Error analysis of the Penalty-Projection method for the time dependent Stokes equations

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Abstract

We address in this paper a fractional-step scheme for the simulation of incompressible flows falling in the class of penalty-projection methods. The velocity prediction is similar to a penalty method prediction step, or, equivalently, differs from the incremental projection method one by the introduction of a penalty term built to enforce the divergence-free constraint. Then, a projection step based on a pressure Poisson equation is performed, to update the pressure and obtain an (approximately) divergence-free end-of-step velocity. An analysis in the energy norms for the model unsteady Stokes problem shows that this scheme enjoys the time convergence properties of both underlying methods: for low value of the penalty parameter r, the splitting error estimates of the so-called rotational projection scheme are recovered, *i.e.* convergence as δt^2 and $\delta t^{3/2}$ for the velocity and the pressure, respectively; for high values of the penalty parameter, we obtain the $\delta t/r$ behaviour for the velocity error known for the penalty scheme, together with a $1/r$ behaviour for the pressure error. Some numerical tests are presented, which substantiate this analysis.

Key words : Finite elements, unsteady Stokes equations, projection methods, penalty methods.

1 Presentation of the penalty-projection scheme

Since the pioneering work of Chorin [5] and Temam [27] in the late sixties, projection methods have received a lot of attention and the fractional-step schemes falling in this category are probably nowadays the most popular ones for the solution of the unsteady Navier-Stokes equations. Indeed, schemes of this type have proved to be extremely efficient, essentially because, at each time step, they reduce without loss of stability the solution of a saddle-point type problem to a sequence of "decoupled" elliptic equations for the velocity and pressure, respectively. This feature makes them particularly attractive for industrial applications, as for instance the simulation of flows encountered in nuclear safety studies which are the context of this work.

The aim of the present paper is to analyse a variant of the projection method the main advantage of which is, concisely speaking, to offer the possibility to reduce the splitting error, *i.e.* the difference between the solution of the fractional-step scheme and the solution of the coupled one, up to make it negligible. The basic idea behind the development of this scheme originates from a paper of Shen in 1992 [22] and consists in adding to the velocity prediction step a term similar to the augmentation term used in the so-called Augmented Lagrangian method (e.g. [7]), which constrains the tentative velocity to remain almost divergence-free. The same idea has been exploited independently later, in 1999, by Caltagirone and Breil [3].

To be more specific, let us consider as model problem the unsteady Stokes problem with homogeneous Dirichlet boundary conditions:

$$\left|\begin{array}{ll} \dfrac{\partial u}{\partial t} - \Delta u + \nabla p = f & \text{in } \Omega\times]0,T[\\ \nabla \cdot u = 0 & \text{in } \Omega\times]0,T[\\ u = 0 & \text{on } \partial\Omega\times]0,T[\\ u(x,0) = u_0(x) & \text{in } \Omega \end{array}\right. \tag{1}$$

where u stands for the velocity, p the pressure, f a forcing term, Ω is a smooth bounded domain of $\mathbb{R}^d$, $d=2$ or $d=3$, of boundary $\partial\Omega$, u_0 is the (divergence-free) initial velocity field and the problem is posed over a finite time interval $]0,T[$.

The semi-discretization in time of this problem by the well-known first-order incremental projection scheme yields the following algorithm, for $0 \leq k \leq N-1$:

$$\left|\begin{array}{ll} (i) & \dfrac{\tilde{u}^{k+1} - u^k}{\delta t} - \Delta \tilde{u}^{k+1} + \nabla p^k = f^{k+1} \\ (ii) & \left\{\begin{array}{l} \dfrac{u^{k+1} - \tilde{u}^{k+1}}{\delta t} + \nabla \phi = 0 \\ -\Delta \phi = -\dfrac{1}{\delta t} \nabla \cdot \tilde{u}^{k+1} \end{array}\right. \\ (iii) & p^{k+1} = p^k + \phi \end{array}\right. \tag{2}$$

where δt is the time step, $\tilde{u}^k$, u^k, p^k and f^k stand respectively for the predicted velocity, the end-of-step velocity, the pressure and the forcing term at time $t^k =$

$k\,\delta t$, and N is such that $t^N = T$. The first step, where the pressure is taken at time t^k and thus known, yields the (non divergence-free) tentative velocity $\tilde{u}^k$, then the (divergence-free) end-of-step velocity is obtained through the so-called pressure Poisson equation and, finally, the pressure is updated. For the prediction step, the boundary condition $\tilde{u}^{k+1} = 0$ on $\partial\Omega$ is retained, while it is changed to $u^{k+1} \cdot n = \nabla\phi \cdot n = 0$ on $\partial\Omega$ in the second one, where n stands for the outward normal vector to $\partial\Omega$. Let then (2) be discretized in space by a finite element method; we then routinely obtain an algebraic system of the form:

$$\left|\begin{array}{ll} \text{(i)} & \dfrac{1}{\delta t}(\mathrm{M}_v \tilde{\mathrm{U}}^{k+1} - \mathrm{M}_v \mathrm{U}^k) + \mathrm{A}\tilde{\mathrm{U}}^{k+1} + \mathrm{B}^t \mathrm{P}^k = \mathrm{F}^{\mathrm{k+1}} \\ \text{(ii)} & \mathrm{L}\Phi = \dfrac{1}{\delta t}\mathrm{B}\tilde{\mathrm{U}}^{k+1}\ , \qquad \mathrm{M}_v \mathrm{U}^{k+1} = \mathrm{M}_v \tilde{\mathrm{U}}^{k+1} - \delta t \mathrm{B}^t \Phi \\ \text{(iii)} & \mathrm{P}^{k+1} = \mathrm{P}^k + \Phi \end{array}\right. \tag{3}$$

where M_v stands for the velocity mass matrix, A, B^t and L for the matrices associated respectively to the velocity laplacian (*i.e.* $-\Delta$), the pressure gradient and the pressure laplacian operators (*i.e.*, here also, $-\Delta$), the vectors $\tilde{\mathrm{U}}^k$, U^k, P^k and F^{k} gather respectively the degrees of freedom of the predicted velocity, the end-of-step velocity, the pressure and the forcing term at time t^k. As the boundary conditions for the predicted velocity and the end-of-step velocity are not the same, the discrete spaces used to approximate these unknowns are *a priori* different. However, from the variational formulation introduced later (section 2), it appears that only the L^2-projection of the end-of-step velocity onto the discrete space associated to the predicted one is useful in the algorithm. Consequently, only this latter quantity appears in the above algebraic system, and the set of degrees of freedom is the same for $\tilde{\mathrm{U}}^k$ and U^k; the discretization of the first relation of (ii) is thus replaced by the equation verified by the projected end-of-step velocity, and the same mass matrix is involved in this relation and in the discretization of equation (i).

