Resurgence, Stokes phenomenon and alien derivatives for level-one linear differential systems.

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Abstract

A precise description of the singularities of the Borel transform of solutions of a level-one linear differential system is deduced from a proof of the summable-resurgence of the solutions by the perturbative method of J. Écalle. Then we compare the meromorphic classification (Stokes phenomenon) from the viewpoint of the Stokes cocycle and the viewpoint of alien derivatives. We make explicit the Stokes-Ramis matrices as functions of the connection constants in the Borel plane and we develop two examples. No assumption of genericity is made.

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1 Introduction

All along the paper we are given an ordinary linear differential system (in short, a differential system or a system) of dimension n with analytic coefficients at 0 in $\mathbb{C}$ and rank one[1]

$$x^2\frac{dY}{dx} = A(x)Y \qquad A(x) \in M_n(\mathbb{C}\{x\}),\ A(0) \neq 0 \tag{1}$$

[1]The rank is the order of the pole $x = 0$ minus 1 in the system written in "solved form" $\dfrac{dY}{dx} = \dfrac{1}{x^2}A(x)\,Y$.

together with a formal fundamental solution at 0

$$\widetilde{Y}(x) = \widetilde{F}(x) x^L e^{Q(1/x)} \tag{2}$$

where

- $\widetilde{F}(x) \in M_n\big(\mathbb{C}((x))\big)$, is an invertible formal meromorphic matrix,
- $L = \bigoplus_{j=1}^{J} (\lambda_j I_{n_j} + J_{n_j})$ where I_{n_j} is the identity matrix of size n_j and $J_{n_j} = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & 1 \\ 0 & \cdots & \cdots & 0 \end{bmatrix}$ is an irreducible Jordan block of size n_j ($J_{n_j} = 0$ if $n_j = 1$),
- $Q(1/x) = \bigoplus_{j=1}^{J} q_j(1/x) I_{n_j}$ where the $q_j's$ are polynomials.

In the very general rank one case the determining polynomials q_j are of maximal degree equal to 1 with respect to $1/x$ but they could be polynomials in a fractional power of $1/x$. Our assumption of "single level equal to 1" implies that the polynomials q_j be monomials of degree 1 in $1/x$, not all equal to a same polynomial q, some of them being possibly zero. We denote

$$Q(1/x) = \bigoplus_{j=1}^{J} \frac{-a_j}{x} I_{n_j}. \tag{3}$$

The system

$$x^2 \frac{dY}{dx} = A_0(x)\, Y \tag{4}$$

with formal fundamental solution $\widetilde{Y}_0(x) = x^L\, e^{Q(1/x)}$ has analytic coefficients and is called *a normal form* of System (1). The fundamental solution $\widetilde{Y}_0(x)$ is called *a normal solution*. It provides all formal invariants of System (1), *i.e.*, invariants under formal gauge transformations $Y \mapsto \widetilde{\Phi}(x)Y$ with $\widetilde{\Phi} \in GL_n\big(\mathbb{C}((x))\big)$ [2].

[2]The formal classification over an extension $\mathbb{C}((t))$ of the base field $\mathbb{C}((x))$ by a ramification $x = t^p$ started with Poincaré and Fabry and later Turrittin [29]. See also Malgrange [8]. Over the base field itself it is due to Balser, Jurkat and Lutz [2]. For a shorter proof and an improvement of the Formal Classification Theorem see [15].

Note that, in the whole paper, we make no other assumption than the assumption of a single level equal to one, case in which the basic theory of resurgence takes place. In particular, no assumption of genericity, such as distinct eigenvalues or diagonal monodromy L, is made and any kind of resonance is allowed.

The paper deals with the analytical properties of the solutions of System (1) underlying the meromorphic classification at 0, also referred to as *Stokes phenomenon*. There exists mainly two "dual" approaches to this phenomenon:

- The first one , related to the theory of summation, is fully developed in the plane of the initial variable x, which we refer to as the *Laplace plane*. Various methods [16, 20, 1, 13] produce a full set of invariants in the form of *Stokes matrices*. Are considered only those Stokes matrices providing the transition between the sums (in our case, Borel-Laplace or 1-sums) of a same formal fundamental solution $\widetilde{Y}(x)$ on each side of its anti-Stokes (singular) directions. We call them *Stokes-Ramis matrices*.

- The second one is strongly related to the *theory of resurgence* [10] and produces invariants in the form of *alien derivations*. The alien derivatives of a series $\widetilde{f}(x)$ have been defined by J. Écalle through an average of various analytic continuations of the Borel transform $\widehat{f}(\xi)$ of $\widetilde{f}(x)$; they have been mostly developed in the plane of the variable ξ which we refer to as the *Borel plane*.

Much have been already said on these questions but either in situations restricted by generic conditions or in very general ones. In particular, the theory of resurgence was developed by J. Écalle in the very general framework of non-linear differential equations, difference equations and so on... where it proves to be very efficient. In this context it seemed to us to be worth to make explicit what is specific to the linear case, what has to be really taken under consideration or can be made more precise, and how the various viewpoints are connected.

The first Section is devoted to proving that the formal gauge transformation $\widetilde{F}(x)$ is summable-resurgent. We first sketch a proof on linear differential equations based on the Newton polygon and Ramis Index Theorems. We develop then a proof by perturbation and majorant solutions following J. Écalle's method [11].

This latter proof allows us to display a precise description of the singularities in the Borel plane (Section 2).

In Section 3 we compare the classification by the Stokes cocycle versus the classification by the alien derivations. Roughly speaking, the first approach selects an element in a unipotent Lie group while the second one provides its "tangent" variant in the associated Lie algebra. Moreover, for theoretical reasons as well as for computational ones, we make explicit the Stokes matrices in terms of the connection constants in the Borel plane.

2 Summable-resurgence

2.1 Prepared system

Before to start the calculations we prepare the system as follows.

A gauge transformation of the form $Y \mapsto T(x)\, x^{-\lambda_1}\, e^{a_1/x} Y$ where the transformation $T(x)$ has explicit computable polynomial entries in x and $1/x$ allows to assume that the following conditions are satisfied:

$$\widetilde{F}(x) = I_n + \sum_{m \geqslant 1} F_m x^m \in GL_n(\mathbb{C}[[x]]) \text{ with initial condition } \widetilde{F}(0) = I_n, \tag{5}$$

$$0 \leqslant \mathrm{Re}(\lambda_j) < 1 \quad \text{for} \quad j = 1, \ldots, J, \tag{6}$$

$$a_1 = \lambda_1 = 0. \tag{7}$$

Conditions (5) and (6) guaranty the unicity of $\widetilde{F}(x)$. Condition (7) is for notational convenience. Still, the a_j's are not supposed distinct.

Any of the J column-blocks of $\widetilde{F}(x)$ associated with the irreducible Jordan blocks of L (matrix of exponents of formal monodromy) can be positioned at the first place by means of a permutation P on the columns of $\widetilde{Y}(x)$. If $\widetilde{Y}(x)$ is given in the form (2) so is the new formal fundamental solution $\widetilde{Y}(x)P = \widetilde{F}(x)Px^{P^{-1}LP}e^{P^{-1}Q(1/x)P}$. Thus, we can restrict our study to the first column-block of $\widetilde{F}(x)$ that we denote by $\widetilde{f}(x)$.

Gauge transformations and permutations of this kind will be referred to as *elementary transformations*.

2.2 Some definitions

For the convenience of the reader we recall some definitions about the notions of resurgence and summation adapted to linear level-one differential systems. It is worth to note that, due to the linearity, it is useless, at least for the moment, to consider convolution algebras and lattices of singularities since no convolution of singularities may occur. For a more general framework we refer to [11, 18, 26].

All along the article, given a matrix M split into blocks fitting the structure of L, we denote

- $M^{j;\bullet}$ the j^{th} row-block of M,
- $M^{\bullet;k}$ the k^{th} column-block of M,
- $M^{j;k}$ the k^{th} column-block in the j^{th} row-block of M,
- $M^{j;(k)}$ the k^{th} column in the j^{th} row-block of M,
- $M^{(j,\ell);\bullet}$ the ℓ^{th} row in the j^{th} row-block of M.

Let $\Omega = \{a_j,\, j = 1, \dots, J\}$ denote the set of *Stokes values* associated with System (1). Theorem 2.7 below asserts, in particular, that all possible singularities of the Borel transform $\widehat{f}(\xi)$ of $\widetilde{f}(x)$ belong to Ω. The directions determined by the elements of $\Omega^* = \Omega\backslash\{0\}$ from 0 are called *anti-Stokes directions associated with* $\widetilde{f}(x)$. Given a direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ let d_θ denote the half line issuing from 0 with argument θ. We denote

- $\Omega_\theta = \Omega^* \cap d_\theta$ the set of non-zero Stokes values of System (1) with argument θ.

The anti-Stokes directions associated with the k^{th} column-block of $\widetilde{F}(x)$ are given by the non-zero elements of $\Omega - a_k$ (to normalize the k^{th} column-block one has to multiply by $e^{-a_k/x}$) and the *anti-Stokes directions of System* (1) (*i.e.*, associated with the full matrix $\widetilde{F}(x)$) are given by the non-zero elements of

- $\mathbf{\Omega} = \{\, a_j - a_k \,;\, 1 \leqslant j, k \leqslant J\}$.

The elements of $\mathbf{\Omega}$ are the Stokes values of the homological system satisfied by $\widetilde{F}(x)$ (*cf.* System (12) below).
We denote

- $\mathbf{\Omega}_\theta = \mathbf{\Omega}^* \cap d_\theta$ the set of non-zero Stokes values of System (12) with argument θ.

The adequate Riemann surface on which the Borel transform $\widehat{f}(\xi)$ of $\widetilde{f}(x)$ lives is the surface $\mathcal{R}_\Omega$ defined as below.

Definition 2.1 Riemann surface $\boldsymbol{\mathcal{R}_\Omega}$.

- The points of $\mathcal{R}_\Omega$ are the homotopy classes with fixed extremities of paths γ issuing from 0 and lying in $\mathbb{C}\backslash\Omega$ (but the starting point 0); in particular, no path except those that are homotopic to the constant path 0 ends at 0.
- The complex structure of $\mathcal{R}_\Omega$ is the pull-back of the usual complex structure of $\mathbb{C}\backslash\Omega^*$ by the natural projection

$$\pi : \begin{cases} \mathcal{R}_\Omega & \longrightarrow & \mathbb{C}\backslash\Omega^* \\ [\gamma] & \mapsto & \text{end-point } \gamma(1). \end{cases}$$

The difference between $\mathcal{R}_\Omega$ and the universal cover of $\mathbb{C}\backslash\Omega$ lies in the fact that $\mathcal{R}_\Omega$ has no branch point at 0 in the first sheet.

Definition 2.2 Resurgence.

- Resurgence in the Borel plane (in Ecalle's language, *convolutive model*). A *resurgent function* with singular support Ω is any function defined and analytic on all of $\mathcal{R}_\Omega$.
- Resurgence in the Laplace plane (in Ecalle's language, *formal model*). A series $\widetilde{f}(x)$ in powers of x is said to be a *resurgent series* with singular support Ω when its Borel transform $\widehat{f}(\xi)$ is a convergent series near $\xi = 0$ which can be analytically continued to $\mathcal{R}_\Omega$ (in short, its Borel transform is resurgent with singular support Ω).
  Let $\widehat{\mathcal{R}es}_\Omega$ and $\widetilde{\mathcal{R}es}_\Omega$ denote the sets of resurgent functions and of resurgent series with singular support Ω respectively.

A resurgent function of $\widehat{\mathcal{R}es}_\Omega$ is analytic at 0 on the first sheet of $\mathcal{R}_\Omega$. To emphasize the special role played by 0 we sometimes denote the singular support by $\Omega, 0$ As a Fourier operator the Borel transform exchanges multiplication by an exponential and translation. We denote by $\mathcal{R}_{\Omega,\omega}$ and $\widehat{\mathcal{R}es}_{\Omega,\omega}$

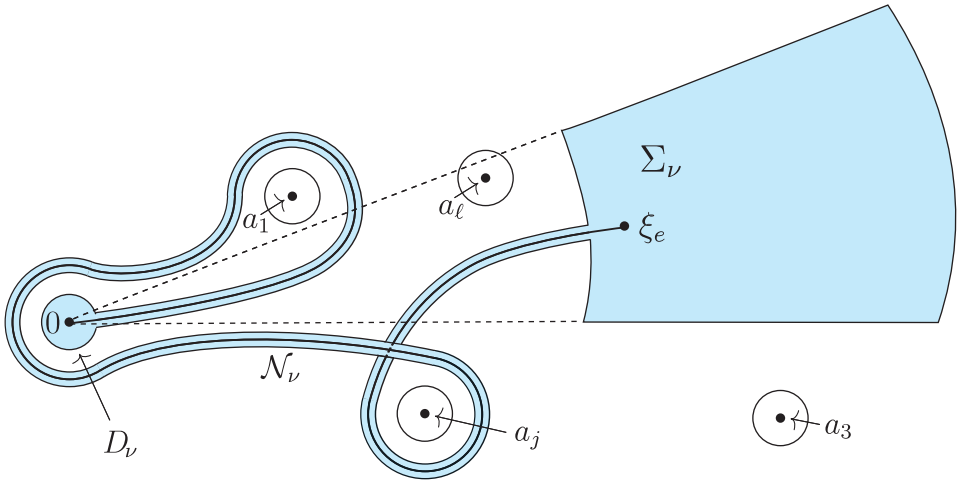
the spaces $\mathcal{R}_\Omega$ and $\widehat{\mathcal{R}es}_\Omega$ translated by ω so that, in particular, $\mathcal{R}_\Omega = \mathcal{R}_{\Omega,0}$ and $\widehat{\mathcal{R}es}_\Omega = \widehat{\mathcal{R}es}_{\Omega,0}$. If a series $\widetilde{f}(x)$ belongs to $\widetilde{\mathcal{R}es}_\Omega$ the Borel transform of $\widetilde{f}(x)e^{-\omega/x}$ belongs to $\widehat{\mathcal{R}es}_{\Omega,\omega}$.

While in non linear situations it is soon necessary to endow $\widehat{\mathcal{R}es}_\Omega$ with a structure of a convolution algebra —usually a quite difficult task— in the linear case under consideration we are going to meet, at least in this section, only the convolution of elements of $\widehat{\mathcal{R}es}_\Omega$ with entire functions over $\mathbb{C}$ growing at most exponentially at infinity. Let $\mathcal{O}^{\leqslant 1}(\mathbb{C})$ denote the convolution algebra of entire functions on $\mathbb{C}$ with exponential growth at infinity. The set $\widehat{\mathcal{R}es}_\Omega$ has a natural structure of a $\mathcal{O}^{\leqslant 1}(\mathbb{C})*$module (the star refers to the fact that multiplication by a scalar is the convolution by an element of $\mathcal{O}^{\leqslant 1}(\mathbb{C})$). There corresponds, on $\widetilde{\mathcal{R}es}_\Omega$, a natural structure of $\mathcal{O}$-module.

Definition 2.3 ν-sectorial region or ν-sector
Given $\nu > 0$, smaller than half the minimal distance between the elements of Ω, we call ν-sectorial region or ν-sector Δ_ν a domain of the Riemann surface $\mathcal{R}_\Omega$ composed of the three following parts:

- an open sector Σ_ν with bounded opening at infinity;
- a neighborhood of 0, say, an open disc D_ν centered at 0;
- a tubular neighborhood $\mathcal{N}_\nu$ of a piecewise-$\mathcal{C}^1$ path γ connecting D_ν to Σ_ν after a finite number of turns around all or part of points of Ω.

Moreover, the distance of D_ν to $\Omega\backslash\{0\}$ and the distance of $\mathcal{N}_\nu \cup \Sigma_\nu$ to Ω has to be greater than ν.



Lemma 2.4 *Let Δ_ν be a sectorial region.*

There exits a constant $K > 0$ so that, for all $\xi \in \Delta_\nu$, there is a piecewise-$\mathcal{C}^1$-path γ_ξ contained in Δ_ν and parameterized by arc length from 0 to ξ such that the arc length s_η of all $\eta \in \gamma_\xi$ satisfies

$$|\eta| \leqslant s_\eta \leqslant K|\eta|. \tag{8}$$

$|\eta|$ denotes the modulus of (the projection of) η in $\mathbb{C}$.

PROOF. The first inequality being trivial we just have to prove that $s_\eta \leqslant K|\eta|$.

Assume first that the path γ is $\mathcal{C}^1$.
Let γ' denote the extension of γ from 0 to its beginning point, say, by a straight line. Let ξ_e denote the end-point of γ in Σ_ν and $\ell|\xi_e|$ denote the length of γ' from 0 to ξ_e. The transversals of γ in $\mathcal{N}_\nu$ have all the same finite length.

- The property is clearly true in D_ν with $K = 1$.
- Given ξ in $\mathcal{N}_\nu$, consider a transversal issuing from ξ and denote ξ_0 its intersection with γ. Let γ_ξ be the path made of γ' from 0 to ξ_0 followed by the arc of transversal $\xi_0\xi$.
  For all $\eta \in \gamma_\xi$, the arc length s_η is less than a constant c independent of ξ (γ and the transversals have finite length) and the euclidian distance $|\eta|$ is greater than ν. Hence, $s_\eta \leqslant \frac{c}{\nu}|\eta|$.
- Given $\xi \in \Sigma_\nu$, consider the point ξ_0 at distance $|\xi|$ from 0 on the ray issued from 0 through ξ_e. Let γ_ξ be the path γ' followed by the segment $(\xi_e\xi_0)$ and the arc $\xi_0\xi$ of the circle centered at 0 with radius $|\xi|$ included in Σ_ν.
  For $\eta \in (\xi_e\xi_0)$ the inequality holds with $K = \ell$. For η belonging to the arc $\xi_0\xi$ we deduce from the inequalities $s_{\xi_0} \leqslant \ell|\xi_0| = \ell|\eta|$ and $s_\eta - s_{\xi_0} \leqslant \alpha|\eta|$, where α denotes the opening angle of Σ_ν, that the inequality holds with $K = \ell + \alpha$.

The case when the path is piecewise-$\mathcal{C}^1$ is similar and left to the reader. l

Definition 2.5 Summable-resurgence.

- A resurgent function $\widehat{f}(\xi) \in \widehat{\mathcal{R}es}_\Omega$ is said to be *summable-resurgent* when it grows at most exponentially at infinity on any ν-sectorial region Δ_ν of $\mathcal{R}_\Omega$.
- A resurgent series $\widetilde{f}(x) \in \widetilde{\mathcal{R}es}_\Omega$ is said to be *summable-resurgent* if its Borel transform is a summable-resurgent function.

We denote respectively $\widehat{\mathcal{R}es}_\Omega^{\,\mathrm{sum}}$ and $\widetilde{\mathcal{R}es}_\Omega^{\,\mathrm{sum}}$ the set of summable-resurgent functions and the set of summable-resurgent series with singular support $\Omega, 0$. We denote $\widehat{\mathcal{R}es}_{\Omega,\omega}^{\,\mathrm{sum}}$ the set of functions of $\widehat{\mathcal{R}es}_\Omega^{\,\mathrm{sum}}$ translated by ω.

In general, the exponential type depends on the ν-sectorial region Δ_ν and is unbounded when the width of Δ_ν goes to infinity (*cf.* Remark 2.11).

The set $\widehat{\mathcal{R}es}_{\Omega,\omega}^{\,\mathrm{sum}}$ is a $\mathcal{O}^{\leqslant 1}(\mathbb{C})*$submodule of $\widehat{\mathcal{R}es}_{\Omega,\omega}$.

Remark 2.6 A summable-resurgent series is both resurgent and summable but *the converse is false.* Indeed, a series is summable when it satisfies the conditions of Definition 2.5 in restriction to the first sheet only.

