The relationship between edge contact forces, double forces and interstitial working allowed by the principle of virtual power

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Mécanique des milieux continus *Continuum Mechanics*

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Abstract- We consider continuous media in which contact edge forces are present. Introducing the notion of quasi-balanced contact force distribution, we are able to prove the conjectures made in [Noll, 1990] concerning the representation of contact edge forces. We first generalise the Noll theorem on Cauchy postulate. Then we adapt the celebrated Cauchy tetrahedron construction to find a representation theorem for stress states. Moreover we find the relationship between double forces Germain [Germain, 1973a,b], interstitial working [Dunn, 1985] and contact edge forces.

Relation entre forces d'arêtes, double forces et flux supplémentaire d'énergie induite par le principe des puissances virtuelles

Résumé- *Nous étudions la modélisation de milieux continus dans lesquels des interactions d'arêtes ont lieu. En introduisant le concept de distribution de forces quasi-équilibrée, nous prouvons les conjectures énoncées dans [*Noll, 1990*] concernant l'expression des forces agissant sur ces arêtes. Nous généralisons le théoreme de Noll et nous adaptons la démonstration du tétraèdre de Cauchy pour obtenir une représentation des états de contrainte. Notre travail montre le lien étroit entre double forces [*Germain, 1973a,b*], flux d'énergie supplémentaire [*Dunn, 1985*] et forces d'arêtes.*

Version française abrégée - Il existe deux manières de modéliser un milieu continu. La première est de postuler une expression pour la puissance virtuelle des efforts intérieurs [Germain, 1973a,b] et d'obtenir ainsi la representation des états de contraintes. La deuxième originellement développée par Cauchy est de postuler une expression pour la puissance virtuelle des efforts extérieurs. Cauchy, raisonnant sur des forces, n'a en fait utilisé que la puissance de ces efforts dans des champs constants. En postulant que les forces surfaciques de contact ne dépendent que de la normale à la surface de contact, il a montré que cette dépendance est linéaire (cf. [Germain, 1973c]). Son postulat a été démontré par [Noll, 1959] sous l'hypothèse d'une densité surfacique de forces uniformement bornée (Cela reste vrai avec des hypothèses plus faibles, voir notre théorème 2). Ces résultats ne sont valables qu'en l'absence de forces d'arêtes. Notre but est de montrer comment un raisonnement analogue à celui de Cauchy permet en présence de forces d'arêtes de définir un second tenseur des contraintes, nous plaçant ainsi dans le cadre d'une théorie de second gradient. Notre hypothèse fondamentale est que la puissance virtuelle des efforts de contact est quasi-équilibrée au sens de [Noll, 1959] [Noll, 1990] (équation (2)).

Nous disons que la forme $\widetilde{(S,x)}$ de S en x est la même que la forme de S' en x' si les deux surfaces coincident localement modulo une translation. Nous appelons coupure plane de S par un plan de normale u (notée $Cut(\widetilde{(S,x)},u)$) la forme d'arête obtenue à l'intersection de S et du plan. Les domaines *admissibles* que nous considérons ont une surface *admissible* formée de faces C^∞ et d'arêtes C^∞. Les sommets sont ignorés et en tout point interieur d'une arête il existe un dièdre tangent à la surface d'angle différent de 0, π ou 2π. Un ensemble de formes contenu dans l'ensemble des formes prises par un nombre fini de surfaces *admissibles* est appelé ensemble de *formes prescrites.* De la même manière les coupures produites par un nombre fini de surfaces admissibles et de vecteurs unitaires sont appelées *coupures planes prescrites.* Ces définitions sont nécessaires pour énoncer des hypothèses de régularité ((i)-(iv) de la partie 2) suffisament faibles pour permettre d'envisager des forces surfaciques non uniformément bornées.

Les forces de contact sont représentées par des densités de surface et de ligne (eq. (1)). Dans la partie 3 (théorème 1) nous montrons que la puissance virtuelle des efforts de contact ne peut se résumer à la somme des puissances de ces forces (eq. (3). La preuve est obtenue en appliquant l'inégalité (3) à un petit domaine autour de l'arête. Nous rappelons les résultats de Noll et Cauchy (théorème 2) valables en l'absence de forces d'arêtes. Notre démonstration, utilisant un petit cylindre de base carrée, se contente d'une hypothèse de majoration des forces de contact pour des formes prescrites.

