Embedding mapping class groups of orientable surfaces with one boundary component

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Abstract

We denote by $S_{g,b,p}$ an orientable surface of genus g with b boundary components and p punctures. We construct homomorphisms from the mapping class groups of $S_{g,1,p}$ to the mapping class groups of $S_{g',1,(b-1)}$, where $b \geq 1$. These homomorphisms are constructed from branched or unbranched covers of $S_{g,1,0}$ with some properties. Our main result is that these homomorphisms are injective. For unbranched covers, this construction was introduced by McCarthy and Ivanov [10]. They proved that the homomorphisms are injective. A particular cases of our embeddings is a theorem of Birman and Hilden that embeds the braid group on p strands into the mapping class group of $S_{(p-2)/2,2,0}$ if p is even, or into the mapping class group of $S_{(p-1)/2,1,0}$ if p is odd. We give a short proof of another result of Birman and Hilden [4] for surfaces with one boundary component.

2000 *Mathematics Subject Classification.* Primary: 20F34; Secondary: 20E05, 20E36, 57M99.

Key words. Mapping class group. Automorphisms of free groups. Ordering. Ends of groups.

1 Main results

We fix non-negative integers g, p and a positive integer b. We denote by $S_{g,b,p}$ an orientable surface of genus g with b boundary components and p punctures.

Our main theorem is the following.

Theorem 1.1. *Suppose $(g,p) \neq (0,2)$. Let $\kappa : S_{g',b,0} \to S_{g,1,0}$ be a finite index regular cover with p branching points in $S_{g,1,0}$ which lift to q points in $S_{g',b,0}$. Suppose every branching point of $S_{g,1,0}$ lifts to the same number of points in*

*The research was funded by Conseil Régional de Bourgogne and the MIC (Spain) through Project MTM2008-01550.

$S_{g',b,0}$. Let $\kappa' : S_{g',b,q} \to S_{g,1,p}$ be the corresponding unbranched cover. Let h be a homeomorphism of $S_{g,1,p}$ which fixes the boundary component pointwise. Suppose h lifts to $S_{g',b,q}$. Let $\hat{h}$ be the lift of h which fixes the b-th boundary component pointwise. Let $\hat{f}$ be the extension of $\hat{h}$ to $S_{g',b,0}$. If the restriction of $\hat{f}$ to $S_{g',1,(b-1)} \subseteq S_{g',b,0}$ is isotopic to the identity relative to the boundary component of $S_{g',1,(b-1)}$, then h is isotopic to the identity relative to the boundary component of $S_{g,1,p}$.

Let $\mathcal{M}_{g,b,p}$ be the mapping class group of $S_{g,b,p}$ relative to the boundary components. That is, $\mathcal{M}_{g,b,p}$ is the group of homeomorphisms of $S_{g,b,p}$ which fix the boundary components pointwise modulo isotopy relative to the boundary components of $S_{g,b,p}$. Since $b \geq 1$, we are restricting ourselves to orientation-preserving homeomorphims of $S_{g,b,p}$.

The following result is immediate from Theorem 1.1.

Corollary 1.2. *Suppose $(g,p) \neq (0,2)$. Let $\kappa : S_{g',b,0} \to S_{g,1,0}$ be a finite index regular cover with p branching points in $S_{g,1,0}$ which lift to q points in $S_{g',b,0}$. Suppose every branching point of $S_{g,1,0}$ lifts to the same number of points in $S_{g',b,0}$. Let $\kappa' : S_{g',b,q} \to S_{g,1,p}$ be the corresponding unbranched cover. If every homeomorphism of $S_{g,1,p}$ which fixes the boundary component pointwise lifts to a homeomorphism of $S_{g',b,q}$, then $\mathcal{M}_{g,1,p}$ embeds in $\mathcal{M}_{g',1,(b-1)}$.*

In the literature there are results about embeddings of mapping class groups constructed from branched covers. Let $\kappa : S_{g',b,0} \to S_{g,c,0}$ be a branched cover with p branching points in $S_{g,c,0}$ which lift to q branching points in $S_{g',b,0}$. Birman and Hilden [4] have results about these covers if $(g,c) = (0,1)$ or $c = 0$. For $(g,c) = (0,1)$, Birman and Hilden [4, Theorem 5] consider the hyperelliptic covers of the disc $\kappa : S_{(p-2)/2,2,0} \to S_{0,1,0}$ if $p \geq 4$ is even, and $\kappa : S_{(p-1)/2,1,0} \to S_{0,1,0}$ if $p \geq 3$ is odd. Birman and Hilden prove that there are embeddings $\mathcal{M}_{0,1,p} \hookrightarrow \mathcal{M}_{(p-2)/2,2,0}$ for $p \geq 4$ even, and $\mathcal{M}_{0,1,p} \hookrightarrow \mathcal{M}_{(p-1)/2,1,0}$ for $p \geq 3$ odd. See Farb and Margalit [9, Section 9.4] for a short proof of these embeddings. We will recover these embeddings in Example 3.1. We see Corollary 1.2 as a generalization of these embeddings. For $c = 0$, Birman and Hilden prove [4, Theorem 2]. From Theorem 1.1 we can prove the following analog of [4, Theorem 2] for surfaces with one boundary component. Notice that we do not need the hypothesis that the group of deck transformations (or covering transformations) is solvable.

Theorem 1.3. *Suppose $(g,p) \neq (0,2)$. Let $\kappa : S_{g',b,0} \to S_{g,1,0}$ be a finite index regular cover with p branching points in $S_{g,1,0}$. Suppose every branching point of $S_{g,1,0}$ lifts to the same number of points in $S_{g',b,0}$. Let $\hat{f}$ be a homeomorphism of $S_{g',b,0}$ which fixes the b-th boundary component pointwise and preserves the fibers of $\kappa : S_{g',b,0} \to S_{g,1,0}$. Then $\hat{f}$ induces a homeomorphism f of $S_{g,1,0}$ such that $\kappa\hat{f} = f\kappa$. If $\hat{f}$ is isotopic to the identity relative to the b-th boundary component, then f is isotopic to the identity relative to the boundary.*

Proof. It is a general fact that if $\hat{f}$ preserves the fibers of $\kappa : S_{g',b,0} \to S_{g,1,0}$, then $\hat{f}$ induces a homeomorphism f of $S_{g,1,0}$ such that $\kappa\hat{f} = f\kappa$. In particular, f sends branching points to branching points.

Let $\kappa' : S_{g',b,q} \to S_{g,1,p}$ be the corresponding unbranched cover. Since f sends branching points to branching points, f restricts to a homeomorphism h of $S_{g,1,p}$. Let $\hat{h}$ be the lift of h which fixes the b-th boundary component of $S_{g',b,q}$ pointwise. Notice $\hat{h}$ extends to a homeomorphism of $S_{g',b,0}$. This extension of $\hat{h}$ coincides with $\hat{f}$. If $\hat{f}$ is isotopic to the identity relative to the b-th boundary component, then the restriction of $\hat{f}$ to $S_{g',1,(b-1)} \subseteq S_{g',b,0}$ is isotopic to the identity relative to the boundary component of $S_{g',1,(b-1)}$. Then, by Theorem 1.1, h is isotopic to the identity relative to the boundary component of $S_{g,1,p}$. This isotopy extends to an isotopy relative to the boundary component of $S_{g,1,0}$ from f to the identity. $\square$

In the literature there are results about embeddings of mapping class groups constructed from unbranched covers. Let $\kappa' : S_{g',0,m} \to S_{g,0,1}$ be a degree m unbranched cover. Ivanov and McCarthy [10] construct embeddings if $g \geq 2$ and the cover $\kappa' : S_{g',0,m} \to S_{g,0,1}$ is characteristic. The condition that the cover $\kappa' : S_{g',0,m} \to S_{g,0,1}$ is characteristic ensures that every homeomorphism f of $S_{g,0,1}$ lifts to a homeomorphism of $S_{g',0,m}$. Then, there is a distinguished lift of f by distinguishing one of the m punctures of $S_{g',0,m}$. This gives a homomorphism $\mathcal{M}_{g,0,1} \to \mathcal{M}_{g',0,m}$. To see that this homomorphism is injective, the fundamental group of $S_{g',0,m}$, denoted $\pi_1(S_{g',0,m})$, is identified via the cover $\kappa' : S_{g',0,m} \to S_{g,0,1}$ with an index m (characteristic) subgroup of $\pi_1(S_{g,0,1})$. Then, the proof is completed by using some properties of $\pi_1(S_{g,0,1})$. The strategy to prove Theorem 1.1 is very close to this point of view: Theorem 1.1 is for surfaces with one boundary component as Ivanov and McCarthy construction is for once punctured surfaces. We can see Corollary 1.2 as a generalization of Ivanov and McCarthy embeddings in the sense that we allow a finite set of branching points. The technical difficulty of a branched cover $\kappa : S_{g',b,0} \to S_{g,1,0}$ is that $\pi_1(S_{g',b,0})$ cannot be identified via the cover with a subgroup of $\pi_1(S_{g,1,0})$.

Aramayona, Leininger and Souto [1] construct embeddings $\mathcal{M}_{g',0,0} \hookrightarrow \mathcal{M}_{g,0,0}$ from unbranched covers $\kappa' : S_{g',0,0} \to S_{g,0,0}$ which satisfy some algebraic properties. These embeddings $\mathcal{M}_{g',0,0} \hookrightarrow \mathcal{M}_{g,0,0}$ follow the construction of Ivanov and McCarthy. Using the algebraic properties of the cover $\kappa' : S_{g',0,0} \to S_{g,0,0}$, they manage to avoid the presence of punctures. Aramayona and Souto [2] prove that every non-trivial homomorphism $\mathcal{M}_{g,c,p} \to \mathcal{M}_{g',b,q}$, where $g \geq 6$, $g' \leq 2g-1$ and $q \geq 1$ if $g' = 2g-1$, is induced by a geometric embedding $S_{g,c,p} \hookrightarrow S_{g',b,q}$, that is, a composition of forgetting punctures, deleting boundary components and subsurfaces embeddings. Corollary 1.2 does not fit in this situation since, in general, g' will be bigger than $2g-1$. Example 3.2.(a) shows that the embeddings of Corollary 1.2 are not simple.

2 The algebraic analog

In this section we translate into algebra Theorem 1.1 and Corollary 1.2. We prove the algebraic analog of Theorem 1.1. Instead of dealing with $S_{g,b,p}$ and homeomorphisms of $S_{g,b,p}$ which fix the boundary components pointwise, we will deal with the fundamental group of $S_{g,b,p}$, denoted $\pi_1(S_{g,b,p})$. Since $b \geq 1$, we choose the base point of $\pi_1(S_{g,b,p})$ in the b-th boundary component. In this way, a homeomorphism of $S_{g,b,p}$ which fixes the boundary components pointwise induces an automorphism of $\pi_1(S_{g,b,p})$.

Notation 2.1. Let G be a group and let g, h be elements of G.

We denote by $\overline{g}$ the inverse of g. We write g^h for the conjugate of g by h, that is, $g^h = \overline{h}gh$. We denote by $[g]$ the conjugacy class of G, that is, $[g] = \{g^a \mid a \in G\}$. We write $[g, h]$ for the element $\overline{g}\overline{h}gh$ of G. Let $g_1, g_2, \ldots, g_k$ be elements of G. We write $\Pi_{i=1}^k g_i$ for the element $g_1 g_2 \cdots g_k$ of G.

We denote by $\mathrm{Aut}(G)$ the automorphism group of G and by $\mathrm{Out}(G)$ the group of outer automorphisms of G. Given $\phi \in \mathrm{Aut}(G)$, we write g^ϕ for the image of g by ϕ.

Notation 2.2. Let $F_{g,b,p}$ be the rank $2g+(b-1)+p$ free group with generating set $\{x_i, y_i\}_{1 \leq i \leq g} \cup \{z_l\}_{1 \leq l \leq (b-1)} \cup \{t_k\}_{1 \leq k \leq p}$. We identify $F_{g,b,p}$ with $\pi_1(S_{g,b,p}, *)$, the fundamental group of $S_{g,b,p}$ based at a point $*$ in the b-th boundary component. In addition, for every $1 \leq l \leq (b-1)$, z_l represents a loop around the l-th boundary component; for every $1 \leq k \leq p$, t_k represents a loop around the k-th puncture, and $(\Pi_{i=1}^g [x_i, y_i] \Pi_{l=1}^{b-1} z_l \Pi_{k=1}^p t_k)^{-1}$ represents a loop around the b-th boundary component.

Let f be a homeomorphim of $S_{g,b,p}$ which fixes the boundary components pointwise. Then f induces an automorphism f_* of $F_{g,b,p}$ which fixes the set of conjugacy classes of $t_1, t_2, \ldots, t_p$. Since f fixes the boundary components of $S_{g,b,p}$ pointwise, we see that f_* fixes $(\Pi_{i=1}^g [x_i, y_i] \Pi_{l=1}^{b-1} z_l \Pi_{k=1}^p t_k)^{-1}$ and the conjugacy class of z_l, for all $1 \leq l \leq b-1$. Two isotopic homeomorphisms of $S_{g,b,p}$ induce the same automorphism of $F_{g,b,p}$. Recall we consider isotopies relative to the boundary components. Notice the Dehn twist with respect a loop around a boundary component is isotopic to the identity, but it is not isotopic to the identity relative to the boundary. To capture this fact, we associate to f an automorphism of $F_{g,b,p} * \langle e_1, e_2, \ldots, e_{(b-1)} \mid \rangle$ which maps $F_{g,b,p}$ to itself and respects the following sets

$$(2.2.1) \qquad \begin{array}{rl} \text{(i)} & \{\Pi_{i=1}^g [x_i, y_i] \Pi_{l=1}^{b-1} z_l \Pi_{k=1}^p t_k\}, \\ \text{(ii)} & \{\overline{z}_1^{e_1}\}, \{\overline{z}_2^{e_2}\}, \ldots, \{\overline{z}_{(b-1)}^{e_{(b-1)}}\}, \\ \text{(iii)} & \{[\overline{t}_k]\}_{1 \leq k \leq p}. \end{array}$$

Recall z_l represents a loop around the l-th boundary component which is based at a point in the b-boundary component. For every $1 \leq l \leq (b-1)$, we view e_l as an arc from the base point in the b-th boundary component to a chosen point in the l-th boundary component. We view $\overline{e}_l z_l e_l = z_l^{e_l}$ as a loop around the l-boundary component and based at the chosen point in the l-boundary component. Since the homeomorphism f fixes the l-boundary component pointwise, the automorphism f_* fixes $\overline{z}_l^{e_l}$. For example, the Dehn twist with respect to the loop represented by $z_l^{e_l}$ gives the following automorphism of $F_{g,b,p} * \langle e_1, e_2, \ldots, e_{(b-1)} \mid \rangle$

$$\begin{cases} e_l & \mapsto & z_l e_l, \\ a & \mapsto & a, \quad a \in \{x_i, y_i\}_{1\leq i\leq g} \cup \{t_k\}_{1\leq k\leq p} \cup \{z_l\}_{1\leq l\leq b} \cup \{e_{l'}\}_{1\leq l'\leq b,\, l'\neq l}. \end{cases}$$

Definition 2.3. We denote by $\mathcal{AM}_{g,b,p}$ the subgroup of $\mathrm{Aut}(F_{g,b,p} * \langle e_1, e_2, \ldots, e_{b-1} \mid \rangle)$ consisting of all the automorphisms of $F_{g,b,p} * \langle e_1, e_2, \ldots, e_{(b-1)} \mid \rangle$ which map $F_{g,b,p}$ to itself and respect the sets of (2.2.1).

