0$, and for any $m \in \{1, \ldots, n\}$, we get by Theorem 13

$$\begin{aligned}
\gamma^{-1}(\overline{\overline{m}}) &= -\sum_{j=0}^{m-1} \binom{m}{j} \gamma^{-1}(\overline{\overline{j}})\, \frac{1}{m-j+1} \\
&= -\frac{1}{m+1} \sum_{j=0}^{m-1} \binom{m+1}{j} \gamma^{-1}(\overline{\overline{j}}),
\end{aligned}$$

which is precisely the definition of the Bernoulli sequence.

By the bijection (11), we finally deduce from ζ^{-1} and γ^{-1} some explicit formulae for the Möbius operator and the Bernoulli operator.

$$Z^{-1}(x,y) := \begin{cases} (-1)^m, & \text{if } [x,y] \cong 2^m, \\ 0, & \text{otherwise}, \end{cases} \qquad x,y \in L,$$

$$\text{and} \;\; \Gamma^{-1}(x,y) := \begin{cases} B_{k(y-x)}, & \text{if } x \trianglelefteq y, \\ 0, & \text{otherwise}, \end{cases} \qquad x,y \in L.$$

7 The interaction operator and its inverse

By means of the expression of the Bernoulli operator and Eq. (10), for any lattice function f, we get

$$m^f(x) = \sum_{y \trianglerighteq x} B_{k(y-x)}\, I^f(y). \tag{13}$$

For any $p \in \mathbb{N}$, and $m = 0, \ldots, p$, we define

$$b^p_m := \sum_{j=0}^{m} \binom{m}{j} B_{p-j},$$

These numbers have been introduced in [7] to express a lattice function f from its interaction transform I^f. It is easy to compute them from the sequence of Bernoulli: $b^p_0 = B_p$, $p \in \mathbb{N}$, and by applying the recursion of the Pascal's triangle:

$$b^{p+1}_{m+1} = b^{p+1}_m + b^p_m, \quad 0 \le m \le p.$$

The coefficients also satisfy the following symmetry:

$$b^p_m = (-1)^p\, b^p_{p-m}, \quad 0 \le m \le p.$$

The values of b^p_m, $p \le 6$, are

	m 0	1	2	3	4	5	6
p 0	1						
1	$-\frac{1}{2}$	$\frac{1}{2}$					
2	$\frac{1}{6}$	$-\frac{1}{3}$	$\frac{1}{6}$				
3	0	$\frac{1}{6}$	$-\frac{1}{6}$	0			
4	$-\frac{1}{30}$	$-\frac{1}{30}$	$\frac{2}{15}$	$-\frac{1}{30}$	$-\frac{1}{30}$		
5	0	$-\frac{1}{30}$	$-\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{30}$	0	
6	$\frac{1}{42}$	$\frac{1}{42}$	$-\frac{1}{105}$	$-\frac{8}{105}$	$-\frac{1}{105}$	$\frac{1}{42}$	$\frac{1}{42}$

We now give an explicit formula for the inverse interaction operator $\mathbb{I}^{-1} = Z \star {}^t\Gamma^{-1}$ (cf. Section 5).

Theorem 15 *For all $x, y \in L$,*

$$\mathbb{I}^{-1}(x,y) = b^{k(x)}_{k(x_y)},$$

where $(x_y)_i := \begin{cases} x_i, & \text{if } x_i \leq y_i \\ \bot_i, & \text{otherwise} \end{cases}$, $i \in N$. *Consequently, for any lattice function f,*

$$f(x) = \sum_{z \in L} b^{k(z)}_{k(z_x)}\, I^f(z), \quad x \in L.$$

PROOF. For all $x \in L$, according to (1) and (13), we have

$$\begin{aligned} f(x) &= \sum_{y \leq x} m^f(y) \\ &= \sum_{y \leq x} \sum_{z \trianglerighteq y} B_{k(z-y)}\, I^f(z) \\ &= \sum_{z \in L} \Bigg(\sum_{\substack{y \trianglelefteq z \\ y \leq x}} B_{k(z-y)} \Bigg) I^f(z). \end{aligned}$$

Note that $y \trianglelefteq z$ and $y \leq x$ iff $y_i \leq x_i$ and ($y_i = \bot_i$ or $y_i = z_i$), $i \in N$, which is equivalent to $y \trianglelefteq z_x$. Let $K(z_x) := \{i \in N \mid (z_x)_i \neq \bot_i\}$. Then

$$\begin{aligned} \sum_{\substack{y \trianglelefteq z \\ y \leq x}} B_{k(z-y)} &= \sum_{y \trianglelefteq z_x} B_{k(z-y)} \\ &= \sum_{Y \subseteq K(z_x)} B_{k(z)-|Y|} \\ &= \sum_{j=0}^{k(z_x)} \binom{k(z_x)}{j} B_{k(z)-j} \\ &= b^{k(z)}_{k(z_x)}, \end{aligned}$$

where the second equality is due to Lemma 12.

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