Randomly colouring graphs (a combinatorial view)

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Draft Version, (2 July 2008), 1–71

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Abstract

The application of the probabilistic method to graph colouring has been yielding interesting results for more than 40 years. Several probabilistic tools are presented in this survey, ranging from the basic to the more advanced. For each of them, an application to a graph colouring problem is presented in detail. In this way, not only is the general idea of the method exposed, but also are the concrete details arising with its application. Further, this allows us to introduce some important variants of the usual graph colouring notion (with some related open questions), and at the same time to illustrate the variety of the probabilistic techniques. The survey tries to be self-contained.

Introduction

Colouring is a core topic of graph theory. It was initiated back in 1852 by the 4-Colour Problem, which has spawned a plethora of research. Fundamental notions, such as nowherezero-flows [184], and useful techniques, e.g. the discharging method, were introduced, studied and developed. The positive answer to the 4-Colour Conjecture given by Appel and Haken [13, 14] in 1977 (the reader can also consult the shorter proof by Robertson, Sanders, Seymour, and Thomas [160], or the dedicated web-page of Robin Thomas [180] for a gentle introduction) did not toll the bell of graph colouring. Indeed, graph colouring is a very generic notion which admits (infinitely?) many variants. Many of them are theoretically interesting, and/or useful to model practical problems. In particular, lots of problems arising in telecommunication networks are closely related to the world of graph colouring.

This survey deals with the probabilistic method applied to graph colouring. The goal is to present some powerful probabilistic techniques used in the context of graph colouring. At the same time, this survey offers a tour in the world of graph colouring, reviewing some variants and generalisations of the usual "chromatic number" along with some related important problems.

The essence of the probabilistic method is as follows. Suppose that we want to prove the existence of a combinatorial object satisfying certain prescribed properties, e.g. a vertex colouring such that no two neighbours are assigned the same colour. The idea is to design a random experiment (in an appropriate discrete probability space) whose outcome is, with positive probability, an object satisfying the desired properties. It then follows that such an object exists.

The first use of this method in combinatorics dates back to 1943, and is due to Szele [177], who proved the existence of a tournament on n vertices with at least $n!2^{1-n}$ Hamiltonian paths—a *tournament* is an orientation of a complete graph, and a *Hamiltonian path* is a directed path traversing once each vertex.

The probabilistic method was then applied by Erdős [50] to obtain a result in Ramsey theory—namely that $R(k,k) > 2^{k/2}$ for $k \geq 3$, where $R(k,k)$ is the smallest integer r such that any 2-edge-colouring of the complete graph on r vertices contains a monochromatic complete subgraph on k vertices. Erdős then developed and widely applied the probabilistic method in combinatorics. His work (and surveys on his work) should be a source of inspiration and learning for those who want to learn the probabilistic method. Regarding the scope of this survey, the most relevant works were done by Noga Alon, Jeff Kahn, Colin McDiarmid, Michael Molloy, Bruce Reed, Joel Spencer, Benjamin Sudakov, and many others.

Several probabilistic tools and techniques are presented in this survey, from the basic to the more advanced (such as McDiarmid's Inequality). For each of them, an application to a graph colouring problem is presented in detail. In this way, not only is the general idea of the method exposed, but also are the concrete details arising with its application. Further, this allows us to introduce some important variants of the usual graph colouring notion (with some related open questions), and at the same time to illustrate the variety of the probabilistic techniques.

The survey is as self-contained as possible. We refer to the books of Alon and Spencer [11], Molloy and Reed [140] and the lecture notes by Matoušek and Vondrák [123] for additional background on discrete probabilities. Not only do those references provide a good introduction to discrete probability theory, but they also cover advanced techniques, and applications to other combinatorial topics than graph colouring. Spencer wrote a historical review of the early probabilistic method [174], and a nice account on some techniques introduced later [173].

One of the main difficulties when applying the probabilistic method is how—and how much—randomness should be added. A complicated random process may be hard to analyse, while a simple one often seems too weak to provide the desired result. A major issue is to actually be able to *combine* probabilistic arguments with more classical techniques from graph theory. The latter provides strong structural properties, which can serve as a starting point for a random process that would fail otherwise or they may help to analyse the random process. The results presented in this survey were chosen to illustrate this fact. Moreover, they should be self-contained (so that the reader sees the whole argument) and with a limited amount of technicalities (so that the essence of the method is not lost among pages of computation). Last, the whole set of applications and examples should also give us the opportunity to define and briefly review some important notions of graph colouring, and state some related conjectures.

1. Some Basics

We provide some definitions, notation, and facts that are needed, along with references.

1.1. Graph Colouring

We define some basic notions of graph colouring. Other variants of graph colourings will be introduced when needed. We refer to the book by Diestel [43] for any notion that is used without being defined. The book by Jensen and Toft [94] gives an in-depth review of many graph colouring notions and problems.

Given a graph $G = (V, E)$, and a vertex $v \in V$, the set of vertices of G adjacent to v is the *neighbourhood* $N_G(v)$ of v. The size of $N_G(v)$ is $\deg(v)$, the *degree* of v. A *colouring* of G is a mapping that assigns to each vertex an integer, called a *colour*.

A colouring c is a *k-colouring* if $f(V) \subseteq \{1, 2, \ldots, k\}$. It is *proper* if no two adjacent vertices are assigned the same colour. Thus, a proper k-colouring of G can be seen as a partition of the vertices into k parts, each being an *independent set* of G, i.e. a set of vertices inducing in G a subgraph with no edge. The *chromatic number* $\chi(G)$ of G is the least integer k for which G admits a proper k-colouring.

The size of a largest complete subgraph of G is $\omega(G)$, the *clique number* of G. Note that $\chi(G) \geq \omega(G)$.

If a graph has maximum degree Δ, then by greedily colouring its vertices we deduce that its chromatic number is at most $\Delta + 1$. This bound is tight, as complete graphs show. However, Brooks [30] proved that complete graphs and odd cycles are the only connected graphs reaching the bound. In other words, the chromatic number of a connected graph G is at most its maximum degree, unless G is a complete graph or an odd cycle.

The decision problem associated with the chromatic number was one of the first problems shown to be NP-complete, in Karp's paper [105]. Two years later, Garey, Johnson, and Stockmeyer [64] proved that the problem remains NP-complete even when restricted to planar graphs of maximum degree 4. Garey and Johnson [63] demonstrated that it is NP-hard to approximate the chromatic number to within any constant less than 2. More recently, Bellare, Goldreich, and Sudan [20] proved that the chromatic number of a graph on n vertices cannot be approximated within $n^{1/7-\varepsilon}$ for any $\varepsilon > 0$, unless P=NP. Let ZPP be the class of languages decidable by a random expected polynomial-time algorithm that makes no error. In other words, ZPP is the class of decision problems L that are decided by algorithms A such that for every input x, the output of A is $L(x)$ with probability 1, and A runs in expected polynomial-time. Equivalently, ZPP can be defined as the class of decision problems L for which there exists a randomised algorithm B that always runs in polynomial time, and on every input x its output $B(x)$ is either $L(x)$ or "`I do not know`", the probability that $B(x)$ equals $L(x)$ being at least $\frac{1}{2}$ for every input x. The chromatic number of a graph G on n vertices cannot be approximated in polynomial time within $n^{1-\varepsilon}$ for any constant $\varepsilon > 0$, unless ZPP=NP [58]. On the other hand, Halldórsson [77] designed a polynomial-time algorithm achieving a performance guarantee of $O\left(n\frac{(\log\log n)^2}{(\log n)^3}\right)$.

The *girth* $g(G)$ of the graph G is the length of a shortest cycle of G. Can a graph of girth at least 4, i.e. a *triangle-free* graph, have an arbitrarily large chromatic number? This was answered positively by Tutte [41] and Zykov [198], and several other authors, namely Ungar and Descartes [185], Kelly and Kelly [106], and Mycielski [146]. (Descartes was also known as Tutte.) However, how much restriction can be put on the girth? In other words, are there graphs with arbitrarily high chromatic number and arbitrarily high girth? It may be expected that such graphs should not exist, since a graph with high girth locally looks like a tree, and trees can be properly 2-coloured. Erdős [51] proved the existence of such graphs by probabilistic means in 1959. To do so, he actually used one of the two virtually unique general lower bounds on the chromatic number of a graph $G = (V, E)$, that is

$$\chi(G) \geq \frac{|V|}{\alpha(G)},$$

where $\alpha(G)$ is the *independence number* of G, i.e. the size of a largest independent set of G. Without providing any further details on the proof (which can be found in almost any monograph or lecture notes on the probabilistic method), let us note that this result is a milestone in the use of the probabilistic method. The approach introduced by Erdős in his proof is now called the *deletion method*. It took about ten more years to be able to exhibit such graphs. Indeed, in 1966, Nešetřil [148] explicitly constructed graphs with arbitrarily high chromatic number and girth 8. Two years later, Lovász [119] was the first to explicitly construct graphs with arbitrarily high girth and arbitrarily high chromatic number. Another short constructive proof was given in 1979 by Nešetřil and Rödl [150].

On the other hand, imposing *both* a high girth *and* planarity—or, more generally, a fixed genus—allows us to improve bounds on the chromatic number. For instance, while general planar graphs are 4-colourable, Grötzsch [74] proved that the chromatic number of any triangle-free planar graph is at most 3. Thus, it is customary, when studying colourings of planar graphs, to impose some restrictions on the girth of the considered graphs.

An *ℓ-list-assignment* of a graph $G = (V, E)$ is a mapping L that assigns to each vertex a list of ℓ colours. An *L-list-colouring* of G is a colouring c such that $c(v) \in L(v)$ for each vertex $v \in V$. The graph G is *ℓ-choosable* if for any ℓ-list-assignment L, there exists a proper L-list-colouring of G. The *choice number* $\mathrm{ch}(G)$ of G is the least integer ℓ for which G is ℓ-choosable. Note that if the lists of all the vertices are the same, then finding a list-colouring amounts to finding a usual colouring.

One may feel that the "harder" case is when all the lists are the same. This is however false. Let us observe that the gap between the chromatic number and the choice number of a graph can be arbitrarily large. A graph is *bipartite* if its vertices can be partitioned into two independent sets, i.e. if it can be properly 2-coloured. The *complete bipartite graph* is composed of two independent sets A and B, and two vertices a and b are adjacent whenever $a \in A$ and $b \in B$. Let $K_{m,m}$ be the complete bipartite graph with parts A and B each of size $m := \binom{2n-1}{n}$. Then, as observed by Erdős, Rubin, and Taylor [53] in their seminal paper about list-colouring,

$$\mathrm{ch}(K_{m,m}) \geq n\,.$$

Thus, perhaps counter-intuitively, list-colouring is indeed harder than usual colouring.

Colouring the edges of a graph is defined analogously as vertex colouring. More precisely, a k-edge-colouring of a graph $G = (V, E)$ is a mapping $c : E \to \{1, 2, \ldots, k\}$. It is *proper* if no two adjacent edges have the same colour. In other words, a proper k-edge-colouring of G is a partition of the edges of G into k matchings—a *matching* of G is a set of edges no two of which are adjacent in G. The *chromatic index* $\chi'(G)$ of G is the minimum k for which G admits a proper k-edge-colouring.

An edge-colouring of a graph G can also be seen as a vertex-colouring of the line graph of G. The *line graph* $\mathscr{L}(G)$ of G is the graph whose vertex-set is E, and two elements e and e' of E are adjacent in $\mathscr{L}(G)$ if and only if e and e' are adjacent edges of G. Thus,

$$\chi'(G) = \chi(\mathscr{L}(G))\,.$$

Vizing's Theorem [188] ensures that the chromatic index of every graph G of maximum Δ is either Δ or $\Delta+1$. On the other hand, it is NP-complete in general to choose between those two values [91]. The *list-chromatic index* $\mathrm{ch}'(G)$ of a graph G is the choice number of the line graph of G. We end this subsection by stating the main open problem is in this area.

CONJECTURE 1.1 (The List-Colouring Conjecture). *For every graph G,*

$$\mathrm{ch}'(G) = \chi'(G)\,.$$

1.2. Discrete Probabilities

We give some formal definitions and basic facts. We refer to the three references given in the introduction [11, 123, 140] for further exposition. For a more general introduction to the theory of probabilities, one can consult the books by Grimmett and Stirzaker [72] and by Grimmett and Welsh [73].

A *sample space* is a finite set Ω. An *event* is a subset of Ω. A *(finite) probability space* consists of a sample space Ω along with a mapping $\mathbf{Pr} : \Omega \to [0,1]$ such that

$$\sum_{\omega\in\Omega} \mathbf{Pr}(\omega) = 1\,.$$

The function $\mathbf{Pr}$ is extended to any event A by setting $\mathbf{Pr}(A) := \sum_{a\in A}\mathbf{Pr}(a)$. It follows that $\mathbf{Pr}(A\cup B) = \mathbf{Pr}(A) + \mathbf{Pr}(B) - \mathbf{Pr}(A\cap B)$ for any two events A and B. For any events $A_1, A_2, \ldots, A_r$,

$$\mathbf{Pr}\left(\cup_{i=1}^{r} A_i\right) \le \sum_{i=1}^{r} \mathbf{Pr}(A_i)\,,$$

with equality if and only if the events A_i are pairwise incompatible, i.e. no two of them may occur simultaneously.

We often use the *uniform distribution* on a sample space Ω, defined by $\mathbf{Pr}(x) = \frac{1}{|\Omega|}$ for every $x \in \Omega$.

The *conditional probability* of an event A given that an event B occurs is

$$\mathbf{Pr}(A|B) := \frac{\mathbf{Pr}(A\cap B)}{\mathbf{Pr}(B)}\,.$$

For every partition $B_1, B_2, \ldots, B_n$ of Ω, and every event A, observe that

$$\mathbf{Pr}(A) = \sum_{i=1}^{n} \mathbf{Pr}(A|B_i)\cdot\mathbf{Pr}(B_i)\,.$$

Two events A and B are *independent* if $\mathbf{Pr}(A|B) = \mathbf{Pr}(A)$, or equivalently if $\mathbf{Pr}(A\cap B) = \mathbf{Pr}(A)\cdot\mathbf{Pr}(B)$. In our considerations, an important notion of independence is the following. An event A is *mutually independent* of a set of events $\mathscr{E}$ if for every $B_1, B_2, \ldots, B_r \in \mathscr{E}$,

$$\mathbf{Pr}\left(A| \cap_{i=1}^{r} B_i\right) = \mathbf{Pr}(A)\,.$$

In most of our considerations, a convenient way to check mutual independence is the *Mutual Independence Principle*: let $\mathscr{X} := X_1, \dots, X_m$ be a sequence of independent random experiments, and suppose that $A_1, \dots, A_n$ are events such that each A_i is determined by the experiments of a set $F_i \subseteq \mathscr{X}$. Then A_i is mutually independent of $\{A_{i_1}, A_{i_2}, \dots, A_{i_k}\}$ provided that $F_i \cap \left(\cup_{j=1}^{k} F_{i_j}\right) = \emptyset$.

Given a probability space $(\Omega, \mathbf{Pr})$, a *random variable* is a function from Ω to $\mathbf{R}$. The *expected value* of a random variable X is

$$\mathbf{E}(X) := \sum_{\omega \in \Omega} \mathbf{Pr}(\omega) X(\omega)\,.$$

It is also called the *Expectation of X*. The Expectation is a linear operator. It also directly follows from the definition that $\mathbf{Pr}\left(X \le \mathbf{E}(X)\right) > 0$. This is called the *first moment principle*. If X is a non-negative integer-valued random variable, and if $\mathbf{E}(X) < 1$, then the first moment principle ensures that $\mathbf{Pr}(X = 0) > 0$.

Expected values are often much easier to estimate than the corresponding random variables. Thus, bounding $|X - \mathbf{E}(X)|$ is an efficient way of bounding X, and results bounding this quantity are known as *concentration bounds*. Some of them are presented in Section 5. Let us state right now an elementary but useful one, which directly follows from the definition of the expectation.

LEMMA 1.2 (Markov's Inequality). *For every non-negative random variable X and every positive real number t,*

$$\mathbf{Pr}(X \ge t) \le \frac{\mathbf{E}(X)}{t}\,.$$

In particular, if X is a non-negative integer-valued random variable, then applying Markov's Inequality with $t = 1$ yields that $\mathbf{Pr}(X > 0) \le \mathbf{E}(X)$. Thus, we obtain

$$\mathbf{Pr}(X = 0) = 1 - \mathbf{Pr}(X > 0) \ge 1 - \mathbf{E}(X)\,,$$

which, in case $\mathbf{E}(X) < 1$, lower bounds the probability that X is 0 (while the first moment principle would just yield that it is positive). The use of the first moment principle and Markov's Inequality is often called the *first moment method*. Four examples of applications (including the probabilistic proof of the existence of triangle-free graphs with arbitrarily high girth) are presented in the book by Molloy and Reed [140, Chapter 3].

We should note here that using discrete probability amounts to counting. In some simple applications, the probability space may even seem artificial, and one could just count without using a probabilistic setting. There is no objection to that. However, one should also see that the use of probability is a very *efficient* way of counting, and allows us to utilise powerful theorems inherited from the probability theory. Knowing how to count is a key issue in combinatorics, and yields powerful results. For instance, the discharging method—used to prove many theorems, including the 4-Colour Theorem—also amounts to counting. Thus, developing efficient ways of counting is a major theme of combinatorics, in a broad sense.

2. Three Glimpses of Probabilistic Method

The historically first two applications of the probabilistic method, mentioned in the previous section, are presented in many places, for instance in the monograph of Alon and Spencer [11]. As a warm-up, we present in this section three more recent applications. All of them make an elementary and clever use of the probabilistic method, mixed with structural theorems on graph colouring. The first one is a bound on the total chromatic number of a graph, derived in the early nineties by McDiarmid and Reed [127]. The second one deals with the choice number of graphs in relation to the minimum degree, and is due to Alon [2]. It illustrates the so-called first moment method. Finally, we introduce the important concept of fractional chromatic number and discuss a recent result of Hatami and Zhu [80].

2.1. Total-colourings of Graphs

Given a graph $G = (V, E)$ and a positive integer k, a *k-total-colouring of G* is a mapping $\lambda : V \cup E \to \{1, 2, \dots, k\}$ such that

(1) $\lambda(u) \neq \lambda(v)$ for every pair (u, v) of adjacent vertices;

(2) $\lambda(v) \neq \lambda(e)$ for every vertex v and every edge e incident to v;

(3) $\lambda(e) \neq \lambda(e')$ for every pair (e, e') of adjacent edges.

This notion was independently introduced by Behzad [18] in his doctoral thesis, and Vizing [190]. It is now a prominent notion in graph colouring, to which a whole book is devoted [194]. Both Behzad and Vizing made the celebrated *Total-colouring Conjecture*, stating that every graph of maximum degree Δ admits a $(\Delta + 2)$-total-colouring. Notice that every such graph cannot be totally-coloured with less than $\Delta + 1$ colours. Moreover, a cycle of length 5 cannot be 3-totally-coloured.

A series of upper bounds of the form $\Delta + o(\Delta)$ were obtained successively by Hind [89], Chetwynd and Häggkvist [37], and McDiarmid and Reed [127]. Next, Hind, Molloy, and Reed [88] proved the first bound of the form $\Delta + \mathrm{poly}(\log \Delta)$. The best general bound so far has been obtained by Molloy and Reed [135]. They established that every graph of maximum degree Δ can be $(\Delta + 10^{26})$-totally-coloured. They used the probabilistic method to obtain this impressive progress on the previously known bounds. Moreover, the Total-colouring Conjecture has been shown to be true for several special cases, namely for $\Delta = 3$ by Rosenfeld [162] and Vijayaditya [186], and then for $\Delta \in \{4, 5\}$ by Kostochka [111].

We prove in this subsection the following result obtained by McDiarmid and Reed [127] in the early nineties.

Theorem 2.1 (McDiarmid and Reed, 1993). *Every graph G on n vertices can be $(\chi'(G) + k + 1)$-totally-coloured for any integer k such that $k! \geq n$.*

This theorem implies a general upper bound of $\Delta + O\left(\frac{\log n}{\log \log n}\right)$ for all the graphs on n vertices with maximum degree Δ. Hence, it is not as good as the currently best bound, found by Molloy and Reed [135]. However, the proof of Theorem 2.1 is much shorter and perfectly fits the purposes of this warm-up. It combines an elementary use of the probabilistic method with Brooks' and Vizing's Theorems.

Proof of Theorem 2.1. We let Δ be the maximum degree of G. We assume that $\Delta \geq 3$, the statement of the theorem being trivially true otherwise. Note that the conclusion holds if $k \geq \chi(G)$, so we may assume that $k < \chi(G)$. Moreover, we may assume that $k \geq 2$ and G is connected. Note that the complete graph on n vertices can be $(n+1)$-totally coloured, so we also assume that G is not complete.

The strategy is to start from a proper edge-colouring of G with $\chi'(G)$ colours. By Vizing's Theorem, $\chi'(G) \in \{\Delta, \Delta+1\}$. Thus, by Brooks' Theorem, we know that the vertices of G can be properly coloured using at most $\chi'(G)$ colours, since $\Delta \geq 3$ and G is not complete. We do so using the same set of colours. Next, we try to combine those two colourings so as to minimise the number of conflicts by permuting the colours of the edges. Finally, we solve the remaining conflicts by using Vizing's Theorem to recolour with new colours the edges involved in conflicts. The existence of the desired permutation is shown by (elementary) probabilistic means.

Let $q := \chi'(G) \in \{\Delta, \Delta+1\}$, and consider a partition $\mathscr{M} = \{M_1, M_2, \ldots, M_q\}$ of the edges of G into q matchings. By Brooks' Theorem, there exists a partition $\mathscr{C} = \{C_1, C_2, \ldots, C_q\}$ of the vertices of G into q independent sets.

To each bijection $\pi : \mathscr{M} \to \mathscr{C}$ we associate the *conflict graph* G_π, which is the subgraph of G spanned by those edges xy such that $x \in \pi(M)$ or $y \in \pi(M)$, where M is the matching in $\mathscr{M}$ containing the edge xy. Thus, if we properly recolour the edges contained in the graph G_π with new colours, then we obtain a total-colouring of G. By Vizing's Theorem, $\chi'(G_\pi) \leq \Delta_\pi + 1$, where Δ_π is the maximum degree of G_π. Therefore, G can be totally coloured using at most

$$s := q + \Delta_\pi + 1$$

colours. So it only remains to prove the existence of a bijection $\pi : \mathscr{M} \to \mathscr{C}$ such that G_π has maximum degree at most k.

Suppose that v is a vertex of G_π of degree larger than k. Among the at least $k+1$ edges of G_π incident with v, at most one has the same colour as v. Thus, there are at least k neighbours w of v whose colour is the same as that of the edge vw. (Note that, consequently, those neighbours have pairwise distinct colours.) Let us exploit this remark.