Dans la partie 4 nous ajoutons aux forces de contact une densité surfacique de double forces G. L'hypothèse (2) s'écrit alors (6). Une démonstration analogue à celle du théorème 2 montre que G ne peut dépendre que de la normale à la surface de contact (théorème 3). Une démonstration analogue à celle de Cauchy nous permet de définir un tenseur d'ordre 3, C (eq. (9)) permettant de représenter G (théorème 4). La représentation des densités de forces (théorème 5) est alors obtenue comme conséquence du théorème 1. Le flux supplémentaire d'énergie q lié a C apparaissant dans l'équation de l'énergie correspond à la somme des puissances des forces d'arêtes, de la partie des forces de surfaces qui dépend de la courbure ainsi que des doubles forces.

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1. INTRODUCTION - This paper is devoted to contact interactions in which contact edge forces are present. Studying such interactions, Noll and Virga pointed out the lack of physical ground for some of the numerous assumptions they used to represent stress state. Here we face this difficulty by considering the virtual power of contact forces. We assume (which is physically reasonable) that, for each velocity field U, the power P_U^c of contact forces is "quasi-balanced" [Noll, 1959]. We show that edge forces can be present only in conjunction with double forces [Germain, 1973a,b] (whose power is related to the "interstitial working" [Dunn, 1985] [Dunn, 1986]). This relationship perfects the relationship between edge forces and the dependence of surface forces on surface curvature pointed out in [Noll, 1990]

2. HYPOTHESES AND NOTATIONS - We say that the shape of the oriented surface S at the point x ($\in S$) is the same than the shape of S' at the point x' ($\in S'$) if and only if it exists a neighbourhood of the origin in which $t_{-x}(S)$ coincides (as oriented surface) with $t_{-x'}(S')$ (t_u denoting the translation of vector u). We call *shape* of S at the point $x \in S$ the equivalence class of (S,x) in this relation and denote it $\widetilde{(S,x)}$. The shape of a plane P of normal n is simply denoted n. Let us consider a dihedron and denote by n_1 and n_2 the normals to the half-planes forming it, by ν_1 and ν_2 the external normals to these half-planes tangent to the planes and by τ the unit vector tangent to the edge such that $\nu_1 = \tau \times n_1$ and $\nu_2 = -\tau \times n_2$. On the edge of this dihedron, the shape (*dihedral shape*) is constant and is determined by n_1, n_2 and τ. This shape is denoted (n_1, n_2, τ). The angle $(-n_1, n_2)$ following τ is called dihedral angle of (n_1, n_2, τ).

Let V and V' be two domains whose boundaries are the surfaces S and S'. Let S'' be the boundary of $V \cap V'$. At each point x in $S \cap S' \cap S''$ the shape of S'' depends only on the shape of S and S'. We denote it $Cut(\widetilde{(S,x)}, \widetilde{(S',x)})$. We will only use cuts of S with surfaces whose shape is a plane shape u. We call them *plane cuts* and denote them $Cut(\widetilde{(S,x)}, u)$.

We only consider as *admissible domains*, bounded domains whose boundary S (*admissible surface*) is a finite union of two-dimensional C^∞ manifolds with boundary (called *faces* of S) and such that the union of the boundaries of these faces is a finite union of one-dimensional C^∞ manifolds with boundary (called *edges* of S). The set of all internal points of the faces (regular points of the surface) is denoted S^r and the set of all internal points of the edges (regular points of the edges) is denoted L^r. Moreover we assume that, everywhere in L^r, S is tangent to a dihedron whose angle is different from 0, π or 2π.

A set of shapes E is called a set of *prescribed shapes* if there exists a finite sequence $(S_i)_{i=1}^m$ of admissible surfaces such that $E \subset \bigcup_{i=1}^m \{(\widetilde{S_i,x}); x \in S_i\}$. It is called a set of *prescribed plane cuts* if there exist a finite sequence (S_i) of admissible surfaces and a finite sequence (u_j) of unit vectors such that $E \subset \bigcup_{i,j}\{Cut((\widetilde{S_i,x}),u_j);\ x \in S_i\}$.