We call $\mathcal{AM}_{g,b,p}$ the *algebraic mapping class group* of $S_{g,b,p}$, an orientable surface of genus g with b boundary components and p punctures.

The mapping class group of $S_{g,b,p}$, denoted $\mathcal{M}_{g,b,p}$, is defined as the group of homeomorphisms of $S_{g,b,p}$ modulo isotopy relative to the boundary components. The above discussion shows that there exists a map $\mathcal{M}_{g,b,p} \to \mathcal{AM}_{g,b,p}$. By Dehn-Nielsen-Baer Theorem for surfaces with boundary, $\mathcal{M}_{g,b,p} \simeq \mathcal{AM}_{g,b,p}$, see [8, Theorem 9.6] and [9, Chapter Eight]. See [8] for a background on algebraic mapping class groups, with some changes of notation.

For $(g, b) = (0, 1)$ and $p \geq 1$, $\mathcal{AM}_{0,1,p}$ is isomorphic to the *p-string braid group.* We have $\mathcal{AM}_{0,1,p} = \langle \sigma_1, \sigma_2, \ldots, \sigma_{p-1} \rangle$, where for all $1 \leq i \leq (p-1)$, $\sigma_i \in \mathrm{Aut}(F_{0,1,p})$ is defined by

$$F_i := \begin{cases} t_i & \mapsto & t_{i+1}, \\ t_{i+1} & \mapsto & t_i^{t_{i+1}}, \\ t_k & \mapsto & t_k, \qquad \text{for all } 1 \leq k \leq p,\ k \neq i, i+1. \end{cases} \tag{2.3.1}$$

Let $d \in \mathbb{Z}$, $d \geq 2$.

Notation 2.4. Let N_d be the normal closure of $t_1^d, t_2^d, \ldots, t_p^d$ in $F_{g,1,p}$. We denote by $F_{g,1,p^{(d)}}$ the group $F_{g,1,p}/N_d$. For every $1 \leq k \leq p$, we denote by τ_k the image of t_k by the natural homomorphism $F_{g,1,p} \to F_{g,1,p^{(d)}}$.

Notice that if $p = 0$, then $N_d = 1$ and $F_{g,1,p^{(d)}} = F_{g,1,p}$.

Definition 2.5. Let $\mathcal{AM}_{g,1,p^{(d)}}$ denote the group of all automorphisms of $F_{g,1,p^{(d)}}$ that respect the sets

$$\{\Pi_{i=1}^{g}[x_i, y_i]\Pi_{k=1}^{p}\tau_k\}, \quad \{[\overline{\tau}_k]\}_{1\leq k\leq p}.$$

Since the sets $\{\Pi_{i=1}^{g}[x_i, y_i]\Pi_{k=1}^{p}t_k\}$, $\{[\bar{t}_k]\}_{1\le k\le p}$ are respected by elements of $\mathcal{AM}_{g,1,p}$, the natural homomorphism $F_{g,1,p} \to F_{g,1,p^{(d)}}$ induces a natural homomorphism

$$\psi : \mathcal{AM}_{g,1,p} \to \mathcal{AM}_{g,1,p^{(d)}}.$$

Notice that if $p = 0$, then $F_{g,1,p} = F_{g,1,p^{(d)}}$ and ψ is the identity.

Theorem 2.6. *The homomorphism $\psi : \mathcal{AM}_{g,1,p} \to \mathcal{AM}_{g,1,p^{(d)}}$ is injective.*

We prove Theorem 2.6 in Section 4.

For the rest of this section we consider a regular cover $\kappa : S_{g',b,0} \to S_{g,1,0}$ of index m with p branching points in $S_{g,1,0}$ which lift to q points in $S_{g',b,0}$ such that every branching point of $S_{g,1,0}$ lifts to the same number of points in $S_{g',b,0}$. We denote by $\kappa' : S_{g',b,q} \to S_{g,1,p}$ the corresponding unbranched cover. Recall that $F_{g,1,p}$ is the fundamental group of $S_{g,1,p}$ with base point $*$ in the boundary component and $F_{g',b,q}$ is the fundamental group of $S_{g',b,q}$ with base point $\hat{*}$ a lift of $*$ in the b-th boundary component. We identify $F_{g',b,q}$ with $\kappa'_*(F_{g',b,q})$. Hence, $F_{g',b,q}$ is a normal subgroup of $F_{g,1,p}$ of index m. In Remark 2.7 we define a basis $\{\hat{x}_i, \hat{y}_i\}_{1\le i\le g'} \cup \{\hat{z}_l\}_{1\le l\le (b-1)} \cup \{\hat{t}_k\}_{1\le k\le p}$ of $F_{g',b,q}$. From this basis, we deduce two technical results: Lemma 2.8 and Proposition 2.10. In Remark 2.11 we discuss the embeddings $S_{g',b,q} \hookrightarrow S_{g',1,(b-1)+q}$ and $S_{g',1,(b-1)+q} \hookrightarrow S_{g',1,(b-1)}$ in terms of fundamental groups and the basis $\{\hat{x}_i, \hat{y}_i\}_{1\le i\le g'} \cup \{\hat{z}_l\}_{1\le l\le (b-1)} \cup \{\hat{t}_k\}_{1\le k\le p}$. Finally, we state and prove the algebraic analog of Theorem 1.1 and we state the algebraic analog of Corollary 1.2.

Remark 2.7. We set $G := F_{g,1,p}/F_{g',b,q}$ the group of deck transformations of the unbranched cover $\kappa' : S_{g',b,q} \to S_{g,1,p}$.

Let ϱ be the image of $\Pi_{i=1}^{g}[x_i, y_i]\Pi_{k=1}^{p}t_k$ by the natural homomorphism $F_{g,1,p} \to F_{g,1,p}/F_{g',b,q} = G$. Let c be the order of ϱ in G. Since $\varrho^c = 1$ in G, we see that $(\Pi_{i=1}^{g}[x_i, y_i]\Pi_{k=1}^{p}t_k)^c \in F_{g',b,q}$. Notice that $(\Pi_{i=1}^{g}[x_i, y_i]\Pi_{k=1}^{p}t_k)^{-c}$ represents a loop around the b-th boundary component. We take a basis $\{\hat{x}_i, \hat{y}_i\}_{1\le i\le g'} \cup \{\hat{z}_l\}_{1\le l\le (b-1)} \cup \{\hat{t}_k\}_{1\le k\le q}$ of $F_{g',b,q}$ such that

$$\Pi_{i=1}^{g'}[\hat{x}_i, \hat{y}_i]\Pi_{l=1}^{b-1}\hat{z}_l\Pi_{k=1}^{q}\hat{t}_k = (\Pi_{i=1}^{g}[x_i, y_i]\Pi_{k=1}^{p}t_k)^c.$$

Recall G has cardinality m. The subgroup $\langle\varrho\rangle \le G$ has index $b = m/c$. For every $1 \le l \le b-1$, we take $w_l \in F_{g,1,p} - F_{g',b,q}$ such that

$$\hat{z}_l = \overline{w}_l(\Pi_{i=1}^{g}[x_i, y_i]\Pi_{k=1}^{p}t_k)^{-c}w_l.$$

Let ρ_l be the image of w_l by the natural homomorphism $F_{g,1,p} \to F_{g,1,p}/F_{g',b,q} = G$. Then $G = \langle\varrho\rangle\rho_1 \cup \langle\varrho\rangle\rho_2 \cdots \cup \langle\varrho\rangle\rho_{(b-1)} \cup \langle\varrho\rangle$. That is, the boundary components of $S_{g',b,p}$ are image by deck transformations of the b-th boundary component.

For every $1 \le k \le p$, let ϱ_k be the image of t_k by the natural homomorphism $F_{g,1,p} \to F_{g,1,p}/F_{g',b,q} = G$. Let d_k be the order of ϱ_k in G. Since t_k corresponds to a branching point, we see $t_k \notin F_{g',b,q}$ and $d_k \geq 2$. Since $\varrho_k^{d_k} = 1$ in $G = F_{g,1,p}/F_{g',b,q}$, we see that $t_k^{d_k} \in F_{g',b,q}$. Notice that $t_k^{d_k}$ represents a loop around a lift of the k-th puncture of $S_{g,1,p}$. The subgroup $\langle \varrho_k \rangle$ has index m/d_k in G. Since all the branching point of $S_{g,1,0}$ lift to the same number of points in $S_{g',b,0}$, $m/d_1 = m/d_k$ for all $2 \leq k \leq p$. Hence, $d_1 = d_k$ for all $2 \leq k \leq p$. Let $d = d_1$. We have $G = \langle \varrho_k \rangle \rho_{1,k} \cup \langle \varrho_k \rangle \rho_{2,k} \cup \cdots \cup \langle \varrho_k \rangle \rho_{m/d,k}$, where $\rho_{j,k} = u_{j,k} F_{g',b,q} \in G = F_{g,1,p}/F_{g',b,q}$ for all $1 \leq j \leq m/d$. Notice that $(t_k^d)^{u_{1,k}}, (t_k^d)^{u_{2,k}}, \ldots, (t_k^d)^{u_{m/d,k}}$ represent loops around the m/d lifts of the k-th puncture. We choose $u_{1,k}, u_{2,k}, \ldots, u_{m/d,k} \in F_{g,1,p}$ such that $\{(t_k^d)^{u_{1,k}}, (t_k^d)^{u_{2,k}}, \ldots, (t_k^d)^{u_{m/d,k}}\} \subseteq \{\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_q\}$. Then

$$\{\hat{t}_1, \hat{t}_2, \ldots, \hat{t}_q\} = \bigcup_{k=1}^{p} \{(t_k^d)^{u_{1,k}}, (t_k^d)^{u_{2,k}}, \ldots, (t_k^d)^{u_{m/d,k}}\}. \tag{2.7.1}$$

Recall N_d is the normal closure of $t_1^d, t_2^d, \ldots, t_p^d$ in $F_{g,1,p}$.

Lemma 2.8. *With the above notation, N_d is equal to the normal closure of $\hat{t}_1, \hat{t}_2, \cdots, \hat{t}_q$ in $F_{g',b,q}$.*

Proof. By (2.7.1), the normal closure of $\hat{t}_1, \hat{t}_2, \cdots, \hat{t}_q$ in $F_{g',b,q}$ is a subgroup of N_d.

Let $1 \leq k \leq p$ and $w \in F_{g,1,p}$. By (2.7.1), it is enough to prove $(t_k^d)^w = (t_k^d)^{u_{j,k}v}$ for some $1 \leq j \leq (m/d)$ and $v \in F_{g',b,q}$. Recall $G = F_{g,1,p}/F_{g',b,q}$, $\varrho_k = t_k F_{g',b,q} \in G$ and $G = \langle \varrho_k \rangle \rho_{1,k} \cup \langle \varrho_k \rangle \rho_{2,k} \cup \cdots \cup \langle \varrho_k \rangle \rho_{m/d,k}$, where $\rho_{j,k} = u_{j,k} F_{g',b,q} \in G$ for all $1 \leq j \leq (m/d)$. Let $1 \leq j \leq (m/d)$ such that $wF_{g',b,q} \in \langle \varrho_k \rangle \rho_{j,k}$. Let $1 \leq r \leq d$ such that $wF_{g',b,q} = \varrho_k^r \rho_{j,k} = t_k^r u_{j,k} F_{g',b,q}$. Then $w = t_k^r u_{j,k} v$, for some $v \in F_{g',b,q}$ and $(t_k^d)^w = (t_k^d)^{t_k^r u_{j,k} v} = (t_k^d)^{u_{j,k} v}$. $\square$

Recall $F_{g,1,p}/N_d = F_{g,1,p^{(d)}}$, and for every $1 \leq k \leq p$, we denote by τ_k the image of t_k by the natural homomorphism $F_{g,1,p} \to F_{g,1,p^{(d)}}$.

Notation 2.9. Let $H \leq F_{g,1,p}$ be a normal subgroup of finite index such that $N_d \leq H$. Notice $H/N_d \leq F_{g,1,p^{(d)}}$. We set

$$\mathcal{AM}_{g,1,p}(H) = \{\phi \in \mathcal{AM}_{g,1,p} \mid H^\phi = H\},$$

and

$$\mathcal{AM}_{g,1,p^{(d)}}(H/N_d) = \{\tilde{\phi} \in \mathcal{AM}_{g,1,p^{(d)}} \mid (H/N_d)^{\tilde{\phi}} = H/N_d\}.$$

Proposition 2.10. *Suppose $(g, p, d) \neq (0, 2, 2)$. Let $H \leq F_{g,1,p}$ be a normal subgroup of finite index such that $N_d \leq H$. Let $\psi : \mathcal{AM}_{g,1,p} \to \mathcal{AM}_{g,1,p^{(d)}}$ be as in Definition 2.5 and $\phi \in \mathcal{AM}_{g,1,p}(H)$. Then $\psi(\phi) \in \mathcal{AM}_{g,1,p^{(d)}}(H/N_d)$. If $\psi(\phi)|_{H/N_d} = 1$, then $\phi = 1$.*

Proof. Since N_d and H are ϕ-invariant, we see H/N_d is $\psi(\phi)$-invariant. Since $\psi(\phi) \in \mathcal{AM}_{g,1,p^{(d)}}$, we have $\psi(\phi) \in \mathcal{AM}_{g,1,p}(H/N_d)$

Since H has finite index in $F_{g,1,p}$, there exists $r \in \mathbb{Z}$, $r \geq 1$, such that

$$(\Pi_{i=1}^{g}[x_i, y_i]\Pi_{k=1}^{p}t_k)^r \in H.$$

Fix $1 \leq k \leq p$. Since H is normal in $F_{g,1,p}$, we see

$$\bar{t}_k(\Pi_{i=1}^{g}[x_i, y_i]\Pi_{k'=1}^{p}t_{k'})^r t_k \in H.$$

If $\psi(\phi)|_{H/N_d} = 1$, in $F_{g,1,p^{(d)}}$,

$$\begin{aligned}
&\overline{\tau}_k(\Pi_{i=1}^{g}[x_i, y_i]\Pi_{k'=1}^{p}\tau_{k'})^r\tau_k \\
=&(\overline{\tau}_k(\Pi_{i=1}^{g}[x_i, y_i]\Pi_{k'=1}^{p}\tau_{k'})^r\tau_k)^{\psi(\phi)} \\
=&\overline{\tau}_k^{\psi(\phi)}(\Pi_{i=1}^{g}[x_i, y_i]\Pi_{k'=1}^{p}\tau_{k'})^r\tau_k^{\psi(\phi)}.
\end{aligned}$$

Then, in $F_{g,1,p^{(d)}}$, $\tau_k^{\psi(\phi)}\overline{\tau}_k$ commutes with $(\Pi_{i=1}^{g}[x_i, y_i]\Pi_{k'=1}^{p}\tau_{k'})^r$. Recall $F_{g,1,p^{(d)}} = F_{g,1,p}/N_d$. Hence, $F_{g,1,p^{(d)}} \simeq F_{g,1,0} * \langle \tau_1, \tau_2, \ldots, \tau_p \mid \tau_1^d, \tau_2^d, \ldots, \tau_p^d \rangle$. Hence, $\tau_k^{\psi(\phi)}\overline{\tau}_k \in \langle \Pi_{i=1}^{g}[x_i, y_i]\Pi_{k'=1}^{p}\tau_{k'} \rangle$, and,

$$\tau_k^{\psi(\phi)} = (\Pi_{i=1}^{g}[x_i, y_i]\Pi_{k'=1}^{p}\tau_{k'})^{r'}\tau_k, \tag{2.10.1}$$

for some $r' \in \mathbb{Z}$. Recall $[\tau_k^{\psi(\phi)}] = [\tau_j]$, for some $1 \leq j \leq p$. If $(g, p) \neq (0, 1)$, and if $(g, p, d) \neq (0, 2, 2)$, then (2.10.1) implies $r' = 0$ and $\tau_k^{\psi(\phi)} = \tau_k$.