2.3 Summable-resurgence theorem

We are now able to state the result in view in this section:

Theorem 2.7 *(J. Écalle [11])*
Assume that System (1) has a single level equal to one (cf. Assumption (3)) and denote $\widetilde{f}(x)$ the first n_1 columns of $\widetilde{F}(x)$.

Then, $\widetilde{f}$ is summable-resurgent with singular support $\Omega, 0$ (recall $a_1 = 0$):

$$\boxed{\widetilde{f}(x) \in \widetilde{\mathcal{R}es}_\Omega^{\,\mathrm{sum}}} \ .$$

By means of elementary transformations (*cf.* Section 2.1) the result can be extended to the full matrix $\widetilde{F}$ replacing however Ω by $\mathbf{\Omega}$. Therefore, we can state:

Corollary 2.8 *Under the same conditions as in Theorem 2.7 the full matrix $\widetilde{F}$ is summable-resurgent with singular support $\mathbf{\Omega}, 0$:*

$$\boxed{\widetilde{F}(x) \in \widetilde{\mathcal{R}es}_{\mathbf{\Omega}}^{\,\mathrm{sum}}} \ .$$

The hypothesis of "single level equal to 1" is central. In a paper to come it will be shown how to generalize this result to the case of a single level not equal to one. In the case of several levels one has to consider several Borel planes simultaneously.

Theorem 2.7 can be proved in different ways. We first sketch a proof based on the Newton polygon and Ramis Index Theorem, the system being given in the form of an equation of order n. Next, we develop a proof following Écalle's method by regular perturbation of the system and majorant series. This second approach will allow us to precisely describe the singularities of the Borel transform $\varphi(\xi) = \widehat{f}(\xi)$ in the Borel plane which are all located in the set Ω.

2.4 Sketch proof of summable-resurgence using linear differential equations

The formal Borel transformation $\widetilde{\mathcal{B}}$ is an isomorphism from the differential algebra $\left(\mathbb{C}[[x]], +, \cdot, x^2\frac{d}{dx}\right)$ to the differential algebra $\left(\delta \oplus \mathbb{C}[[\xi]], +, \star, \xi\cdot\right)$ that changes ordinary product into convolution product and changes derivation $x^2\dfrac{d}{dx}$ into multiplication by ξ. It also changes multiplication by $\dfrac{1}{x}$ into derivation $\dfrac{d}{d\xi}$ allowing thus to extend the isomorphism from the meromorphic series $\mathbb{C}[[x]][1/x]$ to $\mathbb{C}[\delta^{(k)}, k \in \mathbb{N}] \oplus \mathbb{C}[[\xi]]$. Recall that the Borel transform of a monomial reads $\mathcal{B}(x^m) = \dfrac{\xi^{m-1}}{\Gamma(m)}$ for all $m > 0$.

Consider now a differential equation $Dy(x) = 0$ with single level one. By the Birkhoff Algebraization Theorem [28, Th. 3.3.1] we may assume that the operator D has rational coefficients. By means of an elementary, possibly trivial, gauge transformation (*cf.* Section 2.1) we may also assume that the equation admits a formal series solution $\widetilde{f}(x)$.

Multiplying D by a convenient power of $1/x$ if needed, the Borel transformed equation $\widehat{D}\widehat{y}(\xi) = 0$ is again an ordinary linear differential equation with polynomial coefficients. Ramis Index Theorem [22] shows that the series $\widetilde{f}(x)$ is of Gevrey type of order 1. Hence, its Borel transform $\widetilde{\mathcal{B}}(\widetilde{f})(\xi)$ converges in a neighborhood of the origin $\xi = 0$; we denote by $\varphi(\xi)$ its sum. A direct calculation using the characteristic equation associated with slope 1 of the Newton polygon of D shows that the singularities of $\widehat{D}\widehat{y}(\xi) = 0$

belong to the finite set Ω of the Stokes values of $Dy(x) = 0$. From the Cauchy-Lipschitz theorem one can then assert that $\varphi(\xi)$ is resurgent with singular support $\Omega, 0$, *i.e.*, it can be analytically continued along any path issuing from 0 and staying in $\mathbb{C}\backslash\Omega$. To see that $\varphi(\xi)$ has exponential growth at infinity, it suffices to notice that the equation $\widehat{D}\widehat{y}(\xi) = 0$ has rank one at infinity [17, Theorem 1.4].

2.5 Proof of summable-resurgence on systems following J. Écalle's approach

Since, by the Cyclic Vector Theorem, equations and systems are meromorphically equivalent, to get the summable-resurgence of solutions of systems there is no need for a new proof made directly on systems. However, the proof below quoted by J. Écalle [11] has its own interest. In particular, it allows us to give a precise description of the singularities of $\varphi(\xi)$ in the Borel plane. A simpler case where the formal monodromy is assumed to be trivial ($L = I_n$) can be found in [14]; the case of a higher level can be found in [25].

2.5.1 Setting the problem.

The conclusion of Theorem 2.7 being preserved by elementary and meromorphic gauge transformations, we can assume that the system is so prepared that conditions (5), (6) and (7) are all satisfied.

The normal form (4) of System (1) reads

$$x^2\frac{dY}{dx} = A_0(x)Y \quad \text{where} \quad A_0(x) = \bigoplus_{j=1}^{J} a_j I_{n_j} + x(\lambda_j I_{n_j} + J_{n_j}) \tag{9}$$

while the matrix $A(x)$ of System (1) reads

$$A(x) = A_0(x) + B(x) \quad \text{where} \quad B(0) = 0 \tag{10}$$

More precisely, split the matrix $B(x) = \left[B^{j;k}(x)\right]$ into blocks fitting to the Jordan structure of L. Then,

$$B^{j;k}(x) = \begin{cases} O(x) & \text{if} \quad a_j \neq a_k \\ O(x^2) & \text{if} \quad a_j = a_k \text{ (and especially if } j = k). \end{cases} \tag{11}$$

As previously said, we restrict the study to the first column-block $\widetilde{f}$ of $\widetilde{F}$ uniquely determined by the first n_1 columns of the homological system

$$x^2\frac{d\widetilde{F}}{dx} - A_0(x)\widetilde{F} + \widetilde{F}A_0(x) = B(x)\widetilde{F} \tag{12}$$

jointly with the initial condition $\widetilde{f}(0) = I_{n,n_1}$ (Recall that I_{n,n_1} denotes the identity matrix I_n truncated at its first n_1 columns). Hence, the system

$$x^2\frac{d\widetilde{f}}{dx} - A_0(x)\widetilde{f} + x\widetilde{f}J_{n_1} = B(x)\widetilde{f}. \tag{13}$$

(Recall $a_1 = \lambda_1 = 0$).

It is not clear from the Borel transformed system

$$\xi\widehat{f} - \widehat{A}_0 * \widehat{f} + 1 * \widehat{f}J_{n_1} = \widehat{B} * \widehat{f} \tag{14}$$

that the Borel transform $\widehat{f}(\xi)$ of $\widetilde{f}(x)$ satisfies the conditions of Definition 2.5 for $\widetilde{f}(x)$ to be summable-resurgent.

To prove the summable-resurgence of $\widetilde{f}(x)$, J. Écalle suggests in [11] to regularly perturb System (13) by substituting αB for B, next to solve this perturbed system in terms of a power series in the parameter α and then, to proceed by majorant series satisfying a convenient system. There exists, of course, many possible majorant systems. Here below, we make explicit a possible one. We consider the very general case when L is in non diagonal form, covering thus, all possible cases of resonances. The calculation is made more complicated than in the diagonal case since we have to work with packs of equations instead of individual ones but the philosophy keeps the same.

We split $\widetilde{f}(x)$ into row-blocks $\widetilde{f}^{j,\bullet}(x)$ accordingly to the Jordan structure of L and we refer to Section 2.2 for the notations.

2.5.2 The perturbed system.

An identification of equal powers of α shows that the perturbed system

$$x^2\frac{d\widetilde{f}}{dx} - A_0(x)\widetilde{f} + x\widetilde{f}J_{n_1} = \alpha B(x)\widetilde{f} \tag{15}$$

equivalent, for all $j \in \{1, \dots, J\}$, to

$$x^2\frac{d\widetilde{f}^{j,\bullet}}{dx} - (a_j + \lambda_j x)\widetilde{f}^{j,\bullet} - xJ_{n_j}\widetilde{f}^{j,\bullet} + x\widetilde{f}^{j,\bullet}J_{n_1} = \alpha B^{j,\bullet}\widetilde{f} \tag{16}$$

admits a unique formal solution of the form $\widetilde{f}(x,\alpha) = \sum_{m\geqslant 0}\widetilde{f}_m(x)\alpha^m$ satisfying $\widetilde{f}_0(x) = I_{n,n_1}$ and $\widetilde{f}_m(x) \in x\mathbb{C}[[x]]$ for all $m \geqslant 1$. Since there is no possible

ambiguity we keep denoting $\widetilde{f}$ the perturbed solution. The proof proceeds by induction on m. For all $j \in \{1, \dots, J\}$, System (16) reads, for $\ell = 1, \dots, n_j$ and $k = 1, \dots, n_1$,

$$x^2 \frac{d\widetilde{f}_m^{(j,\ell);(k)}}{dx} - (a_j + \lambda_j x)\widetilde{f}_m^{(j,\ell);(k)} = x\widetilde{f}_m^{(j,\ell+1);(k)} - x\widetilde{f}_m^{(j,\ell);(k-1)} + B^{(j,\ell);\bullet}\widetilde{f}_{m-1}^{\bullet;(k)} \tag{17}$$

and can be solved term after term following the alphabetic order on $(k, n_j - \ell)$ (one begins with the first column $k = 1$ and the last row $\ell = n_j$ in each block j).

It turns out actually that for all j and $m \geqslant 1$,

$$\widetilde{f}_{2m-1}^{j;\bullet} = O(x^m) \quad \text{and} \quad \widetilde{f}_{2m}^{j;\bullet} = \begin{cases} O(x^m) & \text{if } a_j = 0 \\ O(x^{m+1}) & \text{if } a_j \neq 0 \end{cases}$$

allowing thus to rewrite the series $\widetilde{f}(x, \alpha) = \sum_{m \geqslant 0} \widetilde{f}_m(x)\alpha^m$ in α as a series in x with coefficients that are polynomial in α. Consequently, the unperturbed solution $\widetilde{f}(x)$ corresponds to $\widetilde{f}(x, 1)$ (unicity of $\widetilde{f}(x)$ and $\widetilde{f}(x, 1)$) and, for all α and in particular for $\alpha = 1$, the series $\widetilde{f}(x, \alpha)$ admits a formal Borel transform $\widehat{f}(\xi, \alpha)$ with respect to x of the form

$$\widehat{f}(\xi, \alpha) = \delta I_{n,n_1} + \sum_{m \geqslant 1} \varphi_m(\xi)\alpha^m \tag{18}$$

where $\varphi_m(\xi)$ denotes the Borel transform of $\widetilde{f}_m(x)$ for all $m \geqslant 1$.

For fixed $j \in \{1, \dots, J\}$ and $m \geqslant 1$ the system satisfied by the $\varphi_m^{(j,\ell);(k)}(\xi)$'s reads, for $\ell = 1, \dots, n_j$ and $k = 1, \dots, n_1$,

$$\begin{cases} (\xi - a_j)\dfrac{d\varphi_m^{(j,\ell);(k)}}{d\xi} - (\lambda_j - 1)\varphi_m^{(j,\ell);(k)} = \\ \qquad \varphi_m^{(j,\ell+1);q} - \varphi_m^{(j,\ell);(k-1)} + \dfrac{d\widehat{B}^{(j,\ell);\bullet}}{d\xi} * \varphi_{m-1}^{\bullet;(k)} + \widehat{B}^{(j,\ell);\bullet}(0)\varphi_{m-1}^{\bullet;(k)} \end{cases} \tag{19}$$

since the Borel transform of $\dfrac{1}{x}B^{(j,\ell);\bullet}(x)\widetilde{f}_{m-1}^{\bullet;(k)}(x)$ is equal to

$$\frac{d}{d\xi}\Big(\widehat{B}^{(j,\ell);\bullet} * \varphi_{m-1}^{\bullet;(k)}\Big)(\xi) = \frac{d\widehat{B}^{(j,\ell);\bullet}}{d\xi} * \varphi_{m-1}^{\bullet;(k)} + \widehat{B}^{(j,\ell);\bullet}(0)\varphi_{m-1}^{\bullet;(k)}$$

($\widehat{B}^{(j,\ell);\bullet}(0)$ is also the coefficient of x in $B^{(j,\ell);\bullet}(x)$ and is not supposed equal to zero).

It can again be solved term after term starting with the first column $k = 1$ and the last row $\ell = n_j$.

Recall that for b an entire function, the convolution $b * \xi^m, m \in \mathbb{N}$ is well defined by the integral $\int_0^\xi b(\xi - t)t^m dt$ as well as $b * \widehat{f}$ by $\int_\gamma b(\xi - t)\widehat{f}(t)dt$ along any path γ avoiding the singularities of $\widehat{f}$ (here, the Borel transforms $\widehat{B}^{(j,k);\bullet}$ of $B^{(j,k);\bullet}$ are entire functions since $B(x)$ is analytic at 0).

Note also that the only singularities of System (19) are the Stokes values $a_j \in \Omega$ and we can conclude that the φ_m's are resurgent functions defined on $\mathcal{R}_\Omega$.

2.5.3 What has to be proved.

Fix now $\nu > 0$ and a ν-sectorial region Δ_ν as described in Definition 2.3.

We are left to prove that

(a) the Borel series $\widehat{f}(\xi) = \widehat{f}(\xi, 1)$ is convergent and can be analytically continued to $\mathcal{R}_\Omega$; we keep denoting $\widehat{f}(\xi)$ the analytic continuation;

(b) $\widehat{f}(\xi)$ grows at most exponentially on Δ_ν at infinity.

These properties could be directly shown to be true for the φ_m's. To prove that they are true for $\widehat{f}$ we use the technique of majorant series.

2.5.4 A candidate majorant system.

Instead of System (16) consider, for $j = 1, \dots, J$, the perturbed linear system

$$(20)\quad \begin{cases} C_j(\widetilde{g}^{j;\bullet} - I^{j;\bullet}_{n,n_1}) = J_{n_j}\widetilde{g}^{j;\bullet} + \widetilde{g}^{j;\bullet}J_{n_1} - 2I^{j;\bullet}_{n,n_1}J_{n_1} + \alpha\dfrac{|B|^{j;\bullet}}{x}\widetilde{g} \text{ if } a_j = 0 \\ (\nu - |\lambda_j - 1|x)\widetilde{g}^{j;\bullet} = J_{n_j}x\widetilde{g}^{j;\bullet} + x\widetilde{g}^{j;\bullet}J_{n_1} + \alpha|B|^{j;\bullet}\widetilde{g} \qquad \text{if } a_j \neq 0 \end{cases}$$

where the unknown $\widetilde{g}$ is, like $\widetilde{f}$, a $n \times n_1$-matrix split into row-blocks $\widetilde{g}^{j;\bullet}$ fitting the Jordan structure of L and where $|B|$ denotes the series B in which the coefficients of the powers of x are replaced by their module. The constants $C_j > 0$ are to be adequately chosen which we will do in Lemma 2.9 below. For now, they are just arbitrary non-zero constants.

System (20), like System (16), admits a unique formal solution $\widetilde{g}(x, \alpha) = \sum_{m \geqslant 0} \widetilde{g}_m(x)\alpha^m$ such that $\widetilde{g}_0(x) = I_{n,n_1}$ and $\widetilde{g}_m(x) \in x\mathbb{C}[[x]]$ for all $m \geqslant 1$. Like

$\widetilde{f}(x,\alpha)$, this series satisfies

$$\widetilde{g}^{j;\bullet}_{2m-1} = O(x^m) \quad \text{and} \quad \widetilde{g}^{j;\bullet}_{2m} = \begin{cases} O(x^m) & \text{if } a_j = 0 \\ O(x^{m+1}) & \text{if } a_j \neq 0 \end{cases}$$

and consequently, $\widetilde{g}(x,\alpha)$ can be seen as a series in powers of x whose coefficients are polynomials in α. But unlike $\widetilde{f}_m(x)$ in general, $\widetilde{g}_m(x)$ has non negative coefficients for all $m \geqslant 1$.

Prove now that $\widetilde{g}(x,\alpha)$ is a convergent series in (x,α) in a domain containing $(x,\alpha) = (0,1)$.

From System (20) written for each individual column k we obtain, for all j, the linear system

$$(21) \quad \begin{cases} (C_j - J_{n_j})\widetilde{g}^{j;(k)} - \alpha\dfrac{|B|^{j;\bullet}}{x}\widetilde{g}^{\bullet;(k)} = \widetilde{g}^{j;(k-1)} + \text{const} & \text{if } a_j = 0 \\ \Big(\nu - |\lambda_j - 1|x - J_{n_j}x\Big)\widetilde{g}^{j;(k)} - \alpha|B|^{j;\bullet}\widetilde{g}^{\bullet;(k)} = x\widetilde{g}^{j;(k-1)} & \text{if } a_j \neq 0. \end{cases}$$

(We set $\widetilde{g}^{j;(0)} = 0$).
For $x = 0$ the system reduces to

$$(22) \quad \begin{cases} (C_j - J_{n_j})\widetilde{g}^{j;(k)} - \alpha|B'|^{j;\bullet}(0)\widetilde{g}^{\bullet;(k)} = \widetilde{g}^{j;(k-1)} + \text{const} & \text{if } a_j = 0 \\ \nu\widetilde{g}^{j;(k)} = 0 & \text{if } a_j \neq 0. \end{cases}$$

However, $|B'|^{j;\bullet}(0)\widetilde{g}^{\bullet;(k)} = \sum_{r\,|\,a_r \neq 0} |B'|^{j;r}(0)\widetilde{g}^{r;(k)}$ since $|B'|^{j;r}(0) = 0$ for all r such that $a_r = a_j = 0$ and an adequate linear combination among equations of this system allows to cancel the terms in α. System (22) is thus equivalent to a constant triangular system whose diagonal terms are either equal to C_j or to ν. Having assumed $C_j \neq 0$ for all j it is then a Cramer system and System (21) is equivalent to a system of the form $(T + xM(x,\alpha))g^{\bullet;(k)} = N(g^{\bullet;(k-1)})$ where T is a constant invertible matrix and M is analytic in (x,α) in a strip around $x = 0$ while the right hand-side is a column vector depending on $g^{j;(k-1)}$ and analytic in the same strip. The determinant of System (21) is analytic in (x,α) and non-zero for $x = 0$ whatever α is equal to. There exists then a bi-disc centered at $(x,\alpha) = (0,0)$ and containing $(x,\alpha) = (0,1)$ on which the determinant does not vanish. On such a bi-disc, System (21) admits an analytic solution $g(x,\alpha)$. By unicity, its Taylor expansion $g(x,\alpha) = \sum g_m(x)\alpha^m$ coincides with $\widetilde{g}(x,\alpha) = \sum \widetilde{g}_m(x)\alpha^m$.

In particular, for $\alpha = 1$, the formal solution $\widetilde{g}(x) = \widetilde{g}(x,1) = \sum \widetilde{g}_m(x)$ converges. Henceforth, its Borel transform $\widehat{g}(\xi) = \sum \phi_m(\xi)$ is an entire function with exponential growth at infinity (ϕ_m denotes the Borel transform of the series $\widetilde{g}_m(x)$ and is also an entire function with exponential growth at infinity).

2.5.5 Majorant series and exponential growth.

Lemma 2.9 below shows that $\widehat{g}(K|\xi|) = \sum \phi_m(K|\xi|)$ is a majorant series for $\widehat{f}(\xi) = \sum \varphi_m(\xi)$ on Δ_ν.

Lemma 2.9

Let $K > 0$ be associated with the chosen ν-sector Δ_ν as in Lemma 2.4.