We assume that: (i) the *contact forces* F^c exerted on V are represented by

$$F^c(V) = \int_{S^r} F(x,(\widetilde{S,x}))\ ds + \int_{L^r} \mathcal{F}(x,(\widetilde{S,x}))\ dl \tag{1}$$

(ii) $F(x,(\widetilde{S,x}))$ or $\mathcal{F}(x,(\widetilde{S,x}))$ depend continuously on x on a given face or edge, (iii) on a given face S, for a given unit vector u nowhere parallel to the normal to S, $\mathcal{F}(x,Cut((\widetilde{S,x}),u))$ is a continuous function of the variable x, (iv) the families $\{F(x,f) : f \in E^r\}$ and $\{\mathcal{F}(x,f) : f \in E^e\}$ of functions of the variable x are equi-continuous on B: (where B is a bounded domain, E^r is a set of regular prescribed shapes, E^e is a set of prescribed edge shapes or prescribed plane cuts).

As a consequence the functions $F(x,(\widetilde{S,y}))$, $\mathcal{F}(x,(\widetilde{S,y}))$ and $\mathcal{F}(x,Cut((\widetilde{S,y}),u))$ are uniformly continuous on $B \times (B \cap S)$, $B \times (B \cap L)$ and $B \times (B \cap S)$. Hence $F(x,(\widetilde{S,x}))$ and $\mathcal{F}(x,(\widetilde{S,x}))$ are uniformly bounded on every family of admissible surfaces included in a bounded domain and whose shapes are prescribed shapes or prescribed plane cuts. Notice that they are not uniformly bounded for all possible shapes. E.g. any continuous dependence on the curvature tensor is possible.

We strengthen the hypothesis of "quasi-balance" of contact forces usually used [Noll, 1990], [Noll, 1973] to get restrictions upon possible contact forces by assuming that the power P_U^c of contact forces distribution in a given C^∞ velocity field $U(x)$ is quasi-balanced (this hypothesis reduces to the previous one when considering constant fields U). Then, for every C^∞ field $U(x)$, we assume the existence of a constant K such that, for every V, the following inequality holds:

$$\mid P_U^c(V) \mid < K|V| \tag{2}$$

3. A NAIVE EXPRESSION FOR THE VIRTUAL POWER OF CONTACT FOR-CES - In this section assuming that contact force distribution is the sum of a surface density on S^r and a line density on L^r we prove that quasi-balance hypothesis

$$\mid P_U^c(V) \mid = \left| \int_{S^r} F(x,(\widetilde{S,x})).U(x)\ ds + \int_{L^r} \mathcal{F}(x,(\widetilde{S,x})).U(x)\ dl \right| < K|V| \tag{3}$$

is not compatible with non vanishing contact edge forces.

Theorem 1: *Inequality (3) implies that, at every regular point of an edge x_0,*

$$\mathcal{F}(x_0,(\widetilde{S,x_0})) = 0 \tag{4}$$

Proof: Let (n_1,n_2,τ) be the dihedral shape tangent to S at x_0 whose angle belongs to $]0,\pi[$ (The proof has to be slightly modified otherwise). We use the coordinate system ($x_0 = 0, e_1, e_2 = \tau, e_3 = \frac{n_1+n_2}{||n_1+n_2||}$). Let $\varepsilon > 0$, $V' = t_{\varepsilon^2 e_3}(V)$, $S' = t_{\varepsilon^2 e_3}(S)$, $L' = t_{\varepsilon^2 e_3}(L)$, $P_\varepsilon = [-c\varepsilon^2, c\varepsilon^2] \times [0, d\varepsilon] \times [0, 2\varepsilon^2]$ and $V^\varepsilon = V' \cap P_\varepsilon$. c and d may be chosen in such a way that, for