We choose a bijection $\pi : \mathscr{M} \to \mathscr{C}$ uniformly at random, i.e. any particular bijection is chosen with probability $\frac{1}{q!}$. Consider a vertex v of G of degree larger than k. We define $\mathscr{K}$ to be the collection of sets $W \subseteq N_G(v)$ of order k such that no two vertices of W have the same colour. For every $W \in \mathscr{K}$, let A_W be the event that for each $w \in W$, the matching $M \in \mathscr{M}$ containing the edge vw is mapped to the stable set containing w. Therefore, if v has degree more than k in G_π, then there exists a set $W \in \mathscr{K}$ such that the event A_W holds.
For every $W \in \mathscr{K}$,

$$\mathbf{Pr}(A_W) = \prod_{i=0}^{k-1} (q-i)^{-1} = \frac{(q-k)!}{q!}\,.$$

As $|\mathscr{K}| \leq \binom{|N_G(v)|}{k}$, it follows that

$$\mathbf{Pr}(|N_{G_\pi}(v)| > k) \leq \binom{|N_G(v)|}{k} \cdot \frac{(q-k)!}{q!}\,. \tag{1}$$

Further, if $|\mathscr{K}| = \binom{|N_G(v)|}{k}$ then all the neighbours of v have distinct colours (recall that $k \geq 2$). Therefore, the events A_W for $W \in \mathscr{K}$ are not incompatible (since $q \geq \Delta$). It follows that the inequality (1) is strict. Consequently,

$$\mathbf{Pr}(|N_{G_\pi}(v)| > k) < \binom{\Delta}{k} \cdot \frac{(\Delta-k)!}{\Delta!} = \frac{1}{k!}$$

since $|W| \leq \Delta \leq q$. This yields that

$$\mathbf{Pr}(\Delta(G') \geq k+1) < \frac{n}{k!} \leq 1\,,$$

which concludes the proof. □

Let us end this subsection by noting that total-colourings of planar graphs have attracted a considerable amount attention. First, Borodin [25] proved that if $\Delta \geq 9$ then every plane graph of maximum degree Δ fulfils the Total-colouring Conjecture. This result can be extended to the case where $\Delta = 8$ by using the 4-Colour Theorem combined with Vizing's Theorem—the reader is referred to the book by Jensen and Toft [94] for further exposition. Sanders and Zhao [163] solved the case where $\Delta = 7$ of the Total-colouring Conjecture for plane graphs. So the only open case regarding plane graphs is $\Delta = 6$. Interestingly, $\Delta = 6$ is also the only remaining open case for Vizing's Edge-colouring Conjecture [189], after Sanders and Zhao [164] resolved the case where $\Delta = 7$. Vizing's Edge-colouring Conjecture states that the chromatic index of every plane graph with maximum degree $\Delta \geq 6$ is Δ. For $\Delta \geq 8$, it was proved to be true by Vizing [189]. The statement cannot be extended to plane graphs with maximum degree smaller than 6 (except the trivial case where $\Delta = 1$). Indeed, as noted by Vizing [189], subdividing one edge in a 4-cycle, the complete graph K_4, the octahedron and the dodecahedron provides examples of plane graphs with maximum degree Δ and chromatic index $\Delta + 1$, for each $\Delta \in \{2, 3, 4, 5\}$, respectively.

An assertion stronger than that of the Total-colouring Conjecture can be proved for plane graphs with high maximum degree. More precisely, Borodin [25] showed that if $\Delta \geq 14$ then every plane graph with maximum degree Δ is $(\Delta+1)$-totally-colourable. He also asked whether 14 could be decreased. Borodin, Kostochka and Woodall extended this result to the case where $\Delta \geq 12$ [27], and later to $\Delta = 11$ [28]. Wang [193] established the result for $\Delta = 10$. Recently, Kowalik, Sereni, and Škrekovski [112] proved the assertion in the case where $\Delta = 9$. On the other hand, this bound is not true if $\Delta \leq 3$. The complete graphs K_2 and K_4 are not 2- and 4-totally-colourable, respectively. As for $\Delta = 2$, a cycle of length $3k + 2$ with $k \geq 1$ cannot be 3-totally-coloured.

2.2. Bounding the Choice Number in Terms of the Minimum Degree

We present in this subsection the following important result of Alon [2]. Molloy and Reed [140, Chapter 3] gave a neat proof of a (slightly) weaker version, following the lines of Alon's proof.

THEOREM 2.2 (Alon, 2000). *Let s be an integer. The choice number of any graph with minimum degree at least*

$$\delta > 2^{2s+2}\frac{(s^2+1)^2}{(\log_2 e)^2} \tag{2}$$

is greater than s.

Alon [2] noted that this result has several interesting consequences. As pointed out in Subsection 1.1,

$$\mathrm{ch}(K_{\delta,\delta}) = (1+o(1))\log_2\delta\,.$$

Thus, the bound in Theorem 2.2 is tight up to a constant factor of $2+o(1)$.

The *colouring number* $\mathrm{col}(G)$ of a graph G is the least integer d such that every subgraph of G contains a vertex of degree smaller than d. Thus, $\mathrm{ch}(G) \le \mathrm{col}(G)$. Theorem 2.2 implies that $\mathrm{ch}(G) \ge \left(\frac{1}{2}-o(1)\right)\log_2 d$ for any graph G whose colouring number exceeds d. Consequently, setting $d := \mathrm{col}(G)$,

$$\left(\frac{1}{2}-o(1)\right)\log_2 d \le \mathrm{ch}(G) \le d\,.$$

As the colouring number of a graph can be computed in linear time, we obtain a linear-time algorithm providing an estimate of the choice number of any graph. Even though the approximation ratio is rough, no analog result is known for the chromatic number of a graph.

Proof of Theorem 2.2. We assume that $s \ge 3$, since the assertion is true when $s \le 2$ thanks to the characterisation of graphs with choice number at most 2, independently proved by Borodin [24] and Erdős, Rubin and Taylor [53] (the reader can also consult a paper of Thomassen [181]).

We define n to be the number of vertices of $G = (V,E)$, and we let $\mathscr{C} := \{1,2,\ldots,s^2\}$ be the set of colours. We show the existence of an s-list-assignment $L : V \to 2^{\mathscr{C}}$ for which G admits no proper colouring c with $c(v) \in L(v)$ for each $v \in V$.

The strategy is as follows. We consider a set $B \subset V$, with lists assigned to its members. There are $s^{|B|}$ different colouring of the subgraph of G induced by B (where each vertex is assigned a colour taken from its list). We would like to show that the lists of (some of) the remaining vertices can be chosen such that none of the colourings of B extends to a proper list-colouring of G. In other words, we seek vertices outside B such that however the vertices of B are coloured, the list of at least one of them will be included in the set of colours assigned to its neighbours in B. To this end, we need first to have a fair amount of vertices outside B, thus we should control the size of B. Then, the vertices outside B that are of interest to us should have neighbours in B whose union of lists is somehow large. This is why the set B and the lists for its vertices should be well-chosen, i.e. they should fulfil certain helpful properties. By analysing random choices, we are able to show that such a good choice exists. After fixing one such choice, we proceed with proving the existence of lists for some vertices outside B such that no proper list-colouring exists. Let us formalise all this.

Each vertex of G is chosen to be a member of B independently at random with probability $\delta^{-1/2}$. Next, each vertex b of B is assigned a list $S(b)$ of s colours taken from $\mathscr{C}$, chosen independently and uniformly at random among all the subsets of cardinality s of $\mathscr{C}$. Note that the expected size of B is $n \cdot \delta^{-1/2}$. Hence, it follows from Markov's Inequality that

$$\mathbf{Pr}(|B| > 2n \cdot \delta^{-1/2}) < \frac{1}{2}\,.$$

A vertex $v \in V \setminus B$ is *good* if for every subset T of $\mathscr{C}$ of cardinality $\lceil s^2/2 \rceil$, there is a neighbour b of v belonging to B and whose list is contained in T.

We assert that the probability that a vertex $v \in V$ is not good is less than $\frac{1}{4}$. Let us take, for a second, this assertion for granted and see how the desired conclusion follows from it. We deduce from the assertion that the expected number of bad vertices is less than $\frac{n}{4}$. Thus, Markov's Inequality implies that the probability that there are less than $\frac{n}{2}$ good vertices is less than $\frac{1}{2}$. Consequently, with positive probability it holds that $|B| \le 2n \cdot \delta^{-1/2}$ *and* the number of good vertices is at least $\frac{n}{2}$.

Let us *fix such a choice of B and S*. Let A be the set of good vertices, so $|A| \ge \frac{n}{2}$. We extend the s-list-assignment S to A by choosing for each $a \in A$ a set $S(a)$ of s colours uniformly at random, and independently. We show now that, with positive probability, there is no proper colouring of $A \cup B$ that assigns to each vertex a colour from its list.

There are $s^{|B|}$ different colourings of B. Let us fix such a colouring and estimate the probability that it can be extended to the vertices of A. For each $a \in A$, let $F(a)$ be the set of colours that appear on its neighbours belonging to B. Note that if a can be properly coloured then $S(a) \nsubseteq F(a)$. Since a is good, $|F(a)| \ge \lceil s^2/2 \rceil$. Therefore, the probability that a can be coloured is at most

$$1 - \frac{\binom{\lceil s^2/2 \rceil}{s}}{\binom{s^2}{s}} \le 1 - 2^{-s-1}\,,$$

where the inequality follows from (3) below.

$$\begin{aligned}
\frac{\binom{\lceil s^2/2\rceil}{s}}{\binom{s^2}{s}} =& \frac{\lceil s^2/2\rceil\left(\lceil s^2/2\rceil-1\right)\dots\left(\lceil s^2/2\rceil-s+1\right)}{s^2(s^2-1)\dots(s^2-s+1)}\\
\geq& 2^{-s}\prod_{i=0}^{s-1}\frac{s^2-2i}{s^2-i}\\
=& 2^{-s}\prod_{i=0}^{s-1}\left(1-\frac{i}{s^2-i}\right)\\
\geq& 2^{-s}\left(1-\sum_{i=0}^{s-1}\frac{i}{s^2-s}\right)\\
=& 2^{-s-1}\,. \qquad (3)
\end{aligned}$$

Since the choice of the lists $S(a)$ for $a \in A$ are independent, we deduce that the probability that a fixed colouring of B can be extended to a proper colouring of $G[A \cup B]$ assigning to each vertex a colour from its list is at most

$$\left(1-2^{-s-1}\right)^{|A|} \leq \left(1-2^{-s-1}\right)^{n/2} \leq \exp\left[-n\cdot 2^{-s-2}\right]\,,$$

since $(1-x)^z \leq e^{-xz}$ for positive real numbers x and z. Consequently, the probability that there is a proper colouring of $G[A\cup B]$ assigning to each vertex a colour from its list is at most

$$s^{|B|}\cdot\exp\left[-n\cdot 2^{-s-2}\right] \leq \exp\left[-2n\cdot\delta^{-1/2}\right] < 1$$

by (2) and the fact that $s \geq 3$. Therefore, there exists an s-list-assignment of A such that $G[A\cup B]$ cannot be properly coloured, as desired.

It remains to prove that the probability that a vertex $v \in V$ is not good is at most $\frac{1}{4}$. Fix a vertex $v \in V$. It belongs to B with probability $\frac{1}{\sqrt{\delta}}$. Suppose now that $v \notin B$. Then, for each set $T \subset \mathscr{C}$ of cardinality $\lceil s^2/2\rceil$, and for each neighbour u of v, it holds that

$$\mathbf{Pr}\left(u\in B \text{ and } S(u)\subset T\right) = \frac{1}{\sqrt{\delta}}\cdot\frac{\lceil s^2/2\rceil\left(\lceil s^2/2\rceil-1\right)\dots\left(\lceil s^2/2\rceil-s+1\right)}{s^2(s^2-1)\dots(s^2-s+1)}\,.$$

There are $\binom{s^2}{\lceil s^2/2\rceil}$ possible choices for the subset T, and at least δ possible choices for the neighbour u. Consequently, the probability that there exists a set T of $\lceil s^2/2\rceil$ colours such that each neighbour of v either is not in B or has a list not contained in T is at most

$$\binom{s^2}{\lceil s^2/2\rceil}\cdot\left(1-\frac{1}{\sqrt{\delta}}\cdot\frac{\lceil s^2/2\rceil\left(\lceil s^2/2\rceil-1\right)\dots\left(\lceil s^2/2\rceil-s+1\right)}{s^2(s^2-1)\dots(s^2-s+1)}\right)^{\delta}\,.$$

In total, the probability that an *arbitrary* vertex $v\in V$ is not good is at most

$$\frac{1}{\sqrt{\delta}}+\left(1-\frac{1}{\sqrt{\delta}}\right)\binom{s^2}{\lceil s^2/2\rceil}\left(1-\frac{1}{\sqrt{\delta}}\cdot\frac{\lceil s^2/2\rceil\left(\lceil s^2/2\rceil-1\right)\dots\left(\lceil s^2/2\rceil-s+1\right)}{s^2(s^2-1)\dots(s^2-s+1)}\right)^{\delta}\,.$$

Hence, from (3) and the fact that $\binom{s^2}{\lceil s^2/2 \rceil} \le 2^{s^2}/4$ for $s \ge 3$, we deduce that

$$\mathbf{Pr}\left(v \text{ is not good}\right) \le \frac{1}{\sqrt{\delta}} + \frac{1}{4} 2^{s^2} \left(1 - \frac{2^{-s-1}}{\sqrt{\delta}}\right)^{\delta} \le \frac{1}{\sqrt{\delta}} + \frac{1}{4} 2^{s^2} \exp\left[-\sqrt{\delta} \cdot 2^{-s-1}\right],$$

which is less than $\frac{1}{4}$ by (2). □

We underline the importance of the two steps in the proof. It is *not* true that, considering the union of two sets A and B with a list-assignment uniformly at random, there will be a pair such that no colouring of B can be extended to A. Actually, the set B (and the list-assignment for the vertices of B) fulfils very particular properties, even though its *existence* was proved by considering sets B at random.

2.3. The Fractional Chromatic Number

The chromatic number of a graph can be viewed as the solution of an integer linear program. Indeed, let $\mathscr{S}(G)$ be the set of all the independent sets of the graph $G = (V, E)$. An r-colouring can be viewed as a mapping $f : \mathscr{S}(G) \to \{0, 1\}$ such that

$$\forall v \in V, \quad \sum_{\substack{S \in \mathscr{S}(G) \\ v \in S}} f(S) \ge 1$$

and

$$\sum_{S \in \mathscr{S}(G)} f(S) \le r\,.$$

If we allow f to take values in $[0, 1]$ instead of $\{0, 1\}$, then we call f a *fractional r-colouring*. The *fractional chromatic number* $\chi_f(G)$ of G is the least r for which G admits a fractional r-colouring.

An equivalent definition of the fractional chromatic number of G is obtained through the concept of weighted colourings. Given integers k and ℓ, a *k-tuple ℓ-colouring* of G is a mapping c that assigns to each vertex v a subset $c(v)$ of $\{1, 2, \ldots, \ell\}$ of order k such that $c(v) \cap c(u) = \emptyset$ whenever $uv \in E$. Then, the fractional chromatic number of G is the infimum of the ratios $\frac{\ell}{k}$ for which G admits a k-tuple ℓ-colouring. Moreover, the infimum of the definition is actually attained [165, p. 24], and thus the fractional chromatic number is a rational number. Note that $\chi_f(G) \le \chi(G)$ by the definition, and the ratio $\frac{\chi(G)}{\chi_f(G)}$ can be arbitrarily large. The book of Scheinerman and Ullman [165] can be consulted for the proof of this fact, and more generally it provides an excellent account on fractional theory of graphs.

Let G be a triangle-free graph on n vertices, with maximum degree at most 3. By Brooks' Theorem, the chromatic number of G is at most 3. Hence, G has an independent set of size at least $\frac{n}{3}$. In 1979, Staton [176] proved that this lower bound can be improved to $\frac{5n}{14}$. This bound is tight since, as noted by Fajtlowicz [54], it is attained by the generalised Petersen graph $P(7, 2)$; see Figure 1. About a decade later, Jones [97] could simplify the proof of Staton's result. In the mid-1990s, Griggs

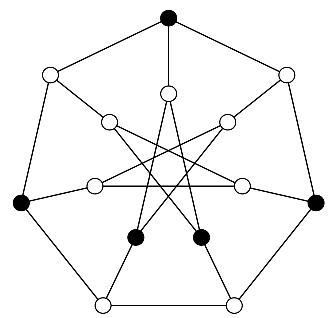

Figure 1: The generalised Petersen graph $P(7,2)$ has fourteen vertices and independence number 5. The black vertices form a maximum independent set.

and Murphy [69] designed a linear-time algorithm to find an independent set of size at least $\frac{5}{14}\cdot(n-k)$, where k is the number of components of G that are 3-regular (i.e. every vertex has degree exactly 3). Heckmann and Thomas [86] provided a new (and simpler) proof of Staton's result. They also designed a linear-time algorithm that finds an independent set of size at least $\frac{5n}{14}$ for every triangle-free graph G of maximum degree at most 3 with n vertices. Moreover, they conjectured that this lower bound can be strengthened to a lower bound on the fractional chromatic number.

Conjecture 2.3 (Heckmann and Thomas, 1998). *The fractional chromatic number of every triangle-free graph of maximum degree at most* 3 *is at most* $\frac{14}{5}=3-\frac{1}{5}$.

The best bound known so far has been obtained by Hatami and Zhu [80], who proved $3-\frac{3}{64}$. Moreover, they also studied the fractional chromatic number of such graphs in relation to their girth. For $k\geq 4$, set

$$\tau_k := \max\{\chi_f(G) : G \text{ is a graph of maximum degree at most 3 and girth at least } k\}\,.$$

Let $\tau := \lim_{k\to\infty}\tau_k$ (note that $(\tau_k)_k$ is a decreasing sequence bounded below). Hatami and Zhu [80] studied the sequence $(\tau_k)_{k\geq 4}$ by means of a sequence $(c_k)_{k\geq 4}$ satisfying $\tau_k\leq c_k$. This sequence is defined below. Numerical studies suggest that $\lim_{k\to\infty}c_k=\frac{8}{3}$, but this is not proved. Finally, a result of McKay [130] implies that $\tau\geq 2.1959$. This result combined with that of Hatami and Zhu [80] yields that $2.1959\leq\tau\leq 2.66681$.

Let us see how Hatami and Zhu [80] obtained the existence of the sequence $(c_k)_k$. Fix a positive integer k and a graph G of maximum degree at most 3 and girth $g\geq 2k+1$. We define two functions $f,F:\{0,1,\dots,g-1\}\times[0,1]\to\mathbf{R}$ as follows.

$$f(0,x):=0 \quad\text{and}\quad F(0,x):=1\,;$$

and for $j\in\{1,2,\dots,g-1\}$,

$$f(j,x):=(1-x)^2+2(1-x)(x-\int_0^x F(j-1,y)\,\mathrm{d}y)+\left(x-\int_0^x F(j-1,y)\,\mathrm{d}y\right)^2\,;$$

$$F(j,x):=(1-x)^2+2(1-x)(x-\int_0^x f(j-1,y)\,\mathrm{d}y)+\left(x-\int_0^x f(j-1,y)\,\mathrm{d}y\right)^2\,.$$

Moreover, we set $A_k := x - \int_0^x F(k-1,y)\,\mathrm{d}y$.

Theorem 2.4 (Hatami and Zhu, 2008). *For every positive integer k, and every graph $G=(V,E)$ of maximum degree at most 3 and girth at least $2k+1$,*

$$\chi_f(G) \le c_{2k+1} := \left(\int_0^1 \left((1-x)^3 + 3(1-x)^2 A_k + 3(1-x)A_k^2 + A_k^3\right)\,\mathrm{d}x\right)^{-1}.$$

Let us define a fractional c_{2k+1}-colouring of G to prove Theorem 2.4. To each ordering $\pi := v_1, v_2, \ldots, v_n$ of the vertices of G, we associate the independent set S_π obtained as follows: *for each i from 1 to n, if S_π contains no neighbour of v_i then add the vertex v_i to S_π.*

For each independent set S, we define $m(S)$ to be the number of orderings π such that $S = S_\pi$. Let f be the mapping that assigns to each independent set S the number $\frac{c_{2k+1}\cdot m(S)}{|V|!}$. So, $\sum_{S\in\mathscr{S}(G)} f(S) = c_{2k+1}$ because $\sum_{S\in\mathscr{S}(G)} m(S) = |V|!$ by the definition. Hence, the conclusion of Theorem 2.4 follows provided that f is indeed a fractional colouring, i.e. for every vertex v of G,

$$\sum_{\substack{S\in\mathscr{S}(G)\\ v\in S}} f(S) \ge 1\,.$$

In other words, it suffices to prove that for every vertex v, the number of orderings π such that $v\in S_\pi$ is at least $\frac{|V|!}{c_{2k+1}}$.

This is achieved by probabilistic means. Let us define a probability space by considering all the possible orderings uniformly at random. To this end, for each vertex v of G we choose uniformly at random (and independently) a *weight* $\omega(v)\in[0,1]$. Note that, with probability 1, no real number is chosen for two distinct vertices. The vertices of G can be ordered according to the increasing order of their weights, yielding the ordering π_ω. Thus, it suffices to prove the following lemma to finish the proof of Theorem 2.4.

Lemma 2.5 (Hatami and Zhu, 2008). *If π is a permutation chosen uniformly at random, then for every vertex v of G,*

$$\mathbf{Pr}(v\in S_\pi) \ge \frac{1}{c_{2k+1}}\,.$$

Let us do some ground work before starting the proof of Lemma 2.5. Given an ordering π_ω and a set X of vertices of $G=(V,E)$, we let $\pi_\omega - X$ be the restriction of π_ω to the vertices of $G-X$. Let u be a vertex of G with neighbours u_1, u_2 and u_3. For any positive integer k, let $N_k(u,u_1)$ be the set of vertices at distance at most k from u in $G-u_1$. Since the girth of G is greater than $2k$, we deduce that $u_3\notin N_{k-1}(u_1,u)$ and $N_{k-1}(u_1,u)\cap N_{k-1}(u_2,u)=\emptyset$.

Choose uniformly at random a weight ω and let $\pi=\pi_\omega$. For every real number $x\in[0,1]$, we set

$$m_k(u,u_1,x) := \min_\sigma \left\{\mathbf{Pr}\left(u\in S_\pi \middle| \pi - \{N_k(u,u_1)\cup\{u_1\}\} = \sigma, \omega(u)=x, \omega(u_1)>x\right)\right\},$$

and

$$M_k(u, u_1, x) := \max_{\sigma} \{\mathbf{Pr}\,(u \in S_\pi | \pi - \{N_k(u, u_1) \cup \{u_1\}\} = \sigma, \omega(u) = x, \omega(u_1) > x)\}\ ,$$

where the minimum and the maximum are taken over all the permutations σ of $V \setminus (N_k(u, u_1) \cup \{u_1\})$. We define $m(k, x)$ and $M(k, x)$ to be the minimum and the maximum of $m_k(u, u_1, x)$ and $M_k(u, u_1, x)$ taken over all the edges uu_1 of G, respectively. We now bound $m(k, x)$ and $M(k, x)$.

Let B be the event that $\omega(u) = x$ and $\omega(u_1) > x$. We partition the set of all events using the following three events. Let B_1 be the event that $\min(\omega(u_2), \omega(u_3)) > x$, let B_2 be the event that either $\omega(u_2) < x < \omega(u_3)$ or $\omega(u_3) < x < \omega(u_2)$, and let B_3 be the event that $\max(\omega(u_2), \omega(u_3)) < x$. Fix an arbitrary permutation σ of $V \setminus (N_k(u, u_1) \cup \{u_1\})$, and let A be the event that $\pi_\omega - (N_k(u, u_1) \cup \{u_1\}) = \sigma$. Then

$$\mathbf{Pr}\,(u \in S_\pi | A, B) = \sum_{i=1}^{3} \mathbf{Pr}\,(u \in S_\pi | A, B, B_i) \cdot \mathbf{Pr}(B_i)\,.$$

Let us estimate $p_i := \mathbf{Pr}\,(u \in S_\pi | A, B, B_i)$ for $i \in \{1, 2, 3\}$. First, $p_1 = 1$. Moreover,

$$\frac{1}{x^2}\left(\int_0^x (1 - M(k-1, y))\,\mathrm{d}y\right)\left(\int_0^x (1 - M(k-1, y))\,\mathrm{d}y\right) \le p_3$$

and

$$p_3 \le \frac{1}{x^2}\left(\int_0^x (1 - m(k-1, y))\,\mathrm{d}y\right)\left(\int_0^x (1 - m(k-1, y))\,\mathrm{d}y\right)\,.$$

Last,

$$\frac{1}{x}\int_0^x (1 - M(k-1, y))\ \mathrm{d}y \le p_2 \le \frac{1}{x}\int_0^x (1 - m(k-1, y))\ \mathrm{d}y\,.$$

Note that $\mathbf{Pr}(B_1) = (1-x)^2$, and $\mathbf{Pr}(B_2) = 2x(1-x)$ and $\mathbf{Pr}(B_3) = x^2$. Therefore, we deduce that

$$m(k, x) \ge (1-x)^2 + 2(1-x)\left(x - \int_0^x M(k-1, y)\,\mathrm{d}y\right) + \left(x - \int_0^x M(k-1, y)\,\mathrm{d}y\right)$$

and

$$M(k, x) \le (1-x)^2 + 2(1-x)\left(x - \int_0^x m(k-1, y)\,\mathrm{d}y\right) + \left(x - \int_0^x m(k-1, y)\,\mathrm{d}y\right)\,.$$

Consequently, $M(k-1, y) \le F(k-1, y)$.