For all $m \geqslant 0$, $\xi \in \Delta_\nu$, $j = 1, \dots, J$ and $q = 1, \dots, n_1$, the following inequalities hold:

$$|\varphi_m^{j;(k)}(\xi)| \leqslant \phi_m^{j;(k)}(s_\xi) \leqslant \phi_m^{j;(k)}(K|\xi|). \tag{23}$$

(φ_m and ϕ_m are the Borel transforms of the "initial" solution $\widetilde{f}_m$ and of the "majorant" solution $\widetilde{g}_m$ respectively).

Proof. Recall that the functions φ_m are defined over $\mathcal{R}_\Omega$ and the functions ϕ_m on all of $\mathbb{C}$. They are then all well defined over Δ_ν.

• The series $\phi_m(\xi)$ have non negative coefficients and by Lemma 2.4 we know that we can connect ξ to 0 by a path so that $s_\xi \leqslant K|\xi|$. Hence, the second inequality.

• Prove the first inequality. For all $m \geqslant 1$, the entries $\varphi_m^{(j,\ell);(k)}$ (row ℓ of row-block j and column k) of φ_m and $\phi_m^{(j,\ell);(k)}$ of ϕ_m satisfy respectively

$$(\xi - a_j)\frac{d}{d\xi}\varphi_m^{(j,\ell);(k)} - (\lambda_j - 1)\varphi_m^{(j,\ell);(k)} = \omega_m^{(j,\ell);(k)} \tag{24}$$

where

$$\omega_m^{(j,\ell);(k)} = \varphi_m^{(j,\ell+1);(k)} - \varphi_m^{(j,\ell);(k+1)} + \frac{d}{d\xi}\widehat{B}^{(j,\ell);\bullet} * \varphi_{m-1}^{\bullet;(k)} + \widehat{B}^{(j,\ell);\bullet}(0)\varphi_{m-1}^{\bullet;(k)}$$

and

$$\begin{cases} \nu\dfrac{d\phi_m^{(j,\ell);(k)}}{d\xi} - |\lambda_j - 1|\phi_m^{(j,\ell);(k)} &= \Omega_m^{(j,\ell);(k)} \quad \text{if } a_j \neq 0 \\ C_j\phi_m^{(j,\ell);(k)} &= \Omega_m^{(j,\ell);(k)} \quad \text{if } a_j = 0 \end{cases} \tag{25}$$

where

$$\Omega_m^{(j,\ell);(k)} = \phi_m^{(j,\ell+1);(k)} + \phi_m^{(j,\ell);(k-1)} + \frac{d}{d\xi}|\widehat{B}|^{(j,\ell);\bullet} * \phi_{m-1}^{\bullet;(k)} + |\widehat{B}|^{(j,\ell);\bullet}(0)\phi_{m-1}^{\bullet;(k)}.$$

Fix $\xi \in \Delta_\nu$ and a path γ_ξ in Δ_ν as in Lemma 2.4 so that

$$|\eta| \leqslant s_\eta \leqslant K|\eta| \quad \text{for all } \eta \in \gamma_\xi.$$

We proceed by recurrence following the alphabetic order on $(m, k, n_j - \ell)$ and we assume that for all $(m', k', n_j - \ell') < (m, k, n_j - \ell)$, the inequality $|\varphi_{m'}^{(j,\ell');(k')}(\eta)| \leqslant \phi_{m'}^{(j,\ell');(k')}(s_\eta)$ holds for all $\eta \in \gamma_\xi$. (It holds for $m' = 0$).
We observe that $|\omega_m^{(j,\ell);(k)}(\xi)| \leqslant \Omega_m^{(j,\ell);(k)}(s_\xi)$. Indeed,

$$\begin{aligned}
\Big|\frac{d\widehat{B}^{(j,\ell);\bullet}}{d\xi} * \varphi_{m-1}^{\bullet;(k)}(\xi)\Big| &= \Big|\int_0^{s_\xi} \frac{d\widehat{B}^{(j,\ell);\bullet}}{d\xi}(\xi - \eta(s))\varphi_{m-1}^{\bullet;(k)}(\eta(s))\eta'(s)ds\Big| \\
&\leqslant \int_0^{s_\xi} \frac{d|\widehat{B}|^{(j,\ell);\bullet}}{d\xi}(|\xi - \eta(s)|)|\varphi_{m-1}^{\bullet;(k)}(\eta(s))|ds \quad (\text{since } |\eta'(s)| = 1) \\
&\leqslant \int_0^{s_\xi} \frac{d|\widehat{B}|^{(j,\ell);\bullet}}{d\xi}(s_\xi - s)\phi_{m-1}^{\bullet;(k)}(s)ds \\
&\qquad\qquad (\text{since } |\widehat{B}|^{(j,\ell);\bullet} \text{ has non negative coefficients}) \\
&= \frac{d|\widehat{B}|^{(j,\ell);\bullet}}{d\xi} * \phi_{m-1}^{\bullet;(k)}(s_\xi)
\end{aligned}$$

and the other three terms of $\omega_m^{(j,\ell);(k)}$ are majored using the recurrence hypothesis.

To conclude in the case when $a_j = 0$ we solve Equation (24). Hence,

$$\varphi_m^{(j,\ell);(k)}(\xi) = \xi^{\lambda_j - 1}\int_0^{s_\xi} \omega_m^{(j,\ell);(k)}(\eta(s))\eta(s)^{-\lambda_j}\eta'(s)ds$$

and then,

$$|\varphi_m^{(j,\ell);(k)}(\xi)| \leqslant |\xi|^{\mathrm{Re}\lambda_j - 1}e^{-\mathrm{Im}(\lambda_j - 1)\arg(\xi)}\int_0^{s_\xi} \Omega_m^{(j,\ell);(k)}(s)|\eta(s)|^{-\mathrm{Re}\lambda_j}e^{\mathrm{Im}(\lambda_j)\arg(\eta(s))}ds.$$

Since points in Δ_ν have bounded arguments there exists a constant $c > 0$ such that $e^{-\mathrm{Im}(\lambda_j - 1)\arg(\xi) + \mathrm{Im}(\lambda_j)\arg(\eta(s))} \leqslant c$ for all ξ and $\eta(s) \in \Delta_\nu$. It results

that

$$\begin{aligned}
|\varphi_m^{(j,\ell);(k)}(\xi)| &\leqslant c|\xi|^{\mathrm{Re}\lambda_j-1}\int_0^{s_\xi}\Omega_m^{(j,\ell);(k)}(s)|\eta(s)|^{-\mathrm{Re}\lambda_j}ds \\
&\leqslant c|\xi|^{\mathrm{Re}\lambda_j-1}\Omega_m^{(j,\ell);(k)}(s_\xi)\int_0^{s_\xi}\Big(\frac{s}{K}\Big)^{-\mathrm{Re}\lambda_j}ds \\
&\qquad \text{(by Lemma 2.4, the fact that } \mathrm{Re}\lambda_j \geqslant 0 \\
&\qquad\quad \text{and that } \Omega_m \text{ has non negative coefficients)} \\
&\leqslant c|\xi|^{\mathrm{Re}\lambda_j-1}\frac{K\Omega_m^{(j,\ell);(k)}(s_\xi)}{1-\mathrm{Re}\lambda_j}\Big(\frac{s_\xi}{K}\Big)^{1-\mathrm{Re}\lambda_j} \\
&\leqslant \frac{cK}{1-\mathrm{Re}\lambda_j}\Omega_m^{(j,\ell);(k)}(s_\xi) \\
&\qquad \text{(using Lemma 2.4 and } 1-\mathrm{Re}\lambda_j>0) \\
&= \phi_m^{(j,\ell);(k)}(s_\xi) \text{ if we choose } C_j=\frac{1-\mathrm{Re}\lambda_j}{cK}.
\end{aligned}$$

To conclude in the case when $a_j \neq 0$ we apply Grönwall Lemma to $\varphi_m^{(j,\ell);(k)}(\gamma_\xi(s))$. This achieves the proof of Lemma 2.9. l

Lemma 2.9 shows that $\widehat{g}(K|\xi|) = \sum \phi_m(K|\xi|)$ is a majorant series for $\widehat{f}(\xi) = \sum \varphi_m(\xi)$ on Δ_ν. Since $\widehat{g}$ is well defined on Δ_ν with exponential growth at infinity the same property holds for $\widehat{f}(\xi)$ which achieves the proof of Theorem 2.7. l

Remark 2.10 It results from the above proof that the series $\sum \varphi_m(\xi)$ converges uniformly to $\widehat{f}(\xi)$ on compact sets of $\mathcal{R}_\Omega$.

Remark 2.11 We see from Lemma 2.9 that when $\widehat{g}$ grows exponentially with type a, *i.e.*, satisfies an inequality $|\widehat{g}(\xi)| \leqslant const.e^{a|\xi|}$ for large ξ then $\widehat{f}(\xi)$ grows exponentially with type Ka. When the width of the domain Δ_ν goes to infinity so does K; hence, the necessity for considering ν-sectorial regions with bounded width. Also the estimates in the proof of Lemma 2.9 would no more be valid on sectors with unbounded width.

Remark 2.12 One should think at reading the previous two sections that the proof with systems is much longer. This is not the case. If the same level of detail were provided the proof on equations like sketched in Section

2.4 would be much longer and also less elementary since it includes the Index Theorem for rank one and the Main Asymptotic Existence Theorem at infinity.

3 Singularities in the Borel plane

Theorem 2.7 tells us that $\widetilde{f}(x)$ is a resurgent series of $\widetilde{\mathcal{R}es_\Omega}^{\rm sum}$. Its Borel transform $\widehat{f}(\xi)$ is then, in particular, analytic on the Riemann surface $\mathcal{R}_\Omega$, its possible singularities being the points $a_1, a_2, \ldots, a_J$ of Ω including $a_1 = 0$ out of the first sheet.

The form of System (19) shows that the singularities of $\widehat{f}$ should at least involve poles since some λ_j are equal to 0, complex powers when some λ_j are not 0 and logarithms. It is already known that, in the case of a system with the unique level one, the singularities all belong to the Nilsson class (*cf.* [3] for instance). In that case, the exponentials $e^{-a_j/x}$ in the formal fundamental solution $\widetilde{Y}(x)$ act as translations in the Borel plane; hence, the location of the singularities at the various points a_j. In the case of higher level or in the case of several levels that, after rank reduction, we could assume to be all $\leqslant 1$ there might also occur exponentials of degree less than one. Their action in the Borel plane would be then transcendental and would generate irregular —no longer in the Nilsson class— singularities.

Our aim in this section is to set up a precise description of the singularities in the Borel plane related to the form of the formal fundamental solution $\widetilde{Y}(x) = \widetilde{F}(x)x^L e^{Q(1/x)}$ of System (1).

As previously, we restrict our study to the first n_1 columns $\widetilde{f}$ of $\widetilde{F}$. We base the analysis on the results of Section 2.5 interpreting the Borel transform $\widehat{f}(\xi)$ of $\widetilde{f}(x)$ as a series $\widehat{f}(\xi) = \delta I_{n,n_1} + \sum_{m\geqslant 1} \varphi_m(\xi)$ which converges uniformly on compact sets of $\mathcal{R}_\Omega$.

Decomposing φ_m into blocks $\left[\varphi_m^{j,\bullet}\right]_{1\leqslant j\leqslant J}$ fitting the Jordan structure of L ($\varphi_m^{j,\bullet}$ has dimension $n_j \times n_1$), System (19) splits into the following J systems:

$$(26)\quad (\xi - a_j)\frac{d\varphi_m^{j,\bullet}}{d\xi} - (\lambda_j - 1)\varphi_m^{j,\bullet} - J_{n_j}\varphi_m^{j,\bullet} + \varphi_m^{j,\bullet} J_{n_1} = \frac{d\widehat{B}^{j,\bullet}}{d\xi} * \varphi_{m-1} + \widehat{B}^{j,\bullet}(0)\varphi_{m-1}$$

where $\dfrac{d\widehat{B}^{j,\bullet}}{d\xi} * \varphi_{m-1} = \displaystyle\sum_{k=1}^{J} \frac{d\widehat{B}^{j,k}}{d\xi} * \varphi_{m-1}^{k,\bullet}$ and $\widehat{B}^{j,\bullet}(0)\varphi_0 = 0$.

Let us first introduce some vocabulary used in resurgence theory. Working locally we place ourselves at the origin of $\mathbb{C}$. As previously, we denote

$\mathcal{O} = \mathbb{C}\{x\}$ the space of holomorphic germs at 0 in $\mathbb{C}$ and $\widetilde{\mathcal{O}}$ the space of holomorphic germs at 0 on the Riemann surface $\widetilde{\mathbb{C}}$ of the logarithm.

Being mostly interested in integrating solutions in the Borel plane on both sides of the singularities, thus enclosing them in a loop, we can neglect holomorphic terms and it is natural to consider the quotient space $\mathcal{C} = \widetilde{\mathcal{O}}/\mathcal{O}$. The elements of $\mathcal{C}$ are called *micro-functions* by B. Malgrange [17] by analogy with hyper- and micro-functions defined by Sato, Kawai and Kashiwara in higher dimensions. They are called *singularities* by J. Ecalle and al. and usually denoted with a nabla, like $\overset{\triangledown}{\varphi}$, for a singularity of the function φ while the space $\mathcal{C} = \widetilde{\mathcal{O}}/\mathcal{O}$ is denoted SING_0 (*cf.* [26]). A representative of $\overset{\triangledown}{\varphi}$ in $\widetilde{\mathcal{O}}$ is often denoted $\breve{\varphi}$ and is called *a major* of φ.
It is worth to consider the two natural maps

$$\text{can}: \quad \widetilde{\mathcal{O}} \longrightarrow \mathcal{C} = \widetilde{\mathcal{O}}/\mathcal{O} \quad \text{the canonical quotient map}$$

$$\text{and var}: \quad \mathcal{C} \longrightarrow \widetilde{\mathcal{O}} \quad \text{the variation map,}$$

action of a positive turn around 0 defined by $\mathrm{var}\overset{\triangledown}{\varphi}(\xi) = \breve{\varphi}(\xi) - \breve{\varphi}(\xi e^{-2\pi i})$ where $\breve{\varphi}(\xi e^{-2\pi i})$ is the analytic continuation of $\breve{\varphi}(\xi)$ along a path turning once clockwise around 0 close enough to 0 for $\breve{\varphi}$ to be defined all along (the result is independent of the choice of a major $\breve{\varphi}$). The germ $\widehat{\varphi} = \mathrm{var}\overset{\triangledown}{\varphi}$ is called *the minor* of $\overset{\triangledown}{\varphi}$.
Let δ denote the Dirac distribution at 0, $\delta^{(m)}$ its m^{th} derivative and $Y = \partial_\xi^{-1}\delta$ the Heaviside (micro-)function. One can make the following identifications:

$$\mathrm{can}\Big(\frac{1}{2\pi i\xi}\Big) = \delta \qquad \mathrm{can}\Big(\frac{(-1)^m m!}{2\pi i\xi^{m+1}}\Big) = \delta^{(m)}$$

$$\mathrm{can}\Big(\frac{\ln\xi}{2\pi i}\Big) = Y \qquad \mathrm{can}\bigg(\Big(\frac{\ln\xi}{2\pi i}\Big)^2\bigg) = \Big(2\frac{\ln\xi}{2\pi i} - 1\Big)Y \quad \text{and so on}\ldots$$

It is sometimes useful not to work at the origin. Given $\omega \neq 0$ in $\mathbb{C}$ we denote $\mathcal{C}_\omega = \mathrm{SING}_\omega$ the space of the singularities at ω, *i.e.*, the space $\mathcal{C} = \mathrm{SING}_0$ translated from 0 to ω. A function $\breve{\varphi}$ is a major of a singularity at ω if $\breve{\varphi}(\omega + \xi)$ is a major of a singularity at 0.

3.1 Simple-moderate singularities

In this Section, we state some properties, used further on, of the singularities —poles, logarithms and complex powers— which should occur in the Borel plane. We shall see (*cf.* Thm 3.7) that poles, logarithms and complex powers

are the only possible singularities arising in the Borel plane for linear systems with the unique level one, far from being any singularity in the Nilsson class.

Definition 3.1 Simple-moderate singularities

- A singularity or micro-function $\overset{\triangledown}{\varphi}$ at 0 is said to be *simple* if it has a major of the form[3]

$$\breve{\varphi}(\xi) = \sum_{p=0}^{N_0} \alpha_p \frac{\ln^p(\xi)}{\xi} + \sum_{p=1}^{N_0+1} \widehat{h}_p(\xi)\, \ln^p(\xi)$$

  where $N_0 \in \mathbb{N}$, $\alpha_p \in \mathbb{C}$ and $\widehat{h}_p \in \mathcal{O}$ for all p.

- A singularity or micro-function $\overset{\triangledown}{\varphi}$ at 0 is said to be *simple-moderate* if it has a major which differs from a simple one by terms of the form

$$\sum_{\lambda\in\Lambda} \sum_{p=0}^{N_\lambda} \alpha_{\lambda,p}\, \xi^{\lambda-1}\, \ln^p(\xi) + \sum_{\lambda\in\Lambda} \sum_{p=0}^{N_\lambda} \widehat{H}_{\lambda,p}(\xi)\, \xi^{\lambda}\, \ln^p(\xi)$$

  where Λ is a finite set of numbers $\lambda \in \mathbb{C}$ satisfying $0 < \mathrm{Re}\lambda < 1$ and for all $\lambda \in \Lambda$, $N_\lambda \in \mathbb{N}$, $\alpha_{\lambda,p} \in \mathbb{C}$ and $\widehat{H}_{\lambda,p} \in \mathcal{O}$ for all p.

- A singularity or micro-function $\overset{\triangledown}{\varphi}$ at ω is said to be *simple* or *simple-moderate* if it has a major $\breve{\varphi}$ such that $\breve{\varphi}(\omega+\xi)$ be of the previous forms respectively.

Let γ be a path from 0 to ξ in $\mathbb{C}$.
We denote $u *_\gamma v(\xi)$ the convolution product along the path γ defined by

$$u *_\gamma v(\xi) = \int_\gamma u(\xi - t)\, v(t)\, dt,$$

when the integral makes sense.

[3]This definition of a simple singularity is less restrictive than the one one can find in the literature (*cf.* [11] or [26] for instance) where N_0 is taken equal to 0. Here, we allow powers of logarithms; still poles are required to be simple but they can be factored by logarithms. We will see that, in the linear case when the system is prepared like in Section 2.1, simple singularities in the restrictive sense would occur only under strong assumptions such as trivial formal monodromy. For a general level-one system, not in prepared form, there could also occur multiple poles.

Lemma 3.2 Convolution with powers and logarithms
Let φ be an entire function on $\mathbb{C}$.
Let $p \in \mathbb{N}$ and $\lambda \in \mathbb{C}$ satisfying $0 \leqslant \operatorname{Re}\lambda < 1$.
Let $\omega \in \mathbb{C}^*$, let ψ be a function satisfying $\psi(\omega) \neq 0$ and holomorphic on a domain containing 0 and ω and let γ_ξ be a path from 0 to ξ avoiding ω and contained in this domain.

Then, the convolution product $\varphi *_{\gamma_\xi} \Big((\xi-\omega)^{\lambda-1}\ln^p(\xi-\omega)\psi\Big)(\xi)$ exists and is, close to ω, of the form

$$\varphi *_{\gamma_\xi} \Big((\xi-\omega)^{\lambda-1}\ln^p(\xi-\omega)\psi\Big)(\xi) = (\xi-\omega)^\lambda P\big((\ln(\xi-\omega)\big) + \mathrm{ent}(\xi)$$

where $P(X) \in \mathbb{C}\{\xi\}[X]$ is a polynomial with holomorphic coefficients at ω and degree

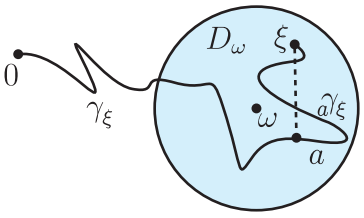
$$\deg(P) = \begin{cases} p & \text{if} \quad \lambda \neq 0 \\ p+1 & \text{if} \quad \lambda = 0 \end{cases}$$

and where ent *stands for an entire function.*

Note that the power of $(\xi-\omega)$ in the right-hand side has increased by 1 unit at least.