Fix $a \in \{x_i, y_i\}_{1 \leq i \leq g}$. Since H has finite index in $F_{g,1,p}$, there exists $s \in \mathbb{Z}, s \geq 1$, such that $a^s \in H$. If $\psi(\phi)|_{H/N_d} = 1$, then $(a^s)^{\psi(\phi)} = a^s$, and, $a^{\psi(\phi)} = a$.

Since $F_{g,1,p^{(d)}} \simeq F_{g,1,0} * \langle \tau_1, \tau_2, \ldots, \tau_p \mid \tau_1^d, \tau_2^d, \ldots, \tau_p^d \rangle$, $a^{\psi(\phi)} = a$ for all $a \in \{x_i, y_i\}_{1 \leq i \leq g}$, and, $\tau_k^{\psi(\phi)} = \tau_k$ for all $1 \leq k \leq p$; we see $\psi(\phi) = 1$. By Theorem 2.6, $\phi = 1$. $\square$

Remark 2.11. Let $\phi \in \mathcal{AM}_{g,1,p}$. Suppose $F_{g',b,q}$ is ϕ-invariant. Then ϕ induces an automorphisms of $F_{g',b,q}$ by restriction. In $F_{g,1,p}$ we have

(i) $\Pi_{i=1}^{g'}[\hat{x}_i, \hat{y}_i]\Pi_{l=1}^{(b-1)}\hat{z}_l\Pi_{k=1}^{q}\hat{t}_k = (\Pi_{i=1}^{g}[x_i, y_i]\Pi_{k=1}^{p}t_k)^c$;

(ii) $\hat{z}_l$ is conjugate to $(\Pi_{i=1}^{g}[x_i, y_i]\Pi_{k=1}^{p}t_k)^{-c}$, for all $1 \leq l \leq (b-1)$;

(iii) $\hat{t}_k$ is conjugate to t_j^d, $1 \leq j \leq p$, for all $1 \leq k \leq q$.

If we identify $F_{g',b,q}$ with $F_{g',1,(b-1)+q}$ by identifying $\hat{z}_l$ with $\hat{t}_l$, for all $1 \leq l \leq (b-1)$, and $\hat{t}_k$ with $\hat{t}_{(b-1)+k}$, for all $1 \leq k \leq q$; then the restriction of ϕ to $F_{g',1,(b-1)+q}$ lies inside $\mathcal{AM}_{g',1,(b-1)+q}$.

Let h be the homeomorphism of $S_{g,1,p}$ which fixes the boundary component pointwise and $h_* = \phi$. Since $F_{g',b,q}$ is ϕ-invariant, h lifts to a homeomorphism

$\hat{h}$ of $S_{g',b,q}$ which fixes the b-th boundary component pointwise. Since $\hat{h}$ may not fix the first $(b-1)$ boundary components pointwise, $\hat{h}$ does not represent an element of $\mathcal{M}_{g',b,q}$, but it represents an element of $\mathcal{M}_{g',1,(b-1)+q}$, that is, we have to convert the first $(b-1)$ boundary components into punctures. If $\hat{h}$ fixes the boundary components pointwise, we can conserve the first $(b-1)$ boundary components. Algebraically, if we want to have an element of $\mathcal{AM}_{g',b,q}$, we have to define the image of $\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_{(b-1)}$. Since $F_{g',b,q}$ is ϕ-invariant, we see ϕ induces an automorphism of $G = F_{g,1,p}/F_{g',b,q}$. If ϕ induces the identity of G, we can define an element of $\mathcal{AM}_{g',b,q}$ from ϕ.

Recall N_d is the normal closure in $F_{g,1,p}$ of $t_1^d, t_2^d, \ldots, t_p^d$. By Lemma 2.8, N_d is the normal closure in $F_{g',b,q}$ of $\hat{t}_1, \hat{t}_2, \cdots, \hat{t}_q$. Hence, $F_{g',b,0} = F_{g',b,q}/N_d$. We identify $F_{g',b,0}$ with $F_{g',1,(b-1)}$ by identifying $\hat{z}_l$ with $\hat{t}_l$ for all $1 \leq l \leq (b-1)$. Hence, $F_{g',1,(b-1)} = F_{g',b,q}/N_d$. Since $F_{g',b,q}$ is ϕ-invariant, by Proposition 2.10, there exists the restriction $\psi(\phi)|_{F_{g',1,(b-1)}} : F_{g',1,(b-1)} \to F_{g',1,(b-1)}$. Recall $\hat{h}$ is a homeomorphism of $S_{g',b,q}$. Since $h_* = \phi$, we have $\hat{h}_* = \phi|_{F_{g',b,q}}$. Notice $\hat{h}$ extends to a homeomorphism $\hat{f}$ of $S_{g',b,0}$. Notice $\hat{f}$ restricts to a homeomorphism of $S_{g',1,(b-1)} \subseteq S_{g',b,0}$. Since $\hat{h}_* = \phi|_{F_{g',b,q}}$ and $F_{g',1,(b-1)} = F_{g',b,q}/N_d$, the restriction of $\hat{f}$ to $S_{g',1,(b-1)} \subseteq S_{g',b,0}$ induces the automorphism $\psi(\phi)|_{F_{g',1,(b-1)}}$.

We, now, can state and prove the algebraic analog of Theorem 1.1.

Theorem 2.12. *Suppose $(g,p) \neq (0,2)$. Let $\kappa : S_{g',b,0} \to S_{g,1,0}$ be a finite index regular cover with p branching points in $S_{g,1,0}$ which lift to q points in $S_{g',b,0}$. Suppose every branching point of $S_{g,1,0}$ lifts to the same number of points in $S_{g',b,0}$. Let $\kappa' : S_{g',b,q} \to S_{g,1,p}$ be the corresponding unbranched cover. Let $\psi : \mathcal{AM}_{g,1,p} \to \mathcal{AM}_{g,1,p^{(d)}}$ be as in Definition 2.5 and $\phi \in \mathcal{AM}_{g,1,p}$. Suppose $F_{g',b,q}$ is ϕ-invariant. If $\psi(\phi)|_{F_{g',1,(b-1)}} = 1$, then $\phi = 1$.*

Proof. Since $F_{g,1,p^{(d)}} = F_{g,1,p}/N_d$, the natural homomorphism $F_{g,1,p} \to F_{g,1,p^{(d)}}$ restricts to the natural homomorphism $F_{g',b,q} \to F_{g',1,(b-1)}$.

Since $\psi : \mathcal{AM}_{g,1,p} \to \mathcal{AM}_{g,1,p^{(d)}}$ is given by the natural homomorphism $F_{g,1,p} \to F_{g,1,p^{(d)}}$, we see $\psi(\phi) : F_{g,1,p^{(d)}} \to F_{g,1,p^{(d)}}$ completes the following commutative square

$$\begin{array}{ccc} F_{g,1,p} & \xrightarrow{\phi} & F_{g,1,p} \\ \downarrow & & \downarrow \\ F_{g,1,p^{(d)}} & \xrightarrow{\psi(\phi)} & F_{g,1,p^{(d)}} \end{array}$$

where the vertical arrows are the natural homomorphisms. Notice $\psi(\phi)|_{F_{g',1,(b-1)}} : F_{g',1,(b-1)} \to F_{g',1,(b-1)}$ completes the following commutative square

$$\begin{array}{ccc} F_{g',b,q} & \xrightarrow{\phi|_{F_{g',b,q}}} & F_{g',b,q} \\ \downarrow & & \downarrow \\ F_{g',1,(b-1)} & \xrightarrow{\psi(\phi)|_{F_{g',1,(b-1)}}} & F_{g',1,(b-1)} \end{array}$$

where the vertical arrows are the natural homomorphisms. By Proposition 2.10, if $\psi(\phi)|_{F_{g',1,(b-1)}} = 1$, then $\phi = 1$. $\square$

We state the algebraic analog of Corollary 1.2.

Corollary 2.13. *Suppose $(g, p) \neq (0, 2)$. Let $\kappa : S_{g',b,0} \to S_{g,1,0}$ be a finite index regular cover with p branching points in $S_{g,1,0}$ which lift to q points in $S_{g',b,0}$. Suppose every branching point of $S_{g,1,0}$ lifts to the same number of points in $S_{g',b,0}$. Let $\kappa' : S_{g',b,q} \to S_{g,1,p}$ be the corresponding unbranched cover. If $F_{g',b,q}$ is $\mathcal{AM}_{g,1,p}$-invariant, then $\mathcal{AM}_{g,1,p}$ embeds in $\mathcal{AM}_{g',1,(b-1)}$. In fact, the embedding is given by $\phi \mapsto \psi(\phi)|_{F_{g',1,(b-1)}}$, where $\psi : \mathcal{AM}_{g,1,p} \to \mathcal{AM}_{g,1,p^{(d)}}$ is as in Definition 2.5.*

3 Examples

We fix g, p such that $(g, p) \neq (0, 2)$. Let $\hat{S}$ be the universal cover of $S_{g,1,p}$. The fundamental group of $S_{g,1,p}$, denoted $F_{g,1,p}$, acts on $\hat{S}$. Let H be a subgroup of $F_{g,1,p}$ of index m. Suppose H is $\mathcal{AM}_{g,1,p}$-invariant. The quotient space $\hat{S}/H$ is an orientable surface, denoted $S_{g',b,q}$. We identify the fundamental group of $S_{g',b,q}$, denoted $F_{g',b,q}$, with H. The cover $\hat{S} \to S_{g,1,p}$ induces a cover $S_{g',b,q} \to S_{g,1,p}$ with group of deck transformation $G := F_{g,1,p}/F_{g',b,q}$. If $t_k \notin F_{g',b,q}$ for all $1 \leq k \leq p$, then the corresponding cover $S_{g',b,0} \to S_{g,1,0}$ has p branching points in $S_{g,1,0}$ which lift to q points in $S_{g',b,0}$. Since H is $\mathcal{AM}_{g,1,p}$-invariant, it can be seen that every branching point of $S_{g,1,p}$ lifts to the same number of points in $S_{g',b,0}$. By Corollary 2.13, we have an embedding $\mathcal{AM}_{g,1,p} \hookrightarrow \mathcal{AM}_{g',1,(b-1)}$. By choosing an appropriated basis of H/N_d, we can compute elements in the image of this embedding from elements of $\mathcal{AM}_{g,1,p}$.

Example 3.1 is Birman and Hilden [4]. In Example 3.2.(a), we give a basis of $F_{g',1,(b-1)}$ and compute elements in the image of $\mathcal{AM}_{g,1,p} \hookrightarrow \mathcal{AM}_{g',1,(b-1)}$.

Example 3.1. Let H be the kernel of the homomorphism $F_{0,1,p} \to \langle \gamma \mid \gamma^2 \rangle$ such that $t_k \mapsto \gamma$ for all $1 \leq k \leq p$. It is standard to see that H is a free group of rank $2p-1$ with basis $t_1^2, t_1t_2, t_1t_3, \ldots, t_1t_p, t_1\bar{t}_2, t_1\bar{t}_3, \ldots, t_1\bar{t}_p$. It is easy to see that H is invariant by the generators of $\mathcal{AM}_{0,1,p}$ given in (2.3.1). For $1 \leq k \leq p$, notice that $\varrho_k = t_kH$ has order 2 in $G := F_{0,1,p}/H \simeq \langle \gamma \mid \gamma^2 \rangle$. Hence, $\langle \varrho_k \rangle$ has index 1 in G and the k-th puncture in $S_{g,1,p}$ lifts to one puncture in $S_{g',b,q}$. Thus, $q = p$.

(a). If p is odd, then $\Pi_{k=1}^{p} t_k \notin H$ and $\varrho = \Pi_{k=1}^{p} t_kH$ has order 2 in G. Hence, $\langle \varrho \rangle$ has index 1 in G and $b = 1$. Since $F_{g',b,q}$ has rank $2g' + b - 1 + q$ and H has rank $2p-1$, we have $2g' + 1 - 1 + p = 2p - 1$ and $g' = (p-1)/2$. Hence, $\mathcal{AM}_{0,1,p} \hookrightarrow \mathcal{AM}_{(p-1)/2,1,0}$, if p is odd. See [3, 9.2 Example] for a basis of $F_{(p-1)/2,1,0}$.

(b). If p is even, then $\Pi_{k=1}^{p} t_k \in H$ and $\varrho = \Pi_{k=1}^{p} t_k H$ has order 1 in G. Hence, $\langle \varrho \rangle$ has index 2 in G and we have $b = 2$. Since $F_{g',b,q}$ has rank $2g' + b - 1 + q$ and H has rank $2p - 1$, we have $2g' + 2 - 1 + p = 2p - 1$ and $g' = (p-2)/2$. Hence, $\mathcal{AM}_{0,1,p} \hookrightarrow \mathcal{AM}_{(p-2)/2,1,1}$, if p is even. See [3, 9.3 Example] for a basis of $F_{(p-2)/2,1,1}$.

