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Four-body Efimov effect

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We study three same spin state fermions of mass \( M \) interacting with a distinguishable particle of mass \( m \) in the unitary limit where the interaction has a zero range and an infinite \( s \)-wave scattering length. We predict an interval of mass ratio \( 13.384 < M/m < 13.607 \) where there exists a purely four-body Efimov effect, leading to the occurrence of weakly bound tetramers without Efimov trimers.

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In a system of interacting particles, the unitary limit corresponds to a zero range \( s \)-wave interaction with infinite scattering length \([1]\). In particular, this excludes any finite energy two-body bound state. Interestingly, in the three-body problem, the Efimov effect may take place \([2]\), leading to the occurrence of an infinite number of three-body bound states, with an accumulation point in the spectrum at zero energy. This effect occurs in a variety of situations, the historical one being the case of three bosons, as recently studied in a series of remarkable experiments with cold atoms close to a Feshbach resonance \([3]\). It can also occur in a system of two same spin state fermions of mass \( m \) and a particle of another species of mass \( M \) and a particle of another species of mass \( m \), in which case the fermions only interact with the third particle, with an infinite \( s \)-wave scattering length: An infinite number of arbitrarily weakly bound trimers then appears in this \( 2 + 1 \) fermionic problem if the mass ratio \( \alpha = M/m \) is larger than \( \alpha_c(2;1) \approx 13.607 \) \([1]\).

The four-body problem has recently attracted a lot of interest \([4]\). The question of the existence of a four-body Efimov effect is however to our knowledge still open. We give a positive answer to this question, by investigating the \( 3 + 1 \) fermionic problem in the unitary limit. We explicitly solve Schrödinger’s equation in the zero range model \([4]\) and we determine the critical mass ratio to have a purely four-body Efimov effect in this system, that is without Efimov trimers.

In the zero-range model, the Hamiltonian reduces to a non-interacting form, here in free space

\[
H = \sum_{i=1}^{4} -\frac{\hbar^2}{2m_i} \Delta r_i, \tag{1}
\]

with \( m_1 = m_2 = m_3 = M \) and \( m_4 = m \). The interactions are indeed replaced by contact conditions on the wavefunction, \( \psi(r_1, r_2, r_3, r_4) \), where \( r_i, i = 1, 2, 3 \) is the position of a fermion and \( r_4 \) is the position of the other species particle: At the unitary limit, for \( i = 1, 2, 3 \), there exist functions \( A_i \) such that

\[
\psi(r_1, r_2, r_3, r_4) = \frac{A_i(r_{4i}, r_k, k \neq i)}{|r_i - r_4|} + O(|r_i - r_4|) \tag{2}
\]

when \( r_i \) tends to \( r_4 \) for a fixed value of the \( i \)-4 centroid \( \mathbf{R}_{4i} \equiv (M r_i + m r_4)/(m + M) \) different from the positions of the remaining particles \( r_k, k \neq i, 4 \). The wavefunction is also subject to the fermionic exchange symmetry with respect to the first three variables \( r_i, i = 1, 2, 3 \).

In what follows, we shall assume that there is no three-body Efimov effect, a condition that is satisfied by imposing \( M/m < \alpha_c(2;1) \approx 13.607 \). The eigenvalue problem \( H \psi = \lambda \psi \) with the contact conditions in Eq.\((2)\) is then separable in hyperspherical coordinates \([4]\). After having separated out the center of mass \( c \) of the system, one introduces the hyperradius \( R = \left[ \sum_{i=1}^{4} m_i(r_i - c)^2/m \right]^{1/2} \), with \( m = (3M + m)/4 \) the average mass, and a set of here 8 hyperangles \( \Omega \) whose expression is not required. For a center of mass at rest, the wavefunction may be taken of the form

\[
\psi(r_1, r_2, r_3, r_4) = R^{-7/2} F(R) f(\Omega). \tag{3}
\]

\( f(\Omega) \) is given by the solution of a Laplacian eigenvalue problem on the unit sphere of dimension 8, which is non-trivial because of the contact conditions. On the contrary, the hyperradial part \( F \) is not directly affected by the contact conditions, due in particular to their invariance by the scaling \( r_i \to \lambda r_i \) \([5]\), and solves the effective 2D Schrödinger equation

\[
EF(R) = \frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial R^2} + \frac{1}{R} \frac{\partial}{\partial R} \right) F(R) + \frac{\hbar^2 s^2}{2m R^2} F(R). \tag{4}
\]

The quantity \( s^2 \) is given by the hyperangular eigenvalue problem. It belongs to a infinite discrete set and is real since there is no Efimov effect on the unit sphere \((R \neq 0)\), that is here no three-body Efimov effect.