Proof. The convolution product $\varphi *_{\gamma_\xi} \Big((\xi-\omega)^{\lambda-1}\ln^p(\xi-\omega)\psi\Big)(\xi)$ is well-defined by the integral $\displaystyle\int_{\gamma_\xi} \varphi(\xi-t)(t-\omega)^{\lambda-1}\ln^p(t-\omega)\psi(t)dt$ since both $\varphi(\xi-t)$ and $\psi(t)$ are holomorphic along γ_ξ; and in the case when $\omega = 0$, the factor $t^{\lambda-1}$ is integrable at 0.

We are interested in the behavior of this function as ξ goes to ω.
Suppose ξ so close to ω that there is a disc D_ω centered at ω, containing ξ and included in the holomorphy domain of ψ.
Fix a on γ_ξ so that the part ${}_a\gamma_\xi$ of γ_ξ from a to ξ belong to D_ω and be homotopic to a straight line in $D_\omega\backslash\{\omega\}$.



Up to an entire function, we can replace the convolution product under consideration by the integral

$$g_a(\xi) = \int_a^\xi \varphi(\xi-t)\,(t-\omega)^{\lambda-1}\,\ln^p(t-\omega)\,\psi(t)dt$$

in $\mathbb{C}$ after a convenient choice of the determinations of argument and logarithm. Expand $\varphi(\xi-t)\psi(t)$ as a Taylor series $\sum_{m\geqslant 0} c_m(\xi)\,(t-\omega)^m$ in D_ω. Since ${}_a\gamma_\xi$ lies in D_ω then, after commutation of sum and integral, $g_a(\xi)$ becomes

$$g_a(\xi) = \sum_{m\geqslant 0} c_m(\xi) \int_a^\xi (t-\omega)^{m+\lambda-1} \ln^p(t-\omega)\, dt.$$

One can check that $\displaystyle\int_a^\xi (t-\omega)^{m+\lambda-1} \ln^p(t-\omega)\, dt = G_{m+\lambda,p}(\xi) - G_{m+\lambda,p}(a)$ where

$$G_{m+\lambda,p}(\xi) = \begin{cases} p!\,(\xi-\omega)^{m+\lambda} \displaystyle\sum_{s=0}^{p} \frac{(-1)^{p-s}}{s!} \frac{\ln^s(\xi-\omega)}{(m+\lambda)^{p-s+1}} & \text{if} \quad m+\lambda \neq 0 \\ \dfrac{1}{p+1} \ln^{p+1}(t-\omega) & \text{if} \quad m=\lambda=0 \end{cases}$$

For all $s = 0,\dots,p$, the series $\displaystyle\sum_{m>0} c_m(\xi)\, \frac{(t-\omega)^{m+\lambda}}{(m+\lambda)^{p-s+1}}$ converges on D_ω. Hence, the result. l

Lemma 3.3 Anti-derivation

Let $Q(X) \in \mathcal{M}_{m,p}(\mathbb{C}\{\xi\}[X])$ be a polynomial matrix with holomorphic coefficients at $\xi = 0$. We assume that the degrees of the successive columns of Q are given by the row matrix $N_Q = [\nu \;\; \nu+1 \;\; \cdots \;\; \nu+p-1]$.
Let $\Lambda \notin -\mathbb{N}^*$.

The matrix function $\xi^\Lambda \xi^{-J_m}\, Q(\ln\xi)\, \xi^{J_p}$ admits a unique anti-derivative of the form

$$K(\xi) = \xi^{\Lambda+1} \xi^{-J_m}\, R(\ln\xi)\, \xi^{J_p}$$

where $R(X) \in \mathcal{M}_{m,p}(\mathbb{C}\{\xi\}[X])$ is a polynomial matrix with holomorphic coefficients at $\xi = 0$ and same column-degrees $N_R = N_Q$ as Q.

Note that the power of ξ which can be factored increases by 1.

Proof. Denote $Q(X) = \displaystyle\sum_{k=0}^{N} Q_k \ln^k X$ and $R(X) = \displaystyle\sum_{k=0}^{N} R_k \ln^k X$ where $N = \nu + p - 1$.

The derivative of K reads

$$\begin{aligned} K'(\xi) &= \xi^{\Lambda}\xi^{-J_m}\Big[\big((\Lambda+1)I_m - J_m\big)(R_N\ln^N\xi + \cdots + R_1\ln\xi + R_0) \\ &\quad +\xi(R'_N\ln^N\xi + \cdots + R'_1\ln\xi + R'_0) + (NR_N\ln^{N-1}\xi + \cdots + R_1) \\ &\quad +(R_N\ln^N\xi + \cdots + R_1\ln\xi + R_0)J_p\Big]\,\xi^{J_p}. \end{aligned}$$

Identifying the powers of $\ln\xi$ we get the $N+1$ systems

$$(*)_k \qquad \xi\,R'_k + \big((\Lambda+1)I_m - J_m\big)R_k + R_kJ_p = Q_k - (k+1)R_{k+1}$$

for $k = 0, 1, \ldots, N$ and $R_{N+1} = 0$ which can be solved inductively starting with $k = N$. Like in Section 2.5, System $(*)_k$ can be solved iteratively from the first to the last column and in each column, from the last row to the first one. The fact that it admits a solution holomorphic at 0 when $\Lambda \notin -\mathbb{N}^*$ results from the fact that this is true for the differential equation $\xi\,y' + \lambda\,y = a(\xi)$ when $a(\xi)$ is holomorphic at 0 and $\lambda \notin -\mathbb{N}$.
When k is greater than ν set $k = \nu+\ell$. The fact that the first ℓ columns in R_k can be chosen equal to 0 results from the fact that the same property holds for Q_k and, by induction, for R_{k+1}. Then, it holds also for the right-hand side of System $(*)_k$ and the condition on the log-degree can be satisfied.

Unicity results from the fact that, since $\xi^{\Lambda+1}$ is neither a pole nor a constant, a non-zero constant cannot be put in such form. l

3.2 Singularities of $\varphi_m, m \geqslant 1$

Recall that the resurgent functions $\varphi_m = \big[\varphi_m^{j,\bullet}\big]_{1\leqslant j\leqslant J}$ for $m \geqslant 1$ are iteratively determined, for all j, as solutions of the system

$$(26) \qquad (\xi - a_j)\frac{d\varphi_m^{j,\bullet}}{d\xi} - (\lambda_j - 1)\varphi_m^{j,\bullet} - J_{n_j}\varphi_m^{j,\bullet} + \varphi_m^{j,\bullet}J_{n_1} = \frac{d}{d\xi}\big(\widehat{B}^{j,\bullet} * \varphi_{m-1}\big)$$

satisfying convenient initial conditions corresponding to those satisfied by the $\widetilde{f}_m$'s ($\varphi_0 = \delta I_{n,n_1}$, $\varphi_m(0) = 0$ for $m \geqslant 3$, $\varphi_2^{j,\bullet}(0) = 0$ when $a_j \neq 0$ and a convenient non necessary 0 constant otherwise).

Lemma 3.3 provides the log-degrees of the successive columns of an anti-derivative of the matrix $\xi^{\lambda_k-1}\,\xi^{J_{n_k}}\,K\,\xi^{-J_{n_1}}$ when K is a generic constant matrix, $0 \leqslant \operatorname{Re}\lambda_k < 1$ and $\lambda_k \neq 0$. One can check that the log-degrees are

increased by 1 when $\lambda_k = 0$. They are given by

$$(27)\quad N[k] = \begin{cases} [(n_k{-}1)\ \ (n_k{-}1)+1\ \cdots\ (n_k{-}1)+(n_1{-}1)] & \text{if } \lambda_k \neq 0 \\ [\ n_k \qquad n_k+1 \qquad \cdots \qquad n_k+(n_1{-}1)\ ] & \text{if } \lambda_k = 0. \end{cases}$$

The behavior of φ_m at a singular point $\omega \in \Omega$ depends on the sheet of the Riemann surface $\mathcal{R}_\Omega$ we are on, *i.e.*, its depends on the path γ of analytic continuation followed from 0 (first sheet) towards ω. Following [26] we denote $\mathrm{cont}_\gamma \varphi_m$ the analytic continuation of φ_m along the path γ.

Definition 3.4

- We say that $\xi \in \mathbb{C}\backslash\Omega$ is *close to* $\omega \in \Omega$ if there is a disc centered at ω which contains ξ and no other element of Ω than ω.
- We call *path from 0 towards* $\omega \in \Omega$ a path $\gamma = \gamma_\xi$ contained in $\mathbb{C}\backslash\Omega$ which goes from 0 to a point ξ close to ω.

Proposition 3.5 *Let $\omega \in \Omega$.*

For any path of analytic continuation γ from 0 towards ω, a major of the singularity $\overset{\nabla_\gamma}{\varphi}{}_m^{j;\bullet}$ of $\mathrm{cont}_\gamma \varphi_m$ at ω exists in the form

$$\breve{\varphi}_m^{j;\bullet}(\omega+\xi) = \xi^{\lambda_j - 1}\, \xi^{J_{n_j}}\, k_{m;(\omega)}^{j;\bullet}\, \xi^{-J_{n_1}} + \mathrm{rem}_m^{j;\bullet}(\xi) \quad \textit{for all } j = 1, \dots, J,$$

with a remainder $\mathrm{rem}_m^{j;\bullet}(\xi) = \displaystyle\sum_{\lambda_k \,|\, a_k = \omega} \xi^{\lambda_k}\, R_{\lambda_k,m}^{j;\bullet}(\ln \xi)$ *where*

- $k_{m;(\omega)}^{j;\bullet}$ *denotes a constant $n_j \times n_1$-matrix (recall that n_k is the size of the k^{th} Jordan block of the matrix L of the exponents of formal monodromy) and $k_{m;(\omega)}^{j;\bullet} = 0$ when $a_j \neq \omega$,*
- $R_{\lambda_k,m}^{j;\bullet}(X)$ *denotes a polynomial matrix with holomorphic coefficients at 0, the columns of which are of degree $N[k]$ (cf. notation just above).*

Of course, $\breve{\varphi}_m^{j;\bullet}, k_{m;(\omega)}^{j;\bullet}$ and $\mathrm{rem}_m^{j;\bullet}$ depend on γ even though, for seek of simplicity, the notations do not show it up.

Note that, in the remainders, the initial factor ξ appears at powers like λ_k and no more $\lambda_k - 1$ so that the power -1 never occurs. Note also that, whatever are the values of λ_j and λ_k, we have $\lambda_j - 1 < \lambda_k$ and this is why the terms in the remainders will always appear as subdominant.

PROOF. For all $m \geqslant 1$, System (26) can be seen as an inhomogeneous linear system in the entries of φ_m. For any j, the general solution of the homogeneous system reads $(\xi-a_j)^{\lambda_j-1}(\xi-a_j)^{J_{n_j}}\, k^{j;\bullet}_{m;(\omega)}\, (\xi-a_j)^{-J_{n_1}}$ with $k^{j;\bullet}_{m;(\omega)}$ an arbitrary constant $n_j \times n_1$ matrix and we have to prove that there is a particular solution of the inhomogeneous system in the form of the remainder. To this end, we integrate the system using the Lagrange method (variation of constants). System (26) is of the form $(\xi - a_j)\dfrac{d\varphi^{j;\bullet}_m}{d\xi}$ + linear terms $=$ Q, the type of Q depending upon m. Looking for a solution in the form $\varphi = (\xi - a_j)^{\lambda_j-1}(\xi - a_j)^{J_{n_j}}\, K\, (\xi - a_j)^{-J_{n_1}}$ we obtain to determine K up to a constant the condition $\dfrac{dK}{d\xi} = (\xi - a_j)^{-\lambda_j}(\xi - a_j)^{-J_{n_j}}\, Q\, (\xi - a_j)^{J_{n_1}}$ and we just have to find anti-derivatives for the various possible Q.

For $m = 1$, the inhomogenuity $Q = \dfrac{d\widehat{B}^{j;\bullet}}{d\xi}$ is an entire function, hence holomorphic at ω. When $a_j \neq \omega$, then ω is an ordinary point for the j^{th} block of System (26) and the inhomogenuity is holomorphic. Hence, there is a holomorphic solution at ω and we can choose $\breve{\varphi}^{j,\bullet}_1 = 0$. When $a_j = \omega$, Lemma 3.3 provides K in the form $K = (\xi - a_j)^{-\lambda_j+1}(\xi - a_j)^{-J_{n_j}}\, R(\xi)\, (\xi - a_j)^{J_{n_1}}$ with R holomorphic at ω and then, a particular solution $\varphi^{j;\bullet}_1(\xi) = R(\xi)$ holomorphic at ω so that we can choose the remainder $\mathrm{rem}^j_1 = 0$.

For $m = 2$, using the superposition principle, we have to consider inhomogenuities Q of the form $Q = \dfrac{d\widehat{B}^{j;k}}{d\xi} * \varphi^{k;\bullet}_1$ for all $k = 1, \ldots, J$. When $a_k \neq \omega$, the solution $\varphi^{k;\bullet}_1$ and then also Q is holomorphic at ω and we can conclude like in the case when $m = 1$. From now, we forget about holomorphic terms. When $a_k = \omega$, then $\varphi^{k;\bullet}_1$ differs from a holomorphic function by terms of the form $(\xi-\omega)^{\lambda_k-1}(\xi-\omega)^{J_{n_k}}\, k^{k;\bullet}_{1;(\omega)}\, (\xi-\omega)^{-J_{n_1}}$. It results from Lemma 3.2 that, modulo a holomorphic function, Q takes the form $(\xi-\omega)^{\lambda_k} P\big(\ln(\xi-\omega)\big)$ with P a $n_j \times n_1$-matrix of polynomials with holomorphic coefficients the columns of which have log-degree $N[k]$. In both cases, $a_j = \omega$ or $a_j \neq \omega$, Lemma 3.3 provides a corresponding solution of the form $(\xi - \omega)^{\lambda_k} R\big(\ln(\xi - \omega)\big)$ where R is a polynomial matrix with holomorphic coefficients at ω and column-log-degrees $N[k]$.

For $m \geqslant 3$, the inhomogenuity contains terms of the form $Q = \dfrac{d\widehat{B}^{j;k}}{d\xi} * \varphi^{k;\bullet}_{m-1}$. The factor $\varphi^{k;\bullet}_{m-1}$ splits into two parts: the first part coming from the general homogeneous solution is treated like in the case when $m = 2$; the second part coming from the remainder is of a similar type but the fact

that all powers $\lambda - 1$ have been changed into λ. Consequently, there appears no factor $(\xi - \omega)^{-1}$ and this insures the stability of the log-degree $N[k]$ since, from now, convolution generates no increase of the log-degree $N[k]$. 1

3.3 Singularities of $\widehat{f}$

We are now ready to make explicit the form of the singularities of $\widehat{f}$ as a consequence of Theorem 2.7, Proposition 3.5 and an iterated application of the variation.

Lemma 3.6 below states, without proof, some useful elementary properties of the variation. We denote $\mathrm{var}^p = \underbrace{\mathrm{var} \circ \mathrm{var} \circ \cdots \circ \mathrm{var}}_{p \text{ times}}$.

Lemma 3.6

1. $\mathrm{var}\Big(\dfrac{\ln \xi}{2\pi i}\Big) = 1$.
2. *For all* $p \in \mathbb{N}$,
$$\begin{aligned}\mathrm{var}\Big(\Big(\frac{\ln \xi}{2\pi i}\Big)^p\Big) &= \sum_{r=0}^{p-1} (-1)^{p-r-1} C_p^r \Big(\frac{\ln \xi}{2\pi i}\Big)^r \\ &= p\Big(\frac{\ln \xi}{2\pi i}\Big)^{p-1} + \textit{ lower log-degree terms.}\end{aligned}$$
Consequently, $\mathrm{var}^p\Big(\Big(\dfrac{\ln \xi}{2\pi i}\Big)^p\Big) = p!$ *and* $\mathrm{var}^{p+1}\Big(\Big(\dfrac{\ln \xi}{2\pi i}\Big)^p\Big) = 0$.
3. *For all* $\lambda \in \mathbb{C}$, $\mathrm{var}(\xi^\lambda) = (1 - e^{-2\pi i\lambda})\xi^\lambda$.
Consequently, $\mathrm{var}^p(\xi^\lambda) = (1 - e^{-2\pi i\lambda})^p \xi^\lambda$ *for all* $p \in \mathbb{N}$ *and* $\mathrm{var}(\xi^\lambda) = 0$ *for all* $\lambda \in \mathbb{Z}$.
4. $\mathrm{var}(fg) = \mathrm{var}(f)\, g + f\, \mathrm{var}(g) - \mathrm{var}(f)\, \mathrm{var}(g)$.
In particular, $\mathrm{var}(fg) = f\, \mathrm{var}(g)$ *when* $\mathrm{var}(f) = 0$.
5. *For all* $\lambda \in \mathbb{C}$ *and* $p \in \mathbb{N}$,
$$\begin{aligned}\mathrm{var}\Big(\xi^\lambda \Big(\frac{\ln \xi}{2\pi i}\Big)^p\Big) &= (1 - e^{-2\pi i\lambda})\xi^\lambda \Big(\frac{\ln \xi}{2\pi i}\Big)^p + e^{-2\pi i\lambda}\xi^\lambda \mathrm{var}\Big(\Big(\frac{\ln \xi}{2\pi i}\Big)^p\Big) \\ &= (1 - e^{-2\pi i\lambda})\xi^\lambda \Big(\frac{\ln \xi}{2\pi i}\Big)^p + \xi^\lambda \times \textit{ lower log-degree terms.}\end{aligned}$$
Consequently,
$$\mathrm{var}^p\Big(\xi^\lambda \Big(\frac{\ln \xi}{2\pi i}\Big)^p\Big) = (1 - e^{-2\pi i\lambda})^p \xi^\lambda \Big(\frac{\ln \xi}{2\pi i}\Big)^p + \xi^\lambda \times \textit{lower log-degree terms.}$$

Theorem 3.7 Singularity of $\widehat{f}$ at $\xi = \omega$
Let $\omega \in \Omega$.
For any path of analytic continuation γ from 0 towards ω, a major of the singularity $\overset{\nabla_\gamma}{f_\omega}$ of $\mathrm{cont}_\gamma \widehat{f}$ at ω exists in the form

$$\boxed{\breve{f}^{j;\bullet}(\omega+\xi) = \xi^{\lambda_j - 1}\, \xi^{J_{n_j}}\, k^{j;\bullet}_{(\omega)}\, \xi^{-J_{n_1}} + \mathrm{Rem}^{j;\bullet}_{(\omega)}(\xi)} \quad \textit{for all } j = 1, \dots, J$$

with a remainder $\mathrm{Rem}^{j;\bullet}_{(\omega)}(\xi) = \displaystyle\sum_{\lambda_\ell | a_\ell = \omega} \xi^{\lambda_\ell}\, R^{j;\bullet}_{\lambda_\ell;(\omega)}(\ln \xi)$ *where*

- $k^{j;\bullet}_{(\omega)}$ *denotes a constant $n_j \times n_1$-matrix (recall that n_k is the size of the k^{th} Jordan block of the matrix L of the exponents of formal monodromy) and $k^{j;\bullet}_{(\omega)} = 0$ when $a_j \neq \omega$,*
- $R^{j;\bullet}_{\lambda_\ell;(\omega)}(X)$ *denotes a polynomial matrix with* summable-resurgent coefficients *in $\widehat{Res}^{\mathrm{sum}}_{\Omega-\omega}$, the columns of which are of log-degree $N[\ell]$ (cf. Section 3.2, Formula (27)).*

Recall that the major $\breve{f}^{j;\bullet}$, the constant matrix $k^{j;\bullet}_{(\omega)}$ and the coefficients of the remainder $R^{j;\bullet}_{\lambda_\ell;(\omega)}$ depend on γ even though, for seek of simplicity, we do not show it up in the notations. Note that, in practice, $k^{j;\bullet}_{(\omega)}$ can be determined as the coefficient of the monomial $\xi^{\lambda_j - 1}$.