The fractional-step scheme studied in this paper is obtained from (3) by adding a penalty term introduced implicitely in the prediction step and modifying consistently the pressure increment. It reads:

$$\left|\begin{array}{ll} \text{(i)} & \dfrac{1}{\delta t}(\mathrm{M}_v \tilde{\mathrm{U}}^{k+1} - \mathrm{M}_v \mathrm{U}^k) + (\mathrm{A} + r\mathrm{B}^t \mathrm{M}_p^{-1}\mathrm{B})\tilde{\mathrm{U}}^{k+1} + \mathrm{B}^t \mathrm{P}^k = \mathrm{F}^{\mathrm{k+1}} \\ \text{(ii)} & \mathrm{L}\Phi = \dfrac{1}{\delta t}\mathrm{B}\tilde{\mathrm{U}}^{k+1}\ , \qquad \mathrm{M}_v \mathrm{U}^{k+1} = \mathrm{M}_v \tilde{\mathrm{U}}^{k+1} - \delta t \mathrm{B}^t \Phi \\ \text{(iii)} & \mathrm{M}_p \mathrm{P}^{k+1} = \mathrm{M}_p(\mathrm{P}^k + \Phi) + r\mathrm{B}\tilde{\mathrm{U}}^{k+1} \end{array}\right. \tag{4}$$

where r is a non-negative parameter (the so-called penalty parameter) and M_p is a scaling matrix used to make the properties of the algorithm as less as possible mesh-dependent. Note that, by taking $r = 0$ in (4), one recovers the incremental projection scheme (3). The standard choice for M_p consists in taking an approximate pressure mass matrix, which, for computational efficiency reasons, can be chosen diagonal. In this case, the term $\mathrm{B}^t \mathrm{M}_p^{-1} \mathrm{B}\, \tilde{\mathrm{U}}^{k+1}$ can be seen as a discrete analogue to $\nabla\nabla \cdot \tilde{u}^{k+1}$; however, we insist on the fact that, as the discrete velocity can only be weakly divergence-free, excepting for very specific spatial discretizations, this penalty term must be built from the weak divergence constraint, that is at the algebraic level, to avoid a loss of accuracy for large values of r. Extensive tests of this scheme for transient Stokes and Navier-Stokes equations and both Dirichlet and open boundary conditions can be found in [16]. Moreover, a generalization for variable density low Mach number flows, for which the Boussinesq approximation is not satisfied, is proposed and numerically experimented in [17].

This scheme differs from the penalty-projection scheme introduced by Shen in [22, section 6] by the computation of the pressure [16, remark p. 506] and also by the fact that r is taken independent of the time step in the present work (instead of r varying as $1/\delta t^2$ in Shen's analysis). It also differs from the so-called "vectorial projection scheme" proposed by Caltagirone and Breil [3] by the whole projection step. Besides these two references, it presents also some analogy with other already presented numerical algorithms, from both the literature of penalty and pressure correction methods. First, instead of being derived from a projection method as performed here, the algorithm (4) can also be obtained by adding a projection step to a penalty (or quasi-incompressibility) method; as far as this latter class of methods is concerned, we refer to [26] for the seminal work and to [24] for an analysis. Setting $r = 0$ in the velocity prediction step and $r = 1$ for the present model problem (r equal to the viscosity for actual Navier-Stokes equations), yields a scheme proposed by Timmermans *et al.* [28] as an alternative to the usual incremental projection method (concerning this latter scheme, see Goda [9] for the original setting, Shen [21, 23, 25] for an analysis in the time semi-discrete case, Guermond and Quartapelle [10, 12, 11] for an analysis of the fully discrete case and Quarteroni *et al.*[19] for the analysis of a variant). The properties of the Timmermans *et al.* scheme were further investigated by Brown *et al.* through a normal mode analysis for a particular problem in [2]. Finally, energy norm estimates for the time-discrete case were obtained by Guermond and Shen [14], which gave to the scheme the name of "rotational pressure-correction projection method". Note also that a term similar to $r\mathrm{B}\tilde{\mathrm{U}}^{k+1}$, with the penalty parameter r once again replaced by the viscosity, was used by Prohl [18, chapter 8] for the pressure update in an algorithm which received the name of "Chorin-Uzawa scheme", since this equation is reminiscent of the pressure update step in the so-called Uzawa method (e.g. [7]).

Our goal here is to perform an analysis in energy norms of the penalty-projection algorithm (4). To this purpose, we will estimate the so-called splitting error, *i.e.* the difference between the results (in velocity and pressure) obtained by the penalty-projection method under consideration and the fully implicit scheme. Indeed, this quantity has been shown by Guermond [11] to be rather insensitive to the order of the time discretization of the unsteady term in the momentum balance equation, so we can hope that the present analysis for a first-order time discretization will also apply to second-order schemes.

This paper is organized as follows. We begin by setting the penalty-projection scheme within a variational framework suitable for an error analysis in energy norms (section 2). Then, after some preliminaries (section 3), the analysis of the penalty-projection method is addressed in section 4. Finally, some numerical tests are presented in section 5.

2 A variational framework

The aim of this section is to provide a variational framework for the two schemes under consideration, namely the implicit Euler method and the penalty-projection scheme (4).

In the first case, the variational setting is standard and reads, for each time step:

$$\text{Find } (\bar{u}^{k+1}, \bar{p}^{k+1}) \in V_h \times M_h \quad \text{such that, } \forall v \in V_h \text{ and } \forall q \in M_h :$$

$$\left| \begin{array}{ll} (i) & \dfrac{1}{\delta t}\,(\bar{u}^{k+1} - \bar{u}^k, v) + (\nabla \bar{u}^{k+1}, \nabla v) - (\nabla \cdot v, \bar{p}^{k+1}) = (f^{k+1}, v) \\ (ii) & -(\nabla \cdot \bar{u}^{k+1}, q) = 0 \end{array} \right. \tag{5}$$

where V_h and M_h are conforming approximations of $\mathrm{H}^1_0(\Omega)^d$ and $\mathrm{L}^2_0(\Omega)$ (*i.e.* the space of square integrable functions with a zero mean value) respectively and the notation $(\cdot,\cdot)$ indifferentlty stands for the $\mathrm{L}^2(\Omega)$ or $\mathrm{L}^2(\Omega)^d$ inner product.