Mathematically, Eq.\((1)\) admits for all energies \( E \) two linearly independent solutions, respectively behaving as \( R^{k+s} \) for \( R \to 0 \). If \( s^2 > 0 \), one imposes \( F(R) \sim R^s \), with \( s > 0 \), which is correct except for accidental, non-universal four-body resonances (see note \([43]\) in \([5]\)), and Eq.\((1)\) then does not support any bound state. On the contrary, if \( s^2 < 0 \), in which case we set \( s = iS \), \( S > 0 \), \( F \) experiences an effective four-body attraction, with a fall
to the center leading to a unphysical continuous spectrum of bound states \([4]\). To make the model self-adjoint, one then imposes an extra contact condition \([5]\), as in the usual three-body Efimov case \([6]\):

\[
F(R) \sim \text{Im} \left[ \frac{R}{R_f} \right]^{iS},
\]

where the four-body parameter \(R_f\) depends on the microscopic details of the true, finite range interaction \([7]\). With the extra condition Eq.\((3)\) one then obtains from Eq.\((4)\) an Efimov spectrum of tetramers:

\[
E_n = -\frac{2\hbar^2}{mR_f^2} \sum_{s \in \mathbb{Z}} \arg(1+iS)e^{-2\pi n/S}, \quad \forall n \in \mathbb{Z}.
\]

The whole issue is thus to determine the values of the exponents \(s\). In particular, the critical mass ratio \(\alpha_3(3;1)\) corresponds to one of the exponents being equal to zero, the other ones remaining positive. To this end, we calculate the zero energy four-body wavefunction with no bound states \([7]\). To make the model self-adjoint, one thus imposes an extra contact condition \([7]\), as in the usual three-body Efimov case \([6]\).

\[
D(k_1, k_2) = \sum_{m_1 = -l}^l [Y_{m_1}^l(\gamma, \delta)]^* e^{im_1\theta/2} f_{m_1}^{(l)}(k_1, k_2, \theta).
\]

Here \(Y_{m_1}^l(\gamma, \delta)\) are the usual spherical harmonics, \(\gamma\) and \(\delta\) are the polar and azimuthal angles of the unit vector \(e_z\) along \(z\) in the direct orthonormal basis \((e_1, e_{\perp 1}, e_{\perp 2})\), with \(e_1 = k_1/k, e_{\perp 1} = k_2/k, e_{\perp 2} = (e_2 - e_1 \cos \theta)/\sin \theta\) and \(e_{\perp 2} = e_1 \wedge e_{\perp 1}/\sin \theta\) \([2]\). The action of parity \(k_1 \rightarrow -k_1\) on this general ansatz is to multiply each term of index \(m_1\) in Eq.\((7)\) by a factor \((-1)^{m_1}\), which allows to decouple the even \(m_1\) terms (even parity) from the odd \(m_1\) terms (odd parity). A relevant example, as we shall see, is the even parity channel with \(l = 1\), where the ansatz reduces to a single term, which is obviously the component along \(z\) of a vectorial spinor:

\[
D(k_1, k_2) \propto e_z \cdot \frac{k_1 + k_2}{\|k_1 \wedge k_2\|} f_{0}^{(1)}(k_1, k_2, \theta).
\]

The last step is to use the scaling invariance of \(D\), see Eq.\((4)\), setting

\[
f_{m_1}^{(l)}(k_1, k_2, \theta) = (k_1^2 + k_2^2)^{-(s+7/2)/2}(\cosh x)^{3/2} \Phi_{m_1}^{(l)}(x, u),
\]

where \(u = \cosh \theta\). The introduction of the logarithmic change of variable \(x = \ln(k_1/k_2)\) is motivated by Efimov physics, and the factor involving the hyperbolic cosine ensures that the final integral equation involves a Hermitian operator. The fermionic symmetry imposes

\[
\Phi_{m_1}^{(l)}(-x, u) = (-1)^{l+1} \Phi_{-m_1}^{(l)}(x, u)
\]

which allows to restrict the unknown functions \(\Phi_{m_1}^{(l)}\) to \(x \geq 0\). Restricting to \(s = iS, S \geq 0\), we finally obtain

\[
0 = \left[ 1 + 2\alpha \left( \frac{k_1^2 + k_2^2}{1 + \alpha^2} \right) + \frac{2\alpha}{1 + \alpha^2} k_1 \cdot k_2 \right]^{1/2} D(k_1, k_2) + \int_{\mathbb{R}_+} \int_1^l \sum_{m_1 = -l}^l K_{m_1}^{(l)}(x, u; x', u') \Phi_{m_1}^{(l)}(x', u').
\]