It is worth to make explicit the following two particular cases.

- **Case with diagonal formal monodromy: $L = \oplus_{j=1}^{n} \lambda_j$**
  In this case, $\breve{f}^{j;\bullet}$ reduces to just one entry which we denote $\breve{f}^{j}$.

$$\boxed{\breve{f}^{j}(\omega+\xi) = \frac{k^{j}_{(\omega)}}{\xi} + h^{j}_{(\omega)}(\xi)\, \ln \xi + \sum_{\lambda_\ell \neq 0 \,|\, a_\ell = \omega} \xi^{\lambda_\ell - 1}\, h^{j}_{\lambda_\ell;(\omega)}(\xi)}$$

  where $k^{j}_{(\omega)}$ is a constant, $h^{j}_{(\omega)}$ and all $h^{j}_{\lambda_\ell;(\omega)}$ are summable-resurgent functions of $\widehat{Res}^{\mathrm{sum}}_{\Omega-\omega}$. Moreover,
  - $k^{j}_{(\omega)} = 0$ when $a_j \neq \omega$ or $\lambda_j \neq 0$,
  - $h^{j}_{(\omega)} = 0$ when there exists no ℓ such that $a_\ell = \omega$ and $\lambda_\ell = 0$.
- **Case with trivial formal monodromy: $L = O_n$**

$$\boxed{\check{f}^j(\omega+\xi) = \frac{k^j_{(\omega)}}{\xi} + h^j_{(\omega)}(\xi)\,\ln\xi}$$

where $k^j_{(\omega)}$ is a constant and $h^j_{(\omega)}$ a summable-resurgent function of $\widehat{Res}^{\rm sum}_{\Omega-\omega}$. Moreover,

- $k^j_{(\omega)} = 0$ when $a_j \neq \omega$,
- $h^j_{(\omega)} = 0$ when there exists no ℓ such that $a_\ell = \omega$.

Thus, under the condition that there exists no $\lambda_\ell \neq 0$ associated with $a_\ell = \omega$, the singularity $\overset{\triangledown}{f}_\omega$ of $\widehat{f}$ at ω is simple in the restrictive sense (*cf.* Definition 3.1 and its footnote).

Proof. — Prove first the weaker result that asserts that the singularity of $\widehat{f}$ at ω has the given form with holomorphic coefficients at 0.
Given $0 < \nu < \nu_1$ and $\varepsilon > 0$, we consider the domains

$$\Delta_\nu = \{\nu < |\xi| < \nu_1\} \text{ in } \mathbb{C}$$

$$\text{and} \quad \Delta'_\nu = \{\nu < |\xi| < \nu_1,\ \theta - \varepsilon - 2\pi N' < \arg(\xi) < \theta + \varepsilon + 2\pi\}$$

a lift of the ring Δ_ν to $N'+1$ consecutive sheets of the Riemann surface of the logarithm at 0. The number N' will be determined later; it has to be finite and large enough. The argument θ fixes the sheet on which $\widehat{f}$ is studied.
We assume ν_1 so small that the disc $|\xi| \leqslant \nu_1$ lies at a distance at least ν from $(\Omega-\omega)\backslash\{0\}$.

We fix j. From Proposition 3.5 we can write

$$\mathrm{cont}_\gamma \varphi^{j,\bullet}_m(\omega+\xi) = \sum_{\lambda\notin\mathbb{Z}} \sum_{p=0}^{p_\lambda} h^{\lambda,p}_m(\xi)\,\xi^\lambda\,\ln^p\xi + \frac{1}{\xi}\sum_{p=0}^{p_0} h^{0,p}_m(\xi)\,\ln^p\xi$$

where the pairs (λ,p) are supposed distinct modulo $\mathbb{Z}$ and finitely many. Denote Λ the set of all exponents λ appearing in these summations. Elements in Λ are equal to either a λ_ℓ or $\lambda_j - 1$ when $\lambda_j \notin \mathbb{Z}$ and $a_j = \omega$. When $\lambda_j = 0$ and $a_j = \omega$, instead of introducing $\lambda = \lambda_j - 1 = -1$ we factor $\dfrac{1}{\xi}$ so that no polar part occurs in the coefficients $h^{0,p}_m(\xi)$. We denote p_λ the highest log-degree p associated with ξ^λ.
From Proposition 3.5 we know that all coefficients $h^{\lambda,p}_m(\xi)$ are holomorphic at 0 and we have to prove that $\widehat{f}(\omega+\xi)$ has the same form as the φ_m's

and holomorphic coefficients at 0. The proof mostly relies on the uniform convergence of the series $\sum_{m\geqslant 1}\varphi_m$ to $\widehat{f}-\delta I_{n,n_1}$ on any compact set avoiding Ω (*cf.* Remark 2.10). For simplicity, we skip writing "cont_γ" although we consider analytic continuations along γ.

Suppose, to begin, that all majors $\breve{\varphi}_m^{j,\bullet}$ are equal to 0 which means that all $\varphi_m^{j,\bullet}(\omega+\xi)$ are holomorphic functions $h_m^{j,\bullet}(\xi)$ at 0 and then, all holomorphic on the same disc $|\xi|<\nu_1$ previously chosen. Since the series $\sum_{m\geqslant 0}\varphi_m^{j,\bullet}(\omega+\xi)=\sum_{m\geqslant 0}h_m^{j,\bullet}(\xi)$ converges uniformly to $\widehat{f}^{j,\bullet}(\omega+\xi)$ on compact sets avoiding $\Omega-\omega$ (Theorem 2.7, Remark 2.10) the function $\widehat{f}^{j,\bullet}(\omega+\xi)=\sum_{m\geqslant 0}h_m^{j,\bullet}(\xi)$ is a holomorphic function on the punctured disc $0<|\xi|<\nu_1$. Its Laurent series is the sum of the Laurent series of the $h_m^{j,\bullet}(\xi)$'s; hence, it displays no polar part and $\widehat{f}^{j,\bullet}(\omega+\xi)$ can be continued into a holomorphic function $h^{j,\bullet}(\xi)$ on the disc $|\xi|<\nu_1$. We can conclude, in this case, that a major of $\widehat{f}$ is also equal to zero: $\breve{f}^{j,\bullet}(\omega+\xi)=0$.

Given (λ,p), to prove that the series $\sum_{m\geqslant 0}h_m^{\lambda,p}(\xi)$ converges to a holomorphic function $h^{\lambda,p}(\xi)$ about ω we proceed as above after having reduced all $\varphi_m^{j,\bullet}(\omega+\xi)$ to $h_m^{\lambda,p}(\xi)$ by means of iterated variations as indicated below. We can then conclude by the same arguments as above since the uniform convergence property keeps valid for the variations as well.

We base the reduction on the properties of the variation stated in Lemma 3.6: the variation of $\ln^p\xi$ iterated p times produces a non-zero constant and it produces 0 in one more step; the iterated variation of $\xi^\lambda\ln^p\xi$ when $\lambda\notin\mathbb{Z}$ generates a dominant term of the same form times a non zero constant.

Fix $\lambda\in\Lambda$.

It is sufficient to consider the case of the monomials $h_m^{\lambda,p_\lambda}(\xi)\,\xi^\lambda\ln^{p_\lambda}\xi$ of highest log-degree. Indeed, we can then proceed iteratively on the descending log-degrees terms after cancellation of the highest log-degree terms.

Here is a possible way to reduce $\varphi_m^{j,\bullet}(\omega+\xi)$ to $h_m^{\lambda,p_\lambda}(\xi)$:

For each $\lambda'\in\Lambda$, $\lambda'\neq\lambda$ successively, multiply by $\xi^{-\lambda'}$, take the variation $p_{\lambda'}+1$ times and multiply by $\xi^{\lambda'}$, thus canceling all terms factored by $\xi^{\lambda'}$. We are left with only terms in $\xi^\lambda\ln^p\xi$, for $p=0,1,\dots,p_\lambda$. Multiply by $\xi^{-\lambda}$ and take the variation p_λ times. We get so $h_m^{\lambda,p_\lambda}(\xi)$ up to a non-zero constant.

We can now estimate a convenient value for N': the process, to be valid, requires that N' be as large as the total number of variations used.

This ends the proof of the fact that the singularity of $\mathrm{cont}_\gamma\widehat{f}$ at ω has the given form with holomorphic coefficients at 0. In particular, the constant matrices $k_{(\omega)}^{j;\bullet}$ are given by $k_{(\omega)}^{j;\bullet}=\sum_{m\geqslant 1}k_{m:(\omega)}^{j;\bullet}$.

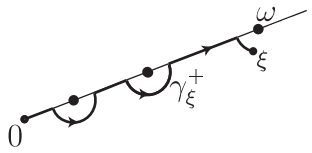
— The proof of the fact that the coefficients $h^{\lambda,p}_{(\omega)}(\xi)$ are actually summable-resurgent functions in $\widehat{\mathcal{R}es}^{\,\mathrm{sum}}_{\Omega-\omega}$ is straightforward from the fact that their germs at the origin are equal to iterated variations of functions themselves in $\widehat{\mathcal{R}es}^{\,\mathrm{sum}}_{\Omega-\omega}$. l

3.4 Principal major and connection constants

Let $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ be an anti-Stokes direction and $\omega \in \Omega_\theta$ a Stokes value in direction θ associated with $\widetilde{f}(x)$.

The constants $k^{j;\bullet}_{(\omega)}$ and the polynomials $R^{j;\bullet}_{\lambda_\ell,(\omega)}$ found in Theorem 3.7 depend, as already said, on the path of analytic continuation γ_ξ and meanwhile, on the chosen determination of the argument around ω.

• We consider a path γ^+ from 0 towards ω defined as follows: $\gamma^+ = \gamma^+_\xi$ goes along the straight line $[0,\omega]$ from 0 towards ω and bypasses all intermediate singular points $\omega' \in \Omega_\theta \cap]0,\omega]$ to the right as shown on the figure.



• We choose the principal determination of the variable ξ around ω[4].

Theorem 3.7 allows to set the following definition.

Definition 3.8 Principal major
We call *principal major of* $\widehat{f}$ at $\omega \in \Omega_\theta$ the major $\breve{f}^{+}(\omega+\xi)$ of $\mathrm{cont}_{\gamma^+}(\widehat{f})$ when the principal determination of the variable ξ is chosen.
According to Theorem 3.7, for $j = 1,\dots,J$ and convenient constant matrices $k^{+j;\bullet}_{(\omega^\star)}$ it reads

$$\breve{f}^{+j;\bullet}(\omega+\xi) = \xi^{\lambda_j-1}\,\xi^{J_{n_j}}\,k^{+j;\bullet}_{(\omega^\star)}\,\xi^{-J_{n_1}} + \sum_{\lambda_\ell | a_\ell=\omega} \xi^{\lambda_\ell}\,R^{+j;\bullet}_{\lambda_\ell;(\omega^\star)}(\ln\xi) \tag{28}$$

Moreover, in case there exists ℓ such that $\lambda_\ell = 0$ and $a_\ell = \omega$, we assume that $R^{+j;\bullet}_{0;(\omega^\star)}(0) = 0$ so that the principal major never contains a holomorphic term.

[4] Any choice is convenient. However, to be compatible, on the Riemann sphere, with the usual choice $0 \leqslant \arg(\zeta = 1/\xi) < 2\pi$ of the principal determination at infinity we suggest to choose $-2\pi < \arg(\xi) \leqslant 0$ as principal determination about 0 as well as about any ω at finite distance.

With the requested conditions the principal major of $\widehat{f}$ at ω is uniquely determined. We denote $k^{+}_{(\omega^\star)}$ with a dot to indicate the choice of the principal determination.

By means of elementary transformations (*cf.* Section 2.1) we can extend the previous results and definitions to any of the k^{th} column-blocks of $\widehat{F}(\xi)$ changing the Stokes values a_ℓ into $a_\ell - a_k$ (actually, with respect to $\widehat{f}(\xi)$, a_ℓ stands for $a_\ell - a_1$ with $a_1 = 0$) and the exponents of formal monodromy λ_ℓ into $\lambda_\ell - \lambda_k$. We can then reformulate Theorem 3.7 as follows.

Corollary 3.9 *Let $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ be an anti-Stokes direction and $\omega \in \mathbf{\Omega}_\theta$ a Stokes value in direction θ associated with System (1).*

To the choice of the principal determination of the variable ξ there is a unique constant matrix $K^{+}_{(\omega^\star)}$ such that the principal major $\breve{F}^{+}$ of $\widehat{F}$ at ω reads

$$\boxed{\breve{F}^{+}(\omega+\xi) = \frac{1}{\xi}\,\xi^{L}\,K^{+}_{(\omega^\star)}\,\xi^{-L} + \mathrm{Rem}^{+}(\ln\xi)} \tag{29}$$

The matrix $K^{+}_{(\omega^\star)}$ is a constant $n\times n$-matrix satisfying $K^{+;j;k}_{(\omega^\star)} = 0$ for all $(j;k)$ such that $a_j - a_k \neq \omega$. In particular, all diagonal blocks are equal to zero. The remainder $\mathrm{Rem}^{+}(\ln\xi)$ is a linear combination of polynomials in $\ln\xi$ with summable-resurgent coefficients weighted by subdominant complex powers of ξ.

Definition 3.10 Matrix of the connection constants
The $n \times n$-matrix

$$\boxed{K^{+}_{\theta^\star} = \sum_{\omega\in\mathbf{\Omega}_\theta} K^{+}_{(\omega^\star)}} \tag{30}$$

is called *(principal) matrix of the connection constants of $\widehat{F}$* in direction θ.

A $n_j \times n_k$-block $K^{+;j;k}_{\theta^\star}$ of $K^{+}_{\theta^\star}$ is equal to 0 when $a_j - a_k$ does not belong to $\mathbf{\Omega}_\theta$. The possibly non-zero entries of $K^{+}_{\theta^\star}$ are also called *principal multipliers of connection in direction θ.*

Note that we still need no other structure on $\widehat{\mathcal{R}es}_\Omega$ and $\mathcal{C}$ than the structure of $\mathcal{O}^{\leqslant 1}(\mathbb{C})*$module and, in particular, there is no need yet to develop a structure of a convolution algebra.

4 Meromorphic invariants: Stokes cocycle or alien derivatives

The classifying set for the local meromorphic classification at 0 of connections endowed with an isomorphism of their formalized (*i.e.*, formal gauge transformations $\widehat{F}$) in a given formal class was given by Y. Sibuya [27, 28] and B. Malgrange [16] in terms of a non-Abelian 1-cohomology set. Actually, this classifying set can be given a structure of a unipotent Lie group and it is isomorphic to the direct product of the Stokes-Ramis groups in each anti-Stokes direction associated with the connection (*cf.* D.G. Babbitt and V.S. Varadarajan [1] for an abstract proof and M. Loday-Richaud [13] for a constructive one). Indeed, in each 1-cohomology class there exists a unique special cocycle, called *Stokes cocycle*, whose components coincide with the Stokes-Ramis automorphisms independently defined as the defects of analyticity of $\widehat{F}$ in each anti-Stokes direction [23]; [13, Prop. III.2.1, Th. III.2.8].

To any formal class and associated anti-Stokes direction the Stokes-Ramis automorphisms form a free Lie group conjugate to a group of unipotent triangular matrices submitted to some vanishing conditions ([13] Def. I.4.12). Its Lie algebra is conjugate to an algebra of nilpotent matrices submitted to similar vanishing conditions and the exponential map sends it homeomorphically onto the Stokes-Ramis group. It is then equivalent to characterize a meromorphic class by giving its image in the Stokes-Ramis groups (*i.e.*, *its Stokes cocycle* or *its Stokes matrices* after the choice of a $\mathbb{C}$-basis of solutions) or its image in the Lie algebra, tangent space of the Stokes-Ramis groups at the identity (*i.e.*, *its alien derivatives*).

Here below, we perform these descriptions in more details and we provide an explicit formula for the Stokes matrices in the Laplace plane in terms of the connection matrices in the Borel plane.

Note. From now, each time a determination of the argument is required, we choose the principal determination fixed similarly in the Laplace and in the Borel plane (*cf.* Footnote 4). Given an anti-Stokes direction $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ we denote $\theta^\star \in \mathbb{R}$ the chosen determination of θ and $\omega^\star$ the Stokes value ω in direction θ with the same determination. We keep denoting the variables $x, \xi, \ldots$ while indicating $\arg(x) \simeq \theta^\star, \arg(\xi) \simeq \theta^\star, \ldots$.

4.1 Stokes cocycle

Let us start with a description of the Stokes-Ramis automorphisms, components of the Stokes cocycle, from the viewpoint of summation.

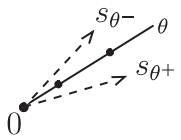
Recall (*cf.* Section 2.2) that the anti-Stokes directions attached to the first n_1 columns $\widetilde{f}$ of the gauge transformation $\widetilde{F}$ are defined as the directions of the various non-zero Stokes values of System (1), *i.e.*, the Stokes values belonging to $\Omega^* = \{a_1, \ldots, a_J\}\backslash\{0\}$ (see notations of Section 2.2). To the ℓ^{th} column-block of $\widetilde{F}$ the set Ω must be translated to $\Omega - a_\ell$. We keep denoting $\mathbf{\Omega} = \{a_j - a_\ell \neq 0\}$ the set of all non-zero Stokes values attached to $\widetilde{F}$.

Given $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ let d_θ denote the half line issuing from 0 with argument θ and set $\Omega_\theta = \Omega^* \cap d_\theta$ and $\mathbf{\Omega}_\theta = \mathbf{\Omega}^* \cap d_\theta$.

4.1.1 Stokes automorphisms as gauge transformations

When θ is not an anti-Stokes direction for $\widetilde{F}$ (*i.e.*, $\mathbf{\Omega}_\theta = \varnothing$) then $\widetilde{F}$ can be applied a Borel-Laplace integral $\displaystyle\int_{d_\theta} \widehat{F}(\xi)e^{-\xi/x}d\xi$ in direction θ (*cf.* Theorem 2.7, for instance) and in neighboring, not anti-Stokes, directions giving thus rise to an analytic function $s_\theta(\widetilde{F})$ defined and 1-Gevrey asymptotic to $\widetilde{F}$ on a sector $\Sigma_{\theta,>\pi}$ bisected by θ with opening larger than π. The function $s_\theta(\widetilde{F})$ is called *1-sum* or *Borel-Laplace sum* of $\widetilde{F}$ in direction θ.

When θ is an anti-Stokes direction for $\widetilde{F}$ the Borel-Laplace integral does not exist anymore in general. However, taking the limit as ε tends to 0 of the Borel-Laplace sums in directions $\theta - \varepsilon$ and $\theta + \varepsilon$ one defines, by analytic continuation, two analytic functions, 1-Gevrey asymptotic to $\widetilde{F}$ on a germ of half-plane $\Sigma_{\theta,\pi}$ bisected by θ.



We call *sum of* $\widetilde{F}$ *to the right* of θ and we denote $s_{\theta+}(\widetilde{F})$ the analytic continuation to $\Sigma_{\theta,\pi}$ of $s_{\theta-\varepsilon}(\widetilde{F})$ [5] as ε tends to 0. We call *sum of* $\widetilde{F}$ *to the left* of θ denoted $s_{\theta-}(\widetilde{F})$ the analytic continuation of $s_{\theta+\varepsilon}(\widetilde{F})$.