Example 3.2. Let $F_3 := \langle a_1, a_2, a_3 \mid \rangle$. Let H be the kernel of the homomorphism $F_3 \to \langle \gamma_1 \mid \gamma_1^2 \rangle \times \langle \gamma_2 \mid \gamma_2^2 \rangle \times \langle \gamma_3 \mid \gamma_3^2 \rangle$ such that $a_k \mapsto \gamma_k$ for all $1 \leq k \leq 3$. It is standard to see that H is a free group of rank 17. It can be shown that H is a characteristic subgroup of F_3.

(a). We identify $F_{0,1,3}$ with F_3 by putting $t_k \leftrightarrow a_k$ for all $1 \leq k \leq 3$. Notice that $\varrho = t_1 t_2 t_3 H$ has order 2 in $G := F_{0,1,3}/H \simeq \langle \gamma_1 \mid \gamma_1^2 \rangle \times \langle \gamma_2 \mid \gamma_2^2 \rangle \times \langle \gamma_3 \mid \gamma_3^2 \rangle$. Hence, $\langle \varrho \rangle$ has index 4 in G and $b = 4$. On the other hand, for all $1 \leq k \leq 3$, $\varrho_k = t_k H$ has order 2 in G. Hence, for all $1 \leq k \leq 3$, $\langle \varrho_k \rangle$ has index 4 in G and the k-th puncture in $S_{0,1,3}$ lifts to 4 punctures in $S_{g',b,q}$. Thus, $q = 12$. Since $F_{g',b,q}$ has rank $2g' + b - 1 + q$ and H has rank 17, we have $2g' + 4 - 1 + 12 = 17$ and $g' = 1$. Hence, $\mathcal{AM}_{0,1,3} \hookrightarrow \mathcal{AM}_{1,1,3}$. It is well-known that $\mathcal{AM}_{0,1,3} = \langle \sigma_1, \sigma_2 \mid \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2 \rangle$, where

$$\sigma_1 := \begin{cases} t_1 & \mapsto & t_2, \\ t_2 & \mapsto & t_1^{t_2}, \\ t_3 & \mapsto & t_3, \end{cases} \qquad \sigma_2 := \begin{cases} t_1 & \mapsto & t_1, \\ t_2 & \mapsto & t_3, \\ t_3 & \mapsto & t_2^{t_3}. \end{cases}$$

Let N_2 be the normal closure of t_1^2, t_2^2, t_3^2 in $F_{0,1,3}$. Then $F_{0,1,3}/N_2 = F_{0,1,3^{(2)}} = \langle \tau_1, \tau_2, \tau_3 \mid \tau_1^2, \tau_2^2, \tau_3^2 \rangle$ and $H/N_2 \simeq F_{1,1,3}$. We take the following basis of H/N_2: $\hat{x} = \tau_2\tau_3\tau_1\tau_3\tau_2\tau_1, \hat{y} = \tau_1\tau_2\tau_1\tau_2, \hat{t}_1 = (\tau_3\tau_2\tau_1\tau_3\tau_2\tau_1)^{\tau_3}, \hat{t}_2 = (\tau_3\tau_2\tau_1\tau_3\tau_2\tau_1)^{\tau_3\tau_1\tau_3}, \hat{t}_3 = (\tau_3\tau_2\tau_1\tau_3\tau_2\tau_1)^{\tau_1\tau_3}$. Then $[\hat{x}, \hat{y}]\hat{t}_1\hat{t}_2\hat{t}_3 = (\tau_1\tau_2\tau_3)^2$ and

$$\hat{\sigma}_1 := \begin{cases} \hat{x} & \mapsto & \hat{y}^{-1}\hat{x}\hat{y}\hat{t}_1\hat{t}_2\hat{t}_1^{-1}\hat{y}^{-1}, \\ \hat{y} & \mapsto & \hat{y}, \\ \hat{t}_1 & \mapsto & \hat{t}_1, \\ \hat{t}_2 & \mapsto & \hat{t}_3, \\ \hat{t}_3 & \mapsto & \hat{t}_2^{\hat{t}_1^{-1}\hat{y}^{-1}\hat{x}^{-1}\hat{y}^{-1}\hat{x}\hat{y}\hat{t}_1\hat{t}_2\hat{t}_3}, \end{cases} \qquad \hat{\sigma}_2 := \begin{cases} \hat{x} & \mapsto & \hat{x}, \\ \hat{y} & \mapsto & \hat{x}\hat{y}\hat{z}_1\hat{t}_2, \\ \hat{t}_1 & \mapsto & \hat{t}_3, \\ \hat{t}_2 & \mapsto & \hat{t}_2^{\hat{t}_1^{-1}\hat{y}^{-1}\hat{x}\hat{y}\hat{t}_1\hat{t}_2\hat{t}_3}, \\ \hat{t}_3 & \mapsto & \hat{t}_1^{\hat{y}^{-1}\hat{x}\hat{y}\hat{t}_1\hat{t}_2\hat{t}_3}. \end{cases}$$

(b). We identify $F_{1,1,1}$ with F_3 by putting $x \leftrightarrow a_1, y \leftrightarrow a_2$ and $t \leftrightarrow a_3$. Notice that $\varrho = [x, y]tH$ has order 2 in $G := F_{1,1,1}/H \simeq \langle \gamma_1 \mid \gamma_1^2 \rangle \times \langle \gamma_2 \mid \gamma_2^2 \rangle \times \langle \gamma_3 \mid \gamma_3^2 \rangle$. Hence, $\langle \varrho \rangle$ has index 4 in G and $b = 4$. On the other hand, $\varrho_1 = tH$ has order 2 in G. Hence, $\langle \varrho_1 \rangle$ has index 4 in G and the puncture in $S_{1,1,1}$ lifts to 4 punctures in $S_{g',b,q}$. Thus, $q = 4$. Since $F_{g',b,q}$ has rank $2g' + b - 1 + q$ and H has rank 17, we have $2g' + 4 - 1 + 4 = 17$ and $g' = 5$. Hence, $\mathcal{AM}_{1,1,1} \hookrightarrow \mathcal{AM}_{5,1,3}$.

4 Proof of Theorem 2.6

Definition 4.1. An element of $F_{g,1,p}$ is said to be *t-squarefree* if, in its reduced expression, no two consecutive terms in $\{t_k, \overline{t}_k\}_{1\leq k\leq p}$ are equal; for example: $x_1x_1t_2t_3$ is t-squarefree; $x_1t_2t_2y_1$ is non-t-squarefree.

To prove Theorem 2.6 we need the following theorem (compare with [3, 7.6 Corollary]).

Theorem 4.2. *For every $\phi \in \mathcal{AM}_{g,1,p}$, the elements of $\{x_i^\phi, y_i^\phi\}_{1\leq i\leq g} \cup \{t_k^\phi\}_{1\leq k\leq p}$ are t-squarefree.*

Proof. (of Theorem 2.6) If $p = 0$, then $\psi : \mathcal{AM}_{g,1,p} \to \mathcal{AM}_{g,1,p^{(d)}}$ is the identity and nothing needs to be said.

Suppose $p \geq 1$. Recall $F_{g,1,p^{(d)}} \simeq F_{g,1,0} * \langle \tau_1, \tau_2, \ldots, \tau_p \mid \tau_1^d, \tau_2^d, \ldots, \tau_p^d\rangle$. Let $a \in \{x_i, y_i\}_{1\leq i\leq g} \cup \{t_k\}_{1\leq k\leq p}$. If ϕ is an element of the kernel of $\psi : \mathcal{AM}_{g,1,p} \to \mathcal{AM}_{g,1,p^{(d)}}$, then $\psi(\phi)$ is the identity of $\mathrm{Aut}(F_{g,1,p^{(d)}})$. Hence, a^ϕ and a have the same image by the natural homomorphism $F_{g,1,p} \to F_{g,1,p^{(d)}}$. On the other hand, by Theorem 4.2, a^ϕ is t-squarefree. Hence, a^ϕ has the same normal form in $F_{g,1,p}$ as in $F_{g,1,p^{(d)}}$. Thus, $a^\phi = a$. $\square$

The rest of the paper is dedicated to prove Theorem 4.2. Notice Theorem 4.2 is trivial for $p = 0$. We will suppose $p \geq 1$. To prove Theorem 4.2 we will use ends of $F_{g,1,p}$, that is, reduced right-infinite words of $F_{g,1,p}$. We will recall that there is an action of $\mathrm{Aut}(F_{g,1,p})$ on the set of ends of $F_{g,1,p}$. In particular, there is an action of $\mathcal{AM}_{g,1,p} \leq \mathrm{Aut}(F_{g,1,p})$ on the set of ends of $F_{g,1,p}$.

The strategy to prove Theorem 4.2 is the following. We define a subset A of the set of ends of $F_{g,1,p}$ such that:

(a) A is $\mathcal{AM}_{g,1,p}$-invariant,

(b) every non-t-squarefree end of $F_{g,1,p}$ (see Definition 7.1) lies in A,

(c) for $(g,p) \neq (0,1)$, $(0,2)$ and $a \in \{\overline{t}_p\} \cup \{x_i, y_i, \overline{x}_i, \overline{y}_i\}_{1\leq i\leq g}$, the end of $F_{g,1,p}$ $a(\Pi_{i=1}^g[x_i,y_i]\Pi_{k=1}^p t_k)^\infty$ does not lie in A.

From (a) and (c) we see that for $(g,p) \neq (0,1), (0,2)$ the set A does not intersect the $\mathcal{AM}_{g,1,p}$-orbit of $a(\Pi_{i=1}^g[x_i,y_i]\Pi_{k=1}^p t_k)^\infty$, where $a \in \{\overline{t}_p\} \cup \{x_i, y_i, \overline{x}_i, \overline{y}_i\}_{1\leq i\leq g}$. Then, by (b), for $(g,p) \neq (0,1)$, $(0,2)$ the elements of the $\mathcal{AM}_{g,1,p}$-orbit of $a(\Pi_{i=1}^g[x_i,y_i]\Pi_{k=1}^p t_k)^\infty$ are t-squarefree. From this fact, and an easy analysis in the special cases $(g,p) = (0,1)$, $(0,2)$, we prove Theorem 4.2.

The subset A is defined via a linear ordering of the set of ends of $F_{g,1,p}$. To prove (a) above we need the fact that this ordering is respected by the $\mathcal{AM}_{g,1,p}$-action. To prove (b) and (c) above, we have to check same inequalities with respect to this ordering.

In Section 5 and Section 6 we define the ordering of the set of ends of $F_{g,1,p}$ and we show that this ordering is respected by the $\mathcal{AM}_{g,1,p}$-action. In Section 7 we prove (a), (b) and (c) above.

5 McCool's Groupoid

For the rest of the paper we suppose $p \geq 1$.

In this section we define McCool's groupoid via Whitehead's graphs, we remark that $\mathcal{AM}_{g,1,p}$ is a subgroup of McCool's groupoid and we recall that McCool's groupoid is generated by Nielsen elements. These facts will be useful to see that the ordering that we will define on the set of ends of $F_{g,1,p}$ is respected by the $\mathcal{AM}_{g,1,p}$-action.

Let $n := 2g + p$, and, let F_n be the free group on X, where X is a set with n elements.

Notation 5.1. Let $w \in F_n$. In this section we will denote by $[w]$ the cyclic word of w.

Definitions 5.2. Let T be a set of words and cyclic words of F_n. Suppose the elements of T are reduced and cyclically reduced, respectively. We define the *Whitehead graph of* T as the graph with vertex set $X \cup \overline{X}$, and, one edge from $a \in X \cup \overline{X}$ to $b \in X \cup \overline{X}$ for every subword $\overline{a}b$ which appears in w or $[u]$, where w and $[u]$ are elements of T. We say that a is the initial vertex and b is the terminal vertex of the edges corresponding to the subword $\overline{a}b$. Repetitions have to be considered. For example, since the subword $\overline{a}b$ appears twice in $\overline{a}b\overline{a}b$, the Whitehead graph of $\{\overline{a}b\overline{a}b\}$ has 2 edges from a to b (and one edge from $\overline{b}$ to $\overline{a}$). A word $\Pi_{i=1}^{k} a_i$ produces $k-1$ edges in the Whitehead graph. A cyclic word $[\Pi_{i=1}^{k} a_i]$ produces k edges in the Whitehead graph. For example, the Whitehead graph of $\{a\}$ does not have any edge and the Whitehead graph of $\{[a]\}$ has one edge from $\overline{a}$ to a.

We say that T is a *surface word set* if the Whitehead graph of T is an oriented segment, that is, the Whitehead graph of T is connected with exactly $2n-1$ edges, every vertex but one is the *initial vertex* of exactly one edge, and, every vertex but one is the *terminal vertex* of exactly one edge.

Example 5.3. Let $F_4 := \langle a, b, c, d \mid \rangle$.

(i). Let $T := \{\overline{a}dc\overline{b}, [\overline{d}b], [\overline{c}a]\}$. The Whitehead graph of T is

$$\overline{a} \to \overline{c} \to \overline{b} \to \overline{d} \to c \to a \to d \to b.$$

Hence, T is a surface word set.

(ii). Let $T := \{\overline{a}dc\overline{b}, \overline{d}b, [\overline{c}a]\}$. The Whitehead graph of T is

$$\overline{a} \to \overline{c} \to \overline{b} \qquad \overline{d} \to c \to a \to d \to b.$$

Hence, T is not a surface word set.

(iii). Let $T := \{\overline{a}dc\overline{b}, dc, [\overline{d}b], [\overline{c}a]\}$. The Whitehead graph of T is

$$\overline{a} \to \overline{c} \to \overline{b} \to \overline{d} \rightrightarrows c \to a \to d \to b.$$

Hence, T is not a surface word set.

We illustrate the following remarks with examples in $F_4 = \langle a, b, c, d \mid \rangle$.

Remarks 5.4. Let T be a surface word set.

(i) The Whitehead graph of T defines a sequence $(a_k)_{1\leq k\leq 2n}$ which lists the element of $X \cup \overline{X}$ such that for all $1 \leq k \leq (2n-1)$, the Whitehead graph of T has exactly one edge with initial vertex a_k and terminal vertex a_{k+1}, equivalently, $\overline{a}_k a_{k+1}$ is a subword that appears exactly once in T. We say that $(a_k)_{1\leq k\leq 2n}$ is the *associated sequence* of T.

In Example 5.3(i), the associated sequence of T is $(\overline{a}, \overline{c}, \overline{b}, \overline{d}, c, a, d, b)$.

(ii) We can recover T from the associated sequence of T. The process to recover T from its associated sequence is the inverse process to construct the Whitehead graph. We give two examples below. From this process, it is easy to see that T has exactly one word, and, all other elements of T are cyclic words.

In F_4, from the sequence $(a, b, c, d, \overline{a}, \overline{b}, \overline{c}, \overline{d})$ we have the surface word set $\{a\overline{b}c\overline{d}\,\overline{a}b\overline{c}d\}$, and, from the sequence $(a, b, c, d, \overline{d}, \overline{c}, \overline{b}, \overline{a})$ we have the surface word set $\{a, [b\overline{a}], [c\overline{b}], [d\overline{c}], [\overline{d}]\}$.