To associate a discrete variational setting to the algebraic formulation of the penalty-projection scheme (4), we face three difficulties, namely to introduce the pressure Poisson problem, to deal with the pressure mass matrix lumping and, finally, to derive a variational analogue of the penalty term added in the velocity prediction step. The first difficulty has been solved by Guermond [10], and its solution consists in searching for the end-of-step velocity in a non $\mathrm{H}^1_0(\Omega)^d$-conforming space X_h which is spanned by the functions of V_h and the gradient of the functions of M_h (which is usually expressed by the notation $X_h = V_h + \nabla M_h$). The second step of (2) or (4) then reads, with obvious notations for the discrete functions:

$$\text{Find } (u^{k+1}, \varphi) \in X_h \times M_h \quad \text{such that :}$$

$$\left| \begin{array}{ll} \dfrac{1}{\delta t}\,(u^{k+1} - \tilde{u}^{k+1}, v) + (\nabla \varphi, v) = 0 & \qquad \forall v \in X_h \\ (u^{k+1}, \nabla q) = 0 & \qquad \forall q \in M_h \end{array} \right.$$

The divergence of the function u^{k+1} does not lie in $\mathrm{L}^2(\Omega)$, and we can no more write the divergence constraint under its standard form $(\nabla \cdot u^{k+1}, q) = 0,\ \forall q \in M_h$; in counterpart, the space M_h is now required to be included in $\mathrm{H}^1(\Omega)$, which gives sense to the substitute $(u^{k+1}, \nabla q) = 0,\ \forall q \in M_h$. The projection step then decomposes in two decoupled sub-problems: choosing $v = \nabla q,\ q \in M_h$ in the first equation and using the second one to eliminate the term $(u^{k+1}, \nabla q)$ yields the usual Poisson problem for the pressure update (first equation of step (*ii*) in (4)); then taking $v \in V_h$ in the first equation gives the variational equation which allows to compute the restriction to V_h (defined as the L^2-projection onto V_h) of the end-of-step velocity (second equation of step (*ii*) in (4)). Let us denote by Π_{V_h} the L^2 orthogonal projection onto V_h. Since we will see that u^{k+1} appears in the prediction step only through its L^2 inner-product with functions of V_h (see equations (8) and (9) below), only $\Pi_{V_h} u^{k+1}$ is important for the algorithm and computed in practice (as observed when stating the algorithm in the algebraic setting (4)). The present definition of u^{k+1} as a function of X_h may just be considered as a trick, however central to obtain convergence results. Note that, as a consequence, these latter must address the convergence of the fields computed in practice, *i.e.* the predicted velocity $\tilde{u}^{k+1}$ and the projection onto V_h of the end-of-step velocity $\Pi_{V_h} u^{k+1}$; remark 3 shows that this is indeed the case.

Let us now associate to the matrix M_p, assumed to be symmetric and positive definite, an approximate $L^2(\Omega)$ scalar product, denoted by $(\cdot,\cdot)_h$ and defined by the following relation:

$$(p,q)_h = (\mathrm{M}_p \mathrm{P}, \mathrm{Q}) \qquad \forall p,q \in M_h$$

where P and Q are the degrees of freedom vectors for the functions p and q, respectively. Let the operator B_h, acting from V_h to M_h, be defined by:

$$u \mapsto B_h u \quad \text{such that} \quad (B_h u, q)_h = (u, \nabla q) \quad \forall q \in M_h$$

We can see that, for any function u in V_h, the vector of degrees of freedom associated to $B_h u$, denoted by $\mathrm{B}_h \mathrm{U}$, reads:

$$\mathrm{B}_h \mathrm{U} = \mathrm{M}_p^{-1} \mathrm{B} \mathrm{U} \tag{6}$$

Consequently, observing that the penalty term in the first step of (4) satisfies the following property:

$$(\mathrm{B}^t \mathrm{M}_p^{-1} \mathrm{B} \mathrm{U}, \mathrm{V}) = (\mathrm{M}_p^{-1} \mathrm{B} \mathrm{U}, \mathrm{B} \mathrm{V}) = (\mathrm{M}_p^{-1} \mathrm{B} \mathrm{U}, \mathrm{M}_p \, [\mathrm{M}_p^{-1} \mathrm{B} \mathrm{V}]) = (\mathrm{M}_p \mathrm{B}_h \mathrm{U}, \mathrm{B}_h \mathrm{V})$$

we obtain that this term stems from the following variational counterpart:

$$c_h(u,v) = (B_h u, B_h v)_h \qquad \forall u,v \in V_h \tag{7}$$

As the matrix M_p is supposed to be symmetric and positive definite, the bilinear form $c_h(\cdot,\cdot)$ is symmetric and positive. Finally, writing the equation (4-(iii)) as:

$$\mathrm{M}_p \mathrm{P}^{k+1} = \mathrm{M}_p \, (\mathrm{P}^k + \Phi + r \mathrm{M}_p^{-1} \mathrm{B} \tilde{\mathrm{U}}^{k+1})$$

and using equation (6), we see that this step is equivalent to $p^{k+1} = p^k + \phi + r B_h \tilde{u}^{k+1}$. We then obtain the following variational algorithm for the penalty-projection scheme:

Find $(\tilde{u}^{k+1}, u^{k+1}, p^{k+1}) \in V_h \times X_h \times M_h$ such that, $\forall v \in V_h,\ \forall w \in X_h$ and $\forall q \in M_h$:

$$\left| \begin{array}{ll} (i) & \dfrac{1}{\delta t} \, (\tilde{u}^{k+1} - u^k, v) + (\nabla \tilde{u}^{k+1}, \nabla v) + r \ c_h(\tilde{u}^{k+1}, v) \\ & \hfill + (\nabla p^k, v) = (f^{k+1}, v) \\ (ii) & \dfrac{1}{\delta t} \, (u^{k+1} - \tilde{u}^{k+1}, w) + (\nabla (p^{k+1} - p^k - r B_h \tilde{u}^{k+1}), w) = 0 \\ (iii) & (u^{k+1}, \nabla q) = 0 \end{array} \right. \tag{8}$$

From the algebraic formulation of the scheme (4), it is clear that the penalty and pressure gradient terms can be recast as $r\mathrm{B}^t\mathrm{M}_p^{-1}\mathrm{B}\tilde{\mathrm{U}}^{k+1} + \mathrm{B}^t\mathrm{P}^k = \mathrm{B}^t\tilde{\mathrm{P}}^{k+1}$ with $\tilde{\mathrm{P}}^{k+1} = \mathrm{P}^k + r\mathrm{M}_p^{-1}\mathrm{B}\tilde{\mathrm{U}}^{k+1}$. We rephrase here this point in the variational setting. Let $\tilde{p}^{k+1}$ be defined by:

$$(\tilde{p}^{k+1}, q)_h = (p^k, q)_h + r(\tilde{u}^{k+1}, \nabla q) \qquad \forall q \in M_h$$