[5] The denominations *to the right* and *to the left* fit the natural orientation around 0 on the Riemann sphere. Our choice of the signs + and − in $s_{\theta+}$ and $s_{\theta-}$ may look inappropriate to such an orientation but our will is to fit the usual notations at infinity. Indeed, positioning the singularity at 0 or at infinity exchanges the orientation on the Riemann sphere. The Borel transform at 0 does not exchange the orientation while it does it at infinity. In both cases, we can refer to the fact that the sums $s_{\theta+}$ and $s_{\theta-}$ correspond to Laplace integrals in the Borel plane along a path following a line d_θ and passing the Stokes values positively for $s_{\theta+}$ and negatively for $s_{\theta-}$ (*cf.* Fig. Section 4.3).

The *Stokes phenomenon* stems from the fact that the two lateral sums $s_{\theta^+}(\widetilde{F})$ and $s_{\theta^-}(\widetilde{F})$ are not analytic continuations from each other in general. The defect of analyticity is quantified by the Stokes automorphisms

$$\boxed{St_{\theta;\widetilde{F}} = s_{\theta^-}(\widetilde{F})^{-1} \circ s_{\theta^+}(\widetilde{F})} \quad . \tag{31}$$

in each anti-Stokes direction θ. Thus defined, the Stokes automorphism of $\widetilde{F}$ in direction θ is an automorphism of the normal form (4); precisely, a gauge transformation, 1-Gevrey flat on $\Sigma_{\theta,\pi}$, which leaves invariant System (4). The Stokes automorphism $St_{\theta;\widetilde{F}}$ depends on $\widetilde{F}$ and θ and it does not depend on the choice of a determination of the argument near $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.

4.1.2 Matrix representations

One can give the Stokes automorphism $St_{\theta;\widetilde{F}}$ matrix representations in $GL(n,\mathbb{C})$ by associating with the (formal) normal solution $\widetilde{Y}_0(x) = x^L e^{Q(1/x)}$ an actual one near θ by means of a choice of a determination of the argument in the Laplace plane:

We change the formal power x^L into an actual function near θ by choosing the principal determination of the argument. We get thus an actual function, still denoted x^L, defined and analytic for $\arg(x)$ close to $\theta^\star$ (denote $\arg(x) \simeq \theta^\star$). In our case of a single level equal to one, the polynomials $q_j(1/x)$ in $Q(1/x)$ do not require any choice of a determination of the argument. However, a formal exponential determines an actual exponential only up to a multplicative constant[6]. We fix such a constant once for all, for instance, by choosing the function denoted $e^{Q(1/x)}$ in the usual analytic meaning. We denote $Y_{0,\theta^\star}(x)$ the fundamental solution such defined for $\arg(x) \simeq \theta^\star$.

To the actual normal solution $Y_{0,\theta^\star}(x)$, there correspond two analytic fundamental solutions $s_{\theta^-}(\widetilde{F})Y_{0,\theta^\star}(x)$ and $s_{\theta^+}(\widetilde{F})Y_{0,\theta^\star}(x)$ of System (1). There exists then a unique constant invertible matrix (called *Stokes-Ramis matrix* or simply *Stokes matrix*[7]), which we denote $I_n + C_{\theta^\star}$, such that

$$s_{\theta^+}(\widetilde{F})\, Y_{0,\theta^\star}(x) = s_{\theta^-}(\widetilde{F})\, Y_{0,\theta^\star}(x)(I_n + C_{\theta^\star}) \tag{32}$$

[6] What is called formal exponential e^p is "the" formal solution of the differential equation $y' - p'y = 0$, which is well defined only up to a multiplicative scalar.

[7] In the literature, a *Stokes matrix* has often a more general meaning where one allows to compare any two asymptotic solutions whose domains of definition overlap. We exclude such a generality here.

and the Stokes automorphism $St_{\theta;\widetilde{F}}$ is given a matrix representation by the formula

$$\boxed{St_{\theta;\widetilde{F}} \equiv s_{\theta^-}(\widetilde{F})^{-1}\, s_{\theta^+}(\widetilde{F}) = x^L e^{Q(1/x)}(I_n + C_{\theta^\star})e^{-Q(1/x)}x^{-L} \quad \text{for } \arg(x) \simeq \theta^\star} \tag{33}$$

Thus, although the Stokes automorphisms, components of the Stokes cocycle, are intrinsically determined from $\widetilde{F}$, their matrix representations as constant invertible matrices are defined only up to the choice of an actual normal solution $Y_{0,\theta^\star}(x)$ in each anti-Stokes direction. As it is clear from Formula (33) this indetermination results in the conjugacy action of the formal monodromy (change of "formal determination" $x \mapsto xe^{2\pi i}$ in the normal solution) and of the exponential torus (see Section 4.4). Temporarily, we forget about the action of the exponential torus by systematically associating with a formal exponential e^q the analytic function denoted the same way (choice of the arbitrary constant equal to 1) so that, a (formal) normal solution being given, our unique freedom lies in the choice of the determination of the argument of the variable x in a neighborhood of the direction θ, *i.e.*, the choice of a determination $\theta^\star$ of θ.

The Stokes automorphism $St_{\theta;\widetilde{F}}$ is unipotent (*i.e.*, its matrix $C_{\theta^\star}$ is nilpotent whatever the choice of a determination of $\theta^\star$) due to the fact that $s_{\theta^-}(\widetilde{F})$ and $s_{\theta^+}(\widetilde{F})$ are both 1-Gevrey asymptotic to the same matrix $\widetilde{F}$. Indeed, a $n_j \times n_\ell$ block $C^{j,\ell}_{\theta^\star}$ is 0 as soon as $a_j - a_\ell \notin \Omega_\theta$. In particular, the diagonal blocks are equal to 0 and one can put $C_{\theta^\star}$ in triangular form by conveniently reordering the Stokes values a_j (*cf.* [13, Consequence I.4.8]).

Formula (33) has an "additive" form

$$\boxed{s_{\theta^+}(\widetilde{F})(x) - s_{\theta^-}(\widetilde{F})(x) = s_{\theta^-}(\widetilde{F})(x)\, x^L\, e^{Q(1/x)}\, C_{\theta^\star}\, e^{-Q(1/x)}\, x^{-L} \quad \text{for } \arg(x) \simeq \theta^\star} \tag{34}$$

which will be used later.

4.1.3 Stokes automorphims acting on formal solutions

The Stokes-Ramis matrix $I_n + C_{\theta^\star}$ was introduced in Formula (32) as the matrix of a linear map

$$s_{\theta^-}(\widetilde{F})\, Y_{0,\theta^\star}(x) \mapsto s_{\theta^+}(\widetilde{F})\, Y_{0,\theta^\star}(x)$$

associating sums to the right with sums to the left in direction $\theta^\star$. The map is defined and bijective in the space of actual solutions of System (1) over a germ of half plane $\Sigma_{\theta,\pi}$.

Coming back from the actual normal solution $Y_{0,\theta^\star}(x)$ to the formal one $\widetilde{Y}_0(x) = x^L e^{Q(x)}$ the previous map can be read as a bijective linear map

$$St_{\theta^\star}: \quad \widetilde{F}(x)\, x^L\, e^{Q(1/x)} \mapsto \widetilde{F}(x)\, x^L\, e^{Q(1/x)}(I_n + C_{\theta^\star}) \tag{35}$$

in the space of formal solutions of System (1). Such a map depends on $\theta^\star$ and no more only on θ.

One also calls the map $St_{\theta^\star}$ a *Stokes automorphism* in direction $\theta^\star$. The Stokes cocycle of System (1) is equivalent to the collection of the Stokes maps $St_{\theta^\star}$ for $\theta^\star$ running over arguments of the anti-Stokes directions in a fundamental domain, say the principal one $-2\pi < \arg \leqslant 0$.

Note that, to the first column-block $\widetilde{f}$ made of formal power series (recall $a_1 = \lambda_1 = 0$), there does not correspond a power series in general. In Section 4.4, in order to define alien derivations, we will extend such a map into an automorphism of a differential algebra containing the formal solutions of System (1).

4.2 Maps $\dot{\Delta}^+_{\omega^\star}$

Given $\omega \in \mathbf{\Omega}^*$ recall that $\omega^\star$ denotes its principal determination with argument $\theta^\star$.

The Stokes automorphism $St_{\theta^\star}$ can be split into a sum of linear maps $\dot{\Delta}^+_{\omega^\star}$, each of them taking into account the contribution of a different Stokes value $\omega \in \mathbf{\Omega}^*$ as follows: their matrices, also denoted $\dot{\Delta}^+_{\omega^\star}$, are obtained from $C_{\theta^\star}$ by keeping unchanged the blocks $C^{j,\ell}_{\theta^\star}$ such that $a_j - a_\ell = \omega$ and equating all other blocks to 0.
Obviously, $C_{\theta^\star} = \sum_{\omega \in \mathbf{\Omega}_\theta} \dot{\Delta}^+_{\omega^\star}$ and Formula (34) reads, for $\arg(x) \simeq \theta^\star$,

$$\boxed{s_{\theta^+}(\widetilde{F})(x) - s_{\theta^-}(\widetilde{F})(x) = s_{\theta^-}(\widetilde{F})(x) \sum_{\omega \in \mathbf{\Omega}_\theta} e^{-\omega/x}\, x^L\ \dot{\Delta}^+_{\omega^\star}\ x^{-L}} \tag{36}$$

In restriction to the first n_1 columns and denoting $\dot{\delta}^+_{\omega^\star}$ the first n_1 columns

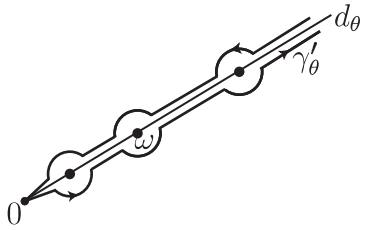
of $\dot{\Delta}^{+}_{\omega^\star}$, we obtain, for $\arg(x) \simeq \theta^\star$,

$$\boxed{s_{\theta^+}(\widetilde{f})(x) - s_{\theta^-}(\widetilde{f})(x) = s_{\theta^-}(\widetilde{F})(x) \sum_{\omega\in\mathbf{\Omega}_\theta} e^{-\omega/x}\, x^L\ \dot{\delta}^{+}_{\omega^\star}\ x^{-J_1}} \quad (37)$$

4.3 Stokes-Ramis versus connection matrices

The left hand side of Formula (37) can be seen as the Laplace integral

$$s_{\theta^+}(\widetilde{f})(x) - s_{\theta^-}(\widetilde{f})(x) = \int_{\gamma'_\theta} \widehat{f}(\zeta) e^{-\zeta/x} d\zeta \quad (38)$$



The two branches of γ'_θ along d_θ are disjointly drawn to point out that they do not belong to the same sheet of the Riemann surface $\mathcal{R}_\Omega$.

where, as shown on the figure, γ'_θ is a Hankel type path going along the straight line d_θ from infinity to 0 and back to infinity passing positively all singularities $\omega \in \Omega_\theta$ on both ways.

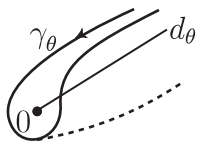
The exponential growth of $\widehat{f}$ in direction θ guaranties the convergence of the integral for x in a disc adherent to 0 with d_θ as a diameter (a Borel disc) (*cf.* [21] for example).

Without changing the value of the integral[8] the path γ'_θ can be deformed into the union $\gamma'_\theta = \cup_{\omega\in\Omega_\theta}\gamma'_\theta(\omega)$ of Hankel type paths $\gamma'_\theta(\omega)$ with asymptotic direction θ around each singular point $\omega \in \Omega_\theta$.

By means of translations from ω to 0 and using the fact that holomorphic functions at ω contributes 0 to the integral around ω we can replace $\widehat{f}$ by its principal majors $\breve{f}^+(\omega+\xi)$ at each ω obtaining so

[8] Contrarily to Formula (37) which only requires the 1-summabilty of the series $\widetilde{f}$, the individual resurgence and 1-summability are not sufficient here. We do need summable-resurgence.

$$(39)\quad s_{\theta^+}(\widetilde{f}) - s_{\theta^-}(\widetilde{f}) = \sum_{\omega\in\Omega_\theta} e^{-\omega/x} \int_{\gamma_\theta} \breve{f}^+(\omega+\xi)e^{-\xi/x}d\xi$$



where, as shown on the figure, γ_θ is a Hankel type path around 0 in direction θ.

We claim that we can identify this linear combination of exponentials $e^{-\omega/x}$ to the linear combination in the right hand side of Formula (37).

Proposition 4.1 *Given $\omega \in \Omega_\theta$, the following identity holds for $\arg(x) \simeq \theta^\star$:*

$$(40)\qquad \int_{\gamma_\theta} \breve{f}^+(\omega+\xi)e^{-\xi/x}d\xi = s_{\theta^-}(\widetilde{F})(x)\, x^L\ \dot{\delta}^{+}_{\omega^\star}\ x^{-J_1}.$$

Proof. The equality is obviously true when there is only one singular point ω on the half line d_θ. Assume then that there are at least two singular points on d_θ.
The Borel transform $\widehat{f}(\xi)$ of $\widetilde{f}(x)$ belongs to $\widehat{\mathcal{R}es}^{\,\mathrm{sum}}_{\Omega}$ and has simple moderate singularities. From Theorem 3.7 we know that any major $\breve{f}(\omega+\xi)$ at ω is a polynomial in $\ln\xi$ with summable-resurgent coefficients, possibly factored by complex powers $\xi^{\lambda-1}$. The Laplace integral $\int_{\gamma_\theta} \breve{f}^+(\omega+\xi)e^{-\xi/x}d\xi$ in the left hand side is then a polynomial in $\ln x$ whose coefficients are 1-Gevrey asymptotic functions on a germ of half plane bisected by the direction θ. Deforming the path γ_θ so as to move the asymptotic direction to a neighboring direction $\theta+\varepsilon$ we get 1-Gevrey asymptotic coefficients on a sector of opening $]\theta-\frac{\pi}{2}, \theta+\frac{\pi}{2}+\varepsilon[$, that is, 1-sums of 1-summable series in direction $\theta+\dfrac{\varepsilon}{2}$ (integer power series, possibly factored by non integer powers x^λ).

The same property holds for the right hand side and therefore, we can conclude by the variant of Watson Lemma below. 1

Lemma 4.2 (A variant of Watson Lemma)

Let $\theta \in \mathbb{R}/2\pi\mathbb{Z}$.

Suppose that, to each $\omega \in \Omega_\theta$, there is a linear combination $P_\omega(\ln x)$ of polynomials in $\ln x$ with summable-resurgent coefficients in $\widetilde{\mathcal{R}es}^{\,\mathrm{sum}}_{\Omega-\omega}$ possibly weighted by complex powers of x, the powers being distinct modulo $\mathbb{Z}$.

Then, an identity $\sum_{\omega\in\Omega_\theta} P_\omega(\ln x)\, e^{-\omega/x} \equiv 0$ implies the nullity of each P_ω, i.e., the nullity of each summable-resurgent coefficient in P_ω.

PROOF. By means of a rotation of the variable x we can assume that $\theta = 0$ so that all ω are positive numbers. We range them by increasing order: $\omega_1 < \omega_2 < \cdots < \omega_r$. If we factor $e^{-\omega_1/x}$ the identity becomes

$$P_{\omega_1}(\ln\, x) + \sum_{\ell=2}^{r} P_{\omega_\ell}(\ln\, x) e^{-(\omega_\ell - \omega_1)/x} \equiv 0.$$

Taking the asymptotic expansion at 0 on the positive real line of both sides of the identity proves that the asymptotic expansion of $P_{\omega_1}(\ln\, x)$ at 0 is 0. The same property holds for the coefficients of each power of $\ln\, x$. These coefficients are 1-summable powers series in which the sequence of exponents is a finite union of arithmetic sequences. Watson Lemma applies to such a situation (*cf.* [19, Thm 2.4.1.4.ii]) implying the unicity of the 1-sum. Hence, $P_{\omega_1} \equiv 0$. The nullity of the other polynomials P_ω follows by recursion. l

Theorem 4.3 Connection-to-Stokes Formula
Given $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ an anti-Stokes direction, the data of the Stokes-Ramis matrix $C_{\theta^\star}$ and of the connection matrix $K^+_{\theta^\star}$ are equivalent and the two matrices are related by the relation

$$\boxed{C_{\theta^\star} = \int_{\gamma_0} \frac{1}{\xi}\, \xi^L\, K^+_{\theta^\star}\, \xi^{-L}\, e^{-\xi} d\xi}$$

where γ_0 is a Hankel type path around $\mathbb{R}^+$ run over by ξ after the choice of the principal determination of its argument.

PROOF. Note that $C_{\theta^\star}$ and $K^+_{\theta^\star}$ have the same block-structure with the same blocks of zeros and same arbitrary non-zero blocks. The map $K^+_{\theta^\star} \mapsto C_{\theta^\star}$ is a linear map between two linear spaces of same dimension and to prove that it is bijective it is enough to prove that it is injective.

Suppose $C_{\theta^\star} = 0$; then, the two lateral sums $s_{\theta^+}(\widetilde{F})$ and $s_{\theta^-}(\widetilde{F})$ glue together and the Borel transform $\widehat{F}(\xi)$ of $\widetilde{F}(x)$ has no singularity on the line d_θ. Hence, $K^+_{\theta^\star} = 0$.

To prove the relation linking $C_{\theta^\star}$ to $K^+_{\theta^\star}$ it is sufficient to compare the first column-blocks $c_{\theta^\star}$ and $k^+_{\theta^\star}$ (*cf.* Section 2.1) and to prove that

$$c_{\theta^\star} = \int_{\gamma_0} \frac{1}{\xi}\, \xi^L\, k^+_{\theta^\star}\, \xi^{-J_1}\, e^{-\xi} d\xi$$

with the choice of the principal determination of argument along γ_0.
Equating the dominant terms in Identity (40) and using Theorem 3.7 we obtain the new identity

$$\begin{aligned} x^L\, c_{\theta^\star}\, x^{-J_1} &= \int_{\gamma_\theta} u^L\, k^+_{\theta^\star}\, u^{-J_1} e^{-u/x} du \\ &= x^L \int_{\gamma_0} \xi^L\, k^+_{\theta^\star}\, \xi^{-J_1} e^{-\xi} d\xi\, x^{-J_1} \quad (\text{setting } u = x\xi) \end{aligned}$$

where for $\arg x = \theta^\star$, the path of integration γ_θ has become the classical Hankel path γ_0 around the non-negative real axis with argument from -2π to 0. Hence the result. l

In restriction to the blocks attached to a given ω we obtain the following statement.

Corollary 4.4 *For all $\omega \in \mathbf{\Omega}^*$, the data of $\dot{\Delta}^+_{\omega^\star}$ and of $K^+_{(\omega^\star)}$ are equivalent and the two matrices are related by*

$$\boxed{\dot{\Delta}^+_{\omega^\star} = \int_{\gamma_0} \frac{1}{\xi}\, \xi^L\, K^+_{(\omega^\star)}\, \xi^{-L}\, e^{-\xi} d\xi}$$

where γ_0 stands for a Hankel type path around $\mathbb{R}^+$ run over by ξ after the choice of the principal determination of its argument.

It can be useful, especially for effective numerical calculations to expand each entry of the Connection-to-Stokes Formula. The following corollary provides such an expanded form.
Let $M^{(j,\ell);(k,r)}$ denote the entry row ℓ of row-block j and column r in column-block k of a $n \times n$-matrix M split into blocks fitting the structure of L.