(iii) Let p be the cardinality of T minus one. We say that T is a (g,p)-surface word set, where $g = (n-p)/2$. By induction on n, it can be seen that $n \geq p$ and $n - p$ is even. Hence, g is a non-negative integer.

Definition 5.5. Let $\phi \in \mathrm{Aut}(F_n)$.

We say that ϕ is a *type*-1 *Nielsen automorphism* if ϕ restricts to a permutation of $X \cup \overline{X}$.

We say that ϕ is a *type*-2 *Nielsen automorphism* if there exist $a, b \in X \cup \overline{X}$ such that $a \neq b, \overline{b}$ and

$$\phi := \begin{cases} a & \mapsto & ab, \\ c & \mapsto & c \quad \text{for all } c \in X,\ c \neq a^{\pm 1}. \end{cases}$$

We denote ϕ by $(a \mapsto ab)$ or $(\overline{a} \mapsto \overline{b}\overline{a})$.

Definition 5.6. Let $\mathcal{G}_{g,p}$ be the groupoid with objects (g,p)-surface word sets, and, given T_1, T_2 two (g,p)-surface word sets

$$\mathrm{Hom}(T_1, T_2) := \{\phi \in \mathrm{Aut}(F_n) \mid T_1^{\phi} = T_2\},$$

where $T_1^{\phi} := \{w^{\phi}, [u^{\phi}] \mid w, [u] \in T_1\}$. Here, w^{ϕ} is reduced and $[u^{\phi}]$ is cyclically reduced. Hence, $[v] = [u^{\phi}]$ means that v and u^{ϕ} are conjugate.

We say that $(T_1, T_2, \phi) \in \mathrm{Hom}(T_1, T_2)$ is a *type*-1 *Nielsen element* of $\mathcal{G}_{g,p}$ if ϕ is a type-1 Nielsen automorphism. Similarly, for type-2 Nielsen automorphisms. We say that $(T_1, T_2, \phi) \in \mathrm{Hom}(T_1, T_2)$ is a *Nielsen element* if it is either a type-1 Nielsen or a type-2 Nielsen.

We illustrate the following remarks with examples in $F_4 = \langle a, b, c, d \mid \rangle$.

Remark 5.7. Let (T_1, T_2, ϕ) be a Nielsen of $\mathcal{G}_{g,p}$.

(i) If (T_1, T_2, ϕ) is a type-1 Nielsen, then the associated sequence of T_2 is obtained from the associated sequence of T_1 by applying the permutation ϕ to every element of the sequence.

In F_4, let $T_1 = \{a\overline{d}\,\overline{b}c, [\overline{a}b], [\overline{c}d]\}$. Notice the associated sequence of T_1 is $(a, b, c, d, \overline{b}, \overline{a}, \overline{d}, \overline{c})$. If $\phi := (a \mapsto \overline{b}, b \mapsto c, c \mapsto \overline{a}, d \mapsto \overline{d})$, then the associated sequence of $T_2 = T_1^{\phi} = \{\overline{b}d\overline{c}\,\overline{a}, [bc], [a\overline{d}]\}$ is $(\overline{b}, c, \overline{a}, \overline{d}, \overline{c}, b, d, a)$.

(ii) Suppose (T_1, T_2, ϕ) is a type-2 Nielsen. Then $\phi = (a_i \mapsto ba_i)$ for some $1 \leq i \leq 2n$, $b \in X \cup \overline{X}$ such that $a_i \neq b, \overline{b}$. Since in the Whitehead graph of T_2 there are exactly $2n-1$ edges, there exists $w \in T_1$ or $[u] \in T_1$ such that applying ϕ to w or $[u]$ produces a cancellation. If the cancellation appears from the subword $\overline{a}_{i-1}a_i$, then $b = a_{i-1}$. If the cancellation appears from the subword $\overline{a}_i a_{i+1}$, then $b = a_{i+1}$. Hence, either $\phi = (a_i \mapsto a_{i-1}a_i)$ for some $2 \leq i \leq 2n$, $a_i \neq \overline{a}_{i-1}$; or $\phi = (\overline{a}_i \mapsto \overline{a}_i\overline{a}_{i+1})$ for some $1 \leq i \leq (2n-1)$, $a_i \neq \overline{a}_{i+1}$. In the former case the associated sequence of T_2 is obtained from the associated sequence of T_1 by moving a_i from immediately after a_{i-1} to immediately before $\overline{a}_{i-1}$. In the later case the associated sequence of T_2 is obtained from the associated sequence of T_1 by moving a_i from immediately before a_{i+1} to immediately after $\overline{a}_{i+1}$.

In F_4, let $T_1 = \{a\overline{b}c\overline{d}\,\overline{a}b\overline{c}d\}$. Notice the associated sequence of T_1 is $(a, b, c, d, \overline{a}, \overline{b}, \overline{c}, \overline{d})$. If $\phi := (b \mapsto ab)$, then the associated sequence of $T_2 = T_1^{\phi}$ is $(a, c, d, b, \overline{a}, \overline{b}, \overline{c}, \overline{d})$. In fact $(a\overline{b}c\overline{d}\,\overline{a}b\overline{c}d)^{(b \mapsto ab)} = a\overline{b}\,\overline{a}c\overline{d}b\overline{c}d$. If $\phi := (\overline{a} \mapsto \overline{a}\overline{b})$, then the associated sequence of $T_2 = T_1^{\phi}$ is $(b, c, d, \overline{a}, \overline{b}, a, \overline{c}, \overline{d})$. In fact $(a\overline{b}c\overline{d}\,\overline{a}b\overline{c}d)^{(\overline{a} \mapsto \overline{a}\overline{b})} = ba\overline{b}c\overline{d}\,\overline{a}\,\overline{c}d$.

Remark 5.8. It is easy to see that $\{\Pi_{i=1}^{g}[x_i, y_i]\Pi_{k=1}^{p}t_k, [\overline{t}_1], [\overline{t}_2], \ldots, [\overline{t}_p]\}$ is a (g,p)-surface word set of $F_{g,1,p}$. Its associated sequence is

$$(\overline{x}_1, y_1, x_1, \overline{y}_1, \overline{x}_2, y_2, x_2, \overline{y}_2, \ldots, \overline{x}_g, y_g, x_g, \overline{y}_g, t_1, \overline{t}_1, t_2, \overline{t}_2, \ldots, t_p, \overline{t}_p).$$

We say that $\{\Pi_{i=1}^{g}[x_i, y_i]\Pi_{k=1}^{p}t_k, [\overline{t}_1], [\overline{t}_2], \ldots, [\overline{t}_p]\}$ is the standard (g,p)- surface word set of $F_{g,1,p}$.

Remark 5.9. $\mathcal{AM}_{g,1,p} = \mathrm{Hom}(T, T)$, where T is the standard (g,p)-surface word set of $F_{g,1,p}$.

Theorem 5.10 (McCool [11],[7]). *$\mathcal{G}_{g,p}$ is generated by Nielsen elements.*

6 Ends of free group

In this section we define, for every (g,p)-surface word set, an ordering of the set of ends of F_n, where $n = 2g + p$. In particular, we define an ordering for the standard (g,p)-surface word set, called the ordering of the set of ends of $F_{g,1,p}$. There is an action of $\mathcal{AM}_{g,1,p}$ on the set of ends of $F_{g,1,p}$. We show that the ordering of the set of ends of $F_{g,1,p}$ is respected by the $\mathcal{AM}_{g,1,p}$-action. We use *shadows* of the set of ends of F_n and results of Section 5.

Recall $n := 2g + p$ and F_n is the free group on X, where X is a set with n elements.

Notation 6.1. Let $\Pi_{i=1}^k a_i$ be the normal form for $w \in F_n$. Then we say that w has length k, denoted $|w| = k$. The set of elements of F_n whose normal forms have $\Pi_{i=1}^k a_i$ as an initial subword is denoted $(w\star)$; and, the set of elements of F_n whose normal forms have $\Pi_{i=1}^k a_i$ as a terminal subword is denoted $(\star w)$. The elements of $(w\star)$ are said to *begin with* w, and the elements of $(\star w)$ are said to *end with* w.

Definition 6.2. An *end* of F_n is a right-infinite word $\Pi_{k\geq 1} a_k = a_1 a_2 \cdots$ where $a_k \in X \cup \overline{X}$ and $a_{k+1} \neq \overline{a}_k$ for every $k \geq 1$.

We denote the set of ends of F_n by ∂F_n.

For each $w \in F_n$, we define the *shadow* of w in ∂F_n to be

$$(w\blacktriangleleft) := \{\Pi_{k\geq 1} a_k \in \partial F_n \mid \Pi_{k=1}^{|w|} a_k = w\}.$$

Thus, for example, $(1\blacktriangleleft) = \partial F_n$.

Definition 6.3. Let T be a surface word set. We now give ∂F_n an ordering, $<_T$, with respect to T as follows. Let $(a_k)_{1\leq k\leq 2n}$ be the associated sequence of T. Recall $(a_k)_{1\leq k\leq 2n}$ is a listing of the elements of $X \cup \overline{X}$. For each $w \in F_n$, we assign an ordering, $<_T$, to a partition of $(w\blacktriangleleft)$ into $2n$ or $2n-1$ subsets, depending as $w = 1$ or $w \neq 1$, as follows. We set

$$(a_1\blacktriangleleft) <_T (a_2\blacktriangleleft) <_T (a_3\blacktriangleleft) <_T \cdots <_T (a_{2n-1}\blacktriangleleft) <_T (a_{2n}\blacktriangleleft).$$

If $1 \leq i \leq n$ and $w \in (\star\overline{a}_i)$, then we set

$$\begin{aligned}&(wa_{i+1}\blacktriangleleft) <_T (wa_{i+2}\blacktriangleleft) <_T (wa_{i+3}\blacktriangleleft) <_T \cdots\\ &\quad \cdots <_T (wa_{2n-1}\blacktriangleleft) <_T (wa_{2n}\blacktriangleleft) <_T (wa_1\blacktriangleleft) <_T (wa_2\blacktriangleleft) <_T (wa_3\blacktriangleleft) <_T \cdots\\ &\quad \cdots <_T (wa_{i-2}\blacktriangleleft) <_T (wa_{i-1}\blacktriangleleft).\end{aligned}$$

Hence, for each $w \in F_n$, we have an ordering $<_T$ of a partition of $(w\blacktriangleleft)$ into $2n$ or $2n-1$ subsets.

If $\Pi_{k\geq 1} b_k$ and $\Pi_{k\geq 1} c_k$ are two different ends, then there exists $j \in \mathbb{Z}, j \geq 0$, such that $\Pi_{k=1}^j b_k = \Pi_{k=1}^j c_k$ and $b_{j+1} \neq c_{j+1}$. Let $w = \Pi_{k=1}^j b_k = \Pi_{k=1}^j c_k$ in

F_n. Then $\Pi_{k\geq 1}b_k$ and $\Pi_{k\geq 1}c_k$ lie in $(w\blacktriangleleft)$, but lie in different elements of the partition of $(w\blacktriangleleft)$ into $2n$ or $2n-1$ subsets. We then order $\Pi_{k\geq 1}b_k$ and $\Pi_{k\geq 1}c_k$ using the ordering of the elements of the partition of $(w\blacktriangleleft)$ that they belong to. This completes the definition of the ordering $<_T$ of ∂F_n.

Remark 6.4. Let w be the non-cyclic element of T. In $(\partial F_n, <_T)$, the smallest element is w^∞ and the largest element is $\overline{w}^\infty$.

For example, in $F_4 = \langle a, b, c, d \mid \rangle$ we take the surface word set $T = \{a\overline{d}\,\overline{b}c, [\overline{a}b], [\overline{c}d]\}$. The associated sequence of T is $(a, b, c, d, \overline{b}, \overline{a}, \overline{d}, \overline{c})$. In $(\partial F_4, <_T)$, the smallest element is $(a\overline{d}\,\overline{b}c)^\infty$, and, the largest element is $(\overline{c}bd\overline{a})^\infty$.

Notation 6.5. We denote by $<$ the ordering of $\partial F_{g,1,p}$ with respect to the standard (g,p)-surface word set of $F_{g,1,p}$.

Review 6.6. Let $\hat{S}$ be the universal cover of $S_{g,1,p}$. Suppose $S_{g,1,p}$ has negative Euler characteristic, that is, $2g+p\geq 2$. Then $\hat{S}$ can be identified with a convex region of the hyperbolic plane $\mathbb{H}^2$. Let $\partial\hat{S}$ be the boundary of $\hat{S}$. Then $\partial\hat{S}$ is a union of geodesic segments of the hyperbolic plane $\mathbb{H}^2$. The union of $\partial\hat{S}$ and the set of geodesic rays of $\hat{S}$, denoted $\partial\hat{S}^-$, can be identified with the boundary of a disc, that is, $\mathbb{R}\cup\{\infty\}$. Let $*$ be the point in $\partial\hat{S}$ corresponding to ∞ by this identification. By work of Nielsen-Thurston [5], [12], there is an action of $\mathcal{M}_{g,1,p}$ on $\partial\hat{S}\cup\partial\hat{S}^-$ which fixes $*\in\partial\hat{S}\cup\partial\hat{S}^-$. This action is defined as follows. There is a bijection between point of $\partial\hat{S}$ and geodesic segments of $\hat{S}$ with starting point $*$ and endpoint in $\partial\hat{S}$. There is a bijection between geodesic rays of $\hat{S}$ and infinite geodesic segments of $\hat{S}$ starting at $*$. Let $\hat{\gamma}$ be such a (finite or infinite) geodesic segment. Let γ be the projection of $\hat{\gamma}$ in S. Let $[f]\in\mathcal{M}_{g,1,p}$. We can suppose that f is an isometry of S. Then, $f(\gamma)$ is a geodesic segment. Define $[f]\cdot\hat{\gamma}$ as the lift of $f(\gamma)$ with starting point $*$. Notice this lift defines a point of $\partial\hat{S}\cup\partial\hat{S}^-$ and $*\in\partial\hat{S}$ is fixed by this action. Hence, there exists an action of $\mathcal{M}_{g,1,p}$ on $\mathbb{R}$. By [12] or [6, Chapter 7], this action respects the usual ordering of $\mathbb{R}$. Corollary 6.8 gives the analog statement for $\mathcal{AM}_{g,1,p}$ and $\partial F_{g,1,p}$.

Let $\phi\in\mathrm{Aut}(F_n)$. It is proved in [5] that $(\Pi_{k\geq 1}a_k)^\phi = \lim_{k\to\infty}(\Pi_{i=1}^k a_i)^\phi$ defines a map $\partial F_n\to\partial F_n$, which we still denote by ϕ.