ε small enough: i) L' meets ∂P_ε only on the lateral surfaces $\{x.e_2 = 0\}$ and $\{x.e_2 = d\ \varepsilon\}$, ii) S' meets ∂P_ε only on these lateral surfaces and on the lower surface $\{x.e_3 = 0\}$. We denote S_ε the boundary of V_ε and by L_ε the upper edge of V_ε ($L_\varepsilon = L' \cap P_\varepsilon$). Let us consider the vectors field $U(x) = (x.e_3)\ U_0$. In V_ε, $\varepsilon^{-2}\ U(x)$ is bounded independently of ε. Considering the measures of each faces and edges, we get from the inequality (3):

$$\lim_{\varepsilon \longrightarrow 0} \varepsilon^{-3} \{ \int_{L_\varepsilon} \mathcal{F}(x, (\widetilde{S_\varepsilon, x})).U(x)\ dl \} = 0$$

The result is obtained recalling our regularity assumptions. □

Theorem 2 *When edge forces are vanishing, there exists a continuous field $T(x)$ such that for all regular shape f tangent to the plane shape n*

$$F(x, f) = F(x, n) = T(x).n. \tag{5}$$

Proof: The first part is known as Noll theorem. As we deal with forces which are not uniformly bounded we modify the proof of Noll theorem [Noll, 1959] by using a cylinder whose basis is a square instead of a circle. Then we use only prescribed shapes. Let x_0 be a regular point of S boundary of V and n_0 the normal to S at x_0. Using the coordinate system ($x_0 = 0, e_1, e_2, e_3 = n_0$). We define $C_\varepsilon = [0, \varepsilon] \times [0, \varepsilon] \times [-c\varepsilon^2, +c\varepsilon^2]$, $V_\varepsilon = V \cap C_\varepsilon$ and $S_\varepsilon = S \cap C_\varepsilon$. A c can be found such that, for ε small enough, S_ε does not meet the lower and upper faces $\{x_3 = \pm c\varepsilon^2\}$ of C_ε. Inequality (3) applied to C_ε and then to V_ε with constant fields U implies $F(x_0, n_0) = -F(x_0, -n_0)$ and then that $F(x_0, (\widetilde{S, x_0})) = -F(x_0, -n_0)$. The second part of (5) is the well known result of Cauchy. □

4. CONTACT DOUBLE FORCES [Germain, 1973a,b] - We assume now that contact force distribution is endowed with a more complex structure: we add a surface double force density G (i.e. the distribution $U \to \int_{S^r} G(x, (\widetilde{S, x})).\frac{\partial U}{\partial n}(x)\ ds$). We assume the same regularity properties for G as we did for F and $\mathcal{F}$. The quasi-balance assumption now reads:

$$\left| \int_{S^r} [G(x, (\widetilde{S, x})).\frac{\partial U}{\partial n}(x) + F(x, (\widetilde{S, x})).U(x)]\ ds + \int_{L^r} \mathcal{F}(x, (\widetilde{S, x})).U(x)\ dl \right| < K|V| \tag{6}$$

4.1. Dependence of double force density on the shape of the contact surface - We are now able to prove a theorem analogous to the theorem of Noll [Noll, 1959].

Theorem 3. *For every shape f tangent to the plane shape n we have*

$$G(x, n) = G(x, -n) \quad ; \quad G(x, f) = G(x, n) \tag{7}$$

Proof : Apply the inequality (6) to the domains C_ε and V_ε defined in the proof of theorem 2, using a vector field $U(x) = x.n\ U_0$. □

4.2. A theorem analoguous to Cauchy theorem.

Theorem 4. *It exists a continuous third order tensor field C such that at every point x_0 and for every plane shape n*

$$G(x_0, n) = (C(x_0).n).n \tag{8}$$

Proof: Using an orthonormal coordinate system $(x_0 = 0, e_1, e_2, e_3)$, we define a tetrahedron V with faces S, S_1, S_2 and S_3 whose normals are n, $-e_1$, $-e_2$ and $-e_3$ and such that $x_0 \in S$. Let f_i be the shapes of the edge L_i. Let V_ε, S_ε, $S_{i\varepsilon}$ and $L_{i\varepsilon}$ be the image of V, S, S_i and L_i under an homothetic transformation of ratio ε. We use the field $U(x) = (x.n)U_0$. We multiply by ε^{-2} the inequality (6) applied to the domain V_ε and evaluate the limit as ε tends to 0. We get