Using the definition of $c_h(\cdot,\cdot)$, the definition of B_h and this relation, we see that, $\forall v \in V_h$:

$$\begin{aligned} r\ c_h(\tilde{u}^{k+1}, v) = r\ (B_h\tilde{u}^{k+1}, B_h v)_h = r\ (\tilde{u}^{k+1}, \nabla B_h v) = \\ (\tilde{p}^{k+1} - p^k, B_h v)_h = (\nabla(\tilde{p}^{k+1} - p^k), v) \end{aligned}$$

Substituting in (8-(i)), we obtain the following equivalent variational formulation:

Find $(\tilde{u}^{k+1}, \tilde{p}^{k+1}, u^{k+1}, p^{k+1}) \in V_h \times M_h \times X_h \times M_h$ such that, $\forall v \in V_h$, $\forall w \in X_h$ and $\forall q \in M_h$:

$$\left| \begin{array}{ll} (i) & \dfrac{1}{\delta t}\ (\tilde{u}^{k+1} - u^k, v) + (\nabla\tilde{u}^{k+1}, \nabla v) + (\nabla\tilde{p}^{k+1}, v) = (f^{k+1}, v) \\ (ii) & (\tilde{p}^{k+1}, q)_h = (p^k, q)_h + r(\tilde{u}^{k+1}, \nabla q) \\ (iii) & \dfrac{1}{\delta t}\ (u^{k+1} - \tilde{u}^{k+1}, w) + (\nabla(p^{k+1} - \tilde{p}^{k+1}), w) = 0 \\ (iv) & (u^{k+1}, \nabla q) = 0 \end{array} \right. \tag{9}$$

Both formulations (8) and (9) will be used in the subsequent analysis.

3 Assumptions and preliminaries

We begin this section by collecting the assumptions relative to the discretization spaces. We suppose that V_h and M_h are conforming approximations in $\mathrm{H}^1_0(\Omega)^d$ and $\mathrm{H}^1(\Omega)$ respectively, satisfying the so-called Babuska-Brezzi or inf-sup condition (e.g. [8, 20]), and that the following approximation property holds for the space M_h:

$$\forall \bar{\varphi} \in \mathrm{H}^1(\Omega), \qquad \inf_{\varphi \in M_h} \|\bar{\varphi} - \varphi\|_0 \ \leq ch\, |\bar{\varphi}|_1 \tag{10}$$

where, here and throughout the remaining of the paper, unless explicitely stated, c stands for a positive real number independent of time or space variables or mesh steps. We assume in addition that the following inverse inequality holds for any function φ in M_h:

$$\|\nabla\varphi\|_0 \ \leq \frac{c}{h}\, \|\varphi\|_0 \tag{11}$$

Both preceding inequalities are valid, for instance, for the usual Lagrange piecewise linear elements and for families of quasi-uniform spatial discretizations [6].

As far as the continous problem is concerned, we suppose that the regularity of the computational domain is such that the Stokes problem is regularizing, in the sense that, if the right hand side lies in $\mathrm{L}^2(\Omega)^d$, the solution lies respectively in $\mathrm{H}^2(\Omega)^d$ for the velocity and in $\mathrm{H}^1(\Omega)$ for the pressure [4, 1].

In the course of this paper, we will make use of the inverse of the discrete Stokes operator, denoted by S_h and defined by:

$$
\begin{aligned}
S_h : \quad & V_h \to V_h \\
& u \mapsto S_h u \quad \text{such that:} \\
& \qquad \left| \begin{array}{ll} (\nabla S_h u, \nabla v) + (\nabla \varphi, v) = (u, v) & \forall v \in V_h \\ (S_h u, \nabla q) = 0 & \forall q \in M_h \end{array} \right.
\end{aligned}
\tag{12}
$$

The above stated assumptions for the discretisation (*i.e. inf-sup* stability, inequality (10) and inequality (11)) and for the regularity of the Stokes problem are used in [11, section 4.1] to prove the following properties of S_h:

Lemma 3.1 Prop. 1: The bilinear form defined over $V_h \times V_h$ by $(u, v) \mapsto (S_h u, v)$ is symmetric positive and defines a semi-norm in V_h which will be denoted by $(S_h u, u) = |u|_s^2$.

Prop. 2: Let us define the space H_h by:

$$H_h = \{v \in X_h \text{ such that } (v, \nabla q) = 0, \ \forall q \in M_h\}$$

Then, for any positive real number α and any function u in V_h, the following inequality holds :

$$(\nabla S_h u, \nabla u) \geq (1 - \alpha) \, \|u\|_0^2 \, - c(\alpha) \, \inf_{w \in H_h} \, \|u - w\|_0^2$$

To simplify the presentation, we will assume that the increments of the pressure obtained with the implicit scheme are such that, $\forall k \geq 0$:

$$
\begin{aligned}
&\forall k \geq 0, \qquad \|\delta \bar{p}^{k+1}\|_0 \equiv \|\bar{p}^{k+1} - \bar{p}^k\|_0 \leq c \, \delta t, \qquad \|\nabla \delta \bar{p}^{k+1}\|_0 \leq c \, \delta t, \\
&\forall k \geq 1, \qquad \|\delta\delta \bar{p}^{k+1}\|_0 \equiv \|\delta \bar{p}^{k+1} - \delta \bar{p}^k\|_0 \leq c \, \delta t^2
\end{aligned}
\tag{13}
$$

where c neither depends on the time nor on the time step. Such an assumption is already used in [10, section 6] and may be seen to follow directly from the time-regularity of the semi-discrete in space unsteady Stokes problem; in this case, the constant c may potentially depend on the mesh size h. Uniform in h estimates, at least for h small enough with respect to the time step, can be obtained through the convergence of the Euler scheme, if the continuous problem itself is regular, which indeed occurs since the initial time only if the forcing term f is regular and the initial condition u_0 satisfies some compatibility conditions, unfortunately of non-local type (for an in-depth discussion on the general (*i.e.* non-regular) case, see [18] and references herein, in particular [15]).