Proposition 6.7. *Let T_1, T_2 be surface word sets of F_n and $(T_1, T_2, \phi)\in\mathrm{Hom}(T_1, T_2)$. Then $\phi : (\partial F_n, \leq_{T_1})\to(\partial F_n, \leq_{T_2})$ respects the orderings.*

Proof. By Theorem 5.10, we can restrict ourselves to the case where (T_1, T_2, ϕ) is a Nielsen.

By Remark 5.7(i), the result is clear if (T_1, T_2, ϕ) is a type-1 Nielsen. Hence, we suppose (T_1, T_2, ϕ) is a type-2 Nielsen.

Let $(a_k)_{1\leq k\leq 2n}$ be the associated sequence of T_1. Then either $\phi = (a_i\mapsto a_{i-1}a_i)$ for some $2\leq i\leq 2n$, $a_i\neq\overline{a}_{i-1}$; or, $\phi = (\overline{a}_i\mapsto\overline{a}_i\overline{a}_{i+1})$ for some $1\leq i\leq (2n-1)$, $a_i\neq\overline{a}_{i+1}$.

Suppose $\phi = (a_i \mapsto a_{i-1}a_i)$ for some $2 \le i \le 2n$, $a_i \ne \overline{a}_{i-1}$.

The following correspondence by the action of $(a_i \mapsto a_{i-1}a_i)$ is clear.

$$\begin{array}{lcl} & (a_i \mapsto a_{i-1}a_i) & \\ (\star\, \overline{a}_i a_{i-1}) & \longrightarrow & (\star\, \overline{a}_i), \\ (\star\, a_{i-1}) - (\star\, \overline{a}_i a_{i-1}) & \longrightarrow & (\star\, a_{i-1}), \\ (\star\, a_i) & \longrightarrow & (\star\, a_i), \\ (\star\, a_k) & \longrightarrow & (\star\, a_k), \quad a_k \ne a_{i-1}^{\pm 1}, a_i^{\pm 1}, \\ (\star\, \overline{a}_{i-1}) & \longrightarrow & (\star\, \overline{a}_{i-1}) - (\star \overline{a}_i \overline{a}_{i-1}), \\ (\star\, \overline{a}_i) & \longrightarrow & (\star\, \overline{a}_i \overline{a}_{i-1}). \end{array}$$

The following correspondence by the action of $(a_i \mapsto a_{i-1}a_i)$ is clear.

$$\begin{array}{lcl} & (a_i \mapsto a_{i-1}a_i) & \\ (a_{i-1} \blacktriangleleft) & \longrightarrow & (a_{i-1} \blacktriangleleft) - (a_{i-1}a_i \blacktriangleleft), \\ (a_i \blacktriangleleft) & \longrightarrow & (a_{i-1}a_i \blacktriangleleft), \\ (a_k \blacktriangleleft) & \longrightarrow & (a_k \blacktriangleleft), \quad a_k \ne a_{i-1}^{\pm 1}, a_i^{\pm 1}, \\ (\overline{a}_{i-1} a_i \blacktriangleleft) & \longrightarrow & (a_i \blacktriangleleft), \\ (\overline{a}_{i-1} \blacktriangleleft) - (\overline{a}_{i-1} a_i \blacktriangleleft) & \longrightarrow & (\overline{a}_{i-1} \blacktriangleleft), \\ (\overline{a}_i \blacktriangleleft) & \longrightarrow & (\overline{a}_i \blacktriangleleft). \end{array}$$

From the first row of the first table and the second table we deduce the following table.

$$\begin{array}{lcl} & (a_i \mapsto a_{i-1}a_i) & \\ (\star\, \overline{a}_i a_{i-1})(a_{i-1} \blacktriangleleft) & \longrightarrow & (\star\, \overline{a}_i)[(a_{i-1} \blacktriangleleft) - (a_{i-1}a_i \blacktriangleleft)], \\ (\star\, \overline{a}_i a_{i-1})(a_i \blacktriangleleft) & \longrightarrow & (\star\, \overline{a}_i)(a_{i-1}a_i \blacktriangleleft), \\ (\star\, \overline{a}_i a_{i-1})(a_k \blacktriangleleft) & \longrightarrow & (\star\, \overline{a}_i)(a_k \blacktriangleleft), \quad a_k \ne a_{i-1}^{\pm 1}, a_i^{\pm 1}, \\ (\star\, \overline{a}_i a_{i-1})(\overline{a}_i \blacktriangleleft) & \longrightarrow & (\star\, \overline{a}_i)(\overline{a}_i \blacktriangleleft). \end{array}$$

Notice the cases $(\star\overline{a}_i a_{i-1})(\overline{a}_{i-1} a_i \blacktriangleleft)$ and $(\star\overline{a}_i a_{i-1})[(\overline{a}_{i-1} \blacktriangleleft) - (\overline{a}_{i-1} a_i \blacktriangleleft)]$ do not have to be considered since they are not in reduced form.

Let $\mathfrak{e}, \mathfrak{f} \in \partial F_n$ such that $\mathfrak{e} = (w\overline{a}_i a_{i-1})\mathfrak{e}'$, $\mathfrak{f} = (w\overline{a}_i a_{i-1})\mathfrak{f}'$ and the first letter of $\mathfrak{e}'$ is different from the first letter of $\mathfrak{f}'$. Let $1 \le j \le 2n$ such that $a_j = \overline{a}_{i-1}$. By the third table, $\mathfrak{e}^{(a_i \mapsto a_{i-1}a_i)} = (u\overline{a}_i)\mathfrak{e}''$, $\mathfrak{f}^{(a_i \mapsto a_{i-1}a_i)} = (u\overline{a}_i)\mathfrak{f}''$ in reduced form. Let $(b_k)_{1 \le k \le 2n}$ be the associated sequence of T_2. Recall $(b_k)_{1 \le k \le 2n}$ is obtained from $(a_k)_{1 \le k \le 2n}$ by moving a_i from immediately after a_{i-1} to immediately before $a_j = \overline{a}_{i-1}$. There are two cases according to $j < i-1$ or $i-1 < j$.

If $j < i-1$, then

$$\begin{array}{ll} (b_k)_{1 \le k \le (j-1)} & = (a_k)_{1 \le k \le (j-1)}, \\ (b_j) & = (a_i), \\ (b_k)_{(j+1) \le k \le i} & = (a_k)_{j \le k \le (i-1)}, \\ (b_k)_{(i+1) \le k \le 2n} & = (a_k)_{(i+1) \le k \le 2n}. \end{array}$$

The partition with respect to $(a_k)_{1\le k\le 2n}$ of $(\overline{a}_j \blacktriangleleft) = (a_{i-1} \blacktriangleleft)$ is $(a_{j+1} \blacktriangleleft)$, $(a_{j+2} \blacktriangleleft), \ldots, (a_{i-1} \blacktriangleleft), (a_i \blacktriangleleft), (a_{i+1} \blacktriangleleft), \ldots, (a_{2n} \blacktriangleleft), (a_1 \blacktriangleleft), (a_2 \blacktriangleleft), \ldots,$ $(a_{j-1} \blacktriangleleft)$. The partition with respect to $(b_k)_{1\le k\le 2n}$ of $(\overline{a}_i \blacktriangleleft)$ is $(a_j \blacktriangleleft)$, $(a_{j+1} \blacktriangleleft), \ldots, (a_{i-1} \blacktriangleleft), (a_{i+1} \blacktriangleleft), (a_{i+2} \blacktriangleleft), \ldots, (a_{2n} \blacktriangleleft), (a_1 \blacktriangleleft), (a_2 \blacktriangleleft), \ldots,$ $(a_{j-1} \blacktriangleleft)$. By the third table,

$$\begin{array}{lcl}
 & (a_i \mapsto a_{i-1}a_i) & \\
(w\overline{a}_i a_{i-1})(a_{j+1} \blacktriangleleft) & \longrightarrow & (u\overline{a}_i)(a_{j+1} \blacktriangleleft), \\
(w\overline{a}_i a_{i-1})(a_{j+2} \blacktriangleleft) & \longrightarrow & (u\overline{a}_i)(a_{j+2} \blacktriangleleft), \\
 & \vdots & \\
(w\overline{a}_i a_{i-1})(a_{i-2} \blacktriangleleft) & \longrightarrow & (u\overline{a}_i)(a_{i-2} \blacktriangleleft), \\
(w\overline{a}_i a_{i-1})(a_{i-1} \blacktriangleleft) & \longrightarrow & (u\overline{a}_i)[(a_{i-1} \blacktriangleleft) - (a_{i-1}a_i \blacktriangleleft)], \\
(w\overline{a}_i a_{i-1})(a_{i} \blacktriangleleft) & \longrightarrow & (u\overline{a}_i)(a_{i-1}a_i \blacktriangleleft), \\
(w\overline{a}_i a_{i-1})(a_{i+1} \blacktriangleleft) & \longrightarrow & (u\overline{a}_i)(a_{i+1} \blacktriangleleft), \\
 & \vdots & \\
(w\overline{a}_i a_{i-1})(a_{2n} \blacktriangleleft) & \longrightarrow & (u\overline{a}_i)(a_{2n} \blacktriangleleft), \\
(w\overline{a}_i a_{i-1})(a_{1} \blacktriangleleft) & \longrightarrow & (u\overline{a}_i)(a_{1} \blacktriangleleft), \\
(w\overline{a}_i a_{i-1})(a_{2} \blacktriangleleft) & \longrightarrow & (u\overline{a}_i)(a_{2} \blacktriangleleft), \\
 & \vdots & \\
(w\overline{a}_i a_{i-1})(a_{j-1} \blacktriangleleft) & \longrightarrow & (u\overline{a}_i)(a_{j-1} \blacktriangleleft).
\end{array}$$

Since $a_j = \overline{a}_{i-1}$, the first column is ordered with respect to T_1. On the other hand, $a_j = \overline{a}_{i-1}$ implies that the partition of $(u\overline{a}_i)(a_{i-1} \blacktriangleleft)$ with respect to T_2 ends with $(u\overline{a}_i)(a_{i-1}a_i \blacktriangleleft)$. Then, the second column of this table is ordered with respect to T_2. Hence, if $(w\overline{a}_i a_{i-1})\mathfrak{e}' <_{T_1} (w\overline{a}_i a_{i-1})\mathfrak{f}'$ then $(u\overline{a}_i)\mathfrak{e}'' <_{T_2} (u\overline{a}_i)\mathfrak{f}''$.

If $i-1<j$, then

$$\begin{array}{ll}
(b_k)_{1\le k\le (i-1)} & = (a_k)_{1\le k\le (i-1)}, \\
(b_k)_{i\le k\le (j-2)} & = (a_k)_{(i+1)\le k\le (j-1)}, \\
(b_{j-1}) & = (a_i), \\
(b_k)_{j\le k\le 2n} & = (a_k)_{j\le k\le 2n}.
\end{array}$$

The partition with respect to $(a_k)_{1\le k\le 2n}$ of $(\overline{a}_j \blacktriangleleft) = (a_{i-1} \blacktriangleleft)$ is $(a_{j+1} \blacktriangleleft)$, $(a_{j+2} \blacktriangleleft), \ldots, (a_{2n} \blacktriangleleft), (a_1 \blacktriangleleft), (a_2 \blacktriangleleft), \ldots, (a_{i-1} \blacktriangleleft), (a_i \blacktriangleleft), (a_{i+1} \blacktriangleleft), \ldots, (a_{j-1} \blacktriangleleft)$. The partition with respect to $(b_k)_{1\le k\le 2n}$ of $(\overline{a}_i \blacktriangleleft)$ is $(a_j \blacktriangleleft), (a_{j+1} \blacktriangleleft), \ldots, (a_{2n} \blacktriangleleft)$, $(a_1 \blacktriangleleft), (a_2 \blacktriangleleft), \ldots, (a_{i-1} \blacktriangleleft), (a_{i+1} \blacktriangleleft), (a_{i+2} \blacktriangleleft), \ldots, (a_{j-1} \blacktriangleleft)$. By the third table,

$$\begin{array}{lcl}
 & (a_i \mapsto a_{i-1}a_i) & \\
(w\overline{a}_i a_{i-1})(a_{j+1} \blacktriangleleft) & \longrightarrow & (u\overline{a}_i)(a_{j+1} \blacktriangleleft), \\
(w\overline{a}_i a_{i-1})(a_{j+2} \blacktriangleleft) & \longrightarrow & (u\overline{a}_i)(a_{j+2} \blacktriangleleft), \\
 & \vdots & \\
(w\overline{a}_i a_{i-1})(a_{2n} \blacktriangleleft) & \longrightarrow & (u\overline{a}_i)(a_{2n} \blacktriangleleft), \\
(w\overline{a}_i a_{i-1})(a_{1} \blacktriangleleft) & \longrightarrow & (u\overline{a}_i)(a_{1} \blacktriangleleft), \\
(w\overline{a}_i a_{i-1})(a_{2} \blacktriangleleft) & \longrightarrow & (u\overline{a}_i)(a_{2} \blacktriangleleft), \\
 & \vdots & \\
(w\overline{a}_i a_{i-1})(a_{i-2} \blacktriangleleft) & \longrightarrow & (u\overline{a}_i)(a_{i-2} \blacktriangleleft), \\
(w\overline{a}_i a_{i-1})(a_{i-1} \blacktriangleleft) & \longrightarrow & (u\overline{a}_i)[(a_{i-1} \blacktriangleleft) - (a_{i-1}a_i \blacktriangleleft)], \\
(w\overline{a}_i a_{i-1})(a_{i} \blacktriangleleft) & \longrightarrow & (u\overline{a}_i)(a_{i-1}a_i \blacktriangleleft), \\
(w\overline{a}_i a_{i-1})(a_{i+1} \blacktriangleleft) & \longrightarrow & (u\overline{a}_i)(a_{i+1} \blacktriangleleft), \\
 & \vdots & \\
(w\overline{a}_i a_{i-1})(a_{j-1} \blacktriangleleft) & \longrightarrow & (u\overline{a}_i)(a_{j-1} \blacktriangleleft).
\end{array}$$

Since $a_j = \overline{a}_{i-1}$, the first column is ordered with respect to T_1. On the other hand, $a_j = \overline{a}_{i-1}$ implies that the partition of $(u\overline{a}_i)(a_{i-1} \blacktriangleleft)$ with respect to T_2 ends with $(u\overline{a}_i)(a_{i-1}a_i \blacktriangleleft)$. Then, the second column of this table is ordered with respect to T_2. Hence, if $(w\overline{a}_i a_{i-1})\mathfrak{e}' <_{T_1} (w\overline{a}_i a_{i-1})\mathfrak{f}'$ then $(u\overline{a}_i)\mathfrak{e}'' <_{T_2} (u\overline{a}_i)\mathfrak{f}''$.