Corollary 4.5 *The various entries of the Connection-to-Stokes Formula read*

$$\boxed{C^{(j,\ell);(k,r)}_{\theta^\star} = \sum_{p=0}^{n_j-\ell+r-1} \kappa_p(\lambda_j - \lambda_k)\, H^{(j,\ell);(k,r)}_p}$$

where $\kappa_p(\lambda_j - \lambda_k) = 2\pi i \dfrac{d^p}{dt^p}\Big(\dfrac{e^{-i\pi t}}{\Gamma(1-t)}\Big)\Big|_{t=\lambda_j-\lambda_k}$ *and*

$$H^{(j,\ell);(k,r)}_p = \sum_{\ell'-r'=p-r+1} (-1)^{r-1-r'} \frac{K^{+\,(j,\ell+\ell');(k,r'+1)}_\theta}{(r-1-r')!\,\ell'!}.$$

4.4 Alien derivations

As already said in the introduction of Section 4, the classifying set of meromorphic classes of formal gauge transformations of a given normal system is naturally endowed with a structure of a Lie group. In this section, we explain how the dotted alien derivations can be defined as infinitesimal generators of this Lie group.

Alien derivations were first given a definition through the Borel plane by J. Écalle ([10, 12], *cf.* also [18, 26]). They have been defined as an average of analytic continuations in the Borel plane followed by a Laplace transform, the weights of the averaging being related to the Campbell-Hausdorf Formula. It is equivalent to see them as the homogeneous components of the logarithm of graded Stokes automorphisms (*cf.* [10], [18, Th I.6.2], [26, Lemma 5 and pp. 35-38]). This is the viewpoint we adopt here, a viewpoint already used in [24, 21]. To perform it we need

- to define an algebra where the Stokes automorphisms make sense and where the alien derivatives will live (to give them a chance of being "derivatives" we cannot keep working in vector spaces),
- to define a graduation on the Stokes automorphisms.

Our aim being to define alien derivations of the solutions of System (1) we proceed as follows.

4.4.1 The algebra

Consider the finitely many Stokes values $\Omega = \{a_1, \dots, a_J\}$ associated with System (1) and the $\mathbb{Z}$-module $\mathbb{Z}\Omega$ they generate. The lattice $\mathbb{Z}\Omega$ may be dense in $\mathbb{C}$ (*cf.* the example of the hypergeometric equation $D_{13,1}$ below) and it contains $\mathbf{\Omega} = \{a_j - a_\ell\}$. Simultaneously, consider the set $\Lambda = \{\lambda_1, \dots, \lambda_J\}$ of exponents of formal monodromy of System (1) and the lattice $\mathbb{Z}\Lambda$ it generates.

We introduce the set $\widetilde{\mathcal{H}}_{\mathbb{Z}\Lambda,\mathbb{Z}\Omega}$ of all series solutions of linear differential equations with meromorphic coefficients, single level 0 or 1, exponents of formal monodromy in $\mathbb{Z}\Lambda$ and Stokes values in $\mathbb{Z}\Omega$. Note that all (convergent) meromorphic series at 0 belong to $\widetilde{\mathcal{H}}_{\mathbb{Z}\Lambda,\mathbb{Z}\Omega}$.

Proposition 4.6 *The set $\widetilde{\mathcal{H}} = \widetilde{\mathcal{H}}_{\mathbb{Z}\Lambda,\mathbb{Z}\Omega}$ is a differential sub-algebra of the algebra of all formal meromorphic series at $x = 0$.*

Consequently, its counter-part $\widehat{\mathcal{H}}$ in the Borel plane is a convolution algebra where multiplication by $1/\xi$ is the derivation; its elements have simple-moderate singularities at finitely many points of $\mathbb{Z}\Omega$.

PROOF. The set $\widetilde{\mathcal{H}}$ is stable with respect to sums and products. Indeed, given two series solutions of linear differential equations $D_1 y = 0$ and $D_2 y = 0$ respectively, their sum satisfies a linear equation $Dy = 0$ where D is the left smaller common multiple of D_1 and D_2, the exponents of formal monodromy and the Stokes values being the union of those of the initial equations. Their product satisfies a linear differential equation $\Delta y = 0$ where Δ is the symmetric tensor product of D_1 and D_2 and where the exponents of formal monodromy and the Stokes values are the two by two sums of those of the initial equations.
The set $\widetilde{\mathcal{H}}$ is stable under derivation with respect to x since the derivative of an element $\widetilde{f}$ of $\widetilde{\mathcal{H}}$ satisfies an equation of the same type obtained by conveniently derivating a differential equation satisfied by $\widetilde{f}$. 1

To prove directly in the Borel plane that $\widehat{\mathcal{H}}$ is a convolution algebra is not so easy. Given $\widehat{f}$ and $\widehat{g}$ in $\widehat{\mathcal{H}}$ with finite singular supports $\Omega(\widehat{f})$ and $\Omega(\widehat{g})$ respectively, the main point is to prove that the convolution $\widehat{f} * \widehat{g}$ which is well defined near 0 can be continued to the whole Riemann surface $\mathcal{R}_{\Omega(\widehat{f})+\Omega(\widehat{g})}$. To this end, one could generalize the technique of two intertwined combs to build $\mathcal{R}$-symmetrically contractile paths (*cf.* [26, Lemma 3, Figures 5 and 10]): one should start here with a comb with upwards nails at $\mathbf{\Omega}(\widehat{f})$, a comb with downwards nails at $-\mathbf{\Omega}(\widehat{g})$, symmetric of $\mathbf{\Omega}(\widehat{g})$ with respect to 0, and an elastic rope tied to the two nails 0 to materialize the convenient paths of analytic continuation when one moves the second comb. We won't formalize such a proof. Exponential growth at infinity and simple-moderate singularities are preserved by convolution.

In order to define alien derivatives of the entries of the gauge transformation $\widetilde{F}(x)$ we limit ourselves to consider the differential sub-algebra $\widetilde{\mathcal{H}}_{\widetilde{F}}$ of $\widetilde{\mathcal{H}}$ generated by the entries of $\widetilde{F}(x)$. Formula (35) as noticed at the end of Section 4.1.3 shows that the Stokes automorphism St_θ does not act inside $\widetilde{\mathcal{H}}_{\widetilde{F}}$ since the image of a formal series may involve complex powers of x, logarithms and exponentials. It is thus natural to extend the differential algebra

$\widetilde{\mathcal{H}}_{\tilde{F}}$ into the polynomial algebra of "resurgent symbols"[9]

$$\widetilde{\mathcal{H}}_{\tilde{F}\tilde{Y}_0} = \widetilde{\mathcal{H}}_{\tilde{F}}\left[(x^\lambda)_{\lambda\in\mathbb{Z}\Lambda}, \ln x, (e^{-\omega/x})_{\omega\in\mathbb{Z}\Omega}\right].$$

This is the differential algebra we are willing to work in. The coefficients are formal 1-summable series, x^λ, $\ln x$, $e^{-\omega/x}$ are formal indeterminates satisfying the usual rules. The derivation is d/dx.

4.4.2 Extended Stokes automorphisms

Let $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ be an anti-Stokes direction associated with System (1) and $\theta^\star \in \mathbb{R}$ its principal determination.

In Section 4.1.3, we described the Stokes automorphism as a map acting on formal solutions by means of a choice of a determination of the argument $\theta^\star$. Such a definition can be extended into an automorphism of the differential algebra $\widetilde{\mathcal{H}}_{\tilde{F}\tilde{Y}_0}$ as follows.

Proposition 4.7 (Extended Stokes automorphism)
The Stokes automorphism $St_{\theta^\star}$ can be extended into a differential unipotent automorphism, still denoted $St_{\theta^\star}$, of the differential algebra $\widetilde{\mathcal{H}}_{\tilde{F}\tilde{Y}_0}$ by setting:

- *power factors x^λ, logarithms and exponentials are kept fixed,*
- *$\widetilde{F}$ is changed into $\widetilde{F}\, x^L\, e^{Q(1/x)}(I_n + C_{\theta^\star})\, e^{-Q(1/x)}\, x^{-L}$.*

Proof. Any resurgent symbol of $\widetilde{\mathcal{H}}_{\tilde{F}\tilde{Y}_0}$ has a unique expression of the form $\sum \widetilde{g}_j\, x^{\lambda_j}\, \ln^{p_j} x\, e^{-\omega_j/x}$ the sum running on finitely many distinct triples $(\lambda_j, p_j, \omega_j) \in \mathbb{Z}\Lambda \times \mathbb{N} \times \mathbb{Z}\Omega$ if we assume, in addition, that $0 \leqslant \Re\lambda_j < 1$ for all j (and $\widetilde{g}_j \neq 0$). It can be isomorphically sent to the actual resurgent symbol $\sum s_{\theta^-}(\widetilde{g}_j)(x)\, x^{\lambda_j}\, \ln^{p_j}(x)\, e^{-\omega_j/x}$ for $\arg x \in \left]\theta^\star - \frac{\pi}{2}, \theta^\star + \frac{\pi}{2}\right[$. With this isomorphism the map $St_{\theta^\star}$ as defined in the proposition reads as the map

$$\sum s_{\theta^-}(\widetilde{g}_j)(x)\, x^{\lambda_j}\, \ln^{p_j}(x)\, e^{-\omega_j/x} \mapsto \sum s_{\theta^+}(\widetilde{g}_j)(x)\, x^{\lambda_j}\, \ln^{p_j}(x)\, e^{-\omega_j/x}$$

in the space of actual resurgent symbols of the form

$$\sum s_{\theta^-}(\widetilde{g}_j)(x)\, x^{\lambda_j} \ln^{p_j}(x) e^{-\omega_j/x} \text{ with } \widetilde{g}_j \in \widetilde{\mathcal{H}}_{\tilde{F}\tilde{Y}_0} \text{ and } \arg x \in \left]\theta^\star - \frac{\pi}{2}, \theta^\star + \frac{\pi}{2}\right[$$

[9] For non-linear situations one has to consider resurgent symbols with infinitely many exponential terms.

if one expands $s_{\theta^+}(\widetilde{g}_j)(x)$ according to Formula (37) (Since $\widetilde{g}_j$ is 1-summable it has a well-defined Stokes phenomenon and the result does not depend on the way $\widetilde{g}_j$ is expanded in terms of the entries of $\widetilde{F}$). Since summations s_{θ^-} and s_{θ^+} are automorphisms of differential algebras so is $St_{\theta^\star}$. The reciprocal map $St_{\theta^\star}^{-1}$ is obtained by keeping the same space of actual resurgent symbols while exchanging the roles of s_{θ^-} and s_{θ^+} and then the Stokes matrix $I_n + C_{\theta^\star}$ by its inverse $(I_n + C_{\theta^\star})^{-1}$.

The extended Stokes automorphism keeps being unipotent. Indeed, this results from the fact that it is already unipotent when acting in the space of formal solutions of System (1) (its matrix $I_n + C_{\theta^\star}$ is unipotent). This can also be seen as follows: if $\widetilde{F}_{j,k}$ is an entry of $\widetilde{F}$ then $St_\theta(\widetilde{F}_{j,k})(x) = \widetilde{F}_{j,k}(x) + \sum g_{\lambda,\ell,\omega}(x)\, x^\lambda \ln^\ell(x)\, e^{-\omega/x}$ where the sum runs on finitely many λ and ℓ and finitely many ω in $\mathbf{\Omega}_\theta$. The coefficients $g_{\lambda.\ell.\omega}$ are themselves elements of $\widetilde{\mathcal{H}}_{\widetilde{F}\widetilde{Y}_0}$. The Stokes values of $g_{\lambda.\ell.\omega}$ are among those of $\widetilde{F}$ translated by $-\omega$ hence none is left on the half-line d_θ after finitely many applications of St_θ. 1

With this extended definition we can now write

$$St_{\theta^\star}(\widetilde{F}(x)) = \widetilde{F}(x)\, x^L\, e^{Q(1/x)}(I_n + C_{\theta^\star})e^{-Q(1/x)}\, x^{-L} \tag{41}$$

instead of $St_{\theta^\star}(\widetilde{F}(x)\, x^L\, e^{Q(1/x)}) = \widetilde{F}(x)\, x^L\, e^{Q(1/x)}(I_n + C_{\theta^\star})$ only (*cf.* (35)).

4.4.3 Graduation on the Stokes automorphisms

The graduation is built so as to discriminate between the different submatrices $\dot{\Delta}^+_{\omega^\star}$ of the Stokes matrix $C_{\theta^\star}$ (*cf.* Section 4.2). This is done using the *exponential torus* $\mathcal{T}$ of System (1).

Here is how to define $\mathcal{T}$ (*cf.* [21, 20]): Let $b_1, \ldots, b_\nu$ be a basis of the lattice $\mathbb{Z}\Omega$. The polynomials $p_1(1/x) = -b_1/x, \ldots, p_\nu(1/x) = -b_\nu/x$ form a basis of the $\mathbb{Z}$-module generated by the determining polynomials $q_1, q_2, \ldots, q_n$ of the diagonal of $Q(1/x)$. The exponential torus takes into account the indetermination of a formal exponential e^p by associating with e^p its complex multiples λe^p (see Footnote 6). To define the exponential torus $\mathcal{T}$ one introduces ν indeterminates $\underline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_\nu)$ and associates $\lambda_j e^{p_j}$ with each e^{p_j}. One can extend $\mathcal{T}$ to the algebra $\widetilde{H}_{\widetilde{F}\widetilde{Y}_0}$ by letting it act trivially on power series, complex powers of x and logarithms. We obtain thus a family of automorphisms of $\widetilde{\mathcal{H}}_{\widetilde{F}\widetilde{Y}_0}$ with ν parameters.

For $j = 1, 2, \ldots, n$, denote $\underline{m}_j = (m_{j,1}, m_{j,2}, \ldots, m_{j,\nu})$ the components of the q_j's with respect to the p_r's. With respect to the formal fundamental solution $\widetilde{F}(x)\, x^L\, e^{Q(1/x)}$ the exponential torus has a matrix representation of the form

$$T_{\underline{\lambda}} = \mathrm{diag}(\underline{\lambda}^{\underline{m}_1}, \underline{\lambda}^{\underline{m}_2}, \ldots, \underline{\lambda}^{\underline{m}_n}) \tag{42}$$

where the notation $\underline{\lambda}^{\underline{m}}$ stands for the product $\lambda_1^{m_1}\, \lambda_2^{m_2} \ldots \lambda_\nu^{m_\nu}$. Its action on the Stokes automorphism $St_{\theta^\star}$ generates a group $G_{\underline{\lambda}}$ of unipotent matrices $I_n + T_{\underline{\lambda}}\, C_{\theta^\star}\, T_{\underline{\lambda}}^{-1}$ with ν parameters. These matrices are polynomials in the parameters λ_r and their inverses. Keep denoting $\mathbf{\Omega}_\theta$ the set of non-zero Stokes values $a_j - a_k$ on the line d_θ (*cf.* Section 2.2). To each Stokes value $\omega \in \mathbf{\Omega}_\theta$ there is a unique collection $\underline{m}(\omega) = (m_1(\omega), \ldots, m_\nu(\omega))$ of weights such that $\displaystyle\sum_{r=1}^{\nu} m_r(\omega)\, p_r(x) = -\frac{\omega}{x}$. One can check that the matrix $\dot{\Delta}^{+}_{\omega^\star}$ (*cf.* Section 4.2) is the coefficient of the monomial $\underline{\lambda}^{\underline{m}(\omega)}$ in $T_{\underline{\lambda}}\, C_{\theta^\star}\, T_{\underline{\lambda}}^{-1}$:

$$T_{\underline{\lambda}}\, C_{\theta^\star}\, T_{\underline{\lambda}}^{-1} = \sum_{\omega \in \mathbf{\Omega}_\theta} \dot{\Delta}^{+}_{\omega^\star}\ \underline{\lambda}^{\underline{m}(\omega)}. \tag{43}$$

4.4.4 Definition of the alien derivations

As a unipotent graded group the group $G_{\underline{\lambda}}$ admits infinitesimal generators $\dot{\Delta}_{\omega^\star}$ in the sense that

$$I_n + T_{\underline{\lambda}}\, C_{\theta^\star}\, T_{\underline{\lambda}}^{-1} = \exp\Big(\sum_{\omega \in \mathbf{\Omega}_\theta} \dot{\Delta}_{\omega^\star}\ \underline{\lambda}^{\underline{m}(\omega)} \Big). \tag{44}$$

Definition 4.8 (dotted and undotted alien derivations)

- The *dotted alien derivations* $\dot{\Delta}_{\omega^\star}$ are the transformations with matrix the coefficient of $\underline{\lambda}^{\underline{m}(\omega)}$ in the expansion of the logarithm $\ln\big(I_n + T_{\underline{\lambda}}\, C_{\theta^\star}\, T_{\underline{\lambda}}^{-1}\big)$ (matrix in the chosen basis of formal solutions $\widetilde{F}(x)\, x^L\, e^{Q(1/x)}$).

  $$\ln\big(I_n + T_{\underline{\lambda}}\, C_{\theta^\star}\, T_{\underline{\lambda}}^{-1}\big) = \sum_{\omega \in \mathbf{\Omega}_\theta} \dot{\Delta}_{\omega^\star}\ \underline{\lambda}^{\underline{m}(\omega)} \tag{45}$$

- The *alien derivation* $\Delta_{\omega^\star}$ is defined by $\Delta_{\omega^\star} = e^{+\omega/x}\ \dot{\Delta}_{\omega^\star}$.

When ω does not belong to $\mathbf{\Omega}_\theta$ for any θ the alien derivations $\dot{\Delta}_{\omega^\star}$ and $\Delta_{\omega^\star}$ are equal to 0.
Like $St^{+}_{\theta^\star}$ the dotted alien derivations act trivially on x^L and $e^{Q(1/x)}$ so that

$$\dot{\Delta}_{\omega^\star}\ \big(\widetilde{F}(x)\big) = \widetilde{F}(x)\, x^L e^{-\omega/x}\ \dot{\Delta}_{\omega^\star}\ x^{-L}.$$

and

$$\Delta_{\omega^\star}\big(\widetilde{F}(x)\big) = \widetilde{F}(x)\, x^L\ \dot{\Delta}_{\omega^\star}\ x^{-L}.$$

The alien derivations are derivations by construction and they commute with the usual derivation d/dx.

We end this section with a remark on the various choices made.
We saw that the meromorphic classifying set is given by the Stokes automorphisms $St_{\theta,\widetilde{F}}$ defined, for all anti-Stokes direction, as gauge transformations of the normal form. To look at the Stokes automorphisms $St_{\theta^\star}$ defined as linear maps on the space of formal solutions of System (1) and get their extended forms we needed to choose a determination of the argument and an actual form of the formal exponentials. With different choices the Stokes matrix is conjugate under the iterated action of the formal monodromy with matrix $\widehat{M} = x^{2\pi i L}$ with respect to $\widetilde{Y}_0 = x^L\, e^{Q(1/x)}$ and under the action of the exponential torus $\mathcal{T}$. The same conjugacy actions must be taken into account when performing the analytic classification with alien derivations.

4.4.5 Bridge equation

The "definition" formula above rewritten in the form

$$\Delta_{\omega^\star}\big(\widetilde{F}(x)x^L\big) = \widetilde{F}(x)\, x^L\ \dot{\Delta}_{\omega^\star} \tag{46}$$

can be seen as Écalle's Bridge Equation. The name "bridge" comes from the fact that the equation links alien derivatives (left hand side of the bridge equation) to ordinary derivatives (right hand side of the bridge equation). Indeed, the right hand side can be seen as an ordinary derivative as follows. We consider only the first n_1 columns, the calculation for the k^{th} block of columns being the same after multiplication by $e^{a_k/x}$. Introduce the general solution (also said formal integral) of System (1) which has the form

$$\widetilde{f}(x)\, x^{L_1} M_0 + \sum_{\omega\in\{a_2,\dots,a_J\}} \phi_{\omega^\star}(x)\, M_{\omega^\star}\, e^{-\omega/x}$$

where the ϕ's are formal-log series and the M's are arbitrary constant matrices. The alien derivative of $\widetilde{f}(x)\, x^{L_1}$ at ω is the derivative of this general solution with respect to $e^{-\omega/x}$ considered as an independent variable[10], with a convenient choice of the matrices M (Écalle's analytic invariants $\dot{\Delta}_{\omega^\star}$).