In addition, we will suppose that the initialisation of the scheme is such that:

$$u^0 = \bar{u}^0 \qquad \text{and } p^0 = \bar{p}^0 \tag{14}$$

The first of these equalities is natural: both the implicit and penalty-projection schemes can be initialized by the same interpolation of the initial condition for the velocity. As no initial pressure $\bar{p}^0$ is used in the implicit scheme, the second one just means that we are able to feed the penalty-projection algorithm with an initial pressure p^0 such that the above assumed bounds for the pressure increments (13) still holds at the first time step:

$$\|\bar{p}^1 - \bar{p}^0\|_0 \;=\; \|\bar{p}^1 - p^0\|_0 \;\leq c\;\delta t, \quad \|\nabla \delta \bar{p}^1\|_0 \;\leq c\;\delta t \quad \text{ and } \|\delta\delta\bar{p}^2\|_0 \;\leq c\;\delta t^2$$

In practice, this is possible when the initial pressure can be estimated (for instance, when the fluid is initially at rest with homogeneous boundary conditions and a forcing term vanishing at $t = 0$ or, as in the numerical experiments performed in this paper, when the solution is known); in the general case, for these inequalities to hold and to circumvent the problem of regularity of the solution at time $t = 0$ (so to allow the regularity assumptions (13) to be satisfied since the first step of the projection algorithm), one may begin the computation by some time steps of the implicit scheme.

Remark 1 (On less restrictive regularity assumptions) Inequalities (13) correspond to an L^∞-in-time control of the time increments of the pressure, which are the discrete equivalent of a L^∞-in-time control of its first and second time derivatives. In fact, a careful examination of the proofs of this paper shows that the following L^2-in-time control would be sufficient:

$$\delta t^2 \sum_{k=0}^{N-1} \delta t \left[\, \|\delta\bar{p}^{k+1}\|_0^2 \;+\; \|\nabla\delta\bar{p}^{k+1}\|_0^2 \,\right] + \sum_{k=1}^{N-1} \delta t\, \|\delta\delta\bar{p}^{k+1}\|_0^2 \;\leq c\;\delta t^4$$

Finally, we will need to assume in the following that the norm associated to the scalar product $(\cdot,\cdot)_h$ is equivalent on V_h to the standard $\mathrm{L}^2(\Omega)$ one:

$$\exists\, \gamma \geq 1 \text{ such that } \quad \frac{1}{\gamma}\;(v,v)_h \leq (v,v) \leq \gamma\;(v,v)_h \qquad \forall v \in V_h \tag{15}$$

This inequality holds in particular for the lumped mass matrix associated to the usual P_1 discretization. Consequently, we have the following result, which is a consequence of the inf-sup stability of the discretization:

LEMMA 3.2 There exists a positive constant c such that, for all $\psi \in M_h$, one can find $v_\psi \in V_h$ such that:

$$(\nabla \cdot v_\psi, q) = (\psi, q)_h \quad \forall q \in M_h, \qquad \|v_\psi\|_1 \;\leq c\,\|\psi\|_0$$

Remark 2 (Another assumption on $(\cdot,\cdot)_h$ in case of Dirichlet boundary conditions) The imbedding $M_h \subset \mathrm{L}^2_0(\Omega)$ never holds in practice for finite element approximation and the restriction of the pressure space to zero mean value functions is obtained through the properties of the algorithms used to solve the discrete problems. In other words, the algorithm employed at the algebraic level let at each step the pressure be an element of $L^2_0(\Omega)$; this property is known to hold, for instance, for the standard Uzawa algorithm, together with some of its variants which can be seen as Krylov methods applied to the pressure Schur complement problem.

In the present case, this property must also hold for the scalar product $(\cdot,\cdot)_h$ for the lemma 3.2 to be valid, in the following sense: the Riez representation in M_h of the linear form $(\psi,\cdot)_h$ must be an element of $L^2_0(\Omega)$ whenever ψ lies in $L^2_0(\Omega)$. This condition simply reads:

$$\forall \psi \in L^2_0(\Omega) \cap M_h, \quad (\psi, 1_h)_h = 0$$

where 1_h stands, in the preceding equation, for the constant function of M_h equal to 1 everywhere. One can easily check that this condition holds when $(\cdot,\cdot)_h$ is associated to the lumped mass matrix, and should be checked for any other choice.

Finally, throughout this paper, we will repeatedly make use of the discrete Gronwall lemma, a version of which reads [20, p. 14]:

LEMMA 3.3 Let $(h_k)_{k=0,\dots,n}$ and $(f_k)_{k=0,\dots,n}$ be two sequences of non-negative real numbers, g_0 a non-negative real number and $(\theta_k)_{k=1,\dots,n}$ a sequence of real numbers. We suppose that:

$$\left|\begin{array}{l} \theta_0 \leq g_0 \\ \theta_k \leq g_0 + \displaystyle\sum_{i=0}^{k-1} f_i + \sum_{i=0}^{k-1} h_i \theta_i \qquad\qquad \forall k = 1,\dots,n \end{array}\right.$$

Then the following bound holds:

$$\theta_k \leq (g_0 + \sum_{i=0}^{k-1} f_i)\ \exp(\sum_{i=0}^{k-1} h_i) \qquad\qquad \forall k = 1,\dots,n$$

4 Error analysis

This section addresses the analysis of the penalty-projection method. We are going to establish that this method inherits the convergence features of both the so-called rotational pressure-correction method analysed in [14] (*i.e.* convergence as δt^2 and $\delta t^{3/2}$ of respectively the velocity and pressure splitting errors) and the penalty method (*i.e.* convergence as $\delta t/r$ of the velocity splitting error); the first estimate is relevant for low values of r, and the second one for high values of r. After a first common part, these results are proven in two separate sub-sections.

4.1 A first set of estimates

Let the splitting errors e^{k+1}, $\tilde{e}^{k+1}$, ϵ^{k+1} stand for the difference between, respectively, the end-of-step velocity, the predicted velocity and the pressure obtained with the penalty-projection method (8) and the velocity and pressure obtained by the coupled algorithm (5):

$$e^{k+1} = u^{k+1} - \bar{u}^{k+1}, \qquad \tilde{e}^{k+1} = \tilde{u}^{k+1} - \bar{u}^{k+1} \qquad \text{and} \qquad \epsilon^{k+1} = p^{k+1} - \bar{p}^{k+1}$$

Remark 3 As $\bar{u}^{k+1}$ is a function of V_h, the L^2 projection of u^{k+1} onto V_h, $\Pi_{V_h} u^{k+1}$, satisfies $\Pi_{V_h} u^{k+1} - \bar{u}^{k+1} = \Pi_{V_h}[u^{k+1} - \bar{u}^{k+1}]$ and thus:

$$\|\Pi_{V_h} u^{k+1} - \bar{u}^{k+1}\|_0 \;=\; \|\Pi_{V_h}[u^{k+1} - \bar{u}^{k+1}]\|_0 \;\leq\; \|u^{k+1} - \bar{u}^{k+1}\|_0 \;=\; \|e^{k+1}\|_0$$

An estimate for $\|e^{k+1}\|_0$ thus also provides an estimate for the difference between the solution of the coupled scheme and the quantity actually computed by the projection algorithm, *i.e.* $\Pi_{V_h} u^{k+1}$.