We are now ready to prove Lemma 2.5.

Proof of Lemma 2.5. Let $\pi = \pi_\omega$ be a random permutation chosen uniformly at random according to a weight ω. It suffices to prove that

$$\mathbf{Pr}(v \in S_\pi | \omega(v) = x) \ge (1-x)^3 + 3(1-x)^2 A_k + 3(1-x) A_k^2 + A_k^3\,. \tag{4}$$

For $i \in \{0,1,2,3\}$, let D_i be the event that i neighbours of v appear before v in the random permutation π_ω. Thus,

$$\mathbf{Pr}(v \in S_\pi | \omega(v) = x) = \sum_{i=0}^{3} \left(\mathbf{Pr}(v \in S_\pi | D_i, \omega(v) = x) \cdot \mathbf{Pr}(D_i | \omega(v) = x)\right) .$$

Note that $P(D_i|\omega(v) = x) = \binom{3}{i}(1-x)^{3-i}x^i$ for each $i \in \{0,1,2,3\}$. Moreover, $\mathbf{Pr}(v \in S_\pi | D_0, \omega(v) = x) = 1$.

Now, fix $i \in \{1,2,3\}$ and assume that $\omega(v_j) < \omega(v) = x$ for each $j \in \{1,\ldots,i\}$. Then, $v \in S_\pi$ if and only if $\{v_1,\ldots,v_i\} \cap S_\pi = \emptyset$. Set $P_j := \mathbf{Pr}(v_j \in S_\pi | \omega(v_j) < x, \omega(v) = x)$. Hence,

$$\mathbf{Pr}(v \in S_\pi | D_i, \omega(v) = x) = \prod_{j=1}^{i}(1 - P_j) .$$

Thus, the conclusion follows provided that

$$\prod_{j=1}^{i}(1 - P_j) \geq \frac{1}{x^i} A_k^i ,$$

which in turn is implied by

$$P_j \leq \frac{1}{x}\int_0^x F(k-1, y)\,\mathrm{d}y .$$

But, by the definition, $P_j \leq \frac{1}{x}\int_0^x M(k-1,y)\,\mathrm{d}y$, which yields the conclusion since $M(k-1,y) \leq F(k-1,y)$ as we noted before starting the proof. □

3. A Few Words on Entropy

We briefly present the concept of entropy, and we illustrate its use in two combinatorial problems. Entropy was introduced by Shannon, and it plays a fundamental role in information theory. It has also proved to be a useful tool for some combinatorial problems, including graph colouring problems [100, 101]. In particular, the ideas of the theory of information can be applied to study counting questions and graph covering issues [156]. It is also a good way to obtain (standard or not) inequalities [61]. (There is also a notion of entropy colouring of graphs [10], though we do not deal with it in this survey.)

We present two applications: a short proof [155] of a theorem of Brègman [29] on the maximum number of different perfect matchings of a bipartite graph, and a result of Kahn [102] about the number of independent sets in bipartite graphs. The general idea illustrated in this section is to express a certain quantity as the (logarithm of the) entropy of a related random variable, and then use tools from the probability theory to derive an upper bound.

Recall that a matching of a graph is a subset of its edges no two of which are adjacent. Thus, a colour class of a proper edge-colouring is a matching, and every matching can be viewed as the colour class of some edge-colouring. A matching M of a graph G is *perfect* if every vertex of G is incident to exactly one edge of M. If G is a Δ-regular bipartite graph, then its edge-chromatic number is Δ, and any proper Δ-edge-colouring of G is a partition of its edge-set into Δ perfect matchings. Consequently, if G has p different perfect matchings, then it has

$$\binom{p}{\Delta} \cdot \Delta! = \frac{p!}{(p-\Delta)!}$$

different proper Δ-edge-colourings. The next theorem [29] gives an upper bound on the number of perfect matchings in any bipartite graph. It was originally conjectured by Minc [131], in terms of permanent of matrices.

THEOREM 3.1 (Brègman, 1973). *Let G be a bipartite graph with parts A and B. The number of perfect matchings of G is at most*

$$\prod_{v \in A} (\deg(v)!)^{1/\deg(v)} .$$

Several proofs of this result are known, the original being combinatorial. In 1978, Schrijver [166] found a short proof. A probabilistic description of this proof is presented in the book of Alon and Spencer [11, Chapter 2]. The one we see in Subsection 3.2 uses the concept of entropy, and was found by Radhakrishnan [155] in the late nineties.

We present in Subsection 3.3 a result related to the number of independent sets of a bipartite graph due to Kahn [102].

THEOREM 3.2 (Kahn, 2001). *The number of independent sets of any Δ-regular bipartite graph on $2n$ vertices is at most*

$$\left(2^{\Delta+1} - 1\right)^{n/\Delta} .$$

As shown by a disjoint union of copies of the complete bipartite graph $K_{\Delta,\Delta}$, this upper bound is tight. Kahn [102] conjectured that this bound is true for general graphs with $2n$ vertices and maximum degree Δ.

3.1. Some Background

We give the definition of entropy along with the basic results that we need to prove Theorems 3.1 and 3.2. We refer to the books by McEliece [128, 129] for a nice exposition of the topic. Simonyi wrote a survey on graph entropy [169], and another one devoted to the links between graph entropy and perfect graphs [170]—thus perfectly fitting our setting.

All the logarithms of this section are in base 2. We now define the entropy of a random variable X. As for the entropy, the *values* taken by X are not relevant, only the *probabilities* with which X takes those values are. This is why, in this section, we slightly deviate from our definition of random variables, by allowing them to take values in *any* set (and not only $\mathbf{R}$). Moreover, we assume that their images is a finite set. We let $0 \cdot \log(1/0) := 0$.

Let X be a random variable taking values in a set $\mathscr{X}$. The *entropy* of X is

$$H(X) := \sum_{x\in\mathscr{X}} \mathbf{Pr}_X(x) \log \frac{1}{\mathbf{Pr}_X(x)}\,,$$

where $\mathbf{Pr}_X(x) := \mathbf{Pr}(X = x)$. If X is a 0–1 random variable being 0 with probability p, then $\mathbf{E}(X)$ is the *binary entropy function*, i.e.

$$\mathbf{E}(X) = H(p) := -p\log p - (1-p)\log(1-p)\,.$$

Let Y be a random variable taking values in a set $\mathscr{Y}$. The *joint entropy* of the two random variables X and Y is

$$H(X,Y) = \sum_{\substack{x\in\mathscr{X}\\ y\in\mathscr{Y}}} \mathbf{Pr}(X = x, Y = y) \log\left(\frac{1}{\mathbf{Pr}(X = x, Y = y)}\right)\,.$$

Thus, $H(X,Y) \le H(X)+H(Y)$ with equality if and only if X and Y are independent.

We can condition the entropy of a random variable on a particular observation, or more generally on the outcome of another random variable. The *conditional entropy of X given that $Y = y$* is

$$H(X|Y = y) = \sum_{x\in\mathscr{X}} \mathbf{Pr}\left(X = x|Y = y\right) \log\left(\frac{1}{\mathbf{Pr}\left(X = x|Y = y\right)}\right)\,.$$

The *conditional entropy of X given Y* is the average of the preceding, i.e., letting Y take values in $\mathscr{Y}$,

$$\begin{aligned} H(X|Y) &:= \sum_{y\in\mathscr{Y}} \mathbf{Pr}(Y = y)H(X|Y = y)\\ &= \sum_{\substack{x\in\mathscr{X}\\ y\in\mathscr{Y}}} \mathbf{Pr}\left(X = x, Y = y\right) \log\left(\frac{1}{\mathbf{Pr}\left(X = x|Y = y\right)}\right)\,. \end{aligned}$$

We deduce directly from the definitions that

$$H(X,Y) = H(X) + H(Y|X) \tag{5}$$

and

$$H(X,Y|Z) = H(X|Z) + H(Y|X,Z)\,. \tag{6}$$

By induction, (5) generalises to the so-called *chain rule*, i.e.

$$H(X_1, X_2, \dots, X_n) = \sum_{i=1}^{n} H(X_i|X_1, \dots, X_{i-1})\,. \tag{7}$$

Moreover,

$$H(X) \le \log|\mathscr{X}|\,, \tag{8}$$

with equality if and only if X is uniformly distributed. Indeed, since log is concave, Jensen's Inequality implies that

$$\begin{aligned} H(X) &\leq \sum_{x\in\mathscr{X}} \mathbf{Pr}_X(x)\log\left(\frac{\sum_{x\in\mathscr{X}}\mathbf{Pr}_X(x)/\mathbf{Pr}_X(x)}{\sum_{x\in\mathscr{X}}\mathbf{Pr}_X(x)}\right) \\ &= \sum_{x\in\mathscr{X}} \mathbf{Pr}_X(x)\log\left(\frac{|\mathscr{X}|}{1}\right) \\ &= \log|\mathscr{X}|\,, \end{aligned}$$

the first line being an equality if and only if X is uniformly distributed. This point is a key ingredient in the proofs presented in the next two subsections.

Finally, we also use the following result, known as Shearer's Lemma [40]. If $X = (X_i)_{i\in\mathscr{I}}$ is a vector and A a subset of $\mathscr{I}$, we set $X_A := (X_i)_{i\in A}$.

Lemma 3.3 (Shearer, 1986). *Let $X = (X_1, X_2, \ldots, X_n)$ be a random variable and let $\mathscr{A} = \{A_i\}_{i\in\mathscr{I}}$ be a collection of subsets of $\{1, 2, \ldots, n\}$ such that each integer $i \in \{1, 2, \ldots, n\}$ belongs to at least k sets of $\mathscr{A}$. Then*

$$H(X) \leq \frac{1}{k}\sum_{i\in\mathscr{I}} H\left(X_{A_i}\right)\,.$$

3.2. Radhakrishnan's Proof of Brègman's Theorem

We prove Theorem 3.1. Let G be a bipartite graphs with parts A and B. We define $\mathscr{M}$ to be the set of all the perfect matchings of G, and we suppose that $\mathscr{M} \neq \emptyset$, otherwise the statement of the theorem holds trivially. In particular, $|A| = |B|$; let us set $n := |A|$. For a perfect matching M and a vertex $a \in A$, we let $M(a)$ be the vertex of B that is adjacent to a in M. Further, for every vertex $b \in B$, we let $M^{-1}(b)$ be the vertex of A that is adjacent to b in M.

We choose a perfect matching $M \in \mathscr{M}$ uniformly at random. Thus, $\log|\mathscr{M}| = H(M)$. Let $a_1, a_2, \ldots, a_n$ be an ordering of the vertices of A. Then, by the chain rule (7),

$$\begin{aligned} H(M) =& H(M(a_1)) + H(M(a_2)|M(a_1)) \\ &+ \ldots + H(M(a_n)|M(a_1), M(a_2), \ldots, M(a_{n-1}))\,. \end{aligned} \tag{9}$$

Note that this equation yields the trivial upper bound $|\mathscr{M}| \leq \prod_{a\in A}\deg(a)$. Indeed, $H(M(a_i)|M(a_1), M(a_2), \ldots, M(a_{i-1})$ is at most $H(M(a_i))$, which in turn is at most $\log\deg(a_i)$. We would obtain a better upper bound on $|\mathscr{M}|$ if we manage to infer a better upper bound on $H(M(a_i)|M(a_1), M(a_2), \ldots, M(a_{i-1}))$.

To this end, note that the range of $M(a_i)$ given $M(a_j)$ for $j \in \{1, 2, \ldots, i-1\}$ is actually contained in $N_G(a_i)\backslash\{M(a_1), M(a_2), \ldots, M(a_{i-1})\}$. So, it may well be smaller than $\deg(a_i)$. Moreover, its size depends on the ordering chosen for the vertices of A.

To exploit this remark, let σ be a permutation of $\{1, 2, \ldots, n\}$, chosen uniformly at random. For each index $i \in \{1, 2, \ldots, n\}$, we set $R_i(M, \sigma) := |N_G(a_i) \setminus$

$\{M(a_{\sigma(1)}), \ldots, M(a_{\sigma(k-1)})\}|$, with $k := \sigma^{-1}(i)$. Observe that, for every integer $j \in \{1, 2, \ldots, \deg(a_i)\}$,

$$\Pr_{M,\sigma}(R_i(M,\sigma) = j) = \frac{1}{\deg(a_i)}. \tag{10}$$

Indeed, for any fixed matching M,

$$\Pr_{\sigma}(R_i(M,\sigma) = j|M) = \frac{1}{\deg(a_i)}, \tag{11}$$

since σ is chosen uniformly at random. In fact, (11) can also be proved, for instance, by counting directly: the number of permutations such that j vertices of $M^{-1}(N_G(a_i))$ occur before a_i is

$$\begin{aligned}
&\sum_{k=1}^{n} \binom{\deg(a_i)-1}{j}\binom{n-\deg(a_i)}{k-j-1}(k-1)!(n-k)! \\
=&(\deg(a_i)-1)!(n-\deg(a_i))! \cdot \sum_{k=1}^{n} \binom{k-1}{j}\binom{n-k}{\deg(a_i)-j-1} \\
=&\frac{n!}{\deg(a_i)\cdot\binom{n}{\deg(a_i)}} \cdot \sum_{k=0}^{n-1} \binom{k}{j}\binom{n-1-k}{\deg(a_i)-j-1} \\
=&\frac{n!}{\deg(a_i)},
\end{aligned}$$

where the last line follows from the following classical binomial identity [67, p. 129].

$$\sum_{k=0}^{n-1} \binom{k}{j}\binom{n-1-k}{d-j-1} = \binom{n}{d}.$$

Now, (11) implies (10) by averaging over all $M \in \mathscr{M}$, i.e.

$$\Pr_{M,\sigma}(R_i(M,\sigma) = j) = \sum_{M} \Pr(M) \cdot \Pr_{\sigma}(R_i(M,\sigma) = j|M) = \frac{1}{\deg(a_i)}.$$

On the other hand, applying (8) we obtain

$$H(M(a_i)|M(a_{\sigma(1)}), \ldots, M(a_{\sigma(\sigma^{-1}(i)-1)})) \le \sum_{j=1}^{\deg(a_i)} \Pr_{M}(R_i(M,\sigma) = j) \cdot \log j. \tag{12}$$

Furthermore, (9) translates to

$$\begin{aligned}
H(M) =&H(M(a_{\sigma(1)})) + H(M(a_{\sigma(2)})|M(a_{\sigma(1)})) \\
&+ \ldots + H(M(a_{\sigma(n)})|M(a_{\sigma(1)}), M(a_{\sigma(2)}), \ldots, M(a_{\sigma(n-1)})).
\end{aligned} \tag{13}$$

Summing (13) over all the permutations σ, we obtain

$$n!H(M) = \sum_{\sigma}\sum_{i=1}^{n} H\left(M(a_{\sigma(i)})|M(a_{\sigma(1)}), \ldots, M(a_{\sigma(i-1)})\right),$$

i.e.

$$H(M) = \mathop{\mathbf{E}}_{\sigma}\left[\sum_{i=1}^{n} H\left(M(a_{\sigma(i)})|M(a_{\sigma(1)}),\ldots,M(a_{\sigma(i-1)})\right)\right].$$

We write the terms of the sum in a different order, and use the linearity of Expectation.

$$\begin{aligned} H(M) &= \sum_{i=1}^{n} \mathop{\mathbf{E}}_{\sigma}\left[H\left(M(a_i)|M(a_{\sigma(1)}),\ldots,M(a_{\sigma(\sigma^{-1}(i)-1)})\right)\right] \\ &\le \sum_{i=1}^{n} \mathop{\mathbf{E}}_{\sigma}\left[\sum_{j=1}^{\deg(a_i)} \mathop{\mathbf{Pr}}_{M}\left(R_i(M,\sigma)=j\right)\cdot\log j\right] && \text{by (12)} \\ &= \sum_{i=1}^{n}\sum_{j=1}^{\deg(a_i)}\sum_{\sigma} \mathbf{Pr}(\sigma)\mathop{\mathbf{Pr}}_{M}\left(R_i(M,\sigma)=j\right)\cdot\log j\,. \end{aligned}$$

Observe that

$$\sum_{\sigma}\mathbf{Pr}(\sigma)\mathop{\mathbf{Pr}}_{M}\left(R_i(M,\sigma)=j\right) = \mathop{\mathbf{Pr}}_{M,\sigma}(R_i(M,\sigma)=j)\,.$$

Thus, (10) implies that

$$\begin{aligned} H(M) &\le \sum_{i=1}^{n}\sum_{j=1}^{\deg(a_i)} \frac{1}{\deg(a_i)}\cdot\log j \\ &= \sum_{i=1}^{n} \log\left(\deg(a_i)!\right)^{1/\deg(a_i)}, \end{aligned}$$

which concludes the proof. □

We end this subsection by mentioning the related problem of lower bounding the number of perfect matchings in regular bipartite graphs with n vertices. A graph is *k-regular* if all the vertices have degree k. The first non-trivial lower bound on the number of perfect matchings in 3-regular bridgeless bipartite graphs was obtained in 1969 by Sinkhorn [171], who proved a bound of $\frac{n}{2}$. He thereby established a conjecture of Marshall. The same year, Minc [132] increased this lower bound by 2 and one year after, Hartfiel [78] obtained $\frac{n}{2}+3$. Next, Hartfiel and Crosby [79] improved the bound to $\frac{3}{2}n-3$. The first exponential bound was obtained in 1979 by Voorhoeve [191], who proved $6\cdot\left(\frac{4}{3}\right)^{n/2-3}$. This was generalised to all regular bipartite graphs in 1998 by Schrijver [167], who thereby proved a conjecture of himself and Valiant [168]. His argument is involved, and as a particular case of a different and more general approach (using hyperbolic polynomials), Gurvits [76] managed to slightly improve the bound, as well as simplify the proof. His main result unifies (and generalises) the conjecture of Schrijver and Valiant with that of van der Waerden on the permanent of doubly stochastic matrices. An $N\times N$-matrix is *doubly stochastic* if it is non-negative entry-wise and every column and every row sums to 1. In 1926, van der Waerden [192] conjectured that the permanent of every $N\times N$-doubly stochastic matrix is at least $\frac{N!}{N^N}$, with equality if and only if each entry of the matrix is $\frac{1}{N}$. The conjecture was proved about sixty years later by Egorychev [47, 48, 49] and, independently, Falikman [55].

The problem of lower bounding the number of perfect matchings is also related to a conjecture of Lovász and Plummer. They conjectured in the mid-1970s that the number of perfect matching of a 3-regular bridgeless graph grows exponentially with the number of vertices (see the book by Lovász and Plummer [121, Conjecture 8.1.8]). Edmonds, Lovász, and Pulleyblank [46] and, independently, Naddef [147], proved that the dimension of the perfect matching polytope of a cubic bridgeless graph with n vertices is at least $n/4+1$. Since the vertices of the polytope correspond to distinct perfect matchings, it follows that any 3-regular bridgeless graph on n vertices has at least $\frac{n}{4}+2$ perfect matchings. Recently, Král', Sereni, and Stiebitz [116] proved a lower bound of $\frac{n}{2}+2$ except for 17 exceptional graphs (one having exactly $\frac{n}{2}$ perfect matchings, the others $\frac{n}{2}+1$). In addition, Chudnovsky and Seymour [39] proved that Lovász and Plummer's conjecture is true for planar graphs.

3.3. The Proof of Kahn's Theorem on the Number of Independent Sets

Let $\mathscr{I}$ be the collection of independent sets of a Δ-regular bipartite graph G with parts A and B. Let us write $|A|=n=|B|$. We want to show that $|\mathscr{I}|\le\left(2^{\Delta+1}-1\right)^{n/\Delta}$.

Let I be an independent set of G chosen uniformly at random among the elements of $\mathscr{I}$. Thus, $H(I)=\log|\mathscr{I}|$ by (8). Write $A=\{v_1,v_2,\ldots,v_n\}$ and $B=\{v_{n+1},v_{n+2},\ldots,v_{2n}\}$. The independent set I can be written as its characteristic vector $x=x(I)=(x_i)_{1\le i\le 2n}$ where $x_i=1$ if $v_i\in I$ and $x_i=0$ otherwise. For a set $W\subseteq\{v_1,v_2,\ldots,v_{2n}\}$, let $x_W=(x_i)_{v_i\in W}$. Thus, $x_A=(x_i)_{1\le i\le n}$ and $x_B=(x_i)_{n+1\le i\le 2n}$. Then, by the chain rule,

$$H(I)=H(x_A|x_B)+H(x_B)\,.$$

Let us consider the part B as the (non-disjoint) union of neighbourhoods of the vertices of A. As G is Δ-regular, each vertex of B belongs to Δ neighbourhoods. Therefore, by applying Lemma 3.3 to $H(x_B)$, we infer that

$$\begin{aligned} H(I) &\le \sum_{i=1}^{n} H\left(x_i|x_B\right)+\frac{1}{\Delta}\sum_{i=1}^{n} H\left(x_{N(v_i)}\right)\\ &\le \sum_{i=1}^{n}\left(H\left(x_i|x_{N(v_i)}\right)+\frac{1}{\Delta}H\left(x_{N(v_i)}\right)\right). \end{aligned} \tag{14}$$

For $i\in\{1,2,\ldots,n\}$, let

$$\mathbf{1}_i:=\begin{cases}0 & \text{if } x_{N(v_i)}=\underline{0}:=(0,0,\ldots,0)\\ 1 & \text{otherwise.}\end{cases}$$

Moreover, set $p:=\mathbf{Pr}\left(\mathbf{1}_i=0\right)$. Then

$$H\left(x_i|x_{N(v_i)}\right)=H(x_i|\mathbf{1}_i)\le p\,. \tag{15}$$

On the other hand,

$$\begin{aligned} H\left(x_{N(v_i)}\right) &= H\left(x_{N(v_i)}, \mathbf{1}_i\right) \\ &= H\left(\mathbf{1}_i\right) + H\left(x_{N(v_i)}|\mathbf{1}_i\right) \qquad \text{by the chain rule.} \end{aligned} \tag{16}$$

By the definitions, $H(\mathbf{1}_i) = H(p)$ and

$$\begin{aligned} H\left(x_{N(v_i)}|\mathbf{1}_i\right) =& pH\left(x_{N(v_i)}|\mathbf{1}_i = 0\right) + (1-p)H\left(x_{N(v_i)}|\mathbf{1}_i = 1\right) \\ =&(1-p)H\left(x_{N(v_i)}|\mathbf{1}_i = 1\right) \\ =&(1-p)\sum_{\omega\in\{0,1\}^\Delta\setminus\{\underline{0}\}} \mathbf{Pr}(x_{N(v_i)} = \omega|\mathbf{1}_i = 1)\log\left(\left(\mathbf{Pr}(x_{N(v_i)} = \omega|\mathbf{1}_i = 1)\right)^{-1}\right) \\ \le&(1-p)\log\left(2^\Delta - 1\right) \quad \text{by Jensen's Inequality, since log is concave.} \end{aligned}$$

Therefore, we infer that

$$H\left(x_{N(v_i)}\right) = H\left(x_{N(v_i)}, \mathbf{1}_i\right) \le H(p) + (1-p)\log\left(2^\Delta - 1\right) . \tag{17}$$

By (14), (15), and (17), we deduce that

$$H(I) \le \sum_{i=1}^{n}\left(p + \frac{1}{\Delta}\left(H(p) + (1-p)\log\left(2^\Delta - 1\right)\right)\right) . \tag{18}$$

A straightforward study of the function $f(x) := x + \frac{1}{\Delta}\left(H(x) + (1-x)\log\left(2^\Delta - 1\right)\right)$ yields that its maximum is $\frac{1}{\Delta}\log\left(2^{\Delta+1} - 1\right)$, which is attained when $x = \frac{2^\Delta}{2^{\Delta+1}-1}$. Consequently, (18) implies that

$$\log|\mathscr{I}| = H(I) \le \frac{n}{\Delta}\log\left(2^{\Delta+1} - 1\right) ,$$

as stated. □

Using similar ideas and techniques, Kahn [103] generalised Theorem 3.2 to the weighted setting.