[10] In the case of scalar solutions of an equation instead of a system, J. Écalle takes derivatives with respect to the constants. Since, here, the constant coefficients of the various exponentials are matrices it is more convenient to derivate with respect to the exponentials themselves.

In Écalle's approach, alien derivatives are defined as an average of analytic continuations in the Borel plane followed by a Laplace transform. The Bridge Equation results from the fact that dotted alien derivatives commute with the derivation d/dx. In this approach, it is nothing more than the definition formula.

4.5 The example of the generalized hypergeometric equation $D_{13,1}$

We consider the generalized hypergeometric equation of order 13

$$\mathbf{D_{13,1}}(\mathbf{y}) \equiv \Big(x\frac{d\mathbf{y}}{dx} - \mu\mathbf{y}\Big) - x\prod_{j=1}^{13}\Big(x\frac{d}{dx} - (\nu_j - 1)\Big)\mathbf{y} = 0 \tag{47}$$

where μ and the ν_j's are complex parameters[11]. Its Newton polygon at 0 has a slope 0 of length 1 and a slope 1/12 of length 12. Putting $x = t^{12}$ the equation becomes

$$D_{13,1}(y) \equiv \Big(\frac{1}{12}\,t\,\frac{dy}{dt} - \mu y\Big) - t^{12}\prod_{j=1}^{13}\Big(\frac{1}{12}\,t\,\frac{d}{dt} - (\nu_j - 1)\Big)\,y = 0. \tag{48}$$

We keep using the equation itself taking benefit of having a quite simple equation but we could as well commute to the companion system.

The hypergeometric equation (48) is of single level 1. Its determining polynomials $q_1, q_2, \ldots, q_{13}$ are calculated in [9] and are

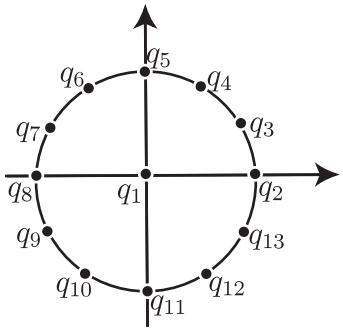
$$\begin{array}{llll} q_1 = 0 & q_2 = -12/t & q_3 = -12\zeta/t & q_4 = -12\zeta^2/t \\ & q_5 = -12\zeta^3/t & q_6 = -12\zeta^4/t & q_7 = -12\zeta^5/t \\ & q_8 = -12\zeta^6/t & q_9 = -12\zeta^7/t & q_{10} = -12\zeta^8/t \\ & q_{11} = -12\zeta^9/t & q_{12} = -12\zeta^{10}/t & q_{13} = -12\zeta^{11}/t \end{array}$$

where ζ stands for the twelfth primitive root of unit $\zeta = e^{2\pi i/12}$. A formal fundamental solution $\widetilde{F}(t)\,t^L e^{Q(1/t)}$ reads

$$\Big[\widetilde{F}^1(t)\,t^{12\mu}\,e^{q_1(1/t)} \quad \widetilde{F}^2(t)\,t^{-12\lambda}\,e^{q_2(1/t)} \quad \ldots \quad \widetilde{F}^{13}(t)\,t^{-12\lambda}\,e^{q_{13}(1/t)}\Big] \tag{49}$$

where $\lambda = \dfrac{1}{12}\Big(\dfrac{13}{2} + \mu - \displaystyle\sum_{j=1}^{13}\nu_j\Big)$.

[11] The irregular singular point, usually put at infinity (*cf.* [9]), is here located at 0.



From the fact that the minimal polynomial of ζ is the cyclotomic polynomial X^4-X^2+1 we deduce that a basis for the lattice built on the q_j's is given by

$$p_1 = q_2 = -12/t \qquad p_2 = q_3 = -12\zeta/t$$
$$p_3 = q_4 = -12\zeta^2/t \qquad p_4 = q_5 = -12\zeta^3/t$$

Note that the lattice built on the coefficients $12, 12\zeta, 12\zeta^2, 12\zeta^3$ is dense in $\mathbb{C}$. However, only finitely many values are Stokes values for $D_{13,1}$.
In the p-basis the determining polynomials read

$$\begin{array}{llllllll} q_1 = 0 & q_2 = p_1 & & q_3 = p_2 & & q_4 = p_3 & & q_5 = p_4 \\ & q_6 = p_3 - p_1 & & q_7 = p_4 - p_2 & & & & \\ & q_8 = -p_1 & & q_9 = -p_2 & & q_{10} = -p_3 & & q_{11} = -p_4 \\ & q_{12} = -p_3 + p_1 & & q_{13} = -p_4 + p_2 & & & & \end{array}$$

so that, denoting $\underline{\lambda} = (\lambda_1, \lambda_2, \lambda_3, \lambda_4)$, the matrix of the exponential torus is given by

$$T_{\underline{\lambda}} = \mathrm{diag}\Big(1, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \frac{\lambda_3}{\lambda_1}, \frac{\lambda_4}{\lambda_2}, \frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \frac{1}{\lambda_3}, \frac{1}{\lambda_4}, \frac{\lambda_1}{\lambda_3}, \frac{\lambda_2}{\lambda_4}\Big).$$

Let us consider the anti-Stokes direction $\theta = 0$ with principal determination $\theta^\star = 0$ and denote C_0 the corresponding Stokes matrix.
Let $E_{j,\ell}$ denote the elementary 13×13-matrix the entries of which are all 0 but the one at row j and column ℓ which is equal to 1.
The Stokes values ω supported by the half line d_0, issued from 0 with argument 0 (*i.e.*, $\omega \in \mathbf{\Omega}_0$), are

$\omega = 12$ associated with $q_1 - q_8 = q_2 - q_1 = q_4 - q_6 = q_{12} - q_{10} = p_1$

$\omega = 12\sqrt{3}$ associated with $q_3 - q_7 = q_{13} - q_9 = \sqrt{3}\, p_1$

$\omega = 24$ associated with $q_2 - q_8 = 2\, p_1$

and consequently, the Stokes matrix satisfies

$$\begin{aligned} T_{\underline{\lambda}} C_0 T_{\underline{\lambda}}^{-1} = \quad & \Big(c^{(1,8)} E_{1,8} + c^{(2,1)} E_{2,1} + c^{(4,6)} E_{4,6} + c^{(12,10)} E_{12,10} \Big)\, \lambda_1 \\ & + \Big(c^{(3,7)} E_{3,7} + c^{(13,9)} E_{13,9} \Big)\, \frac{\lambda_2^2}{\lambda_4} \\ & + \Big(c^{(2,8)} E_{2,8} \Big)\, \lambda_1^2 \end{aligned}$$

since one has also the relations $q_3 - q_7 = q_{13} - q_9 = 2p_2 - p_4$. In other words, we obtain:

$$\begin{aligned} \dot{\Delta}_{12}^{+} &= c^{(1,8)} E_{1,8} + c^{(2,1)} E_{2,1} + c^{(4,6)} E_{4,6} + c^{(12,10)} E_{12,10}, \\ \dot{\Delta}_{12\sqrt{3}}^{+} &= c^{(3,7)} E_{3,7} + c^{(13,9)} E_{13,9}, \\ \dot{\Delta}_{24}^{+} &= c^{(2,8)} E_{2,8}. \end{aligned}$$

Note that $T_{\underline{\lambda}} C_0 T_{\underline{\lambda}}^{-1}$ does not depend on the parameter λ_3 which, in turn, would appear in $T_{\underline{\lambda}} C_{-\pi} T_{\underline{\lambda}}^{-1}$. The Stokes multipliers $c^{(j,\ell)}$ are made explicit in term of Barnes integrals in [9].

We know from the previous section that the alien derivations in the various Stokes values belonging to $\mathbf{\Omega}_0$ are given in terms of the Stokes multipliers above by taking the logarithm of $I_{13} + T_{\underline{\lambda}} C_0 T_{\underline{\lambda}}^{-1}$. From the relations $(\dot{\Delta}_{12}^{+})^2 = c^{(2,1)}\, c^{(1,8)}\, E_{2,8}$ and $\dot{\Delta}_{12}^{+} \dot{\Delta}_{12\sqrt{3}}^{+} = \dot{\Delta}_{12\sqrt{3}}^{+} \dot{\Delta}_{12}^{+} = \dot{\Delta}_{12}^{+} \dot{\Delta}_{24}^{+} = \cdots = (\dot{\Delta}_{24}^{+})^2 = 0$ we obtain

$$\begin{aligned} & \ln\left(I_{13} + T_{\underline{\lambda}} C_0 T_{\underline{\lambda}}^{-1} \right) \\ & \quad = \Big(\dot{\Delta}_{12}^{+}\, \lambda_1 + \dot{\Delta}_{12\sqrt{3}}^{+}\, \frac{\lambda_2^2}{\lambda_4} + \dot{\Delta}_{24}^{+}\, \lambda_1^2 \Big) - \frac{1}{2} \Big(\dot{\Delta}_{12}^{+}\, \lambda_1 + \cdots \Big)^2 + \cdots \\ & \quad = \dot{\Delta}_{12}^{+}\, \lambda_1 + \dot{\Delta}_{12\sqrt{3}}^{+}\, \frac{\lambda_2^2}{\lambda_4} + \Big(\dot{\Delta}_{24}^{+} - \frac{1}{2} (\dot{\Delta}_{12}^{+})^2 \Big) \lambda_1^2. \end{aligned}$$

Hence,

(50)
$$\boxed{\begin{aligned} \dot{\Delta}_{12} &= \dot{\Delta}_{12}^{+} = c^{(1,8)} E_{1,8} + c^{(2,1)} E_{2,1} + c^{(4,6)} E_{4,6} + c^{(12,10)} E_{12,10} \\ \dot{\Delta}_{12\sqrt{3}} &= \dot{\Delta}_{12\sqrt{3}}^{+} = c^{(3,7)} E_{3,7} + c^{(13,9)} E_{13,9} \\ \dot{\Delta}_{24} &= \dot{\Delta}_{24}^{+} - \frac{1}{2} (\dot{\Delta}_{12}^{+})^2 = \Big(c^{(2,8)} - \frac{1}{2} c^{(2,1)}\, c^{(1,8)} \Big) E_{2,8} \end{aligned}}$$

We can develop these formulæ by writing $\dot{\Delta}_{12}\,(\widetilde{F}) = \widetilde{F}\,t^L\,e^{-12/t}\;\dot{\Delta}_{12}\;t^{-L}$ or $\Delta_{12}(\widetilde{F}) = \widetilde{F}\,t^L\;\dot{\Delta}_{12}\;t^{-L}$ and so on…, *i.e.*,

$$(51)\qquad \begin{cases} \Delta_{12}(\widetilde{F}^1) = \widetilde{F}^2\,t^{-12(\lambda+\mu)}\,c^{(2,1)} & \Delta_{12}(\widetilde{F}^6) = \widetilde{F}^4\,c^{(4,6)} \\ \Delta_{12}(\widetilde{F}^8) = \widetilde{F}^1\,t^{12(\lambda+\mu)}\,c^{(1,8)} & \Delta_{12}(\widetilde{F}^{10}) = \widetilde{F}^{12}\,c^{(12,10)} \\ \Delta_{12\sqrt{3}}(\widetilde{F}^7) = \widetilde{F}^3\,c^{(3,7)} & \Delta_{12\sqrt{3}}(\widetilde{F}^9) = \widetilde{F}^{13}\,c^{(13,9)} \\ \Delta_{24}(\widetilde{F}^8) = \widetilde{F}^2\left(c^{(2,8)} - \dfrac{1}{2}c^{(2,1)}c^{(1,8)}\right) & \end{cases}$$

all other alien derivatives on the real positive line being 0.

Equations (51) can be seen as *Écalle bridge equations*.

Remark 4.9 We end this example with a comment on, for instance, the last formula in (50) compared to those of [26, Lemma 5] deduced directly from the definition of alien derivations by analytic continuation in the Borel plane. If we consider that there is on $d_0 = \mathbb{R}^+$ the three singularities $\omega_1 = 12, \omega_2 = 12\sqrt{3}$ and $\omega_3 = 24$ then our formula does not fit the expression given in Lemma 5 of [26] for Δ_{ω_3}. To fit it in, we should actually re-introduce the missing singular point $\omega_1 = 24 - 12\sqrt{3}$ attached to the monomial $\dfrac{\lambda_1^2\lambda_4}{\lambda_2^2}$ — although with a null coefficient $\dot{\Delta}^{+}_{\omega_1} = 0$— which combined with the monomial $\dfrac{\lambda_2^2}{\lambda_4}$ attached to $12\sqrt{3}$ gives the monomial λ_1^2 attached to 24. Set $\omega_2 = 12, \omega_3 = 12\sqrt{3}, \omega_4 = 24$ and the calculation, in both cases, gives the same result

$$\begin{aligned}\Delta_{\omega_4} = \Delta^+_{\omega_4} - \frac{1}{2}(\Delta^{+\,2}_{\omega_2} + \Delta^+_{\omega_1}\Delta^+_{\omega_3} + \Delta^+_{\omega_3}\Delta^+_{\omega_1}) \\ + \frac{1}{3}(\Delta^{+\,2}_{\omega_1}\Delta^+_{\omega_2} + \Delta^+_{\omega_1}\Delta^+_{\omega_2}\Delta^+_{\omega_1} + \Delta^+_{\omega_2}\Delta^{+\,2}_{\omega_1}) - \frac{1}{4}\Delta^{+\,4}_{\omega_1}\end{aligned}$$

if one takes into account the fact that $\Delta^+_{\omega_1} = 0$.

4.6 An example with resonance

We consider the system

$$(\mathfrak{S})\qquad x^2\frac{dY}{dx} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ x^2 & 1 & x & 0 \\ x^2 & 0 & 1 & x \\ x^2 & 0 & 0 & 1 \end{bmatrix} Y$$

and its formal fundamental solution $\widetilde{Y}(x) = \widetilde{F}(x)\, x^L\, e^{Q(1/x)}$ where

- $Q(1/x) = \mathrm{diag}(0, -1/x, -1/x, -1/x)$, (Hence, the system has the unique level 1 and the Stokes values ± 1),
- $L = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (L is not diagonal; hence the resonance),
- $\widetilde{F}(x) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \widetilde{f}_2 & 1 & 0 & 0 \\ \widetilde{f}_3 & 0 & 1 & 0 \\ \widetilde{f}_4 & 0 & 0 & 1 \end{bmatrix}$ is a power series satisfying $\widetilde{F}(x) = I_4 + O(x^2)$.

The system admits the two anti-Stokes directions $\theta = 0$ and $\theta = \pi$. Obviously, the Stokes matrix in direction π is trivial: $I_4 + C_\pi = I_4$.

We consider the anti-Stokes direction $\theta = 0$ supporting the unique Stokes value $\omega = 1$. Our aim is the calculation of the alien derivation Δ_1 in terms of the Stokes multipliers in direction $\theta = 0$. Actually, although System $(\mathfrak{S})$ is quite a little bit involved since it exhibits resonance, it is simple enough to allow an exact calculation of the Stokes multipliers as below. We will then be able to give an exact calculation for Δ_1.

One can check that the series $\widetilde{f}_j$'s are the unique solutions of the system

$$\left\{ \begin{array}{rcl} x^2 \dfrac{d\widetilde{f}_2}{dx} - \widetilde{f}_2 &=& x^2 + x\widetilde{f}_3 \\ x^2 \dfrac{d\widetilde{f}_3}{dx} - \widetilde{f}_3 &=& x^2 + x\widetilde{f}_4 \\ x^2 \dfrac{d\widetilde{f}_4}{dx} - \widetilde{f}_4 &=& x^2 \end{array} \right. \tag{52}$$

satisfying the condition $\widetilde{f}_j(x) = O(x^2)$. It results that their Borel transforms $\widehat{f}_j$ are given by

$$\left\{ \begin{array}{rcl} \widehat{f}_2(1+\xi) &=& \dfrac{1}{\xi}\Big(\dfrac{1}{2}(6 - \pi^2 + 4\pi i) + (2+\pi i)\ln\xi + \dfrac{1}{2}\ln^2\xi\Big) + 3 \\ \widehat{f}_3(1+\xi) &=& \dfrac{1}{\xi}\Big((2+\pi i) + \ln\xi\Big) + 2 \\ \widehat{f}_4(1+\xi) &=& \dfrac{1}{\xi} + 1 \end{array} \right. \tag{53}$$

and consequently, the connection matrix $K^+_{(1)}$ is given by

$$K^+_{(1)} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ k_2 = \frac{1}{2}(6 - \pi^2 + 4\pi i) & 0 & 0 & 0 \\ k_3 = 2 + \pi i & 0 & 0 & 0 \\ k_4 = 1 & 0 & 0 & 0 \end{bmatrix}.$$

From Corollary 4.5 we deduce that the Stokes multipliers $C_0^{(2,1)}, C_0^{(3,1)}$ and $C_0^{(4,1)}$ are

$$\begin{cases} C_0^{(2,1)} &= \kappa_0(0)\, k_2 + \kappa_1(0)\, k_3 + \frac{1}{2}\kappa_2(0)\, k_4 \\ C_0^{(3,1)} &= \kappa_0(0)\, k_3 + \kappa_1(0)\, k_4 \\ C_0^{(4,1)} &= \kappa_0(0)\, k_4 \end{cases}$$

Recall that $\kappa_p(\lambda) = 2\pi i \dfrac{d^p}{dt^p}\Big(\dfrac{e^{-i\pi t}}{\Gamma(1-t)}\Big)\Big|_{t=\lambda}$ and then,

$$\begin{cases} \kappa_0(0) &= 2\pi i \\ \kappa_1(0) &= 2\pi^2 - 2\pi i\gamma \\ \kappa_2(0) &= -4\pi^2\gamma - \frac{7\pi^3 i}{3} + 2\pi\gamma^2 i \end{cases}$$

where $\gamma = 0.5772\ldots$ is the Euler constant. We obtain

$$\boxed{\begin{array}{lcl} C_0^{(2,1)} &=& (6\pi - \frac{1}{6}\pi^3 - 4\pi\gamma + \pi\gamma^2)\, i \\ C_0^{(3,1)} &=& 2\pi(2-\gamma)\, i \\ C_0^{(4,1)} &=& 2\pi i \end{array}}.$$

The lattice built on the unique polynomial $q(1/x) = -1/x$ is generated by q itself and the matrix of the exponential torus is $T_\lambda = \mathrm{diag}(1, \lambda, \lambda, \lambda)$. The action of the exponential torus on $I_4 + C_0$ results in $T_\lambda(I_4 + C_0)T_\lambda^{-1} =$

$I_4+\lambda\, C_0$ and $\ln(I_4+T_\lambda C_0 T_\lambda^{-1}) = \lambda C_0$. Thus, $\dot\Delta_1^{+} = C_0$ and the alien derivation $\dot\Delta_1 = C_0$. We can write $\dot\Delta_1\ (\widetilde F) = \widetilde F\, x^L\, C_0\, x^{-L}\, e^{-1/x}$ or $\Delta_1(\widetilde F) = \widetilde F\, x^L\, C_0\, x^{-L}$, *i.e.*,

$$\boxed{\begin{array}{rcl} \Delta_1(\widetilde f_2) &=& C_0^{(2,1)} + C_0^{(3,1)}\ \ln x + \dfrac{1}{2} C_0^{(4,1)}\ \ln^2 x \\ \Delta_1(\widetilde f_3) &=& C_0^{(3,1)} + C_0^{(4,1)}\ \ln x \\ \Delta_1(\widetilde f_4) &=& C_0^{(4,1)} \end{array}}\ .$$

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