The symmetrized kernel \(K_{m_1}^{(l)}(x, u; x', u') = \sum_{e, e' = \pm 1}(ee')^{l+1}K_{m_1}^{(l)}(x, u; e'e', u')\) is expressed
quite close to the 2 + 1 critical value α_c(2; 1) ≃ 13.384. (16)

To gain some insight on this result, we have studied analytically an important feature of the spectrum of \( M_{s=0} \) the lower border of its continuum. When \( x, x' \to +\infty \), which corresponds physically to having \( k_2 \gg k_1 \) in the function \( D(k_1, k_2) \), both the symmetrized and non-symmetrized kernels reduce to the asymptotic form

\[
K_{m_l, m_l'}^{(l)}(x, u; x', u') \sim e^{iS(x-x')e^{-im_0/2}}e^{im_0/2i}e^{ikx/2} \times \frac{d\phi}{4\pi^2 \cosh(x-x')} \tag{17}
\]

Since \( D \) is independent of \( x \) and \( x' \), this is invariant by translation over the \( x \) coordinates, leading to a continuous spectrum of asymptotic plane wave eigenfunctions. In the even sector of angular momentum \( l = 1 \), we found that \( \Phi_0^{(l)}(x, u) \sim e^{ikx/2}e^{i\pi/2} \) gives rise to an eigenfunction in the continuous spectrum of \( M_{s=0} \) with the real eigenvalue \( \Lambda(k = S, \alpha) \) where

\[
\Lambda(k, \alpha) = \cos 2\beta + \frac{1 - ik \sin[2\beta(1 + ik)]}{2(1 + k^2) \sin^2 2\beta \sin(ik\pi/2)} - c.c. \tag{18}
\]

In Eq. (18) we have set for convenience \( \sin 2\beta = \alpha/(1 + \alpha) \) with \( \beta \in [0, \pi/2] \). For real \( k \), this function \( \Lambda(k, \alpha) \) has a global minimum in \( k = 0 \). We expect that \( \Lambda(k = 0, \alpha) \) is the lower border of the continuous spectrum of \( M_{s=0} \). Since \( \Lambda(0, \alpha) \) exactly vanishes for the three-body critical mass ratio \( \alpha_c(2; 1) \approx 13.607 \), our asymptotic analysis amounts to uncovering the three-body problem as a limit \( k_2/k_1 \to +\infty \) of the four-body problem.

We tested this prediction against the numerics, plotting in Fig. 2 the quantity \( \Lambda(k = 0, \alpha) \) as a function of \( \alpha \) in dotted line. Except for the even sector of \( l = 1 \), the minimal numerical eigenvalues are close to \( \Lambda(k = 0, \alpha) \); the fact that they are slightly above is due to a finite \( x_{\max} \) truncation effect, that indeed decreases for increasing \( x_{\max} \) (not shown). This implies that the eigenfunctions corresponding to these minimal eigenvalues are extended, that is not square integrable. The numerics agrees with this analysis. In the even sector of \( l = 1 \), the minimal numerical eigenvalue is clearly below \( \Lambda(0, \alpha) \), for all values of \( \alpha \) in Fig. 1. This indicates that the corresponding eigenvector must be a bound state of \( M_{s=0} \), with a square integrable eigenfunction \( \Phi_0^{(1)}(x, u) \). This is confirmed by the numerics, which shows that at large \( x \), \( \Phi_0^{(1)}(x, u) \propto \sqrt{1 - u^2}e^{-\kappa x} \tag{19} \). The analytical reasoning even predicts the link between the minimal eigenvalue \( \Lambda_{\min} \) of \( M_{s=0} \) and the decay constant \( \kappa \): The plane wave \( e^{ikx} \) is analytically continued into a decreasing exponential if one sets \( k = ik \), so that

\[
\Lambda_{\min} = \Lambda(ik, \alpha). \tag{19}
\]