THEOREM 3.4 (Kahn, 2002). *Let G be a bipartite graph with parts A and B such that*

$$\begin{cases} \forall a \in A, & \deg(a) \le k \quad \textit{and} \\ \forall b \in B, & \deg(b) \ge k\,. \end{cases}$$

Let $\alpha, \beta \in [1, \infty)$ and set

$$\lambda_v := \begin{cases} \alpha & \textit{if } v \in A, \\ \beta & \textit{if } v \in B. \end{cases}$$

Then,

$$\sum_{S\in\mathscr{I}(G)}\prod_{v\in S}\lambda_v \le \left((1+\alpha)^k + (1+\beta)^k - 1\right)^{|A|/k} . \tag{19}$$

Again, a disjoint union of copies of the complete bipartite graph shows the tightness of the given bound. Moreover, Kahn [103] conjectured that the statement of the theorem actually holds for $\alpha, \beta \in [0, \infty)$.

4. The Lovász Local Lemma

We state a version of the Lovász Local Lemma in each of the following three subsections. The first two versions are equivalent, and an application to an edge-colouring problem is given for both of them. The last version is weaker, yet very handy to work with. We use it several times in Section 6.

4.1. Asymmetric Version

LEMMA 4.1 (The Lovász Local Lemma, 1975). *Let $\mathscr{A} = \{A_1, A_2, \dots, A_n\}$ be a set of events in a probability space Ω, and let $G = (V, E)$ be a graph with $V = \{1, 2 \dots, n\}$ such that for each $i \in \{1, 2, \dots, n\}$, the event A_i is mutually independent of $\{A_j : ij \notin E\}$. Suppose that there exist real numbers $x_1, x_2, \dots, x_n \in (0, 1)$ such that*

$$\mathbf{Pr}(A_i) < x_i \prod_{ij \in E} (1 - x_j)$$

for every $i \in \{1, 2, \dots, n\}$. Then the probability that no event in $\mathscr{A}$ occurs is positive.

The Lovász Local Lemma [52] is a beautiful and powerful result, which has been extensively applied to solve many different problems. Its philosophy is that if there is a set of events, each having low probability and being mutually independent of many others, then with positive probability none of the events occur. Consequently, the events $A_1, A_2, \dots, A_n$ are usually the "bad" events, i.e. all the situations we would like to avoid. Then, provided our setting fulfils the hypothesis of the lemma, we are ensured that a configuration avoiding all the bad events exists. We present an application of the Asymmetric Lovász Local Lemma to acyclic edge-colourings due to Alon, Sudakov, and Zaks [12].

Recall that a proper edge-colouring of a graph G is an assignment of colours to the edges of G such that no two adjacent edges are assigned the same colour. An edge-colouring is *acyclic* if G has no 2-coloured cycles, i.e. the subgraph of G induced by the union of any two colour classes is a forest. The *acyclic edge-chromatic number* of G is $a'(G)$, the least number of colours in an acyclic edge-colouring of G.

Acyclic colourings were introduced by Grünbaum [75]. The acyclic edge-chromatic number (and its vertex analog) can be used to obtain bounds on other colouring parameters, such as the oriented chromatic number or the circular chromatic number which are of particular interest to model various practical problems.

Being a proper edge-colouring, any acyclic edge-colouring of a graph of maximum degree Δ uses at least Δ colours. Alon, Sudakov, and Zaks [12] made the following conjecture.

CONJECTURE 4.2 (Alon, Sudakov, and Zaks, 2001). *For any graph G of maximum degree Δ,*

$$a'(G) \leq \Delta + 2\,.$$

The first upper bound on the acyclic edge-chromatic number was obtained by Alon, McDiarmid, and Reed [8], who proved that $a'(G) \leq 60\Delta$ for any graph G of maximum degree Δ. This bound was later decreased to 16Δ by Molloy and Reed [135], and this is the best bound known so far—let us note here that Muthu, Narayanan, and Subramanian [145] observed a flaw in the sketch of an argument yielding an upper bound of 9Δ, given by Molloy and Reed [140, Chapter 19, p. 226].

Alon, Sudakov, and Zaks [12] proved that Conjecture 4.2 is true for "almost all" Δ-regular graphs. This was improved by Nešetřil and Wormald [151] who obtained the upper bound $\Delta + 1$ for a random Δ-regular graph.

Further, Alon, Sudakov, and Zaks [12] proved that Conjecture 4.2 holds for graphs with sufficiently high girth (in terms of the maximum degree). This latter result is stated and proved below. Muthu, Narayanan, and Subramanian [145] showed that $a'(G) \leq 4.52\Delta$ for every graph G of maximum degree Δ and girth at least 220.

THEOREM 4.3 (Alon, Sudakov, and Zaks, 2001). *For every graph G of maximum degree Δ and girth at least $2000\Delta \log \Delta$,*

$$a'(G) \leq \Delta + 2 \,.$$

The proof illustrates a useful strategy: given a non-valid colouring with few conflicts, introduce some randomness to solve the conflicts and obtain the desired properties. The non-valid colouring may be obtained by a random colouring procedure—designed such that the number of conflicts can be bounded—or it can, as here, be given by a known theorem.

Proof of Theorem 4.3. Let $G = (V, E)$ be a graph of maximum degree Δ and girth $x \geq 2000\Delta \log \Delta$. By Vizing's Theorem [188], let $c : E \to \{1, 2, \ldots, \Delta + 1\}$ be a proper edge-colouring of G. Each edge is recoloured with the new colour $\Delta + 2$ randomly and independently with probability $\frac{1}{32\Delta}$. We assert that, with positive probability, the obtained colouring is proper (i.e. no pair of adjacent edges are recoloured) and acyclic (i.e. every cycle of G contains at least three different colours).

So as to use the Asymmetric Lovász Local Lemma, we now have to design a suitable set of "bad" events. Let us define them according to three types. An even cycle C half of whose edges are assigned the same colour by the colouring c is *half-monochromatic*. We let $H(C)$ be the set of those edges (so $H(C)$ induces a perfect matching of C). Note that if C is a 2-coloured cycle, then there are two choices for $H(C)$.

- Type I For each pair B of adjacent edges, let E_B be the event that both the edges of B are recoloured.
- Type II For each cycle C of G that is 2-coloured by c, let E_C be the event that no edge of C is recoloured.
- Type III For each half-monochromatic cycle D, let E_D be the event that every edge not in $H(D)$ is recoloured.

Our aim now is to apply the Asymmetric Lovász Local Lemma to show that, with positive probability, no event of type I, II or III holds. This would imply that the obtained edge-colouring is acyclic. Indeed, the colouring would be proper since no event of type I holds. Moreover, let C be an even cycle of G. Since no event of type II holds, C is not 2-coloured unless one of the two colours is $\Delta+2$. Thus, as the obtained colouring is proper, the edges not coloured $\Delta+2$ cannot be monochromatic, because no event of type III holds.

So it remains to show that our setting satisfies the conditions of the Asymmetric Lovász Local Lemma. Let us first look at the dependencies. We have defined an event E_H for each subgraph H of G composed of either two adjacent edges, or a 2-coloured cycle, or an half-monochromatic cycle. Observe that any edge e is adjacent to less than 2Δ edges, and it is contained in less than Δ two-coloured cycles.

We assert that any edge e is contained in less than $2\Delta^{k-1}$ half-monochromatic cycles D of length $2k$. Indeed, let $D := v_1v_2\dots v_{2k}$ with $e = v_1v_{2k}$. Suppose first that $e \in H(D)$. Then (recalling that the colouring c is proper), there is at most one choice for each vertex v_{2i-1} with $i \in \{2,3,\dots,k\}$. Further, there are at most Δ choices for each vertex v_{2i}, where $i \in \{1,2,\dots,k-1\}$. Hence, in total, there are at most Δ^{k-1} such cycles. If $e \notin H(D)$, a similar argument applied to the edge v_2v_3 gives an upper bound of Δ^{k-1}, which ends the proof of the assertion.

Consider the dependency graph described in the Asymmetric Lovász Local Lemma. Note that an event E_H is mutually independent of the set all the events $E_{H'}$, where H' does not share an edge with H. Thus, in the dependency graph, each event E_H where H contains x edges is adjacent to at most $2x\Delta$ events of type I, at most $x\Delta$ events of type II and at most $2x\Delta^{|H(D)|-1}$ events E_D of type III (where D is an half-monochromatic cycle).

We now have to bound the probability of each event, and find appropriate real constants x_i to be able to finish the proof. The following bounds readily follow from the definition of the events.

(1) $\mathbf{Pr}(E_B) = \frac{1}{1024\Delta^2}$ for each event E_B of type I;

(2) $\mathbf{Pr}(E_C) = \left(1-\frac{1}{32\Delta}\right)^x \le e^{-x/(32\Delta)}$ for each event E_C of type II, where C is a cycle of length x; and

(3) $\mathbf{Pr}(E_D) \le \frac{2}{(32\Delta)^x}$ for each event E_D of type III, where D is an half-monochromatic cycle of length $2x$.

To each event of type I, we associate the real constant $\frac{1}{512\Delta^2}$. To each event of type II is associated the real constant $\frac{1}{128\Delta^2}$, and $\frac{1}{(2\Delta)^{|H(D)|}}$ is associated to each event E_D of type III. Thus, it only remains to show the following three inequalities.

$$\frac{1}{1024\Delta^2} \le \frac{1}{512\Delta^2}\left(1-\frac{1}{512\Delta^2}\right)^{4\Delta}\left(1-\frac{1}{128\Delta^2}\right)^{2\Delta}\prod_k\left(1-\frac{1}{(2\Delta)^k}\right)^{4\Delta^{k-1}} \tag{20}$$

$$e^{-\frac{x}{32\Delta}} \le \frac{1}{128\Delta^2}\left(1-\frac{1}{512\Delta^2}\right)^{2x\Delta}\left(1-\frac{1}{128\Delta^2}\right)^{x\Delta}\prod_k\left(1-\frac{1}{(2\Delta)^k}\right)^{2x\Delta^{k-1}} \tag{21}$$

$$\frac{2}{(32\Delta)^x} \le \left(\frac{1}{2\Delta}\right)^x\left(1-\frac{1}{512\Delta^2}\right)^{4x\Delta}\left(1-\frac{1}{128\Delta^2}\right)^{2x\Delta}\prod_k\left(1-\frac{1}{(2\Delta)^k}\right)^{4x\Delta^{k-1}} \tag{22}$$

with $x \ge 4$ in (21) and $x \ge 2$ in (22).

We prove them using some standard estimates. For every real $x \geq 2$ it holds that $\left(1-\frac{1}{x}\right)^x \geq \frac{1}{4}$. Thus, for every $x, d \geq 2$,

$$\begin{aligned}
\prod_k \left(1-\frac{1}{(2\Delta)^k}\right)^{2x\Delta^{k-1}} &= \prod_k \left(1-\frac{1}{(2\Delta)^k}\right)^{(2\Delta)^k \cdot 2^{1-k} \cdot x/\Delta} \\
&\geq \prod_k \left(\frac{1}{4}\right)^{2^{1-k} \cdot x/\Delta} \\
&\geq \left(\frac{1}{4}\right)^{2x \cdot \sum_k 2^{-k}/\Delta} \\
&\geq 4^{-x/(256\Delta)}\,, \qquad (23)
\end{aligned}$$

where the last inequality uses that $2k \geq g(G) \geq 20$. Similarly,

$$\left(1-\frac{1}{512\Delta^2}\right)^{2x\Delta} \geq 4^{-x/(256\Delta)}\,, \qquad (24)$$

and

$$\left(1-\frac{1}{128\Delta^2}\right)^{x\Delta} \geq 4^{-x/(128\Delta)}\,. \qquad (25)$$

Therefore, we deduce from (23), (24), and (25) that

$$\left(1-\frac{1}{512\Delta^2}\right)^{2x\Delta} \left(1-\frac{1}{128\Delta^2}\right)^{x\Delta} \prod_k \left(1-\frac{1}{(2\Delta)^k}\right)^{2x\Delta^{k-1}} \geq 2^{-x/(32\Delta)}\,.$$

Consequently, (20) holds because $2^{\left(1-\frac{1}{16\Delta}\right)} \geq 1$, and so does (22) since $2^{\left(1-5x+x+\frac{x}{16\Delta}\right)} \leq 1$ for all $x \geq 1$. Finally, since $x \geq 2000\Delta\log\Delta \geq 32\Delta\frac{\log(128\Delta^2)}{\log(e/2)}$ and $\Delta > 2$, we infer that

$$e^{-x/(32\Delta)} \leq \frac{1}{128\Delta^2} \cdot 2^{-x/(32\Delta)}\,,$$

which implies (21), thereby completing the proof. □

In the previous proof, we started from a proper colouring using at most $\Delta + 1$ colours, whose existence is ensured by Vizing's Theorem. Then, a new colour was used uniformly at random on the edges. The Asymmetric Lovász Local Lemma guaranteed that with positive probability, the recolouring destroyed all 2-coloured cycles without violating the properness.

The requirement on the girth is natural: in this setting, short 2-coloured cycles have a much larger probability of surviving the recolouring than long cycles.

The approach used recently by Muthu, Narayanan, and Subramanian [145] (to prove a weaker upper bound, but with a much weaker girth assumption, as we saw earlier) is similar. In particular they use the Lovász Local Lemma in the same way. However, instead of starting from a proper colouring and destroying 2-coloured cycles, they colour all the edges randomly.

4.2. Multiple Version

Here is the so-called *multiple version* of the Lovász Local Lemma. It is equivalent to the asymmetric version, and we refer the reader to the monograph of Alon and Spencer [11] for further details.

Lemma 4.4 (The Multiple Lovász Local Lemma, 1975). *Let $\mathscr{A}$ be a finite set of events, partitioned into parts $\mathscr{A}_1, \mathscr{A}_2, \dots, \mathscr{A}_r$ such that $\mathbf{Pr}(A) \le p_i$ for every $A \in \mathscr{A}_i$ and each $i \in \{1, 2, \dots, r\}$. Suppose that there exist real numbers $a_i \in (0,1)$ and $\Delta_{ij} \ge 0$ for every $(i,j) \in \{1,2,\dots,r\}^2$ such that*

(1) *for any event $A \in \mathscr{A}_i$, there exists a set $\mathscr{D}_A \subseteq \mathscr{A}$ such that A is mutually independent of $\mathscr{A} \setminus (\mathscr{D}_A \cup \{A\})$ and $|\mathscr{D}_A \cap \mathscr{A}_j| \le \Delta_{ij}$ for every $j \in \{1,2,\dots,r\}$; and*

(2) $p_i \le a_i \prod_{j=1}^{r} (1-a_j)^{\Delta_{ij}}$ *for every $i \in \{1,2,\dots,r\}$.*

Then with positive probability none of the events in $\mathscr{A}$ holds.

Let us see an application of this lemma to non-repetitive colourings. Given a finite set $\mathscr{S}$ of symbols, a finite sequence of elements of $\mathscr{S}$ is *non-repetitive over $\mathscr{S}$* if it does not contain a subsequence of the form xx, where x is a finite sequence of symbols of $\mathscr{S}$. Thue [182, 183] proved the existence of arbitrarily long non-repetitive sequences provided that $\mathscr{S}$ contains (at least) three different symbols. Several generalisations of this concept have been introduced, and the one we focus on concerns graph colouring.

Alon *et al.* [4] introduced the concept of non-repetitive colouring of graphs. Let $G = (V, E)$ be a graph. An edge-colouring c of G is *non-repetitive* if for any path $v_1 v_2 \dots v_r$ of G (where all the vertices are distinct), the sequence $(c(v_i v_{i+1}))_{1 \le i \le r-1}$ is non-repetitive over $c(E)$. The smallest number of colours needed in a non-repetitive edge-colouring of G is $\pi(G)$, the *Thue number* of G.

In this setting, Thue's Theorem states that the Thue number of any path (of length at least 3) is 3. Consequently, the Thue number of any cycle is at most 4 (and it can be 4 as shown by a cycle of length 5). Thus, $\pi(G) \le 4$ for any graph G of maximum degree at most 2. It is natural to look for an upper bound on $\pi(G)$ in terms of the maximum degree of G. Alon *et al.* [4] proved the following.

Theorem 4.5 (Alon, Grytczuk, Hałuszczak, and Riordan, 2002). *For every graph G of maximum degree Δ,*

$$\pi(G) \le 2e^{16}\Delta^2 .$$

They moreover conjectured a linear bound in terms of the number of vertices.

Conjecture 4.6 (Alon, Grytczuk, Hałuszczak, and Riordan, 2002). *There exists an integer c such that*

$$\pi(G) \le c \cdot n$$

for every graph G on n vertices.

Proof of Theorem 4.5. Let G be a graph of maximum degree Δ, and let $\mathscr{C}$ be a set of $2e^{16}\Delta^2$ colours. For each edge e of G, we choose uniformly at random a colour

$c(e)$ from $\mathscr{C}$, independently from the choices already made. We aim at applying the Multiple Lovász Local Lemma to prove that, with positive probability, the obtained edge-colouring of G is non-repetitive.

For any path $P := v_1v_2 \dots v_{2s}$ of G of even length, let A_P be the event that $(c(v_iv_{i+1}))_{1\le i\le s-1} = (c(v_iv_{i+1}))_{s\le i\le 2s-1}$. Let

$$\mathscr{A}_s := \{A_P : P \text{ path of length } 2s\}\,.$$

Thus, c is non-repetitive if and only if no event in $\mathscr{A} := \cup_s \mathscr{A}_s$ occurs.

For any positive integer s and any event $A_P \in \mathscr{A}_s$, the probability that A_P occurs is at most $|\mathscr{C}|^{-s}$. Moreover, A_P is mutually independent of all the events A_Q where Q has no common edge with P. Since a path of length $2s$ shares an edge with at most

$$(2s)\cdot(2t)\Delta^{2t} = 4st\Delta^{2t}$$

paths of length $2t$, we set $\Delta_{st} := 4st\Delta^{2t}$.

It remains to define the real numbers a_s for $s \in \{1, 2, \dots, r\}$ so that the condition (2) of the Multiple Lovász Local Lemma is satisfied. Set $a_s := a^{-s}$ with $a := 2\Delta^2$. Since $a_s \le \frac{1}{2}$, it follows that

$$1 - a_s \ge e^{-2a_s}\,.$$

So the condition (2) is fulfilled if

$$\begin{aligned}|\mathscr{C}|^{-s} \le& a_s \prod_t e^{-2a_t\Delta_{st}}\\ =& a_s \prod_t \exp\left[-8\cdot 2^{-t}\cdot st\right]\,,\end{aligned}$$

i.e. if

$$\begin{aligned}|\mathscr{C}| \ge& a\cdot\exp\left[8\sum_t 2^{-t}\cdot t\right]\\ =& 2\Delta^2 e^{8\cdot 2}\,,\end{aligned}$$

since $\sum_{t=1}^{\infty} t2^{-t} = 2$. This is the case by the choice of $\mathscr{C}$. Consequently, the Multiple Lovász Local Lemma applies and yields the sought conclusion. $\square$

We end this subsection with an open problem about non-repetitive colouring of cycles. As mentioned in the introduction, the Thue number of any cycle is at most 4, and this upper bound is attained by the 5-cycle. Let C_n be the cycle with n vertices. Alon *et al.* [4] verified by numerical experiment that if $n \le 2001$, then $\pi(C_n) = 4$ if and only if $n \in \{5, 7, 9, 10, 14, 17\}$. This is why they made the following conjecture.

CONJECTURE 4.7 (Alon, Grytczuk, Hałuszczqk, and Riordan, 2002). *Every cycle of length at least* 18 *has Thue number at most* 3.

4.3. Symmetric Version

Let us state the so-called *symmetric version* of the Lovász Local Lemma [52]. It is less general than the versions seen previously. However, it is very handy to work with, and it is sufficient in many situations. It is applied in Section 6.

Lemma 4.8 (The Symmetric Lovász Local Lemma, 1975). *Let $\mathscr{A} = \{A_1, \ldots, A_n\}$ be a set of random events so that, for each $i \in \{1, 2, \ldots, n\}$,*

(1) $\mathbf{Pr}(A_i) \le p$ *and*

(2) A_i *is mutually independent of all but at most d other events of $\mathscr{A}$.*

If $pd \le \frac{1}{4}$ then the probability that no event of $\mathscr{A}$ occurs is positive.

5. Concentration Inequalities

As mentioned in the introduction, concentration inequalities bound the deviation between a random variable and its expected value (or its median). Thus, they allow us to translate bounds on the expected value of a random variable to bounds on the random variable. It is useful since the expected value of a random variable is often easier to bound than the random variable itself.

We present three concentration bounds in this section, namely Chernoff's Bound, Talagrand's Inequality and McDiarmid's Inequality. We do not state them in their full generality. We rather give weaker (but handy) versions.

5.1. The Chernoff Bound

The *binomial random variable* $\mathrm{Bin}(n, p)$ is the sum of n independent 0–1 variables, each being 1 with probability p. Thus, $\mathbf{E}(\mathrm{Bin}(n, p)) = np$. The well-known Chernoff Bound [11, 36, 124] bounds the probability that $\mathrm{Bin}(n, p)$ deviates from its expected value np. It appears in the literature under many guises, and we give several formulations. Further details and proofs can be found, for instance, in the book by Janson, Luczak, and Ruciński [93, Chapter 2].

Lemma 5.1 (Chernoff's Bound, 1952).

(1) *For every $t \ge 0$,*

$$\mathbf{Pr}(\mathrm{Bin}(n, p) \ge np + t) \le \exp\left[-\frac{t^2}{2(np + t/3)}\right].$$

(2) *For every $t \ge 0$,*

$$\mathbf{Pr}(\mathrm{Bin}(n, p) \le np - t) \le \exp\left[-\frac{t^2}{2np}\right].$$

(3) *For every $t \in [0, np]$,*

$$\mathbf{Pr}\left(|\mathrm{Bin}(n, p) - np| > t\right) < 2\exp\left(-\frac{t^2}{3np}\right).$$

Let us give a somehow less friendly but slightly more general version that is used in Subsection 6.2. We also refer to it as to the Chernoff Bound.

LEMMA 5.2 (Chernoff's Bound, 1952). *For every $t > 0$,*

$$\mathbf{Pr}\left(|\operatorname{Bin}(n,p) - np| > t\right) < 2\exp\left(t - \ln\left(1+\frac{t}{np}\right)(np+t)\right).$$

There are many applications of Chernoff's Bound in the literature, and the one we present now allows us to introduce a very interesting circular variant of list-colouring.

Let $G = (V, E)$ be a graph. If p and q are two integers, a (p,q)*-colouring* of G is a function $c : V \to \{0, \ldots, p-1\}$ such that for each edge $uv \in E$,

$$q \le |c(u) - c(v)| \le p - q\,.$$

The *circular chromatic number* of the graph G is

$$\chi_c(G) := \inf\left\{p/q : G \text{ admits a } (p,q)\text{-colouring}\right\}.$$

The circular chromatic number was introduced by Vince [187] under a different terminology. He proved in particular that the infimum in the definition is always attained. Hence, the circular chromatic number is always a rational number. Furthermore, $\chi(G) - 1 < \chi_c(G) \le \chi(G)$ for every graph G. Thus, the chromatic number of a graph is the ceiling of its circular chromatic number. We refer to the survey of Zhu [196] for an in-depth review of this fundamental notion.