The splitting errors are controlled by the following system of equations, valid for $0 \leq k \leq N-1$:

$$\left|\begin{array}{lll} (i) & \dfrac{1}{\delta t}\,(\tilde{e}^{k+1} - e^k, v) + (\nabla \tilde{e}^{k+1}, \nabla v) + r\; c_h(\tilde{e}^{k+1}, v) & \\ & \qquad\qquad\qquad\qquad +(\nabla \psi^k, v) = 0, & \forall v \in V_h \\ (ii) & \dfrac{1}{\delta t}\,(e^{k+1} - \tilde{e}^{k+1}, v) + (\nabla(\epsilon^{k+1} - \psi^k - r\; B_h \tilde{e}^{k+1}), v) = 0, & \forall v \in X_h \\ (iii) & (e^{k+1}, \nabla q) = 0, & \forall q \in M_h \end{array}\right. \tag{16}$$

where ψ^k is defined by $\psi^k = p^k - \bar{p}^{k+1} = \epsilon^k - \delta\bar{p}^{k+1}$. Note that the first equation is valid in particular because the bilinear form $c_h(\bar{u}^k, v)$ vanishes for any $k \in [1, N]$ and any v in V_h (see (7)), which is a consequence of the fact that we use an algebraic formulation of the penalty term. In the opposite case, an additional error, decreasing with the mesh size and growing with the penalty parameter, would appear.

By taking the difference of this system of equations at two consecutive time steps, we obtain the equations which control the splitting error increments, for $1 \leq k \leq N-1$:

$$\left|\begin{array}{lll} (i) & \dfrac{1}{\delta t}\,(\delta\tilde{e}^{k+1} - \delta e^k, v) + (\nabla \delta\tilde{e}^{k+1}, \nabla v) & \\ & \qquad\qquad +r\; c_h(\delta\tilde{e}^{k+1}, v) + (\nabla \delta\psi^k, v) = 0, & \forall v \in V_h \\ (ii) & \dfrac{1}{\delta t}\,(\delta e^{k+1} - \delta\tilde{e}^{k+1}, v) & \\ & \qquad +(\nabla(\delta\epsilon^{k+1} - \delta\psi^k - r\; B_h \delta\tilde{e}^{k+1}), v) = 0, & \forall v \in X_h \\ (iii) & (\delta e^{k+1}, \nabla q) = 0, & \forall q \in M_h \end{array}\right. \tag{17}$$

where $\delta\tilde{e}^{k+1} = \tilde{e}^{k+1} - \tilde{e}^k$, $\delta e^{k+1} = e^{k+1} - e^k$, $\delta\epsilon^{k+1} = \epsilon^{k+1} - \epsilon^k$ and $\delta\psi^k = \psi^k - \psi^{k-1}$.

Schematically speaking, the estimates for the pressure splitting error are to be derived from an estimate of the increment of the velocity error δe^k ; its seems that this fact was first evidenced in [23] in the time semi-discrete setting and to [10] for the fully discrete unsteady Stokes problem (see also [12] for the Navier-Stokes equations). For the penalty-projection scheme, we will need an additional result, namely the bound for the divergence of $\tilde{e}^k$, that is $\|B_h \tilde{e}^k\|_0$.

On the other hand, it appears from the analysis of second order schemes presented in [11] that the key to obtain second order estimates for the velocity is to bound the quantity $\|e^k - \tilde{e}^k\|_0$; we will see here that this estimate is also crucial to recover, for the penalty-projection scheme, the same bounds as the penalty method. Note that $\|e^k - \tilde{e}^k\|_0$ can be written equivalently $\|u^k - \tilde{u}^k\|_0$, so this quantity can be seen to measure the distance between $\tilde{u}^k$ and its projection on the divergence-free velocities space; thus it is not surprising that $\|e^k - \tilde{e}^k\|_0$ be closely linked to the splitting error.

The aim of the first step of the analysis presented in this section is to provide bounds for these three quantities of interest, namely $\|\delta e^k\|_0$, $\|B_h \tilde{e}^k\|_0$ and $\|e^k - \tilde{e}^k\|_0$. To this purpose, we follow basically the same lines as the theory presented in [14].

The first step is to prove the following result.

Lemma 4.1 The following bounds hold for $1 \leq n \leq N$:

$$\left[\sum_{k=1}^{n} \delta t \, \|\delta \tilde{e}^k - \delta e^k\|_0^2\right]^{1/2} \leq c \; \delta t^{5/2}$$

$$\|B_h \tilde{e}^n\|_0 \leq c \; \frac{\delta t^{3/2}}{r^{1/2}}$$

$$\|\nabla(\delta \epsilon^n - r \; B_h \tilde{e}^n)\|_0 \leq c \; \delta t$$

Proof First, we take $v = 2\delta t \, \delta \tilde{e}^{k+1}$ in (17-(i)) to obtain, using the definition of the bilinear form $c_h(\cdot,\cdot)$:

$$\begin{aligned} &\|\delta \tilde{e}^{k+1}\|_0^2 + \|\delta \tilde{e}^{k+1} - \delta e^k\|_0^2 - \|\delta e^k\|_0^2 + 2\delta t \, \|\nabla \delta \tilde{e}^{k+1}\|_0^2 \\ &\qquad + 2r\delta t \, \|B_h \delta \tilde{e}^{k+1}\|_0^2 + 2\delta t \, (\nabla \delta \psi^k, \delta \tilde{e}^{k+1}) = 0 \end{aligned} \tag{18}$$

Then, reordering the terms in (17-(ii)) yields:

$$([\frac{\delta e^{k+1}}{\delta t} + \nabla(\delta \epsilon^{k+1} - r \; B_h \tilde{e}^{k+1})] - [\frac{\delta \tilde{e}^{k+1}}{\delta t} + \nabla(\delta \psi^k - r \; B_h \tilde{e}^k)], v) = 0, \qquad \forall v \in X_h$$