Numerically, we have successfully tested this relation for various values of \( \alpha \), and we also found that \( M_{s=0} \) has no other bound state in the even sector of \( l = 1 \).
Finally, we completed our study of the four-body Efimov effect by calculating, as a function of the mass ratio \( \alpha \), the exponent \( s = i \mathcal{L} \) as a function of \( \alpha \) in a vicinity of the critical value \( \alpha_c(3; 1) \). The result is shown in Fig. 2. Close to the \( 2 + 1 \) critical mass ratio \( \alpha_c(2; 1) \approx 13.607 \), the values of \( |s| \) are not far from the three-boson Efimov exponent \( |s_0| \approx 1 \) proved to have observable effects\([5]\). Close to the \( 3 + 1 \) critical mass ratio \( \alpha_c(3; 1) \), \( |s| \) varies as expected as \( (\alpha - \alpha_c)^{1/2} \) (see dashed line). Low values of \( |s| \) may lead to extremely low Efimov tetramer binding energies: For an interaction of finite range \( b \), setting \( R_f \approx b \) and \( n = 1 \) in Eq. \([1]\), we estimate the ground state Efimov tetramer energy for \( |s| \ll 1 \) as \( E_{\text{E}}(s) \approx -e^{-2\pi/|s|}\hbar^2/(2\hbar^2)\). For \( |s| = 0.5 \), taking the mass of \( ^4\text{He} \) for \( m \) and a few nm for \( b \) gives \( E_{\text{E}}(s)/k_B \) in the nK range, accessible to cold atoms. Moreover, for a large but finite scattering length \( a \), successive Efimov tetramers come in for values of \( a \) in geometric progression of ratio \( e^{2\pi/|s|} \), so that too low values of \( |s| \) require unrealistically large values of the scattering length. Another experimental issue is the narrowness of the mass interval. Several pairs of atomic species have a mass ratio in the desired interval, e.g. \(^3\text{He}^\ast\) and \(^{41}\text{Ca} \sim 13.58\), and with exotic species, \(^{11}\text{B}^\ast\) and \(^{149}\text{Sm} \sim 13.53\), \(^7\text{Li}^\ast\) and \(^{93}\text{Mo} \sim 13.53\). A more flexible solution is to start with usual atomic species having a slightly off mass ratio, such as \(^3\text{He}^\ast\) and \(^{40}\text{K} \sim 13.25\), and to use a weak optical lattice to finely tune the effective mass of one of the species\([6]\).

To conclude, in the zero range model at unitarity, we studied the interaction of three same spin state fermions of mass \( M \) with another particle of mass \( m \). For \( M/m < 13.384 \), no Efimov effect was found. Over the interval \( 13.384 < M/m < 13.607 \), remarkably a purely four-body Efimov effect takes place, in the sector of even parity and angular momentum \( l = 1 \), that may be observed with a dedicated cold atom experiment. For \( M/m > 13.607 \), the three-body Efimov effect sets in, and the zero range model has to be supplemented by three-body contact conditions that break its separability. The intriguing question of whether the Efimov tetramers then survive as resonances, decaying in a trimer plus a free atom, is left for the future. F. Werner is warmly thanked for discussions.

[6] Another point is that \( R \) varies to second order in \( |r_2 - r_1| \) when \( r_1 \) tends to \( r_2 \) for a fixed \( R_{\text{ext}} \), cf. note \([38]\).
[9] For a narrow resonance, \( R_\text{min} \) may be proportional to the effective range as for the three-body Efimov effect \([1]\).
[12] One thus has \( \cos \gamma = e_{\text{e}_1} \cdot e_{\text{e}_2} \sin \cos \delta = e_{\text{e}_1} \cdot e_{\text{e}_3}, \sin \gamma \sin \delta = e_{\text{e}_2} \cdot e_{\text{e}_3}, \gamma \in [0, \pi] \) and \( \delta \in [0, 2\pi] \).
[13] The matrix elements \( \langle l, m_1 | e^{iG_L}/\hbar | l, m_2 \rangle \) are evaluated by insertion of a closure relation in the eigenbasis of \( L_z \). One then faces integrals \( J_n = \int_0^{\pi} \! \! d\phi \langle \text{e}^{iG_L} \rangle_{\text{e}^{iG_L} e^{iG_L}} = 2\pi |n| \sqrt{b_0 - b_1^2} \). With \( z_0 = -(b_0/b_1) + \sqrt{(b_0/b_1)^2 - 1} \).
[14] To gain in precision, we used \( \theta \) rather than \( u = \cos \theta \) as a variable, with \( x_{\text{max}} \) up to 40, \( dx = d\theta/\pi \) down to 1/40. Taking as unknown functions \( (x, \theta) \rightarrow \sqrt{\sin \theta \Phi(x)} (x, \cos \theta) \) preserves the operator hermiticity.
[15] This is more rapidly obtained by taking the large \( k_L \) limit directly in Eq. \([1]\) and inserting the ansatz \( D(a_k, k_2) \approx e_{\text{e}_1} \wedge e_{\text{e}_2} e_{\text{e}_2}^{-1} (k_2/k_1)_{\Delta x}^{1/2} \).
[16] The corresponding eigenvalue of \( M_{\text{ext}} \) is thus weakly sensitive to truncation effects.
[17] Imposing \( F(R = b) = 0 \) also gives this estimate. Cutting \( s^2/R^2 \) to \( s^2/b^2 \) for \( R < b \), as in §35 of Landau and Lifschitz, Quantum Mechanics, gives \( E_{\text{E}}(s) \approx -e^{-\pi/|s|} \hbar^2/(2\hbar^2) \).