The concept of *circular choosability*, introduced by Mohar [133] and Zhu [197], combines the concepts of circular colouring and list-colouring, respectively, in a natural way.

A list-assignment L is a t-(p,q)*-list-assignment* if $L(v) \subseteq \{0, \ldots, p-1\}$ and $|L(v)| \ge tq$ for each vertex $v \in V$. The graph G is (p,q)*-L-colourable* if there exists a (p,q)*-L-colouring* c, i.e. c is both a (p,q)-colouring and an L-colouring. For any real number $t \ge 1$, the graph G is t-(p,q)*-choosable* if it is (p,q)-L-colourable for every t-(p,q)-list-assignment L. Last, G is *circularly t-choosable* if it is t-(p,q)-choosable for any p, q. The *circular choice number* of G is

$$\operatorname{cch}(G) := \inf\{t \ge 1 : G \text{ is circularly } t\text{-choosable}\}.$$

Zhu [197] proved that $\operatorname{cch}(G) \ge \max\{\operatorname{ch}(G) - 1, \chi_c(G)\}$ for every graph G. He also raised several very interesting questions, including whether the infimum of the definition is always attained. This was answered negatively by Norine [152]. To this end, he proved that the complete bipartite graph $K_{2,4}$ has circular choice number 2, and yet it is not circularly 2-choosable. In other words, $K_{2,4}$ is circularly t-choosable if and only if $t > 2$. Thus, it is natural to ask, as Zhu did, whether the circular choice number is always a rational number. This latter question was answered in the affirmative by Müller and Waters [142].

Another problem concerns the link between the circular choice number, and the (usual) choice number. Zhu [197] asked whether $\operatorname{cch}(G) = O(\operatorname{ch}(G))$ for every graph G (and he observed that there are graphs G for which $\operatorname{cch}(G) \ge 2\operatorname{ch}(G)$). So far, the only general link between those two graph invariants is the following, which unfortunately depends on the number of vertices of the considered graph [81].

Theorem 5.3 (Havet, Kang, Müller, and Sereni, 2006). *For every graph G with n vertices,*

$$\mathrm{cch}(G) \le 36 \cdot (\mathrm{ch}(G) + \ln n) + 3 .$$

The proof of Theorem 5.3 is probabilistic, and we present it as an application of the Chernoff Bound. The way Chernoff's Bound is used is typical: the probability of an event is upper bounded by that of the deviation of a binomial random variable from its expected value.

Proof of Theorem 5.3. Fix two integers p and q and set $t := 36 \cdot (\mathrm{ch}(G) + \ln n) + 3$. Suppose that lists $L(v) \subseteq \mathbf{Z}_p$ of size at least $\lfloor tq \rfloor + 1$ are given. If $q = 1$, then we can certainly (p, q)-L-colour G, as $t > \mathrm{ch}(G)$. So we assume that $q \ge 2$.

Let us partition $\left\{0, \ldots, \left\lfloor \frac{p-1}{q-1} \right\rfloor\right\}$ into groups $g_i := \{3i, 3i+1, 3i+2\}$ of three consecutive numbers, where the last group may contain less than three numbers. Out of each group of three numbers but the very last one, we pick one element at random, but in such a way that we never pick two consecutive numbers. To be more precise, for $i = 0$ we simply pick one of $0, 1, 2$ uniformly at random. Once a choice has been made for g_{i-1}, we pick one of $3i, 3i+1, 3i+2$ uniformly at random provided we did not choose $3(i-1)+2$ from g_{i-1}. Otherwise, we choose one of $3i+1, 3i+2$ at random each with probability $\frac{1}{2}$. The set of selected indices is $\mathcal{K} := \{k : k \text{ was chosen}\}$. With each index $k \in \left\{0, \ldots, \left\lfloor \frac{p-1}{q-1} \right\rfloor\right\}$, we associate an interval $I_k = \{k(q-1), \ldots, (k+1)(q-1)-1\}$ of $\mathbf{Z}_p$. Notice that the I_k are disjoint intervals of length $q-1$. A crucial observation for the sequel is that if k and l are two distinct elements of $\mathcal{K}$, then $|a-b|_p \ge q$ for every $a \in I_k$ and every $b \in I_l$.

Let us set $\mathcal{I} := \bigcup_{k \in \mathcal{K}} I_k$. For each $v \in V$, we let $S(v) := \{k \in \mathcal{K} : I_k \cap L(v) \ne \emptyset\}$. The idea for the rest of the proof is to show that t was chosen in such a way that $\mathbf{Pr}(|S(v)| < \mathrm{ch}(G)) < \frac{1}{n}$ for all v. Then it follows that

$$\mathbf{Pr}(|S(v)| < \mathrm{ch}(G) \text{ for some } v \in V) < n \cdot \frac{1}{n} = 1 .$$

In other words, there exists a choice of non-adjacent intervals, one from each group of three, for which $|S(v)| \ge \mathrm{ch}(G)$ for all $v \in V$. By the definition of the choice number, there exists a proper colouring c of G with $c(v) \in S(v)$. Let us define a new colouring f by choosing $f(v) \in I_k \cap L(v)$ if $c(v) = k$. This can be done for each v, by the definition of $S(v)$. Now f is a (p, q)-L-colouring, because if $vw \in E(G)$ then $c(v)$ and $c(w)$ are distinct elements of $\mathcal{K}$. Consequently, $f(v)$ and $f(w)$ have been chosen from non-adjacent intervals $I_{c(v)}$ and $I_{c(w)}$, and hence $|f(v) - f(w)|_p \ge q$.

It remains to show that t is chosen such that $\mathbf{Pr}(|S(v)| < \mathrm{ch}(G)) < \frac{1}{n}$. We first assert that the probability that $|S(v)| < \mathrm{ch}(G)$ is bounded above by

$$\mathbf{Pr}\left(\mathrm{Bin}\left(s, \frac{1}{6}\right) \le \mathrm{ch}(G)\right) ,$$

where $s := \left\lceil \frac{t}{3} \right\rceil - 1$. To prove the assertion, we "thin" the lists $L(v)$ to get sublists $L'(v) \subseteq L(v)$ with

$$|L'(v)| \ge \left\lceil \frac{|L(v)|}{3(q-1)} \right\rceil - 1 > \frac{t}{3} - 1 ,$$

and a distance of at least $3(q-1)$ between elements of $L'(v)$. Indeed, we can construct $L'(v)$ by taking the first, the $(3(q-1)+1)^{\text{th}}$, the $(6(q-1)+1)^{\text{th}}$, and so on up to (and including) the $((M-1)(q-1)+1)^{\text{th}}$ element of $L(v)$, where $M := \left\lceil \frac{tq}{3(q-1)} \right\rceil$, and we discard the $(M(q-1)+1)^{\text{th}}$ element, to avoid possible wrap-around effects. Let $L'(v) := \{a_1, \ldots, a_l\}$ with $a_i \le a_{i+1}$. For $J \subseteq \{1, \ldots, i-1\}$, let $A(i,J)$ be the event that $a_j \in \mathcal{I}$ for $j \in J$ and $a_j \notin \mathcal{I}$ for all $j \in \{1, \ldots, i-1\} \setminus J$. We assert that for every $J \subseteq \{1, \ldots, i-1\}$,

$$\mathbf{Pr}(a_i \in \mathcal{I} | A(i,J)) \ge \frac{1}{6}\,. \tag{26}$$

To see this, observe that if $a_i \in I_{3k+1}$ or $a_i \in I_{3k+2}$ for some k, then the probability that a_i is covered by $\mathcal{I}$ given that $A(i,J)$ holds is at least $\frac{1}{3}$. Indeed, regardless of which element of g_{k-1} was selected, the probability that $3k+1$ (respectively $3k+2$) is selected is at least $\frac{1}{3}$. Now, supposing that $a_i \in I_{3k}$ for some k, it follows that $a_{i-1} \notin I_{3(k-1)+1} \cup I_{3(k-1)+2}$. Therefore, the probability that a_i is covered given that $A(i,J)$ holds is at least the minimum of two probabilities: the probability that $3k$ is chosen given that $3(k-1)$ was chosen from g_{k-1}; and the probability that $3k$ is chosen given that $3(k-1)$ was not chosen from g_{k-1}. This minimum is $\frac{1}{6}$, which proves the assertion.

We use the following Chernoff Bound.

$$\forall r \ge 0, \quad \mathbf{Pr}(\mathrm{Bin}(k,p) \le kp - r) \le \exp\left[-2\frac{r^2}{k}\right].$$

Setting $r := \frac{s}{6} - \mathrm{ch}(G) \ge 0$, it follows that

$$\mathbf{Pr}(|S(v)| < \mathrm{ch}(G)) \le \mathbf{Pr}\left(\mathrm{Bin}\left(s, \frac{1}{6}\right) \le \frac{s}{6} - r\right) \le \exp\left[-3\frac{r^2}{s}\right].$$

This yields the conclusion provided that

$$3\left(\mathrm{ch}(G) - \frac{s}{6}\right)^2 > s \ln n\,,$$

i.e.,

$$s^2 - 12 \cdot (\mathrm{ch}(G) + \ln n)\, s + 36\, \mathrm{ch}^2(G) > 0\,.$$

This is certainly true if

$$\begin{aligned} s &> \frac{1}{2}\left(12\,\mathrm{ch}(G) + 12 \ln n + \sqrt{144\,(\mathrm{ch}(G) + \ln n)^2 - 144\,\mathrm{ch}(G)^2}\right) \\ &= 6(\mathrm{ch}(G) + \ln n) + 6\sqrt{(2\,\mathrm{ch}(G) + \ln n) \cdot \ln n}\,. \end{aligned}$$

Thus, since $2\sqrt{ab} < a + b$ for any two distinct positive real numbers a and b, it suffices that

$$s \ge 12 \cdot (\mathrm{ch}(G) + \ln n)\,,$$

which is the case because

$$3s \ge t - 3 = 36 \cdot (\mathrm{ch}(G) + \ln n)\,.$$

□

Let us note that known results about the choice number yield an upper bound on cch that is exponential in ch. Given a graph G, let $\delta^*(G)$ be the *degeneracy* of G, i.e. the smallest integer k such that each induced subgraph of G contains a vertex of degree at most k. Hence, the greedy algorithm shows that $\mathrm{ch}(G) \le \delta^* + 1$. The results of Alon [2] studied in Subsection 2.2 yield that $\mathrm{ch}(G) = \Omega(\ln(\delta^*(G)))$. On the other hand, as Zhu [197] noted, $\mathrm{cch}(G) \le 2\delta^*(G)$. Therefore, we deduce that

$$\mathrm{cch}(G) \le e^{\beta\,\mathrm{ch}(G)},$$

for some $\beta > 0$.

5.2. Talagrand's and McDiarmid's Inequalities

In the mid-1990s, Talagrand [178] exhibited a very useful concentration inequality. The version we give, though not quite as powerful as the original one, is usually sufficient for our purposes and at the same time handy to use. Further exposition can be found in the survey written by McDiarmid [125], the lecture notes of Lugosi [122], and in the book by Molloy and Reed [140]. Examples of applications of concentration inequalities in computer science are presented in the survey of Díaz, Petit, and Serna [42].

Lemma 5.4 (Talagrand's Inequality, 1995). *Let X be a non-negative random variable determined by the independent trials $T_1, \dots, T_n$. Suppose that for every set of possible outcomes of the trials*

(1) *changing the outcome of any one trial can affect X by at most c; and*
(2) *for each $s > 0$, if $X \ge s$ then there is a set of at most rs trials whose outcomes certify that $X \ge s$.*

Then for every $t \in \left[60c\sqrt{r\,\mathbf{E}(X)}, \mathbf{E}(X)\right]$,

$$\mathbf{Pr}\left(|X - \mathbf{E}(X)| > t\right) \le 4\exp\left(-\frac{t^2}{32c^2 r\,\mathbf{E}(X)}\right).$$

Let us now see an application of Talagrand's Inequality, which deals with the chromatic number of graph powers. It is due to Alon and Mohar [9]. We note that the way we stated Talagrand's Inequality permits a slightly less technical application than that of the original proof.

The *k^{th}-power G^k* of the graph $G = (V, E)$ is the graph on V where two vertices are adjacent whenever their distance in G is at most k. Alon and Mohar [9] defined the parameter $f_k(\Delta, g)$ to be the maximum of the values $\chi(G^k)$ taken over all the graphs G with maximum degree Δ and girth g. Since the maximum degree of the square of a graph with maximum degree Δ is at most Δ^2, it follows that $f_2(\Delta, g) \le \Delta^2 + 1$. By Brooks' Theorem, this upper bound can be attained only if $g \le 5$ and if there exists a graph of diameter 2, maximum degree Δ and $\Delta^2 + 1$ vertices. As shown by Hoffman and Singleton [90], such graphs exist only for $\Delta \in \{2, 3, 7\}$ and possibly $\Delta = 57$. Alon and Mohar [9] observed that $f_2(2, g) = 4$ if $g \ge 6$, and proved the following theorem which determines the behaviour of $f_2(\Delta, g)$.

THEOREM 5.5 (Alon and Mohar, 2002).

(i) For all $g \leq 6$*,*

$$(1 - o(\Delta))\Delta^2 \leq f_2(\Delta, g) \leq \Delta^2 + 1\,.$$

(ii) There are absolute constants c_1 *and* c_2 *such that for every* $\Delta \geq 2$ *and every* $g \geq 7$*,*

$$c_1 \frac{\Delta^2}{\log \Delta} \leq f_2(\Delta, g) \leq c_2 \frac{\Delta^2}{\log \Delta}\,.$$

Thus, there is a phase transition as g grows: $f_2(\Delta, g)$ stays roughly the same when g grows from 3 to 6, decreases significantly when g grows from 6 to 7 and then stays essentially the same as g increases.

Let us also note that the upper bounds are obtained using a result of Alon, Krivelevich, and Sudakov [7] about the chromatic number of sparse graphs, which was proved using a similar approach to that in Subsection 6.1. However, this last result is stronger than the one we present in Subsection 6.1, and the proof more involved.

It is natural to ask for the behaviour of the functions f_k when $k \neq 2$. The complete graph on $\Delta + 1$ vertices shows that $f_1(\Delta, 3) = \Delta + 1$ for any integer Δ. In addition, well-known results about random graphs [21] ensure the existence of a constant c_1 such that

$$f_1(\Delta, g) \geq c_1 \frac{\Delta}{\log \Delta}$$

for every $g \geq 4$. Obtaining an upper bound can be seen as improving Brooks' Theorem for graphs with no short cycles. The first non-trivial result was obtained, independently, by Borodin and Kostochka [26], Catlin [34], and Lawrence [117]. Their results imply that

$$f(\Delta, 4) \leq \frac{3}{4} \cdot (\Delta + 2)\,.$$

It took almost ten years until Kostochka [110] improved this upper bound to $\frac{2}{3}\Delta + 2$. In the mid-1990s, Kim [107] and, independently, Johansson [95] made a dramatic breakthrough by proving that

$$f(\Delta, 4) \leq (1 + o(1)) \frac{\Delta}{\log \Delta}\,.$$

Kim obtained this result by the naive colouring procedure. The proof requires a very detailed and highly technical analysis of the deviations of random variables from their means. In that regards, recent concentration results (as Talagrand's and McDiarmid's Inequalities) may help to simplify the analysis. Kim's proof is algorithmic, and both him and Johansson actually obtained the upper bound for the choice number (and hence for the chromatic number as well). To sum-up, there exist two constants c_1 and c_2 such that for every $\Delta \geq 2$ and every $g \geq 4$,

$$c_1 \frac{\Delta}{\log \Delta} \leq f_1(\Delta, g) \leq c_2 \frac{\Delta}{\log \Delta}\,.$$

As for $k \geq 3$, Alon and Mohar [9] proved the following.

THEOREM 5.6 (Alon and Mohar, 2002).

- *There exists a constant c such that for all integers $k \geq 1$, $\Delta \geq 2$ and $g \geq 3k+1$*

$$f_k(\Delta, g) \leq \frac{c}{k} \cdot \frac{\Delta^k}{\log \Delta}.$$

- *For every positive integer k, there exists a positive number b_k such that for every $\Delta \geq 2$ and every $g \geq 3$*

$$f_k(\Delta, g) \geq b_k \frac{\Delta^k}{\log \Delta}.$$

The upper bound is proved by an analog argument as for the case where $k = 2$. Let us prove the lower bound. The approach used has the same flavor as Erdős' proof that there are graphs with arbitrary high girth and chromatic number [51].

To lower bound the chromatic number of a graph $G = (V, E)$, the usual lower bound used is $\chi(G) \geq \frac{|V|}{\alpha(G)}$, where $\alpha(G)$ is the independence number of G (defined in Subsection 1.1). We first obtain a graph satisfying some properties, which allow us to remove from it some vertices in order to obtain the desired graph.

Fix a positive integer k and an integer $g \geq 3$. We assume that Δ is sufficiently large compared to k. Let $V' = \{1, 2, \ldots, n\}$ with $n \gg \Delta^{\max(2k,g)}$. We let G' be the *random graph* $G_{n,p}$ with $p := \frac{\Delta}{2n}$, i.e. G' has vertex-set V', and each pair of distinct elements of V' is chosen to be an edge randomly and independently with probability p. We first prove two properties about G', using the first moment method.

(A) *The probability that G' has at most $10\Delta^g$ cycles of length less than g is at least* 0.9.

By the linearity of Expectation, the expected number of cycles of length less than g in G' is

$$\sum_{i=3}^{g-1} \binom{n}{i} \cdot \frac{(i-1)!}{2} \cdot p^i < \frac{1}{2} \sum_{i=3}^{g-1} \left(\frac{\Delta}{2}\right)^i < \Delta^g.$$

The desired property follows by Markov's Inequality.

(B) *The probability that G' has at most $10n \cdot 2^{-\Delta/10}$ vertices of degree more than Δ is at least* 0.9.

To see this, we assert that the expected number of vertices of G' of degree greater than Δ is at most $n \cdot 2^{-\Delta/10}$. The conclusion then follows by Markov's Inequality. To prove the assertion, notice that the degree of any fixed vertex is the random binomial variable $\text{Bin}(n-1, p)$. Consequently, we deduce from the Chernoff Bound that the probability that any fixed vertex has degree more than Δ is less than $2^{-\Delta/10}$. The linearity of Expectation now yields the assertion.

The following lemma is a key ingredient of Alon and Mohar's proof. Let $x := c_k \frac{n}{\Delta^k} \log \Delta$, where $c_k > 0$ (to be made precise later).

LEMMA 5.7. *The following holds with probability $1 - o(1)$. For every set $U \subseteq V'$ of cardinality x, there are at least $\frac{c_k^2 n \log^2 \Delta}{2^{k+5} \Delta^k}$ internally vertex-disjoint paths of length k, both of whose endpoints are in U and whose other vertices are in $V' \setminus U$.*

Before proving Lemma 5.7, let us see how it allows us to obtain the desired lower bound.

We choose a graph G' satisfying Properties (A) and (B), and Lemma 5.7 (Δ and n being sufficiently large). We define G to be obtained from G' by removing all the vertices of degree greater than Δ, and in addition one vertex chosen from each cycle of length less than g. Thus, G has maximum degree at most Δ, girth at least g and more than $\frac{n}{2}$ vertices. Let us be more precise: the number of vertices we removed is at most

$$10n \cdot 2^{-\Delta/10} + 10\Delta^g < \frac{c_k^2 \log^2 \Delta}{2^{k+5}\Delta^k} .$$

Thus, since G' satisfies Lemma 5.7, every set of x vertices of G contains at least one path of length at most k both of whose endvertices are in U (and internally disjoint from U). Therefore, we deduce that G^k contains no independent set of size x. Consequently,

$$\chi\left(G^k\right) \geq \frac{n}{2} \cdot \frac{1}{x} = \frac{\Delta^k}{2c_k \log \Delta} .$$

By adding to G pendant edges and a disjoint cycle of length g (if needed), we obtain the desired conclusion for an appropriately defined constant $b_k > 0$.

It remains to prove Lemma 5.7. Given a subset U of vertices of a graph G, a *U-path* is a path of G of length k both of whose endvertices are in U, and whose internal vertices are outside U.

Proof of Lemma 5.7. Let us fix a set U of size x. Let X be the maximum number of internally vertex-disjoint U-paths in G. We obtain the desired result by first lower bounding the expected value of X, and then we prove, thanks to Talagrand's Inequality that X is concentrated.

Let us show that

$$\mathbf{E}(X) \geq c_k^2 \cdot \frac{n \log^2 \Delta}{2^{k+2} \cdot \Delta^k} . \tag{27}$$

The expected number of U-paths is

$$\mu := \binom{x}{2}(n-x)(n-x-1)\dots(n-x-k+2)p^k .$$

Since $\Delta \gg k$, we deduce that

$$\mu > 0.49 \cdot c_k^2 n \log^2 \Delta \cdot 2^{-k}\Delta^{-k} .$$

Let ν be the expected number of pairs of U-paths that share at least one common internal vertex. By the linearity of Expectation, $\mathbf{E}(X) \geq \mu - \nu$. We assert that $\nu < \frac{\mu}{3}$, which hence will yield (27). Indeed, we can classify pairs of internally intersecting U-paths into several types, according to the number of vertices they share. Note that the number of types is upper bounded by a function of k. Moreover, the number of pairs of any given type is at most

$$\mu x n^{k-2} p^{k-1} ,$$

since $n \gg \Delta \gg k$. The assertion follows.

The random variable X is determined by the outcomes of the random process determining the edges of G. The change of any single outcome changes the value of X by at most 1, by the definition of X. Moreover, if $X = s$ then there are at most ks edges whose presence certify this fact. Thus, we may apply Talagrand's Inequality with $c = 1$ and $r = k$. Set $t := \frac{\mathbf{E}(X)}{2}$. Talagrand's Inequality ensures that

$$\begin{aligned}\mathbf{Pr}(|\,\mathbf{E}(X) - X| > t) \leq& 4 \cdot \exp\left[-\frac{\mathbf{E}(X)}{128 \cdot k}\right]\\ \leq& 4 \cdot \exp\left[-c_k^2 \cdot \frac{n \log^2 \Delta}{2^{k+9} \cdot k \cdot \Delta^k}\right].\end{aligned}$$

Note that if $X < \frac{c_k^2 n \log^2 \Delta}{2^{k+5} \Delta^k}$ then $|\,\mathbf{E}(X) - X| > t$ by (27).

On the other hand, the number of choices for the set U is

$$\binom{n}{x} \leq \left(\frac{en}{x}\right)^x \leq \left(\frac{e\Delta^k}{c_k \log \Delta}\right)^{c_k n \Delta^{-k} \log \Delta} \leq e^{c_k k n \Delta^{-k} \log^2 \Delta}.$$

Consequently, the statement of the lemma follows provided that

$$\lim_{n\to\infty} \exp\left[-c_k^2 \cdot \frac{n \log^2 \Delta}{2^{k+9} \cdot k \cdot \Delta^k}\right] \cdot \exp\left[c_k \cdot \frac{k \cdot n \log^2 \Delta}{\Delta^k}\right] = 0\,.$$

This holds if

$$\frac{c_k^2}{2^{k+9} \cdot k} > c_k \cdot k\,,$$

which is true if

$$c_k > 2^{k+9} k^2\,.$$

$\square$

McDiarmid [126] extended Talagrand's Inequality to the setting where X depends on independent trials and permutations. We state a useful corollary rather than the original inequality. The derivation can be found in the book by Molloy and Reed [140].