Choosing:

$$v = \delta t^2 \; ([\frac{\delta e^{k+1}}{\delta t} + \nabla(\delta \epsilon^{k+1} - r \; B_h \tilde{e}^{k+1})] + [\frac{\delta \tilde{e}^{k+1}}{\delta t} + \nabla(\delta \psi^k - r \; B_h \tilde{e}^k)]) \in X_h$$

we get:

$$\begin{aligned} &\|\delta e^{k+1}\|_0^2 + \delta t^2 \, \|\nabla(\delta \epsilon^{k+1} - r \; B_h \tilde{e}^{k+1})\|_0^2 - \|\delta \tilde{e}^{k+1}\|_0^2 \\ &\qquad - \delta t^2 \, \|\nabla(\delta \psi^k - r \; B_h \tilde{e}^k)\|_0^2 - 2\delta t \, (\nabla(\delta \psi^k - r \; B_h \tilde{e}^k), \delta \tilde{e}^{k+1}) = 0 \end{aligned}$$

Developping the last term yields:

$$\begin{aligned}\|\delta e^{k+1}\|_0^2 &+ \delta t^2 \,\|\nabla(\delta\epsilon^{k+1} - r\; B_h\tilde{e}^{k+1})\|_0^2 - \|\delta\tilde{e}^{k+1}\|_0^2 \\ &-2\delta t\,(\nabla\delta\psi^k, \delta\tilde{e}^{k+1}) + 2r\delta t\,(\nabla B_h\tilde{e}^k, \delta\tilde{e}^{k+1}) = \delta t^2\,\|\nabla(\delta\psi^k - r\; B_h\tilde{e}^k)\|_0^2\end{aligned} \tag{19}$$

As $B_h\tilde{e}^k$ belongs to M_h, by the definition of the operator B_h, we have:

$$(\nabla B_h\tilde{e}^k, \delta\tilde{e}^{k+1}) = (B_h\tilde{e}^k, B_h\delta\tilde{e}^{k+1})_h$$

Developping, we get:

$$-2r\delta t\,(\nabla\cdot\delta\tilde{e}^{k+1}, B_h\tilde{e}^k) = r\delta t\left[\,\|B_h\tilde{e}^{k+1}\|_h^2 - \|B_h\delta\tilde{e}^{k+1}\|_h^2 - \|B_h\tilde{e}^k\|_h^2\,\right] \tag{20}$$

On the other hand, the right-hand side in (19) reads:

$$\begin{aligned}\delta t^2\,\|\nabla(\delta\psi^k - r\; B_h\tilde{e}^k)\|_0^2 &= \delta t^2\,\|\nabla(\delta\epsilon^k - r\; B_h\tilde{e}^k) - \nabla\delta\delta\bar{p}^{k+1}\|_0^2 \\ &\le \delta t^2(1+\delta t)\,\|\nabla(\delta\epsilon^k - r\; B_h\tilde{e}^k)\|_0^2 + \delta t^2(1+\frac{1}{\delta t})\,\|\nabla\delta\delta\bar{p}^{k+1}\|_0^2 \\ &\le \delta t^2(1+\delta t)\,\|\nabla(\delta\epsilon^k - r\; B_h\tilde{e}^k)\|_0^2 + c\;\delta t^5\end{aligned}$$

Combining these two latter estimates with the equations (18) and (19), we observe that the term $r\delta t\,\|B_h\delta\tilde{e}^{k+1}\|_h^2$ in (20) can be absorbed in the penalty term in (18) (see remark below), and we get for $1 \le k \le N-1$:

$$\begin{aligned}\|\delta e^{k+1}\|_0^2 + \|\delta\tilde{e}^{k+1} - \delta e^k\|_0^2 - \|\delta e^k\|_0^2 + 2\delta t\,\|\nabla\delta\tilde{e}^{k+1}\|_0^2 + r\delta t\,\|B_h\delta\tilde{e}^{k+1}\|_0^2 \\ +\delta t^2\,\|\nabla(\delta\epsilon^{k+1} - r\; B_h\tilde{e}^{k+1})\|_0^2 + 2r\delta t\,\|B_h\tilde{e}^{k+1}\|_0^2 - 2r\delta t\,\|B_h\tilde{e}^k\|_0^2 \\ \le \delta t^2(1+\delta t)\,\|\nabla(\delta\epsilon^k - r\; B_h\tilde{e}^k)\|_0^2 + c\;\delta t^5\end{aligned}$$

To apply the Gronwall lemma, we need an estimate for $\|\delta e^1\|_0^2$, $\|B_h\tilde{e}^k\|_0^2$ and $\|\nabla(\delta\epsilon^1 - r\; B_h\tilde{e}^1)\|_0^2$, *i.e.*, as $e^0 = 0$ and $\epsilon^0 = 0$, $\|e^1\|_0^2$, $\|B_h\tilde{e}^k\|_0^2$ and $\|\nabla(\epsilon^1 - r\; B_h\tilde{e}^1)\|_0^2$. Using once again $e^0 = 0$ and $\epsilon^0 = 0$, the system of equations controlling the splitting error at the first time step reads:

$$\left|\begin{array}{lll}(i) & \dfrac{1}{\delta t}\,(\tilde{e}^1, v) + (\nabla\tilde{e}^1, \nabla v) + r\; c_h(\tilde{e}^1, v) = (\nabla\delta\bar{p}^1, v), & \forall v \in V_h \\ (ii) & \dfrac{1}{\delta t}\,(e^1 - \tilde{e}^1, v) + (\nabla(\epsilon^1 - r\; B_h\tilde{e}^1), v) = (\nabla\delta\bar{p}^1, v), & \forall v \in X_h \\ (iii) & (e^1, \nabla q) = 0, & \forall q \in M_h\end{array}\right.$$

Since, by assumption, $\|\nabla\delta\bar{p}^1\|_0 \le c\;\delta t$, taking $v = \delta t\,\tilde{e}^1$ in the first relation yields using Young's inequality:

$$\frac{1}{2}\,\|\tilde{e}^1\|_0^2 + \delta t\,\|\nabla\tilde{e}^1\|_0^2 + r\delta t\,\|B_h\tilde{e}^1\|_h^2 \le c\;\delta t^4$$