For every row of the first table, there is a case which needs to be considered. Similarly, in all these cases, it can be shown that if $\mathfrak{e} <_{T_1} \mathfrak{f}$, then $\mathfrak{e}^{(a_i \mapsto a_{i-1}a_i)} <_{T_2} \mathfrak{f}^{(a_i \mapsto a_{i-1}a_i)}$.

The case $\phi = (\overline{a}_i \mapsto \overline{a}_i\overline{a}_{i+1})$ for some $1 \leq i \leq (2n-1)$, $a_i \neq \overline{a}_{i+1}$, is similar. $\square$

Since $\mathcal{AM}_{g,1,p}$ is the subgroup of McCool's groupoid based at the standard (g,p)-surface word set, see Remark 5.9, we have the following.

Corollary 6.8. *The $\mathcal{AM}_{g,1,p}$ acts on $(\partial F_{g,1,p}, \leq)$ respecting the ordering.*

7 t-squarefreeness

In this section we define a subset A of $\partial F_{g,1,p}$ such that A is $\mathcal{AM}_{g,1,p}$-invariant, every non-t-squarefree end of $F_{g,1,p}$ (see Definition 7.1) lies in A and for $(g,p) \neq (0,1)$, $(0,2)$ the end $a(\Pi_{i=1}^{g}[x_i,y_i]\Pi_{k=1}^{p}t_k)^{\infty}$, where $a \in \{\overline{t}_p\} \cup \{x_i, y_i, \overline{x}_i, \overline{y}_i\}_{1\leq i\leq g}$, does not lie in A. From these, and studying the special cases $(g,p) = (0,1)$, $(0,2)$, we complete the proof of Theorem 4.2. We use the ordering of $\partial F_{g,1,p}$ and results of Section 6.

Recall $2g+p=n$ and $F_{g,1,p}$ is the free group on $\{x_i,y_i\}_{1\leq i\leq g} \cup \{t_k\}_{1\leq k\leq p}$.

The following definition extends Definition 4.1 to $F_{g,1,p} \cup \partial F_{g,1,p}$.

Definition 7.1. An element of $F_{g,1,p} \cup \partial F_{g,1,p}$ is said to be *t-squarefree* if, in its reduced expression, no two consecutive terms in $\{t_k, \overline{t}_k\}_{1 \le k \le p}$ are equal.

Notation 7.2. Recall that if G is a group and $g_1, g_2, \ldots, g_k \in G$, then $\Pi_{i=1}^{k} g_i = g_1 g_2 \cdots g_k$. We use the notation $\Pi_1^{i=k} g_i = g_k g_{k-1} \cdots g_1$.

In the standard surface word set, we denote

$$\begin{aligned} \overline{z}_1 &= \Pi_{i=1}^{g}[x_i, y_i]\Pi_{k=1}^{p} t_k = [x_1, y_1][x_2, y_2] \cdots [x_g, y_g] t_1 t_2 \cdots t_p, \\ z_1 &= \Pi_1^{k=p} \overline{t}_k \Pi_1^{i=g}[y_i, x_i] = \overline{t}_p \overline{t}_{p-1} \cdots \overline{t}_1 [y_g, x_g][y_{g-1}, x_{g-1}] \cdots [y_1, x_1]. \end{aligned}$$

From Remark 6.4, the smallest element of $(\partial F_{g,1,p}, <)$ is $\overline{z}_1^{\infty}$ and the largest element of $(\partial F_{g,1,p}, <)$ is z_1^{∞}. We denote by $\min(\partial F_{g,1,p}) = \overline{z}_1^{\infty}$ and $\max(\partial F_{g,1,p}) = z_1^{\infty}$ these facts.

Given two ends $\mathfrak{e}, \mathfrak{f} \in \partial F_{g,1,p}$, we write

$$[\mathfrak{e}, \mathfrak{f}] := \{\mathfrak{g} \in \partial F_{g,1,p} \mid \mathfrak{e} \le \mathfrak{g} \le \mathfrak{f}\}.$$

Definition 7.3. For every $1 \le k \le p$, let

$$A_k = \bigcup_{w \in F_{g,1,p} - (\star t_k) - (\star \overline{t}_k)} [w t_k \overline{w}(\overline{z}_1^{\infty}), w \overline{t}_k \overline{w}(z_1^{\infty})] \subseteq \partial F_{g,1,p}.$$

Let

$$A = \bigcup_{1 \le k \le p} A_k \subseteq \partial F_{g,1,p}.$$

Lemma 7.4. *The set $A \subseteq \partial F_{g,1,p}$ is $\mathcal{AM}_{g,1,p}$-invariant.*

Proof. Let $\phi \in \mathcal{AM}_{g,1,p}$. By definition, ϕ permutes the set of conjugacy classes $[\overline{t}_1], [\overline{t}_2], \ldots, [\overline{t}_p]$. Hence, ϕ fixes the sets

$$\{w t_k \overline{w} \mid 1 \le k \le p,\ w \in F_{g,1,p} - (\star t_k) - (\star \overline{t}_k)\} \subseteq F_{g,1,p},$$

and

$$\{w \overline{t}_k \overline{w} \mid 1 \le k \le p,\ w \in F_{g,1,p} - (\star t_k) - (\star \overline{t}_k)\} \subseteq F_{g,1,p}.$$

By definition, ϕ fixes $\overline{z}_1 = \Pi_{i=1}^{g}[x_i, y_i]\Pi_{k=1}^{p} t_k$. Hence, ϕ fixes the sets

$$\{w t_k \overline{w}(\overline{z}_1^{\infty}) \mid 1 \le k \le p,\ w \in F_{g,1,p} - (\star t_k) - (\star \overline{t}_k)\} \subseteq \partial F_{g,1,p},$$

and

$$\{w \overline{t}_k \overline{w}(z_1^{\infty}) \mid 1 \le k \le p,\ w \in F_{g,1,p} - (\star t_k) - (\star \overline{t}_k)\} \subseteq \partial F_{g,1,p}.$$

Now, Corollary 6.8 completes the proof. □

Lemma 7.5. *Let $1 \le k_0 \le p$ and $w \in F_{g,1,p} - (\star t_{k_0}) - (\star \overline{t}_{k_0})$. Then the following hold in $(\partial F_{g,1,p}, \le)$:*

(i). $wt_{k_0}\overline{w}(\overline{z}_1^\infty) \le wt_{k_0}((\Pi_{k=k_0}^{p} t_k \Pi_{i=1}^{g}[x_i, y_i]\Pi_{k=1}^{k_0-1} t_k)^\infty) = \min(wt_{k_0}t_{k_0} ◀)$;

(ii). $\max(wt_{k_0}t_{k_0} ◀) < \min(w\overline{t}_{k_0}\overline{t}_{k_0} ◀)$;

(iii). $\max(w\overline{t}_{k_0}\overline{t}_{k_0} ◀) = w\overline{t}_{k_0}((\Pi_1^{k=k_0}\overline{t}_k\Pi_1^{i=g}[y_i, x_i]\Pi_{k_0+1}^{k=p}\overline{t}_k)^\infty) \le w\overline{t}_{k_0}\overline{w}(z_1^\infty)$.

Hence, $(wt_{k_0}t_{k_0} ◀) \cup (w\overline{t}_{k_0}\overline{t}_{k_0} ◀) \subseteq [wt_{k_0}\overline{w}(\overline{z}_1^\infty), w\overline{t}_{k_0}\overline{w}(z_1^\infty)]$. *In particular* A_{k_0} *contains all non-*t_{k_0}*-squarefree ends of* $F_{g,1,p}$ *and* A *contains all non-t-squarefree ends of* $F_{g,1,p}$.

Proof. Recall $<$ is the ordering with respect to the sequence

$$(\overline{x}_1, y_1, x_1, \overline{y}_1, \overline{x}_2, y_2, x_2, \overline{y}_2, \cdots, \overline{x}_g, y_g, x_g, \overline{y}_g, t_1, \overline{t}_1, t_2, \overline{t}_2, \cdots, t_p, \overline{t}_p).$$

(i). It is straightforward to see that

$$wt_{k_0}((\Pi_{k=k_0}^{p} t_k \Pi_{i=1}^{g}[x_i, y_i]\Pi_{k=1}^{k_0-1} t_k)^\infty) = \min(wt_{k_0}t_{k_0} ◀).$$

Let $a \in X \cup \overline{X}$ be such that $\overline{w}((\Pi_{i=1}^{g}[x_i, y_i]\Pi_{k=1}^{p} t_k)^\infty) \in (a ◀)$. Note $a \ne \overline{t}_{k_0}$.
If $a \ne t_{k_0}$, then $(wt_{k_0}a ◀) < (wt_{k_0}t_{k_0} ◀)$, and we have

$$wt_{k_0}\overline{w}(\overline{z}_1^\infty) = wt_{k_0}\overline{w}((\Pi_{i=1}^{g}[x_i, y_i]\Pi_{k=1}^{p} t_k)^\infty) < \min(wt_{k_0}t_{k_0}).$$

If $a = t_{k_0}$, then $\overline{w}$ is completely canceled in $\overline{w}((\Pi_{i=1}^{g}[x_i, y_i]\Pi_{k=1}^{p} t_k)^\infty)$, and, moreover,

$$\begin{aligned} wt_{k_0}\overline{w}(\overline{z}_1^\infty) &= wt_{k_0}\overline{w}((\Pi_{i=1}^{g}[x_i, y_i]\Pi_{k=1}^{p} t_k)^\infty) \\ &= wt_{k_0}((\Pi_{k=k_0}^{p} t_k \Pi_{i=1}^{g}[x_i, y_i]\Pi_{k=1}^{k_0-1} t_k)^\infty) \\ &= \min(wt_{k_0}t_{k_0} ◀). \end{aligned}$$

(ii). It is clear.

(iii). It is straightforward to see that

$$\max(w\overline{t}_{k_0}\overline{t}_{k_0} ◀) = w\overline{t}_{k_0}((\Pi_1^{k=k_0}\overline{t}_k\Pi_1^{i=g}[y_i, x_i]\Pi_{k_0+1}^{k=p}\overline{t}_k)^\infty).$$

Let $a \in X \cup \overline{X}$ be such that $\overline{w}((\Pi_1^{k=p}\overline{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) \in (a ◀)$. Note $a \ne t_{k_0}$.
If $a \ne \overline{t}_{k_0}$, then $(w\overline{t}_{k_0}\overline{t}_{k_0} ◀) < (w\overline{t}_{k_0}a ◀)$, and we have

$$\max(w\overline{t}_{k_0}\overline{t}_{k_0} ◀) < w\overline{t}_{k_0}\overline{w}((\Pi_1^{k=p}\overline{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) = w\overline{t}_{k_0}\overline{w}(z_1^\infty).$$

If $a = \overline{t}_{k_0}$, then $\overline{w}$ is completely canceled in $\overline{w}((\Pi_1^{k=p}\overline{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty)$, and, moreover,

$$\begin{aligned} w\overline{t}_{k_0}\overline{w}(z_1^\infty) &= w\overline{t}_{k_0}\overline{w}((\Pi_1^{k=p}\overline{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) \\ &= w\overline{t}_{k_0}((\Pi_1^{k=k_0}\overline{t}_k\Pi_1^{i=g}[y_i, x_i]\Pi_p^{k=k_0+1}\overline{t}_k)^\infty) \\ &= \max(w\overline{t}_{k_0}\overline{t}_{k_0} ◀). \end{aligned}$$

□

Lemma 7.6. *Let $1 \leq k_0 \leq p$ and $w \in F_{g,1,p} - (\star t_{k_0}) - (\star \overline{t}_{k_0})$. Suppose $(g,p) \neq (0,1)$, $(0,2)$. Then one of the followings holds in $(\partial F_{g,1,p}, \leq)$:*

(i) $\overline{t}_p(\overline{z}_1^\infty) > w\overline{t}_{k_0}\overline{w}(z_1^\infty)$;

(ii) $\overline{t}_p(\overline{z}_1^\infty) < wt_{k_0}\overline{w}(\overline{z}_1^\infty)$.

Hence, $\overline{t}_p(\overline{z}_1^\infty) \notin [wt_{k_0}\overline{w}(\overline{z}_1^\infty), w\overline{t}_{k_0}\overline{w}(z_1^\infty)]$. *In particular,* $\overline{t}_p(\overline{z}_1^\infty) \notin A_{k_0}$ *and* $\overline{t}_p(\overline{z}_1^\infty) \notin A$.

Proof. Recall $<$ is the ordering with respect to the sequence

$$(\overline{x}_1, y_1, x_1, \overline{y}_1, \overline{x}_2, y_2, x_2, \overline{y}_2, \cdots, \overline{x}_g, y_g, x_g, \overline{y}_g, t_1, \overline{t}_1, t_2, \overline{t}_2, \cdots, t_p, \overline{t}_p).$$

By Lemma 7.5,

$$wt_{k_0}\overline{w}((\Pi_{i=1}^g[x_i, y_i]\Pi_{k=1}^p t_k)^\infty) < w\overline{t}_{k_0}\overline{w}((\Pi_1^{k=p}\overline{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty).$$

Case 1. $w = 1$. Since $(\overline{t}_p\overline{x}_1 \blacktriangleleft) \cup (\overline{t}_p t_1 \blacktriangleleft) > (\overline{t}_{k_0}\overline{t}_p \blacktriangleleft)$, we see

$$\overline{t}_p(\overline{z}_1^\infty) = \overline{t}_p((\Pi_{i=1}^g[x_i, y_i]\Pi_{k=1}^p t_k)^\infty) > \overline{t}_{k_0}((\Pi_1^{k=p}\overline{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) = \overline{t}_{k_0}(z_1^\infty).$$

Thus, (i) holds.

Case 2. $w \notin (\overline{t}_p\star) \cup \{1\}$. Since $(\overline{t}_p \blacktriangleleft) > (w\overline{t}_{k_0} \blacktriangleleft)$, we see

$$\begin{aligned}\overline{t}_p(\overline{z}_1^\infty) &= \overline{t}_p((\Pi_{i=1}^g[x_i, y_i]\Pi_{k=1}^p t_k)^\infty) \\ &> w\overline{t}_{k_0}\overline{w}((\Pi_1^{k=p}\overline{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) = w\overline{t}_{k_0}\overline{w}(z_1^\infty).\end{aligned}$$

Thus, (i) holds.

Case 3. $w \in (\overline{t}_p\overline{t}_p\star)$. Since $(\overline{t}_p\overline{x}_1 \blacktriangleleft) \cup (\overline{t}_p t_1 \blacktriangleleft) > (w\overline{t}_{k_0} \blacktriangleleft)$, we see

$$\begin{aligned}\overline{t}_p(\overline{z}_1^\infty) &= \overline{t}_p((\Pi_{i=1}^g[x_i, y_i]\Pi_{k=1}^p t_k)^\infty) \\ &> w\overline{t}_{k_0}\overline{w}((\Pi_1^{k=p}\overline{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) = w\overline{t}_{k_0}\overline{w}(z_1^\infty).\end{aligned}$$

Thus, (i) holds.