Lemma 5.8 (McDiarmid's Inequality, 2002). *Let X be a non-negative random variable determined by the independent trials $T_1, \ldots, T_n$ and m independent permutations $\Pi_1, \ldots, \Pi_m$. Suppose that for every set of possible outcomes of the trials*

(1) *changing the outcome of any one trial can affect X by at most c;*

(2) *interchanging two elements in any one permutation can affect x by at most c; and*

(3) *for each $s > 0$, if $X \geq s$ then there is a set of at most rs trials whose outcomes certify that $X \geq s$.*

Then for every $t \in \left[60c\sqrt{r\,\mathbf{E}(X)}, \mathbf{E}(X)\right]$,

$$\mathbf{Pr}\left(|X - \mathbf{E}(X)| > t\right) \leq 4 \exp\left(-\frac{t^2}{32c^2 r\,\mathbf{E}(X)}\right).$$

McDiarmid's Inequality is used in Subsection 6.2.

6. Reed's Lemma (Cutting Graphs into Pieces)

When it comes to showing that a graph can be coloured using at most a certain number of colours, a possible probabilistic approach is as follows. The graph is decomposed into several parts, and each part is coloured one after the other to finally obtain a colouring of the whole graph. Each part is coloured using a random procedure, which is analysed to show that, with positive probability, the partial colouring obtained so far is valid (regarding the particular sought conditions). It is customary to colour not *any* graph, but rather a supposed counter-example to the statement to establish. Such a counter-example is often assumed to be minimal regarding some parameters—e.g. graphs with less vertices all fulfil the statement to establish—which permits to prove some useful structural properties.

In that regards, it is important to cut the graphs into pieces having helpful properties to design or analyse the random colouring procedures. An efficient tool towards this goal is a lemma due to Reed [157], that we present next. It also appears in Chapter 15 of the book by Molloy and Reed [140, Lemma 15.2].

Let $G = (V, E)$ be a graph of maximum degree Δ. A vertex of G is *d-sparse* if the subgraph induced by its neighbourhood contains fewer than $\binom{\Delta}{2} - d\Delta$ edges. A vertex of G is *d-dense* if it is not d-sparse. Note that a vertex v can be d-sparse even if its neighbourhood induces a clique, provided that the degree of v is sufficiently small.

LEMMA 6.1 (Reed, 1998). *Let $G = (V, E)$ be a graph of maximum degree Δ and let $d \le \frac{\Delta}{100}$. The vertices of G can be partitioned into sets $D_1, D_2, \ldots, D_\ell, S$ such that*

(1) $\Delta + 1 - 8d \le |D_i| \le \Delta + 4d$ *for every* $i \in \{1, 2, \ldots, \ell\}$*;*

(2) *for each* $i \in \{1, 2, \ldots, \ell\}$*, at most* $8d\Delta$ *edges of* G *join* D_i *to* $V \setminus D_i$*;*

(3) *for each* $i \in \{1, 2, \ldots, \ell\}$*, a vertex belongs to* D_i *if and only if it has at least* $\frac{3}{4}\Delta$ *neighbours in* D_i*; and*

(4) *every vertex of* S *is* d*-sparse.*

The decomposition of Lemma 6.1 can be built in linear time, in a greedy fashion. We refer to the book of Molloy and Reed [140] for further exposition (including a proof of the lemma).

Lemma 6.1 has proved to be a key ingredient in several results obtained via the probabilistic method [83, 136, 138, 139, 157, 158, 159]. Thanks to the partition given by the lemma, one can design appropriate (random) colouring procedures for different parts of the graph, regarding their density. We present techniques used to colour sparse vertices in the next subsection, and see an approach to colour "big" cliques (for instance contained in the sets D_i) in Subsection 6.2. The remaining vertices are usually coloured after the sparse ones and before the big cliques. Because big cliques are removed when we colour them, a greedy procedure can suffice. However, one often wants to have more control on the obtained colouring, in particular to ensure some extra-properties that will help when it comes to colouring big cliques. To this end, it may be useful (i.e. powerful enough) to design an iterative random colouring procedure, and colour those vertices in many iterations. At each iteration, the colouring obtained so far is randomly extended to some more vertices, and the Lovász Local Lemma is used (at each iteration) to show that, with positive probability, the obtained colouring fulfils the required properties.

6.1. Colouring Sparse Graphs

As already mentioned, sparse vertices are those whose neighbourhood induces a graph with "few" edges. If the neighbourhood of a (sparse) vertex v is small enough, then however the neighbours of v are coloured, the colouring can be greedily extended to v. Otherwise, we know that the subgraph induced by the neighbourhood of this sparse vertex v has "many" pairs of non-adjacent vertices. If we can obtain a partial colouring of a graph such that every uncoloured vertex has "many" repeated colours in its neighbourhood, then we can finish the colouring greedily. The meaning of "many" depends on the kind of colouring we want. Generally speaking, if we have c colours, and an uncoloured vertex has at most $f \geq c$ neighbours, then we would like to have at least $f + 1 - c$ repeated colours in the neighbourhood of v.

This strategy is widely used to deal with the sparse vertices of Lemma 6.1, or when the whole graph considered is sparse itself. The reader is referred to the references given after Lemma 6.1 for many applications. However, proofs using Lemma 6.1 are too long to be presented here. This is why we focus on an application that uses the same approach, but deals only with sparse graphs.

It is a result by Molloy and Reed [134] from 1997, which deals with a generalisation of the chromatic index suggested by Erdős and Nešetřil back in 1985. More details can be found in the papers of Faudree *et al.* [56, 57] and Horák [92].

In a proper edge-colouring, every edge is adjacent to at most two edges of any given colour. A proper edge-colouring is *strong* if every edge is adjacent to at most one edge of each colour. The *strong chromatic index* $s\chi'(G)$ of a graph G is the minimum number of colours for which G admits a strong edge-colouring.

Another way to define the strong chromatic index is by using the line graph of a graph. Recall that the square G^2 of the graph $G = (V, E)$ is the graph with vertex-set V, and an edge between any two vertices that are at distance at most 2 in G. Then

$$s\chi'(G) = \chi(\mathscr{L}(G)^2)\,.$$

Let Δ be the maximum degree of G. Then, $\mathscr{L}(G)^2$ has maximum degree $2\Delta^2-2\Delta$. Thus, by Brooks' Theorem $s\chi(G) \leq 2\Delta^2 - 2\Delta$.

As reported by Faudree *et al.* [56], Erdős and Nešetřil exhibited, for any even integer Δ, a graph G_Δ of maximum degree Δ and such that $s\chi(G_\Delta) = \frac{5}{4}\Delta^2$. Indeed, one can take for G_Δ the graph obtained from a cycle of length 5 by replacing each vertex with an independent set of size $\frac{\Delta}{2}$; see Figure 1. The line graph of G_Δ has diameter 2 and $\frac{5}{4}\Delta^2$ edges, thus the desired property follows. They moreover conjectured the following.

Conjecture 6.2 (Erdős and Nešetřil, 1985). *For every graph G of maximum degree Δ,*

$$s\chi'(G) \leq \frac{5}{4}\Delta^2\,.$$

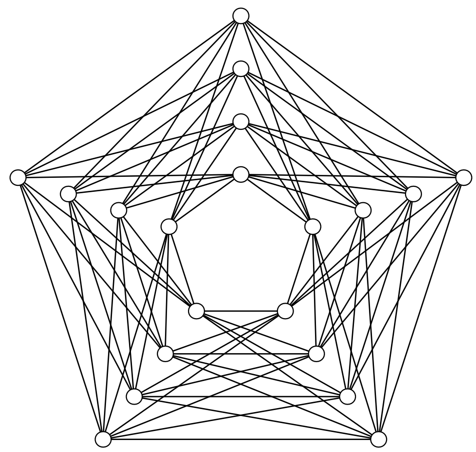

Figure 1: The graph G_8 has maximum degree 8 and strong chromatic number $\frac{5}{4} \cdot 8^2 = 80$.

Actually, they even asked if an upper bound of $c\Delta^2$ for any constant c smaller than 2 could be proved. Molloy and Reed [134] answered this question in the affirmative.

THEOREM 6.3 (Molloy and Reed, 1997). *There is a constant ε such that for every graph G of maximum degree Δ,*

$$s\chi'(G) \le (2-\varepsilon)\Delta^2 \, .$$

Theorem 6.3 directly follows from the following theorem.

THEOREM 6.4 (Molloy and Reed, 1997). *There is a constant Δ_0 such that if G has maximum degree $\Delta \ge \Delta_0$ then $s\chi'(G) \le 1.99995\Delta^2$.*

We use the following lemma to obtain Theorem 6.4. Its proof consists of a careful case analysis [134], and we omit it.

LEMMA 6.5. *If G has a sufficiently large maximum degree Δ, then for each edge e of G the subgraph of $\mathscr{L}(G)^2$ induced by the neighbourhood of e has at most $\left(1-\frac{1}{36}\right)\binom{2\Delta^2}{2}$ edges.*

Using Lemma 6.5, the next result directly implies Theorem 6.4. Its proof illustrates the probabilistic approach used to colour sparse graphs. It also includes several other common techniques. In particular, to upper bound the probability of a given event it is customary to actually consider a less restrictive event that is easier to upper bound. The result we propose is stronger that what we really need, and a close version—whose proof is omitted—is given in the book of Molloy and Reed [140, Chapter 10]. The approach of the proof is basically the original one. However, the presentation is closer to what is used in some other papers [83, 159].

THEOREM 6.6 (Molloy and Reed, 1997). *There exists Δ_0 such that if G has maximum degree $\Delta \ge \Delta_0$ and for each vertex v of G, the neighbourhood of v in G induces* *a subgraph with at most* $\binom{\Delta}{2} - B$ *edges where* $B \geq e^{12} \cdot 7680 \cdot \Delta \log \Delta$*, then*

$$\chi(G) \leq \Delta + 1 - \frac{B}{e^6 \Delta} .$$

Proof . In the sequel, we suppose that Δ is sufficiently large so that the asserted inequalities are satisfied. Let $\mathscr{C} := \{1, 2, \ldots, \Delta + 1 - Be^{-6}\Delta^{-1}\}$ be the set of colours. Note that $\Delta \log \Delta \leq B \leq \frac{\Delta(\Delta-1)}{2}$, which implies that $|\mathscr{C}| > \frac{3}{4}\Delta$. We randomly colour the graph G according to the following procedure.

(1) For each vertex v, we choose a colour $r(v) \in \mathscr{C}$, independently and uniformly at random.

(2) For each vertex v, if $r(v) \notin \{r(u) : u \in N_G(v)\}$ then the colour $r(v)$ is *assigned* to v.

Thus, the procedure yields a partial proper colouring c of G. Set $C := Be^{-6}\Delta^{-1}$.

Note that if an uncoloured vertex has degree at most $|\mathscr{C}| - 1 = \Delta - C$, then we can colour it greedily however its neighbours are coloured. So, we only deal in the sequel with the set V' of uncoloured vertices with at least $|\mathscr{C}|$ neighbours. More precisely, we aim at showing that with positive probability, the obtained colouring is such that each uncoloured vertex of V' at least C colours appearing at least twice in its neighbourhood. Then, we can finish the colouring of G greedily, since for each uncoloured vertex the number of available colours will be at least

$$\Delta + 1 - \frac{B}{e^6 \Delta} - \Delta + C = 1 .$$

For $v \in V'$, let E_v be the event that fewer than C colours are assigned by c to at least two neighbours of v. Each event E_v is mutually independent of all the events E_u where u is at distance at least 4 from v. Hence, each event E_v is mutually independent of all but at most Δ^4 events. Therefore, the Symmetric Lovász Local Lemma yields the sought conclusion provided that

$$\forall v \in V', \quad \mathbf{Pr}(E_v) < \frac{1}{4\Delta^4} . \tag{28}$$

Fix an arbitrary vertex $v \in V'$. Since v has more than $\Delta - C$ neighbours in G, the vertex v has at least $\binom{\Delta}{2} - C\Delta$ pairs of neighbours. Hence, v has at least $B(1 - e^{-6}) > \frac{B}{2}$ pairs of non-adjacent neighbours. Let Ω be a collection of $\frac{B}{2}$ pairs of non-adjacent neighbours of v. We consider the random variable X_v defined as the number of pairs $(u, w) \in \Omega$ such that

(i) $r(u) = r(w)$;

(ii) $r(s) \neq r(u)$ if $s \in N_G(v) \setminus \{u, w\}$; and

(iii) $r(s) \neq r(u)$ if $s \in N_G(u) \cup N_G(w)$, i.e. both u and v are assigned their colour.

Thus, X_v is at most the number of colours appearing at least twice in $N_G(v)$. The probability that a given pair $(u, w) \in \Omega$ satisfies *(i)* is $\frac{1}{|\mathscr{C}|}$. In total, the number of neighbours of v, u, w in G is at most 3Δ. Therefore, given that they satisfy *(i)*, the vertices u and w also satisfy (ii) and (iii) with probability at least $\left(1-\frac{1}{|\mathscr{C}|}\right)^{3\Delta}$. Consequently, by the linearity of Expectation,

$$\begin{aligned}\mathbf{E}(X_v) \geq & \frac{B}{2}\cdot\frac{|\mathscr{C}|}{|\mathscr{C}|^2}\left(1-\frac{1}{|\mathscr{C}|}\right)^{3\Delta}\\ > & \frac{B}{2|\mathscr{C}|}\exp\left(-\frac{3\Delta}{|\mathscr{C}|}\right)\\ \geq & \frac{2B}{\Delta}\cdot e^{-6} = 2C\,,\end{aligned}$$

where we used that $1-\frac{1}{x} > e^{-x}$ if $x \geq 2$ and $\frac{3}{4}\Delta < |\mathscr{C}| < \Delta$.

Hence, if A_v holds then X_v is smaller than its expected value by more than C. But we assert that

$$\mathbf{Pr}\left(\mathbf{E}(X_v) - X_v > C\right) < \frac{1}{4\Delta^4}\,, \tag{29}$$

which will yield the desired result.

To establish (29), we apply Talagrand's Inequality. We set X_1 to be the number of colours chosen for at least two vertices in $N(v)$, including both members of at least one pair in Ω. In other words, a colour $i \in \mathscr{C}$ is counted by X_1 if and only if there exists a pair $(u,w) \in \Omega$ such that $r(u) = i = r(w)$. We define X_2 to be the number of colours that

(i) are chosen for both members of at least one pair in Ω; and

(ii) are chosen also for one of their neighbours, or for a third vertex of $N_G(v)$.

Note that $X_v = X_1 - X_2$. Therefore, by what precedes, if A_v holds then either X_1 or X_2 differs from its expected value by more than $\frac{C}{2}$. Notice that, since $|\mathscr{C}| > \frac{3}{4}\Delta$,

$$\mathbf{E}(X_2) \leq \mathbf{E}(X_1) \leq |\mathscr{C}|\cdot\frac{B}{2}\cdot\frac{1}{|\mathscr{C}|^2} \leq \frac{2}{3}\cdot\frac{B}{\Delta} < e^6\cdot C\,.$$

If $X_1 \geq s$, then there is a set of at most $2s$ trials whose outcomes certify this, namely the choices of colours for s pairs of variables. Moreover, changing the outcome of any random trial can only affect X_1 by at most 2, since it can only affect whether the old colour and the new colour are counted or not. Thus Talagrand's Inequality applies and, since $2C < \mathbf{E}(X) \leq \mathbf{E}(X_1) < e^6\cdot C$, we obtain

$$\begin{aligned}\mathbf{Pr}\left(|X_1 - \mathbf{E}(X_1)| > \frac{C}{2}\right) &\leq 4\exp\left(-\frac{C^2}{e^6\cdot 1024\cdot C}\right)\\ &\leq \frac{1}{8\Delta^4}\,,\end{aligned}$$

because $B \geq e^{12}\cdot 7680\cdot\Delta\log\Delta$ and hence $C > 5\cdot e^6\cdot 1024\cdot\log\Delta$.

Similarly, if $X_2 \geq s$ then there is a set of at most $3s$ trials whose outcomes certify this fact, namely the choices of colours of s pairs of vertices and, for each of these pairs, the choice of the (same) colour of a neighbour of a vertex of the pair or of another neighbour of v. As previously, changing the outcome of any random trial can only affect X_2 by at most 2. Therefore by Talagrand's Inequality, if $\mathbf{E}(X_2) \geq \frac{C}{2}$ then

$$\begin{aligned}\mathbf{Pr}\left(|X_2 - \mathbf{E}(X_2)| > \frac{C}{2}\right) &\leq 4\exp\left(-\frac{C^2}{e^6 \cdot 1536 \cdot C}\right)\\ &\leq \frac{1}{8\Delta^4}\,.\end{aligned}$$

If $\mathbf{E}(X_2) < \frac{C}{2}$, then we consider a binomial random variable that counts each vertex of $N_G(v)$ independently with probability $\frac{C}{4|N_G(v)|}$. We let X_2' be the sum of this random variable and X_2. Note that $\frac{C}{4} \leq \mathbf{E}(X_2') \leq \frac{3C}{4}$ thanks to the linearity of Expectation. Moreover, observe that if $|X_2 - \mathbf{E}(X_2)| > \frac{C}{2}$ then $|X_2' - \mathbf{E}(X_2')| > \frac{C}{4}$. Therefore, by applying Talagrand's Inequality to X_2' with $c = 2, r = 3$ and $t = \frac{C}{4} \in [60c\sqrt{r\,\mathbf{E}(X_2')}, \mathbf{E}(X_2')]$, we also deduce in this case that

$$\begin{aligned}\mathbf{Pr}\left(|X_2 - \mathbf{E}(X_2)| > \frac{C}{2}\right) &\leq \mathbf{Pr}\left(|X_2' - \mathbf{E}(X_2')| > \frac{C}{4}\right)\\ &\leq 4\exp\left(-\frac{C^2}{4608 \cdot C}\right)\\ &\leq \frac{1}{8\Delta^4}\,.\end{aligned}$$

Consequently, we infer that $\mathbf{Pr}\left(\mathbf{E}(X_v) - X_v > C\right) \leq \frac{1}{4}\Delta^{-4}$, as asserted. □

The approach just introduced to prove Theorem 6.6 is often used to colour the sparse vertices of the decomposition obtained by Reed's Lemma. In the next subsection, we turn our attention to vertices whose neighbourhood induces a large clique.

6.2. Where Friends Solve Conflicts

When using Reed's Lemma, it is important to be able to colour both vertices whose neighbourhood induces a sparse graph—i.e. with relatively few edges—and vertices whose neighbourhood induces a clique. The former case is usually achieved using the tools introduced in Subsection 6.1. Let us now see in more details an example of the latter case.

We consider a setting appearing in a recent proof about the channel assignment problem, more precisely concerning $L(p, 1)$-labellings of graphs [82, 83].

In the channel assignment problem, transmitters at various nodes within a geographic territory must be assigned channels or frequencies in such a way as to avoid interferences. A model for the channel assignment problem developed wherein channels or frequencies are represented with integers, "close" transmitters must be assigned different integers and "very close" transmitters must be assigned integers that differ by at least 2. This quantification led to the definition of an $L(p, q)$-labelling of a graph $G = (V, E)$ as a function f from the vertex set to the integers such that $|f(x) - f(y)| \geq p$ if $\mathrm{dist}(x, y) = 1$ and $|f(x) - f(y)| \geq q$ if $\mathrm{dist}(x, y) = 2$, where $\mathrm{dist}(x, y)$ is the distance between the two vertices x and y in the graph G. The notion of $L(2, 1)$-labelling first appeared in 1992 [71]. Since then, a large number of articles has been published devoted to the study of $L(p,q)$-labellings. We refer the reader to the surveys of Calamoneri [31] and Yeh [195].

Generalisations of $L(p,q)$-labellings in which for each $i \geq 1$, a minimum gap of p_i is required for channels assigned to vertices at distance i, have also been studied (see for example the recent survey of Griggs and Kráľ [70], and consult also [15, 113, 114, 118]).

In the context of the channel assignment problem, the main goal is to minimise the number of channels used. Hence, we are interested in the *span* of an $L(p,q)$-labelling f, which is the difference between the largest and the smallest labels of f. The $\lambda_{p,q}$*-number* of G is $\lambda_{p,q}(G)$, the minimum span over all $L(p,q)$-labellings of G. In general, determining the $\lambda_{p,q}$-number of a graph is NP-hard [65]. In their seminal paper, Griggs and Yeh [71] observed that a greedy algorithm yields $\lambda_{2,1}(G) \leq \Delta^2+2\Delta$, where Δ is the maximum degree of the graph G. Moreover, they conjectured that this upper bound can be decreased to Δ^2.

CONJECTURE 6.7 (Griggs and Yeh, 1992). *For every $\Delta \geq 2$ and every graph G of maximum degree Δ,*

$$\lambda_{2,1}(G) \leq \Delta^2 .$$

The bound offered by the conjecture, if true, would be tight. Jonas [96] improved slightly on Griggs and Yeh's upper bound by showing that every graph of maximum degree Δ admits an $L(2,1)$-labelling with span at most $\Delta^2 + 2\Delta - 4$. Subsequently, Chang and Kuo [35] provided the upper bound $\Delta^2+\Delta$ which remained the best general upper bound for about a decade. Kráľ and Škrekovski [115] brought this upper bound down by 1 as the corollary of a more general result. And, using the algorithm of Chang and Kuo [35], Gonçalves [32] decreased this bound by 1 again, thereby obtaining the upper bound $\Delta^2+\Delta-2$. As for planar graphs, Conjecture 6.7 is still open for $\Delta = 3$, but is known to be true for other values of Δ. For $\Delta \geq 7$ it follows from a result of van den Heuvel and McGuinness [87], and Bella *et al.* [19] proved it for the remaining cases (for planar graphs).

The following approximate version of the generalisation of Conjecture 6.7 to $L(p,1)$-labelling was proved recently [82, 83].

THEOREM 6.8 (Havet, Reed, and Sereni, 2007). *For any fixed integer p, there exists a constant C_p such that for every integer Δ and every graph of maximum degree Δ,*

$$\lambda_{p,1}(G) \leq \Delta^2 + C_p .$$

This result is obtained by combining the bound given by a greedy labelling (when Δ is small) with the next theorem which, in particular, settles Conjecture 6.7 for sufficiently large Δ.

THEOREM 6.9 (Havet, Reed, and Sereni, 2007). *For any fixed integer p, there is a Δ_p such that for every graph G of maximum degree $\Delta \geq \Delta_p$,*

$$\lambda_{p,1}(G) \leq \Delta^2 .$$

The proof of this theorem makes intensive use of the probabilistic method. The Lovász Local Lemma is applied many times, and so are the concentration bounds presented so far. The proof also relies on structural results and techniques (such as Lemma 6.1). Random colouring procedures are designed to obtain an $L(p,1)$-labelling of a (supposed) minimal (in terms of the number of vertices) counter-example to the theorem, thereby giving the sought contradiction. The vertices are coloured in three steps. The first one concerns sparse vertices and is close to what we saw in Subsection 6.1. The last step concerns big cliques, and this is the point we now focus on.

We define a formal setting close to the one appearing in the third step of the proof of Theorem 6.9. The goal is to illustrate an approach to extend a partial colouring to big cliques of a graph (i.e. of size close to the maximum degree), provided that the partial colouring fulfils some conditions. We do not pretend that these extra conditions are easy to obtain. On the contrary, this is one of the main challenges in the proof of Theorem 6.9. Its solution is provided in part by structural arguments (by building and randomly colouring a different graph than the original in the first step of the proof), and next by an iterative quasi-random procedure, where the Lovász Local Lemma is applied at each iteration (this is an application of the so-called naive colouring procedure). The analysis of this procedure is too long and technical to be presented in this survey.

Let $G=(V,E)$ be a graph. We assume that the maximum degree Δ of G is large enough to fulfil the inequalities asserted in the sequel. Let $\mathscr{K}=\{A_1,A_2,\ldots,A_\ell\}$ be a set of vertex-disjoint cliques of G, each being of size at least $\Delta-c\Delta^{3/4}$ for a fixed positive real number c. We assume that each vertex of the clique has at most $\sqrt{\Delta}$ neighbours outside the clique. Those neighbours are referred to as *external neighbours*. Hence, at most $\Delta^{3/2}$ edges leave each given clique of $\mathscr{K}$.