$$2 \mid S \mid G(0,n) = \sum_{i=1}^{3}\{\mathcal{F}(0,f_i)(n.e_i) \mid L_i \mid^2\} + 2\sum_{i=1}^{3}\{G(0,-e_i) \mid S_i \mid (n.e_i)\}$$

which reads $G(0,n) = (C(0).n).n$, C being the third order tensor defined by

$$C(x) = \frac{1}{2}\mathcal{F}(x,f_1) \otimes (e_2 \otimes e_3 + e_3 \otimes e_2) + \frac{1}{2}\mathcal{F}(x,f_2) \otimes (e_3 \otimes e_1 + e_1 \otimes e_3) + \frac{1}{2}\mathcal{F}(x,f_3) \otimes (e_1 \otimes e_2 + e_2 \otimes e_1) + \sum_{i=1}^{3}\{G(x,e_i) \otimes e_i \otimes e_i\} \tag{9}$$

Remarks: (i) Tensor C is not uniquely determined as only its right side products by symmetric two order tensors are determined. (ii) If $\mathcal{F}$ is vanishing, there exists a vector field W such that $C(x) = W(x) \otimes Id$.

4.3. Representation theorem for contact forces - Assuming that $G(.,f)$ and $\mathcal{F}(.,d)$ are functions of class C^1 we can now solve the open problem pointed out by Noll and Virga when stating their assumption III on page 21 of [Noll, 1990].

Theorem 5. *Let x be a regular point of an edge of S. Denoting by (n_1, n_2, τ) the tangent dihedral shape, $\nu_1 = \tau \times n_1$ and $\nu_2 = -\tau \times n_2$ we have*

$$\mathcal{F}(x,(\widetilde{S,x})) = (C(x).n_1).\nu_1 + (C(x).n_2).\nu_2 \tag{10}$$

Moreover it exists a second order tensor field T such that (n denoting the normal to S at a regular point x, Π the projector on the tangent plane and ∇^s the surface gradient)

$$F(x,(\widetilde{S,x})) = T(x).n - \nabla^s.((C(x).n).\Pi) \tag{11}$$

Remark: the arbitrariness in C has no influence on the representation formula (10) as it can be easily verified that $n_1 \otimes \nu_1 + n_2 \otimes \nu_2$ is a symmetric tensor.

Proof: Because of divergence theorem $\int_{S^r} \nabla U : (C.n)\ ds$ is quasi-balanced. Subtracting this term to the inequality (6), using (8) and using the divergence theorem on each face of S we find that the following quantities verify inequality (3):

$$\begin{aligned} \mathcal{F}'(x,(\widetilde{S,x})) &= \mathcal{F}(x,(\widetilde{S,x})) - (C(x).n_1).\nu_1 - (C(x).n_2).\nu_2 \\ F'(x,(\widetilde{S,x})) &= F(x,(\widetilde{S,x})) + \nabla^s.((C(x).n).\Pi). \end{aligned}$$

Theorem 3 and 4 state that $\mathcal{F}'(x,(\widetilde{S,x}))$ is vanishing and $F'(x,(\widetilde{S,x})) = T(x).n$. □

5.CONCLUSION - Our representation theorems show that a continuous medium in which edge contact forces are present has to be described in the framework of second gradient theory [Germain, 1973a,b] which needs two stress tensors to represent contact forces and mechanical energy transport. In such a medium the interstitial working, due to the flux $q = \nabla U : C$ can be interpreted as the sum of three terms: i) the power of edge contact forces, ii) the power of a part of surface forces (the part depending on the curvature), iii) the power expended by double forces. The remark (ii) of section 4.2 makes explicit the possible expressions for interstitial work flux when edge forces are vanishing.

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Résumé:
Nous étudions la modélisation de milieux continus dans lesquels des interactions d'arêtes ont lieu. En introduisant le concept de distribution de forces quasi-équilibrée, nous prouvons les conjectures énoncées dans [Noll, 1990] concernant l'expression des forces agissant sur ces arêtes. Nous généralisons le théoreme de Noll et nous adaptons la démonstration du tétraèdre de Cauchy pour obtenir une représentation des états de contrainte. Notre travail montre le lien étroit entre double forces [Germain, 1973a,b], flux d'énergie supplémentaire [Dunn, 1985] et forces d'arêtes.

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