Case 4. $w \in (\overline{t}_p\star) - (\overline{t}_p\overline{t}_p\star)$.
Here,

$$wt_{k_0}\overline{w}(\overline{z}_1^\infty) = wt_{k_0}\overline{w}((\Pi_{i=1}^g[x_i, y_i]\Pi_{k=1}^p t_k)^\infty) \in (wt_{k_0} \blacktriangleleft) \subset (\overline{t}_p \blacktriangleleft) - (\overline{t}_p\overline{t}_p \blacktriangleleft).$$

Hence,

$$\begin{aligned}\overline{t}_p((\Pi_{i=1}^g[x_i, y_i]\Pi_{k=1}^p t_k)^\infty) &= \min((\overline{t}_p \blacktriangleleft) - (\overline{t}_p\overline{t}_p \blacktriangleleft)) \\ &\leq wt_{k_0}\overline{w}((\Pi_{i=1}^g[x_i, y_i]\Pi_{k=1}^p t_k)^\infty).\end{aligned}$$

To prove (ii) holds, it remains to show that

$$\overline{t}_p((\Pi_{i=1}^g[x_i,y_i]\Pi_{k=1}^p t_k)^\infty) \neq wt_{k_0}\overline{w}((\Pi_{i=1}^g[x_i,y_i]\Pi_{k=1}^p t_k)^\infty),$$

that is, $(\Pi_{i=1}^g[x_i,y_i]\Pi_{k=1}^p t_k)^\infty \neq t_p w t_{k_0}\overline{w}((\Pi_{i=1}^g[x_i,y_i]\Pi_{k=1}^p t_k)^\infty)$, that is, $t_p w t_{k_0}\overline{w} \notin \langle \Pi_{i=1}^g[x_i,y_i]\Pi_{k=1}^p t_k\rangle$. We can write $w = \overline{t}_p u$ where $u \notin (t_p\star)$. Then $t_p w t_{k_0}\overline{w} = u t_{k_0}\overline{u} t_p$, in normal form. Thus it suffices to show

$$u t_{k_0}\overline{u} t_p \notin \langle \Pi_{i=1}^g[x_i,y_i]\Pi_{k=1}^p t_k\rangle.$$

If $u = 1$, then $u t_{k_0}\overline{u} t_p \notin \langle \Pi_{i=1}^g[x_i,y_i]\Pi_{k=1}^p t_k\rangle$, since $(g,p) \neq (0,1)$, $(0,2)$.

If $u \neq 1$, then $u t_{k_0}\overline{u} t_p \notin \langle \Pi_{i=1}^g[x_i,y_i]\Pi_{k=1}^p t_k\rangle$, since $u t_{k_0}\overline{u} t_p$ does not lie in the submonoid of $F_{g,1,p}$ generated by $\Pi_{i=1}^g[x_i,y_i]\Pi_{k=1}^p t_k$, nor in the submonoid generated by $\Pi_1^{k=p}\overline{t}_k\Pi_1^{i=g}[y_i,x_i]$. □

Lemma 7.7. *Let $1 \leq k_0 \leq p$, $w \in F_{g,1,p} - (\star t_{k_0}) - (\star\overline{t}_{k_0})$ and $1 \leq i_0 \leq g$. If $a \in \{x_{i_0}, \overline{x}_{i_0}, y_{i_0}, \overline{y}_{i_0}\}$, then one of the following holds in $(\partial F_{g,1,p}, \leq)$:*

(i). $a(z_1^\infty) > w\overline{t}_{k_0}\overline{w}(z_1^\infty)$;

(ii). $a(z_1^\infty) < wt_{k_0}\overline{w}(\overline{z}_1^\infty)$.

Hence, $a(z_1^\infty) \notin [wt_{k_0}\overline{w}(\overline{z}_1^\infty), w\overline{t}_{k_0}\overline{w}(z_1^\infty)]$. In particular, $a(z_1^\infty) \notin A_{k_0}$ and $a(z_1^\infty) \notin A$.

Proof. Recall $<$ is the ordering with respect to the sequence

$$(\overline{x}_1, y_1, x_1, \overline{y}_1, \overline{x}_2, y_2, x_2, \overline{y}_2, \cdots, \overline{x}_g, y_g, x_g, \overline{y}_g, t_1, \overline{t}_1, t_2, \overline{t}_2, \cdots, t_p, \overline{t}_p).$$

Let $a \in \{x_{i_0}, \overline{x}_{i_0}, y_{i_0}, \overline{y}_{i_0}\}$.

Case 1. $w = 1$. Since $(a \blacktriangleleft) < (t_{k_0} \blacktriangleleft)$, we see

$$a(z_1^\infty) = a((\Pi_1^{k=p}\overline{t}_k\Pi_1^{i=g}[y_i,x_i])^\infty) < t_{k_0}((\Pi_{i=1}^g[x_i,y_i]\Pi_{k=1}^p t_k)^\infty) = t_{k_0}(\overline{z}_1^\infty).$$

Thus, (ii) holds.

Case 2. $w \notin (a\star) \cup \{1\}$.

If $(a \blacktriangleleft) > (w \blacktriangleleft)$, then $(a \blacktriangleleft) > (w \blacktriangleleft) \supset (w\overline{t}_{k_0} \blacktriangleleft)$ and

$$\begin{aligned} a(z_1^\infty) &= a((\Pi_1^{k=p}\overline{t}_k\Pi_1^{i=g}[y_i,x_i])^\infty) \\ &> w\overline{t}_{k_0}\overline{w}((\Pi_1^{k=p}\overline{t}_k\Pi_1^{i=g}[y_i,x_i])^\infty) = w\overline{t}_{k_0}\overline{w}(z_1^\infty). \end{aligned}$$

Thus, (i) holds.

If $(a \blacktriangleleft) < (w \blacktriangleleft)$, then $(a \blacktriangleleft) < (w \blacktriangleleft) \supset (wt_{k_0} \blacktriangleleft)$ and

$$\begin{aligned} a(z_1^\infty) &= a((\Pi_1^{k=p}\overline{t}_k\Pi_1^{i=g}[y_i,x_i])^\infty) \\ &< wt_{k_0}\overline{w}((\Pi_{i=1}^g[x_i,y_i]\Pi_{k=1}^p t_k)^\infty) = wt_{k_0}\overline{w}(\overline{z}_1^\infty). \end{aligned}$$

Thus, (ii) holds.

Case 3. $w \in (a\bar{t}_p\star)$.
Since $a((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) = \max(a\bar{t}_p \blacktriangleleft)$, we see

$$\begin{aligned} a(z_1^\infty) &= a((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) \\ &\geq w\bar{t}_{k_0}\overline{w}((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) = w\bar{t}_{k_0}\overline{w}(z_1^\infty). \end{aligned}$$

To prove (i) holds, it remains to show that

$$a((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) \neq w\bar{t}_{k_0}\overline{w}((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty),$$

that is, $(\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty \neq \overline{a}w\bar{t}_{k_0}\overline{w}((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty)$, that is $\overline{a}w\bar{t}_{k_0}\overline{w} \notin \langle\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i]\rangle$. We can write $w = a\bar{t}_pu$ where $u \notin (t_p\star)$. Then $\overline{a}w\bar{t}_{k_0}\overline{w} = \bar{t}_pu\bar{t}_{k_0}\overline{u}t_p\overline{a}$, in normal form. Thus it suffices to show that

$$\bar{t}_pu\bar{t}_{k_0}\overline{u}t_p\overline{a} \notin \langle\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i]\rangle,$$

which is clear since $\bar{t}_pu\bar{t}_{k_0}\overline{u}t_p\overline{a}$ does not lie in the submonoid of $F_{g,1,p}$ generated by $\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i]$, nor in the submonoid generated by $\Pi_{i=1}^g[x_i, y_i]\Pi_{k=1}^p t_k$.

Case 4. $w \in (a\star) - (a\bar{t}_p\star)$, $|w| \geq 2$.
If $(a\bar{t}_p \blacktriangleleft) > (w \blacktriangleleft)$, then $(a\bar{t}_p \blacktriangleleft) > (w \blacktriangleleft) \supset (w\bar{t}_{k_0} \blacktriangleleft)$ and

$$\begin{aligned} a(z_1^\infty) &= a((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) \\ &> w\bar{t}_{k_0}\overline{w}((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) = w\bar{t}_{k_0}\overline{w}(z_1^\infty). \end{aligned}$$

Thus, (i) holds.

If $(a\bar{t}_p \blacktriangleleft) < (w \blacktriangleleft)$, then $(a\bar{t}_p \blacktriangleleft) < (w \blacktriangleleft) \supset (wt_{k_0} \blacktriangleleft)$ and

$$\begin{aligned} a(z_1^\infty) &= a((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) \\ &< wt_{k_0}\overline{w}((\Pi_{i=1}^g[x_i, y_i]\Pi_{k=1}^p t_k)^\infty) = wt_{k_0}\overline{w}(\overline{z}_1^\infty). \end{aligned}$$

Thus, (ii) holds.

Case 5. $w = a$.
Since $a(z_1^\infty) = \max(a\bar{t}_p \blacktriangleleft)$, $(a\bar{t}_p \blacktriangleleft) \supset (a\bar{t}_p\overline{y}_g\overline{x}_g \blacktriangleleft)$ and $(a\bar{t}_p\overline{y}_g\overline{x}_g \blacktriangleleft) > (a\bar{t}_{k_0}\overline{a}\bar{t}_p \blacktriangleleft)$, we see

$$\begin{aligned} a(z_1^\infty) &= a((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) \\ &> a\bar{t}_{k_0}\overline{a}((\Pi_1^{k=p}\bar{t}_k\Pi_1^{i=g}[y_i, x_i])^\infty) = a\bar{t}_{k_0}\overline{a}(z_1^\infty). \end{aligned}$$

Thus, (i) holds. □

Proposition 7.8. *If $(g, p) \neq (0, 1), (0, 2)$ then the following hold for each $\phi \in \mathcal{AM}_{g,1,p}$:*

(i). $\overline{t}_p^{\phi}(\overline{z}_1^{\infty})$ *is a t-squarefree end,*

(ii). for every $1 \leq i_0 \leq g$ *and every* $a \in \{x_{i_0}, \overline{x}_{i_0}, y_{i_0}, \overline{y}_{i_0}\}$*,* $a^{\phi}(z_1^{\infty})$ *is a t-squarefree end.*

Proof. (i). By Lemma 7.4, A is $\mathcal{AM}_{g,1,p}$-invariant. By Lemma 7.6, $\overline{t}_p(\overline{z}_1^{\infty}) \notin A$. By Lemma 7.5, A contains all non-t-squarefree ends of $\partial F_{g,1,p}$. Thus, $\overline{t}_p^{\phi}(\overline{z}_1^{\infty}) = (\overline{t}_p(\overline{z}_1^{\infty}))^{\phi}$ is a t-squarefree end of $F_{g,1,p}$.

(ii). By Lemma 7.4, A is $\mathcal{AM}_{g,1,p}$-invariant. By Lemma 7.7, $a(\overline{z}_1^{\infty}) \notin A$. By Lemma 7.5, A contains all non-t-squarefree ends of $\partial F_{g,1,p}$. Thus, $a^{\phi}(\overline{z}_1^{\infty}) = (a(\overline{z}_1^{\infty}))^{\phi}$ is a t-squarefree end of $F_{g,1,p}$. □

Proof. (of Theorem 4.2) The case $(g,p) = (0,1)$ is clear since $\mathcal{AM}_{0,1,1} = 1$.

Recall (2.3.1). $\mathcal{AM}_{0,1,2} = \langle \sigma_1 \rangle$, and

$$\{t_2^{\phi} \mid \phi \in \mathcal{AM}_{0,1,2}\} = \{t_2^{\sigma_1^{2m}}, t_2^{\sigma_1^{2m+1}} \mid m \in \mathbb{Z}\} = \{t_2^{(t_1 t_2)^m}, t_1^{(t_1 t_2)^m} \mid m \in \mathbb{Z}\}$$

Thus, every element of $\{t_2^{\phi} \mid \phi \in \mathcal{AM}_{0,1,2}\}$ is t-squarefree.

Suppose, now, $(g,p) \neq (0,1), (0,2)$. Let $1 \leq i_0 \leq g$ and $a \in \{x_{i_0}, y_{i_0}\}$. By Proposition 7.8(ii), $a^{\phi}(z_1^{\infty}) = a^{\phi}((\Pi_1^{k=p}\overline{t}_k \Pi_1^{i=g}[y_i, x_i])^{\infty})$ is a t-squarefree end. Hence, either a^{ϕ} is t-squarefree or $a^{\phi} = ut_k t_k v$ in normal form, and $t_k v$ is canceled in $a^{\phi}(z_1^{\infty}) = ut_k t_k v(z_1^{\infty})$; moreover $ut_k, t_k v$ are t-squarefree. By Proposition 7.8(ii),

$$\overline{a}^{\phi}(z_1^{\infty}) = \overline{a}^{\phi}((\Pi_1^{k=p}\overline{t}_k \Pi_1^{i=g}[y_i, x_i])^{\infty})$$

is a t-squarefree end. Hence, $\overline{a}^{\phi} \neq \overline{v}\overline{t}_k\overline{t}_k\overline{u}$.

Since ϕ permutes the set $\{[\overline{t}_k]\}_{1 \leq k \leq p}$, we can write $\overline{t}_p^{\phi} = \overline{t}_{p^{\pi}}^{w_p}$, where π is a permutation of $\{1, 2, \ldots, p\}$ and $w_p \in F_{g,1,p} - (t_{p^{\pi}} \star) - (\overline{t}_{p^{\pi}} \star)$. It is not difficult to see that

$$\overline{t}_p^{\phi}(\overline{z}_1^{\infty}) = \overline{w}_p \overline{t}_{p^{\pi}} w_p((\Pi_{i=1}^{g}[x_i, y_i]\Pi_{k=1}^{p} t_k)^{\infty}) \in (\overline{w}_p \blacktriangleleft).$$

By Proposition 7.8(i), $\overline{t}_p^{\phi}(\overline{z}_1^{\infty})$ is a t-squarefree end. Hence, $\overline{w}_p$ is t-squarefree. Since $\overline{w}_p$ is t-squarefree, $\overline{t}_p^{\phi} = \overline{w}_p \overline{t}_p w_p$ is also t-squarefree. Hence, t_p^{ϕ} is t-squarefree.

Suppose, now, $p \geq 2$. Let $1 \leq k \leq p$. Since t_k is in the $\mathcal{AM}_{g,1,p}$-orbit of t_p, we see t_k^{ϕ} is t-squarefree for all $\phi \in \mathcal{AM}_{g,1,p}$. □

Acknowledgments

The author is grateful to Warren Dicks and Luis Paris for many interesting conversations.

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