We assume that there is a partial proper k-colouring of the vertices outside the cliques of $\mathscr{K}$, and we want to extend it to the whole graph G. We hence suppose that k is at least the size of the biggest clique of $\mathscr{K}$. We assume that the colouring has the following property. For each clique $A\in\mathscr{K}$ and each colour j, the number of vertices of A with a neighbour outside A coloured j is at most $\frac{4}{5}\Delta$.

A crucial fact to exploit to colour cliques of size near the maximum degree is that they have few edges linking them to the rest of the graph. Thus, a possible approach is to properly colour each of them one by one, randomly and independently of each other, and of the rest of the graph. The obtained colouring has conflicts, since a vertex in a clique may well have an external neighbour with the same colour as itself. However, we are able to keep the number of conflicts small (i.e. we can show that, with positive probability, the obtained colouring of G does not create too many conflicts). All the conflicts are solved simultaneously by swapping the colours of some vertices. More precisely, each badly coloured vertex of a clique of $\mathscr{K}$ chooses, inside its clique, a vertex called a *friend*. Friends are defined so that, when simultaneously swapping the colour of each badly coloured vertex with the one of its friend, the resulting colouring is proper. The existence of a colouring such that each badly coloured vertex has a friend is obtained by analysing our colouring procedure. Let us now see this precisely.

Phase 1. For each clique $A \in \mathscr{K}$, we choose uniformly at random a subset of $|A|$ colours among the whole set of colours. Those choices are made independently one of each other. Then, we assign a random permutation of those colours to the vertices of A. Again, the choices for different cliques are made independently. This yields a proper colouring of each clique of $\mathscr{K}$. We let Temp_A be the set of vertices of A with an external neighbour of the same colour.

LEMMA 6.10. *With positive probability, the following hold.*

(i) $|\mathrm{Temp}_i| \le 3\sqrt{\Delta}$ *for each* $i \in \{1, 2, \dots, \ell\}$*; and*

(ii) for each clique $A_i \in \mathscr{K}$ *and each colour* j*, at most* $\Delta^{9/10}$ *vertices of* A_i *have a neighbour in* $\cup_{k \ne i} A_k$ *coloured* j*.*

Proof . We use the Symmetric Lovász Local Lemma. For every index $i \in \{1, 2, \dots, \ell\}$, we let $E_1(i)$ be the event that $|\mathrm{Temp}_i|$ is greater than $3\sqrt{\Delta}$. For each index $i \in \{1, 2, \dots, \ell\}$ and each colour j, we define $E_2(i, j)$ to be the event that condition (ii) is not fulfilled. Each event is mutually independent of all the events involving cliques at distance greater than 2, so each event is mutually independent of all but at most Δ^5 other events. According to the Lovász Local Lemma, it is enough to show that each event has probability at most Δ^{-6}, since $\Delta^5 \times \Delta^{-6} < \frac{1}{4}$.

Our first goal is to upper bound $\mathbf{Pr}(E_1(i))$. We may assume that both the colour assignments for all cliques other than A_i, and the choice of the $|A_i|$ colours to be used on A_i have already been made. Thus it only remains to choose a random permutation of those $|A_i|$ colours onto the vertices of A_i. Since every vertex $v \in A_i$ has at most $\sqrt{\Delta}$ external neighbours, the probability that $v \in \mathrm{Temp}_i$ is at most $\frac{\sqrt{\Delta}}{|A_i|}$. So we deduce that $\mathbf{E}(|\mathrm{Temp}_i|) \le \sqrt{\Delta}$. We define a binomial random variable B that counts each vertex of A independently with probability $\frac{\sqrt{\Delta}}{|A_i|}$. We set $X := |\mathrm{Temp}_i| + B$. By the linearity of Expectation,

$$\sqrt{\Delta} \le \mathbf{E}(X) = \mathbf{E}(|\mathrm{Temp}_i|) + \sqrt{\Delta} \le 2\sqrt{\Delta}\,.$$

Moreover, if $|\mathrm{Temp}_i| > 3\sqrt{\Delta}$ then $|\mathrm{Temp}_i| - \mathbf{E}(|\mathrm{Temp}_i|) > 2\sqrt{\Delta}$, and hence $X - \mathbf{E}(X) > \sqrt{\Delta}$. We now apply McDiarmid's Inequality to show that X is concentrated. Note that if $|\mathrm{Temp}_i| \ge s$, then the colours to $2s$ vertices (that is, s members of Temp_i and one neighbour for each) certify this fact. Moreover, switching the colours of two vertices in A_i may only affect whether those two vertices are in Temp_i. So we may apply McDiarmid's Inequality to X with $c = 2 = r$ and $t = \sqrt{\Delta} \in \left[60c\sqrt{r\,\mathbf{E}(X)}, \mathbf{E}(X)\right]$. We deduce that the probability that the event $E_1(i)$ holds is at most

$$\begin{aligned}\mathbf{Pr}\left(|X - \mathbf{E}(X)| > \sqrt{\Delta}\right) &< 4\exp\left(-\frac{\Delta}{32 \cdot 8 \cdot 2\sqrt{\Delta}}\right)\\ &< \Delta^{-6}\,.\end{aligned}$$

We now upper bound $\mathbf{Pr}(E_2(i, j))$. Recall that the vertices of A_i are assigned pairwise distinct colours. Every vertex $v \in A_i$ has at most $\sqrt{\Delta}$ external neighbours. We let $S(v)$ be the set of all external neighbours of v in $\cup_k A_k$. Hence $|S(v)| \le \sqrt{\Delta}$.

Note that each vertex of $\cup_{k\neq i} A_k$ is in at most $\sqrt{\Delta}$ sets $S(v)$ for $v \in A_i$. We want to show that the probability that the number of sets $S(v)$, for $v \in A_i$, containing a vertex coloured j is greater than $\Delta^{9/10}$ is at most Δ^{-6}. In other words, we aim at proving that

$$\mathbf{Pr}\left(|\{v \in A_i : S(v) \text{ contains a vertex coloured } j\}| > \Delta^{9/10}\right) < \Delta^{-6}\,.$$

Each vertex of a set $S(v)$ is assigned the colour j with probability at most

$$\max_{A\in\mathscr{K}} \frac{1}{|A|} \leq \Delta^{-9/10}\,,$$

because $\min|A_i| \geq \Delta - c\Delta^{3/4}$ by our assumptions. Moreover, for any set $M \subseteq \cup_{v\in A_i} S(v)$,

$$\mathbf{Pr}\,(\text{all the vertices of } M \text{ are coloured } j) \leq \Delta^{-9|M|/10}\,,$$

since the choices of colours and colour assignments are made independently for different cliques.

Let us partition the vertices of the sets $S(v)$ regarding the number of sets to which they belong: for $s \in \{1,2,3,4\}$, let T_s be the vertices of $\cup_{v\in A_i} S(v)$ that belong to between $\Delta^{(s-1)/8}$ and $\Delta^{s/8}$ sets. Further, let E_s be the event that at least $\frac{1}{4}\Delta^{9/10}$ sets $S(v)$ contain a vertex coloured j. Note that if more than $\Delta^{9/10}$ sets $S(v)$ contains a vertex coloured j, then at least one of the events E_i holds.

Since $|\cup_{v\in A_i} S(v)| \leq \Delta^{3/2}$, we deduce that $|T_s| \leq \frac{\Delta^{3/2}}{\Delta^{(s-1)/8}}$. Moreover, if E_s holds then at least $\frac{1}{4}\Delta^{9/10-s/8}$ vertices of T_s are coloured j. Therefore,

$$\begin{aligned}
\mathbf{Pr}(E_i) &\leq \binom{\Delta^{3/2}/\Delta^{(s-1)/8}}{\frac{1}{4}\Delta^{9/10-s/8}} \cdot \left(\Delta^{-9/10}\right)^{\frac{1}{4}\Delta^{9/10-s/8}}\\
&\leq \left(\frac{e\Delta^{3/2}/\Delta^{(s-1)/8}}{\frac{1}{4}\Delta^{9/10-s/8}\cdot\Delta^{9/10}}\right)^{\frac{1}{4}\Delta^{9/10-s/8}} && \text{by Stirling's Formula}\\
&\leq \left(\frac{4e}{\Delta^{1/10}}\right)^{\frac{1}{4}\Delta^{9/10-s/8}}\,.
\end{aligned}$$

Since $\frac{1}{4}\Delta^{9/10-s/8} \geq \frac{1}{4}\Delta^{1/10}$, the probability that E_i holds is at most $\frac{1}{4}\exp\left(-\Delta^{1/10}\right)$, which is less than $\frac{1}{4}\Delta^{-6}$. Thus the probability that at least one of the events E_i holds is at most Δ^{-6}. The sought conclusion follows. □

Phase 2. We consider a colouring γ satisfying the conditions of Lemma 6.10. For each index $i \in \{1,2,\ldots,\ell\}$ and each vertex $v \in \mathrm{Temp}_i$, we let $\mathrm{Swappable}_v$ be the set of vertices u such that

(*a*) $u \in A_i \setminus \mathrm{Temp}_i$;

(*b*) $\gamma(u)$ does not appear on an external neighbour of v; and

(*c*) $\gamma(v)$ does not appear on an external neighbour of u.

LEMMA 6.11. *For every $v \in \mathrm{Temp}_i$, the set $\mathrm{Swappable}_v$ contains at least $\frac{\Delta}{10}$ vertices.*

Proof . Let us upper bound the number of vertices that are not in $\mathrm{Swappable}_v$. By Lemma 6.10(i), at most $3\sqrt{\Delta}$ vertices of A_i violate condition (a) and at most $\sqrt{\Delta}$ vertices violate condition (b) by the definition of A_i. According to our assumption on the original partial colouring, the number of vertices of A_i violating condition (c) because of a neighbour not in $\cup_{k\neq i} A_k$ is at most $\frac{4}{5}\Delta$. Finally, the number of vertices violating conditions (c) because of a colour assigned during Phase 1 is at most $\Delta^{9/10}$ thanks to Lemma 6.10(ii). Therefore, we deduce that the size of $\mathrm{Swappable}_v$ is at least

$$|A_i| - \frac{4}{5}\Delta - \Delta^{9/10} - 4\sqrt{\Delta} \geq \frac{1}{10}\Delta\,,$$

as $|A_i| \geq \Delta - c\Delta^{3/4}$ by hypothesis. □

For each clique $A_i \in \mathscr{K}$ and each vertex $v \in \mathrm{Temp}_i$, we choose 100 uniformly random members of $\mathrm{Swappable}_v$. These vertices are called *candidates* of v.

Definition 6.12. A candidate u of v is *unkind* if either

(a) u is a candidate for some other vertex;

(b) v has an external neighbour w that has a candidate w' with the same colour as u;

(c) v has an external neighbour w that is a candidate for a vertex w' with $\gamma(w') = \gamma(u)$;

(d) u has an external neighbour w that has a candidate w' with the same colour as v; or

(e) u has an external neighbour w that is a candidate for a vertex w' with the same colour as v.

A candidate of v is *kind* if it is not unkind.

Lemma 6.13. *With positive probability, for each $i \in \{1, 2, \ldots, \ell\}$ every vertex of* Temp_i *has a kind candidate.*

We choose candidates satisfying the preceding lemma. For each vertex $v \in \cup_{i=1}^{\ell} \mathrm{Temp}_i$, we swap the colour of v and one of its kind candidates. The obtained colouring is the desired one. So to finish our proof, it remains to prove Lemma 6.13.

Proof of Lemma 6.13. For every vertex v in some Temp_i, let $E_1(v)$ be the event that v does not have a kind candidate. Each event is mutually independent of all the events involving cliques at distance greater than 2. So each event is mutually independent of all but at most Δ^5 other events. We prove that the probability of each event is at most Δ^{-6}. Then, the conclusion follows from the Symmetric Lovász Local Lemma, since $\Delta^{-6} \cdot \Delta^5 < \frac{1}{4}$.

Observe that the probability that a particular vertex of $\mathrm{Swappable}_v$ is chosen is $100/|\mathrm{Swappable}_v|$, which is at most $1000\Delta^{-1}$.

We wish to upper bound $\mathbf{Pr}(E_1(v))$ for an arbitrary vertex $v \in \mathrm{Temp}_i$, so we can assume that all the vertices but v have already chosen candidates. Recall that the vertex v has at most $\sqrt{\Delta}$ external neighbours, each having at most 100 candidates. By Lemma 6.10(i), the number of vertices that satisfy condition (a) of Definition 6.12 is at most $300\sqrt{\Delta}$. Since each colour appears on at most one member of $\mathrm{Swappable}_v$, we deduce that the number of vertices satisfying one of the conditions (b) and (c) is at most $101\sqrt{\Delta}$.

We now deal with the remaining two conditions, starting with condition (d). The number of vertices of A_i that satisfy condition (d) is at most the number of edges with an endvertex in A_i and an endvertex in A_k with $k \neq i$, and such that the external endvertex has chosen a candidate with the colour of v. For each vertex $w \in \cup_{k\neq i} A_k$, we let N_w be the number of neighbours of w in A_i. So, $N_w \leq \sqrt{\Delta}$. Note that $\sum N_w \leq \Delta^{3/2}$ since at most $\Delta^{3/2}$ edges leave the clique A_i. We define the random variable F_w to be N_w if w has a candidate with the colour of v, and 0 otherwise. Thus, the number of vertices of A_i that satisfy condition (d) is at most the sum σ of the variables F_w for $w \in \cup_{k\neq i} A_k$. We aim at showing that

$$\mathbf{Pr}\left(\sigma > 2\Delta^{3/5}\right) < \frac{1}{4}\Delta^{-6}. \tag{30}$$

Since each vertex in some set Temp_k chooses its candidates independently, the variables F_w are independent. Set $s := \left\lceil \log_2\left(\sqrt{\Delta}\right)\right\rceil$. For each $r \in \{0, 1, \dots, s\}$, let S_r be the set of vertices w of $\cup_{k\neq i} A_k$ such that $2^{r-1} < N_w \leq 2^r$. So

$$\sigma \leq \sum_{r=0}^{s} \sum_{w\in S_r} F_w \leq \sum_{r=0}^{s} 2^r \sigma_r$$

where $\sigma_r := |\{w \in S_r : F_w \neq 0\}|$. Consequently, to prove (30) it suffices to show that for every index $r \in \{0, 1, \dots, s\}$,

$$\mathbf{Pr}\left(\sigma_r > t\right) < \frac{\Delta^{-6}}{4\left(s+1\right)}$$

where

$$t := \frac{2\Delta^{3/5}}{2^r\left(s+1\right)}.$$

Fix an index r. Note that $|S_r| < 2^{1-r}\Delta^{3/2}$ since at most $\Delta^{3/2}$ edges leave A_i. As the variables F_w are independent, the probability that σ_r is more than t is no more than the probability that the binomial random variable $\mathrm{Bin}(n, p)$ with $n := 2^{1-r}\Delta^{3/2}$ and $p := 1000\Delta^{-1}$ is more than t. Therefore, we deduce from Chernoff's Bound that

$$\begin{aligned}
\mathbf{Pr}\left(\sigma_r > t\right) &\leq \mathbf{Pr}\left(\mathrm{Bin}(n,p) - np > \frac{t}{2}\right)\\
&< 2\exp\left(\frac{t}{2} - \left(np + \frac{t}{2}\right)\ln\left(1 + \frac{t}{2np}\right)\right)\\
&< \frac{\Delta^{-6}}{4\left(s+1\right)},
\end{aligned}$$

as wanted.

We now consider condition (e) using a similar approach. A vertex u of A_i satisfies condition (e) if it has an external neighbour that was chosen as a candidate for a vertex with the same colour as v. We actually consider the number of edges with an endvertex in A_i and the other in some A_k with $k \neq i$, and such that the endvertex not in A_i is a candidate for a vertex with the same colour as v. We express this as the sum of several random variables.

Recall that N_w is the number of neighbours of w in A_i, for every $w \in \cup_{k \neq i} A_k$. So, $N_w \leq \sqrt{\Delta}$. We define X_w to be N_w if w is a candidate for a vertex with the colour of v, and 0 otherwise. Thus, the probability that $X_w = N_w$ is at most $1000\Delta^{-1}$. The number of vertices of A_i satisfying condition (e) is at most the sum τ of the variables X_w for $w \in \cup_{k \neq i} A_k$. Our aim is to show that

$$\mathbf{Pr}\left(\tau > 2\Delta^{3/5}\right) < \frac{1}{4}\Delta^{-6}. \tag{31}$$

Recall that

$$S_r = \{w \in \cup_{k \neq i} A_k : 2^{r-1} < N_w \leq 2^r\}$$

for every $r \in \{0, 1, \ldots, s\}$. Hence,

$$\tau \leq \sum_{r=0}^{s} \sum_{w \in S_r} X_w \leq \sum_{r=0}^{s} 2^r \tau_r$$

where $\tau_r := |\{w \in S_r : X_w \neq 0\}|$. Consequently, to prove (31) it suffices to show that for every index $r \in \{0, 1, \ldots, s\}$,

$$\mathbf{Pr}\left(\tau_r > t\right) < \frac{\Delta^{-6}}{4\,(s+1)} \tag{32}$$

where

$$t := \frac{2\Delta^{3/5}}{2^r\,(s+1)}.$$

Let us fix an index r. Observe that τ_r is at most $100 \sum_{k \neq i} Z_r^k$ where each Z_r^k is a 0–1 random variable, which is 1 if there is a vertex of $S_r \cap A_k$ that is a candidate for a vertex with the same colour as v, and 0 otherwise. In particular, $Z_r^k = 1$ with probability at most $1000|S_r \cap A_k|\Delta^{-1}$. Moreover, if $\tau_r > t$ then $\sum_{k \neq i} Z_r^k > \frac{t}{100}$. Let $R_r := 2^{1-r} \cdot \Delta^{3/2}$. By our assumptions, for every $k \neq i$ the size of $S_r \cap A_k$ is at most $M_r := \min\left(\Delta, R_r\right)$. We set

$$T_m := \{k \neq i : 2^{m-1} < |S_r \cap A_k| \leq 2^m\}$$

for every integer $m \in \{0, 1, \ldots, \lceil \log_2(M_r) \rceil\}$. Hence, $|T_m| \leq 2^{2-m-r} \cdot \Delta^{3/2}$, and

$$\tau_r \leq 100 \sum_{m=0}^{\lceil \log_2(M_r) \rceil} \sum_{k \in T_m} Z_r^k.$$

To prove (32), it suffices to show that

$$\forall m \in \{0, 1, \ldots, \lceil \log_2(M_r) \rceil\}, \quad \mathbf{Pr}\left(\sum_{k \in T_m} Z_r^k > t'\right) < \frac{\Delta^{-6}}{4(s+1)\left(\lceil \log M_r \rceil + 1\right)} \tag{33}$$

where
$$t' := \frac{t}{100 \cdot (\lceil \log_2(M_r) \rceil + 1)} .$$

Let us fix an index m. The variables Z_r^k for $k \in T_m$ are independent 0–1 random variables, each being 1 with probability at most $2^m \cdot 1000\Delta^{-1}$. Observe that if $2^m \geq \Delta/1000$, then $|T_m| \leq 4 \cdot 10^3 \cdot 2^{-r}\sqrt{\Delta} \leq t'$ and hence (33) holds. Thus, we assume in the sequel that $2^m \leq \Delta/1000$. We define Y_m to be the sum of $2^{2-m-r} \cdot \Delta^{3/2}$ independent 0–1 random variables, each being 1 with probability $2^m \cdot 1000\Delta^{-1}$. Thus, $\sum_{k\in T_m} Z_r^k \leq Y_m$. The expected value of Y_m is
$$\mathbf{E}(Y_m) = 4000 \cdot 2^{-r}\sqrt{\Delta} < \Delta^{4/7} .$$

We deduce from Chernoff's Bound that
$$\begin{aligned}\mathbf{Pr}\left(Y_m - \mathbf{E}(Y_m) > \frac{t'}{2}\right) &< 2\exp\left(\frac{t'}{2} - \left(\mathbf{E}(Y_m) + \frac{t'}{2}\right) \cdot \ln\left(1 + \frac{t'}{2\,\mathbf{E}(Y_m)}\right)\right)\\ &< \frac{\Delta^{-5}}{4\,(s+1)\,(\lceil \log_2 (M_r) \rceil + 1)} .\end{aligned}$$

This yields (33), and thus (32), which in turn implies (31), as desired.

Therefore, with probability at least $1 - \frac{1}{2}\Delta^{-6}$ the number of unkind members of Swappable_v is at most
$$4\Delta^{3/5} + 300\sqrt{\Delta} + 101\sqrt{\Delta} < \Delta^{3/4}.$$

In this case, the probability that no candidate is kind is at most
$$\left(\frac{\Delta^{3/4}}{\Delta/10}\right)^{100} < \frac{1}{2}\Delta^{-6}.$$

Consequently, the probability that $E_1(v)$ holds is at most $\frac{1}{2}\Delta^{-6} + \frac{1}{2}\Delta^{-6} = \Delta^{-6}$, as desired. This concludes the proof. $\square$

To prove Theorem 6.9, a more general setting than the one of $L(p,1)$-labellings was actually considered. Being more general, the setting used is also more flexible. It allowed the authors to use techniques inspired from usual graph colouring.

We conclude this section about Reed's Lemma with an important conjecture of Reed, for the study of which he developed Lemma 6.1. As mentioned in Subsection 1.1, $\omega(G) \leq \chi(G) \leq \Delta + 1$ for any graph G of maximum degree Δ. Reed [157] conjectured that the ceiling of the average of those two quantities is an upper bound for the chromatic number.

Conjecture 6.14 (Reed, 1998). *For every graph G of maximum degree Δ,*
$$\chi(G) \leq \left\lceil \frac{1}{2} \cdot \omega(G) + \frac{1}{2} \cdot (\Delta + 1) \right\rceil .$$

Reed [157] proved the following.

THEOREM 6.15 (Reed, 1998). *There exists a positive constant a such that for every graph G*

$$\chi(G) \leq \lceil a \cdot \omega(G) + (1-a) \cdot (\Delta + 1) \rceil ,$$

where Δ *is the maximum degree of* G.

As for particular classes of graphs, the result of Johansson [95] mentioned in Subsection 5.2 implies Conjecture 6.14 restricted to triangle-free graphs (i.e. $\omega(G) = 2$) with large enough maximum degree. Reed [157] observed that the matching theory can be used to prove the conjecture for graphs with a universal vertex—a vertex is *universal* if it is adjacent to all the other vertices. King, Reed, and Vetta [109] proved Conjecture 6.14 restricted to line graphs. This was later improved by King and Reed [108], who showed that quasi-line graphs satisfy the conjecture—a *quasi-line graph* is a graph in which every neighbourhood can be covered by two cliques, so that any line graph is a quasi-line graph. (Since the maximum degree of a quasi-line graph G is at most $2\omega(G) - 2$, this last result is stronger than the bound $\chi(G) \leq \frac{3}{2} \cdot \omega(G)$ for every quasi-line graph G, obtained by Chudnovsky and Ovetsky [38].) Rabern [154] proved that every graph G on n vertices satisfies Conjecture 6.14 provided that $\Delta \geq n + 2 - \alpha(G) - \sqrt{n + 5 - \alpha(G)}$, where Δ is the maximum degree of G (recall that $\alpha(G)$ is the independence number of G).

7. Fourier Analysis

We end this paper with techniques whose flavour slightly differs from what was presented so far. During the last twenty years, Fourier analysis has been used to study Boolean functions in combinatorics and computer science [5, 60, 62, 104, 120]. It is often convenient to interpret the information obtained about the Fourier transform of a (Boolean) function by probabilistic means, which permits the use of the first moment method. As an illustration, we present a result of Alon, Dinur, Friedgut, and Sudakov [3]. It deals with the size of independent sets in weak products of complete graphs, and hence is related to the usual notion of graph colouring. Before that, we give the definitions and theorems that we need.

7.1. Some Background

Harmonic analysis is a tool to study spaces of functions taking values in the complex field $\mathbf{C}$. It takes its simplest form when the functions are from a finite Abelian group. In this context, it has many applications in combinatorics. The reader is referred to the monograph by Terras [179] for a gentle introduction and an in-depth exposition. Basic and advanced exposition on harmonic analysis can also be found in lecture notes provided on MIT Open Course Ware [33].

For our purposes, it suffices to consider a fixed group $G := \mathbf{Z}_r^n$ with $r \geq 2$. Since $\mathbf{C}$ is a field, $\mathbf{C}^G$ is a vectorial-space of dimension $|G|$. A basis is $\{\delta_S : S \in G\}$ where $\delta_S(x)$ is 1 if $x = S$ and 0 otherwise.

A natural Hermitian form of $\mathbf{C}^G$ is given by

$$\forall f,g : G \to \mathbf{C}, \quad \langle f,g\rangle := \frac{1}{|G|}\sum_{S\in G} f(S)\overline{g(S)}\,,$$

where $\overline{z}$ is the conjugate of the complex number z. Endowing $\mathbf{C}^G$ with this product turns it into a Hilbert space $L^2(G)$. The associated norm is given by

$$\|f\|_2 := \sqrt{\langle f,f\rangle} = \left(\frac{1}{|G|}\sum_{S\in G}|f(S)|^2\right)^{1/2}\,,$$

and more generally the *p-norm* is defined by

$$\|f\|_p := \sqrt{\langle f,f\rangle} = \left(\frac{1}{|G|}\sum_{S\in G}|f(S)|^p\right)^{1/p}\,.$$

Note that G endowed with the discrete topology can be viewed as a discrete probability space: for a function $f : G \to \mathbf{C}$, we set $\int_G f(x)\,\mathrm{d}x := \frac{1}{|G|}\sum_{S\in G} f(S)$.

For each element $S \in G$, we let S_i be the i^{th} coordinate of S. Let $\underline{0} := (0,0,\dots,0)$, $\underline{1} := (1,1,\dots,1)$, and $|S| := \{i \in \{1,2,\dots,n\} : S_i \neq 0\}$ for $S \in G$. The set $\{\delta_S : S \in G\}$ forms an orthogonal basis of $\left(\mathbf{C}^G, \langle\cdot,\cdot\rangle\right)$. We now define another basis of $\mathbf{C}^G$ by considering the characters of G. Information on a function $f \in \mathbf{C}^G$ is usually obtained by comparing its decompositions on the two basis.

A *character* of G is an homomorphism $\mu : G \to \mathbf{C}^\times$, where $\mathbf{C}^\times$ is the multiplicative group on $\mathbf{C}\setminus\{0\}$. By Lagrange's Theorem, $\mu^{|G|} \equiv 1$. Actually, the set of characters of G is a group, written $\widehat{G}$ and called the *dual* group of G. Since G is a finite Abelian group, it is isomorphic to its dual, i.e. $\widehat{G} = \mathbf{Z}_r^n$. More precisely, the dual of a cyclic group $\mathbf{Z}_r$ is the group of r^{th}-roots of unity. The dual of the direct product of two finite Abelian groups H and K is isomorphic to the direct product $\widehat{H}\times\widehat{K}$. The isomorphism is given by

$$\begin{array}{rcccccc} f: & \widehat{H}\times\widehat{K} & \longrightarrow & \widehat{H\times K} & & & \\ & (\mu,\mu') & \longmapsto & f(\mu,\mu'): & H\times K & \longrightarrow & \mathbf{C}^\times \\ & & & & (h,k) & \longmapsto & \mu(h)\cdot\mu(k)\,. \end{array}$$

In particular, G has $|G|$ characters, and we write them μ_S for $S \in G$. Hence, $\mu_{S+T} = \mu_S\cdot\mu_T$ and $\mu_{-S} = \mu_S^{-1} = \overline{\mu_S}$. Although we do not really need it, we can explicit μ_S as follows.

$$\forall T \in \mathbf{Z}_r^n, \quad \mu_S(T) = \exp\left[\frac{2\pi i}{r}\cdot\sum_{j=1}^n S_j\cdot T_j\right]\,.$$

The set of characters of G form an orthonormal basis of $\left(\mathbf{C}^G, \langle\cdot,\cdot\rangle\right)$, since the roots of unity sum to 0 and $\mu_{\underline{0}} \equiv 1$. Consequently, every function $f : G \to \mathbf{C}$ has a unique expansion of the form $\sum_{S\in G}\widehat{f}(S)\mu_S$ where

$$\widehat{f}(S) := \langle f,\mu_S\rangle = \frac{1}{|G|}\sum_{T\in G} f(T)\cdot\overline{\mu_S(T)}\,.$$

This expansion is called the *Fourier transform* of f, and $\left\{\widehat{f}(S) : S \in G\right\}$ are the *Fourier coefficients* of f. We note that the Fourier transform is usually defined as a function from $\widehat{G}$ to $\mathbf{C}$. However, in our setting it is equivalent via the (non-canonical) isomorphism between G and $\widehat{G}$.

From the orthonormality of the basis, we directly infer Parseval's Equality, i.e.

$$\begin{aligned} \forall f : G \to \mathbf{C}, \quad \|f\|_2^2 &= \sum_{S \in G} \sum_{T \in G} \widehat{f}(S)\overline{\widehat{f}(T)}\langle \mu_S, \mu_T \rangle \\ &= \sum_{S \in G} \left|\widehat{f}(S)\right|^2 . \end{aligned} \tag{34}$$

Many applications of Fourier analysis to Boolean functions use the Bonami-Becker Inequality [23, 17], which deals with functions from $\{0,1\}^n$ to $\mathbf{C}$. Alon *et al.* [3] proved the following two lemmas, which are tailored for applications using Fourier analysis of $\mathbf{Z}_r^n$. We omit their proofs in this survey.

Lemma 7.1 (Alon, Dinur, Friedgut, and Sudakov, 2004). *Let $f : \mathbf{Z}_r^n \to \{0,1\}$ be a function such that*

$$\forall S \in G, \quad |S| > 1 \Rightarrow \widehat{f}(S) = 0 .$$

Then either f is constant or it depends on precisely one coordinate.

Lemma 7.2 (Alon, Dinur, Friedgut, and Sudakov, 2004). *For every $r \geq 2$, there exists $K > 0$ such that the following holds for every $\varepsilon > 0$. Let $f : \mathbf{Z}_r^n \to \mathbf{C}$ be a function such that*

$$\widehat{f}(0) = \alpha = \sum_{S \in G} |\widehat{f}(S)|^2 \quad \textit{and} \quad \sum_{\substack{S \in G \\ |S|>1}} \left|\widehat{f}(S)\right|^2 = \varepsilon .$$

Then, there exists a function $g : \mathbf{Z}_r^n \to \{0,1\}$ depending on at most one coordinate and such that

$$\|f - g\|_2^2 < \frac{K}{\alpha - \alpha^2 - \varepsilon} \cdot \varepsilon .$$

7.2. Independent Sets

The *weak product* of two graphs $G = (V, E)$ and $H = (V', E')$ is the graph $G \times H$ with vertex-set $V \times V'$ and an edge between two vertices (g_1, h_1) and (g_2, h_2) whenever $g_1 g_2 \in E$ and $h_1 h_2 \in E'$. The weak product is also called the *direct* or *categorical* product.

We are interested in the n^{th} weak power K_r^n of the complete graph K_r on r vertices. (Note that the power symbol used in Subsection 2.2 was associated to a different kind of graph product.)

The graph K_r^n can be equivalently defined as follows. The vertex set of K_n^r is $\mathbf{Z}_r^n$ and there is an edge between two vertices $v = (v_1, v_2, \dots, v_n)$ and $u = (u_1, u_2, \dots, u_n)$ if $v_i \neq u_i$ for each $i \in \{1, 2, \dots, n\}$.

The chromatic number of K_r^n is r, since we can duplicate any r-colouring of K_r. Are there other r-colourings? To answer, let us investigate the maximum independent sets of K_r^n. One can partition the graph K_r^n into disjoint cliques of size r, thereby showing that every independent set has size at most r^{n-1}. We do not provide details, as this will be proved in another way later. For each $i \in \{1, 2, \ldots, n\}$ and each $k \in \mathbf{Z}_r$, the set $\{(v_1, v_2, \ldots, v_n) \in \mathbf{Z}_r^n : v_i = k\}$ is an independent set of K_n^r of size r^{n-1}. The next theorem states that they are the only maximum independent sets of K_r^n.

Theorem 7.3. *Let $G = K_r^n$ with $r \geq 3$. Let I be an independent set of size r^{n-1}. Then there exists a coordinate $i \in \{1, 2, \ldots, n\}$ and an integer $k \in \mathbf{Z}_r$ such that*

$$I = \{v : v_i = k\} .$$

Consequently, the only r-colourings of G are those induced by colourings of one of the factors K_r.

This theorem was first proved in a stronger form by Greenwell and Lovász [68], and, independently, by Müller [143, 144]. Their motivation was a conjecture of Nešetřil [149] asserting the existence of graphs of arbitrarily high chromatic number and arbitrarily high girth, which moreover admit only one optimal colouring (up to permutations of colours). The first case, i.e. that of triangle-free graphs, had been settled by Nešetřil [149].

The proof of Alon *et al.* [3] of Theorem 7.3 is different than the previously known ones. It allowed them to also obtain the next "stability" result. For two sets I and J, the *symmetric difference* of I and J is the set

$$I \triangle J := (I \setminus J) \cup (J \setminus I) .$$

Theorem 7.4 (Alon, Dinur, Friedgut, and Sudakov, 2004). *For every $r \geq 3$, there exists a constant M such that for any $\varepsilon > 0$ the following holds. If J is an independent set of K_r^n of size $r^{n-1} - \varepsilon r^n$, then there exists an independent set I of size r^{n-1} such that*

$$|J \triangle I| < M\varepsilon \cdot r^n .$$

This last theorem states that, for constant r and arbitrary n, any independent set of K_r^n of size close to the maximum is close to some independent set of maximum size. Let us note that Theorem 7.4 was recently improved by Ghandehari and Hatami [66], who showed that the theorem is actually true for arbitrary r and n. Their proof also uses Fourier analysis of $\mathbf{Z}_r^n$.

Let us use the Fourier transform to study indicator functions of independent sets. The following lemma provides useful information about such an indicator, and its proof consists of routine calculations. Given a set $I \subseteq \mathbf{Z}_r^n$, the *indicator function* of I is the mapping $f : \mathbf{Z}_r^n \to \{0, 1\}$ such that $f(x) = 1$ if and only if $x \in I$. We set $D := (\mathbf{Z} \setminus \{0\})^n$ and $d := |D| = (r-1)^n$.

Lemma 7.5. *Let I be an independent set of $G := K_r^n$, and let $f : \mathbf{Z}_r^n \to \{0, 1\}$ be its indicator function. Then*

$$\sum_{S \in G} \left|\widehat{f}(S)\right|^2 \left(\frac{-1}{r-1}\right)^{|S|} = 0 .$$

Proof . For $\tau \in D$, we set $f_\tau(x) := f(x+\tau)$. Let

$$A(f) := \frac{1}{d}\sum_{\tau\in D} f_\tau\,.$$

Note that $N_G(S) = \{S+\tau : \tau \in D\}$ for every $S \in G$. So, as I is an independent set, f and $A(f)$ are orthogonal, i.e. $\langle f, A(f)\rangle = 0$.

Let us compute the Fourier coefficients of $A(f)$ in terms of those of f. We assert that

$$\widehat{A(f)}(S) = \widehat{f}(S)\left(\frac{-1}{r-1}\right)^{|S|}\,.$$

Indeed,

$$\begin{aligned}
\widehat{A(f)}(S) &= \frac{1}{r^n}\sum_{T\in\mathbf{Z}_r^n}\frac{1}{d}\sum_{\tau\in D} f(T+\tau)\overline{\mu_S(T)} = \frac{1}{d}\sum_{\tau\in D}\frac{1}{r^n}\sum_{T\in\mathbf{Z}_r^n} f(T+\tau)\overline{\mu_S(T)}\\
&= \frac{1}{d}\sum_{\tau\in D}\frac{1}{r^n}\sum_{T\in\mathbf{Z}_r^n} f(T)\overline{\mu_S(T)}\cdot\mu_S(\tau) = \frac{1}{d}\widehat{f}(S)\sum_{\tau\in D}\mu_S(\tau)\\
&= \frac{1}{d}\widehat{f}(S)\sum_{\tau\in D}\prod_{j=1}^{n}\mu_{S_j}(\tau_j) = \frac{1}{d}\widehat{f}(S)\prod_{j=1}^{n}\sum_{k=1}^{r-1}\mu_{S_j}(k)\\
&= \frac{1}{d}\widehat{f}(S)\prod_{j:S_j=0}(r-1)\prod_{j:S_j\neq 0}(-1) = \widehat{f}(S)\left(\frac{-1}{r-1}\right)^{|S|}\,,
\end{aligned}$$

since $d = (r-1)^n$.

Therefore, by the orthogonality of f and $A(f)$, we infer that

$$0 = \sum_{S\in G}\widehat{f}(S)\overline{\widehat{A(f)}(S)} = \sum_{S\in G}\left|\widehat{f}(S)\right|^2\left(\frac{-1}{r-1}\right)^{|S|}\,.$$

□

We are now ready to prove Theorems 7.3 and 7.4.

Proof of Theorems 7.3 and 7.4. Let $\alpha := \frac{|I|}{r^n}$. So $\|f\|_2^2 = \alpha$, and hence Parseval's Equality implies that

$$\sum_{S\in\mathbf{Z}_r^n}\left|\widehat{f}(S)\right|^2 = \alpha\,.$$

Moreover, note that

$$\widehat{f}(\underline{0}) = \frac{1}{r^n}\sum_{T\in\mathbf{Z}_r^n} f(T) = \alpha\,.$$

Consequently,

$$\sum_{S\in\mathbf{Z}_r^n\setminus\{\underline{0}\}}\left|\widehat{f}(S)\right|^2 = \alpha - \alpha^2\,. \tag{35}$$

By Lemma 7.5,

$$\sum_{S\in\mathbf{Z}_r^n\setminus\{\underline{0}\}}\left|\widehat{f}(S)\right|^2\left(\frac{-1}{r-1}\right)^{|S|}=-\alpha^2\,. \tag{36}$$

We now exploit the information given by (35) and (36) by probabilistic means. Let T be a random variable taking values in $\mathbf{Z}_r^n\setminus\{\underline{0}\}$ with

$$\forall S\in\mathbf{Z}_r^n\setminus\{\underline{0}\},\quad \mathbf{Pr}(T=S)=\frac{\left|\widehat{f}(S)\right|^2}{\alpha-\alpha^2}\,.$$

Let $X(T):=\left(\frac{-1}{r-1}\right)^{|T|}$. Thus, by (36),

$$\mathbf{E}(X)=\sum_{S\in\mathbf{Z}_r^n\setminus\{\underline{0}\}}\mathbf{Pr}(T=S)\cdot X(T)=\frac{\alpha^2}{\alpha^2-\alpha}=\frac{\alpha}{\alpha-1}\,.$$

Observe also that for all T, it holds that $X(T)\geq\frac{-1}{r-1}$ with equality if and only if $|T|=1$. We consider three cases regarding the value of α.

1. $\alpha>\frac{1}{r}$. Then $\mathbf{E}(X)<\frac{-1}{r-1}$, a contradiction. This in particular implies that K_r^n does not have an independent set of size larger than r^{n-1}.

2. $\alpha=\frac{1}{r}$. Then $\mathbf{E}(X)=\frac{-1}{r-1}$ and hence $X(T)=\frac{-1}{r-1}$ for all T. So, $\left|\widehat{f}(S)\right|^2=0$ unless $|S|=1$. Consequently, by Lemma 7.1, since f is not constant, we deduce that f is the indicator function of a set of the form $\{v\in\mathbf{Z}_r^n : v_j=k\}$ for some integers j and k, as wanted. This ends the proof of Theorem 7.3.

3. $\alpha=\frac{1}{r}-\varepsilon$. Notice that for every S with $|S|>1$,

$$X(S)\geq\frac{-1}{(r-1)^3}>\frac{-1}{r-1}\,.$$

Let $Y:=X+\frac{1}{r-1}$, so $Y\geq 0$. Further, when $Y>0$ then $Y\geq\frac{-1}{(r-1)^3}+\frac{1}{r-1}=\frac{r(r-2)}{(r-1)^3}$. Therefore, Markov's Inequality yields that

$$\mathbf{Pr}(Y>0)\leq\mathbf{E}(Y)\cdot\frac{(r-1)^3}{r(r-2)}\,. \tag{37}$$

Since $\alpha=\frac{1}{r}-\varepsilon$, it follows from the definition of Y and the linearity of Expectation that

$$\mathbf{E}(Y)=\frac{\varepsilon r^2}{(r+r\varepsilon-1)(r-1)}\,.$$

Thus, as $r\geq 3$, we deduce from (37) that

$$\mathbf{Pr}(Y>0)\leq\frac{\varepsilon r}{r+r\varepsilon-1}\cdot\frac{(r-1)^2}{r-2}\leq\frac{\varepsilon r\cdot(r-1)}{r-2}\leq 2\varepsilon r\,.$$

Recall that $Y > 0$ if and only if $|S| > 1$, so that

$$\mathbf{Pr}(Y > 0) = \mathbf{Pr}(T = S \text{ for some } S \in G \text{ with } |S| > 1) = \sum_{\substack{S \in G \\ |S|>1}} \frac{|\widehat{f}(S)|^2}{\alpha - \alpha^2}\,.$$

It follows that

$$\sum_{\substack{S \in G \\ |S|>1}} \left|\widehat{f}(S)\right|^2 = (\alpha - \alpha^2)\,\mathbf{Pr}(Y > 0) \le 2\varepsilon\,.$$

Therefore, by Lemma 7.2, there is a function g depending on at most one coordinate such that $\|f - g\|_2 < \frac{2K\varepsilon}{\alpha - \alpha^2 - \varepsilon}$. This ends the proof of Theorem 7.4. □

A recent paper by Dinur, Friedgut, and Regev [44] provides further study of independent sets in weak powers of connected non-bipartite graphs. The approach also uses Fourier analysis, combined with spectral techniques and the Invariance Principle of Mossel, O'Donnell, and Oleszkiewicz [141].

Concluding Remarks

An Apology of Naiveness: The Naive Colouring Method

The naive colouring method is actually a powerful tool. Recall a general approach we used several times, for instance to colour sparse graphs in Subsection 6.1. We colour uniformly at random the vertices, independently one of each other. Then, vertices creating conflicts are uncoloured. As was already hinted at—and maybe surprisingly—iterating this simple procedure gives considerably more power. It was first introduced by Kahn [99] to prove that the list-colouring conjecture is asymptotically correct—which is a very strong result that dramatically improved the upper bounds known at that time. The idea is to randomly build a colouring (or, more generally, a combinatorial object), in several steps (the number of steps may well depend on a parameter, e.g. the maximum degree of the graph). After each iteration, one *particular* colouring is chosen. Its existence is proved by showing that it occurs at the end of the iteration with positive probability—for instance, by using the Lovász Local Lemma. Thus, the colouring is built via a series of extensions of previously built colourings. Each partial colouring fulfils some particular properties, and its existence is obtained by the probabilistic method. Then, the last iteration is usually slightly different from the general procedure, allowing us to finish the desired colouring thanks to the properties of the partial colouring obtained so far. Iterating random procedures is a powerful tool, but unfortunately proofs using it are too long and technical to be presented here. The reader is referred to the papers already cited [82, 83, 99, 139].

We note that the general method (not necessarily for colourings), is often called the guided method, the incremental method, the pseudo-random method, the semi-random method, or the Rödl Nibble. It has been successfully used many times [6, 59, 98, 99, 153, 161].

Hard-core Distributions

Hard-core distributions have been successfully used by Kahn [100, 101] to prove the following strong result.

Theorem 7.6 (Kahn, 2000). *For multigraphs G,*

$$\chi_f(\mathscr{L}(G)) \simeq \mathrm{ch}'(G) \quad \text{as } \chi_f(\mathscr{L}(G)) \to \infty.$$

They also are a key ingredient in an important result of Havet, van den Heuvel, McDiarmid and Reed [84, 85] that the list chromatic number of every planar graph of sufficiently large maximum degree Δ is at most $\frac{3}{2}\Delta\,(1+o(1))$.

Let us just say a few words on hard-core distributions, and refer to the book of Molloy and Reed [140, Chapter 22] for a good exposition and further references on their use, for instance in statistical physics.

In the three cited papers, independent sets of line graphs, i.e. matching of graphs, play an important role. The goal is to show that certain properties hold in the neighbourhood of a vertex, regardless of what the matching is far away from this vertex. In other words, we would like to be able to condition on the matching far away from a vertex. Let $\mathscr{M}(G)$ be the set of all the matchings of the graph G. A probability distribution p on the matchings of G is *hard-core* if it is obtained by associating a positive real $\lambda(e)$ to each edge e of G so that for every matching $M \in \mathscr{M}(G)$,

$$p(M) = \frac{\prod_{e\in M}\lambda(e)}{\sum_{T\in\mathscr{M}(G)}\prod_{e\in T}\lambda(e)}\,.$$

Thus, the probability that a matching M is chosen is proportional to $\prod_{e\in M}\lambda(e)$. The real numbers $\lambda(e)$ are the *activities* of p.

An important property of hard-core distributions is that they allow us to select random matchings by choosing one edge at a time. A proof of the following lemma can be found in the book of Molloy and Reed [140, Lemma 22.4].

Lemma 7.7. *Let $e = uv$ be an edge of a graph G and let M be a matching chosen according to a hard-core distribution on $\mathscr{M}(G)$. Let M_1 and M_2 be matchings of $G-e$ and $G-\{u,v\}$ chosen using the hard-core distribution with the same activities, respectively. Then,*

$$\text{for every } N \in \mathscr{M}(G-e), \quad \mathbf{Pr}(M_1 = N) = \mathbf{Pr}(M = N | e \notin M)\,,$$

and

$$\text{for every } N \in \mathscr{M}(G-\{u,v\}), \quad \mathbf{Pr}(M_2 = N) = \mathbf{Pr}(M = N+e | e \in M)\,.$$

Edmonds' characterisation of the matching polytope [45] can be used to prove the existence of hard-core distributions on matchings with certain independence properties. They thereby become a useful tool in conjunction with the Lovász Local Lemma.

Algorithmic Angle

The essence of the probabilistic method is to ensure the *existence* of an object, by showing that it occurs with positive probability in an appropriate probabilistic space.

However, it is also interesting to be able to *construct* those objects whose existence was proved by the probabilistic method. In particular, a lot of efforts have been made during the last twenty years towards algorithmic versions of the Lovász Local Lemma. Beck [16], Alon [1] and many others developed efficient algorithmic versions of the Lovász Local Lemma. Molloy and Reed [137, 138] designed a method allowing us to obtain efficient algorithms from virtually any application of the Symmetric Lovász Local Lemma. Recently, Srinivasan [175] obtained further nice improvements on this topic, in particular regarding the running-time of the algorithms.

Some More Topics

Needless to say, there are many tools (e.g. Janson's Inequality, Suen's Lemma, Azuma's Inequality) that were neither presented nor mentioned here. We end this survey by pointing out two related fields where the probabilistic method yields striking results, namely Ramsey theory and hypergraph colouring. We refer the reader to papers and lecture notes by Spencer [172, 174], and more generally to the book by Alon and Spencer [11] for a good account on those.

A natural field as for probabilistic techniques, which we have not dealt with, is that of random graphs. We conclude by pointing out two specific references on random graphs, namely the book by Janson, Łuczak, and Ruciński [93] and the monograph of Bollobás [22].

Acknowledgements. The author thanks Professor Jaroslav Nešetřil for his continuous support and his communicative enthusiasm regarding this survey.

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