Remarking $\|e^1\|_0^2 = \|\tilde{e}^1\|_0^2 - \|e^1 - \tilde{e}^1\|_0^2 \le \|\tilde{e}^1\|_0^2$, this relation gives the first two estimates. Then taking $v = \nabla(\epsilon^1 - r\; B_h\tilde{e}^1)$ in the second relation and using the third one, we obtain:

$$\|\nabla(\delta\epsilon^1 - r\; B_h\tilde{e}^1)\|_0^2 \le c\;\delta t^2$$

The desired estimates then follow by applying the discrete Gronwall lemma and remarking that, by orthogonality of the velocity correction with the discrete divergence-free fields, we have $\|\delta\tilde{e}^{k+1} - \delta e^k\|_0^2 = \|\delta\tilde{e}^{k+1} - \delta e^{k+1}\|_0^2 + \|\delta e^{k+1} - \delta e^k\|_0^2$ and thus $\|\delta\tilde{e}^{k+1} - \delta e^{k+1}\|_0^2 \leq \|\delta\tilde{e}^{k+1} - \delta e^k\|_0^2$. □

Remark 4 This proof is the analogue of the first step in the analysis of rotational pressure-correction methods [14, lemma 4.1]. By comparison, one may note that the penalization in the velocity prediction step yields two improvements. First, we do not need to absorb the term proportional to $\|B_h\delta\tilde{e}^{k+1}\|_h^2$ by the term proportional to $\|\nabla\delta\tilde{e}^{k+1}\|_0^2$ at the left-hand side; it means that the method is stable whatever the value of r may be and, thinking about Stokes equations with variable viscosity, whatever the value of the viscosity may be. Second, as expected, we obtain a better control of the divergence of $\tilde{e}^{k+1}$ and $\delta\tilde{e}^{k+1}$ (division by $r^{1/2}$).

Remark 5 We may note that this bound (and the rest of the analysis) shows that the method will work with a pressure update of the form $p^{k+1} = p^k + \rho B_h\tilde{e}^{k+1} + \phi$, provided that $\rho \leq 2r$. However, $\rho = r$ seems to allow an optimal control of both the divergence of $\delta\tilde{e}^{k+1}$ and $\tilde{e}^{k+1}$.

We are now in position to give two estimates for $\|e^k - \tilde{e}^k\|_0$.

LEMMA 4.2 The following bound hold for $1 \leq n \leq N$:

$$\|e^n - \tilde{e}^n\|_0 \leq c \ \min\,(\delta t^2, \frac{\delta t^{3/2}}{r^{1/2}})$$

Proof Let ϕ be the pressure increment in equation (16-(ii)), *i.e.*:

$$\phi = \epsilon^{k+1} - \psi^k - r\ B_h\tilde{e}^{k+1} = \delta\epsilon^{k+1} + \delta\bar{p}^{k+1} - r\ B_h\tilde{e}^{k+1}$$

We have from (16-(ii)), for $0 \leq k \leq N-1$:

$$\|\tilde{e}^{k+1} - e^{k+1}\|_0 \leq \delta t\,\|\nabla\phi\|_0$$

Thanks to lemma 4.1, we are going now to derive two different estimates for $\|\nabla\phi\|_0$. On one hand, we have:

$$\|\nabla\phi\|_0 \leq \|\nabla(\delta\epsilon^{k+1} - rB_h\tilde{e}^{k+1})\|_0 + \|\nabla\delta\bar{p}^{k+1}\|_0 \leq c\ \delta t \tag{21}$$

On the other hand, choosing $v = \nabla\phi$ in (16-(ii)) yields:

$$\|\nabla\phi\|_0^2 = \frac{1}{\delta t}\ (\tilde{e}^{k+1}, \nabla\phi) = \frac{1}{\delta t}\ (B_h\tilde{e}^{k+1}, \phi)_h$$

Thus, by a generalized Poincaré-Friedrichs inequality, since $\phi \in \mathrm{L}_0^2(\Omega)$:

$$\|\nabla\phi\|_0 \leq \frac{c}{\delta t}\ \|B_h\tilde{e}^{k+1}\|_h \leq c\ \frac{\delta t^{1/2}}{r^{1/2}} \tag{22}$$

□

The following lemma provides an estimate for $\|\delta e^k\|_0$ and concludes this section.

LEMMA 4.3 The following bound holds for $1 \leq n \leq N$:

$$\left[\sum_{k=1}^{n} \delta t \, \|\delta \tilde{e}^k\|_0^2\right]^{1/2} + \left[\sum_{k=1}^{n} \delta t \, \|\delta e^k\|_0^2\right]^{1/2} \leq c \; \delta t^{5/2}$$

Proof The starting point is the sum of equations (17-(i)) and (17-(ii)) respectively written at time k and $k-1$:

$$\frac{1}{\delta t} \, (\delta \tilde{e}^{k+1} - \delta \tilde{e}^k, v) + (\nabla \delta \tilde{e}^{k+1}, \nabla v) + (\nabla \xi, v) = 0 \quad \forall v \in V_h$$

where ξ is an element of M_h (note that, for all $v \in V_h$, $c_h(\delta \tilde{e}^{k+1}, v) = (\nabla B_h \delta \tilde{e}^{k+1}, v)$). Choosing $v = 2\, \delta t \, S_h \delta \tilde{e}^{k+1}$ then yields, by the first result of lemma 3.1:

$$|\delta \tilde{e}^{k+1}|_s^2 \, + \, |\delta \tilde{e}^{k+1} - \delta \tilde{e}^k|_s^2 \, - \, |\delta \tilde{e}^k|_s^2 \, + 2\delta t \, (\nabla \delta \tilde{e}^{k+1}, \nabla S_h \delta \tilde{e}^{k+1}) = 0$$

Making use of the second result of lemma 3.1 with $\alpha = 1/2$, we then get:

$$|\delta \tilde{e}^{k+1}|_s^2 \, + \, |\delta \tilde{e}^{k+1} - \delta \tilde{e}^k|_s^2 \, - \, |\delta \tilde{e}^k|_s^2 \, + \delta t \left[\, \|\delta \tilde{e}^{k+1}\|_0^2 \, - c \inf_{w \in H_h} \, \|\delta \tilde{e}^{k+1} - w\|_0^2 \right] \leq 0$$

Finally, choosing $w = \delta e^{k+1}$, we obtain:

$$|\delta \tilde{e}^{k+1}|_s^2 \, + \, |\delta \tilde{e}^{k+1} - \delta \tilde{e}^k|_s^2 \, - \, |\delta \tilde{e}^k|_s^2 \, + \delta t \, \|\delta \tilde{e}^{k+1}\|_0^2 \, \leq c \, \delta t \, \|\delta \tilde{e}^{k+1} - \delta e^{k+1}\|_0^2$$

For $1 \leq n \leq N$, summing up these equations from $k = 0$ to $k = n$ yields:

$$\sum_{k=0}^{n} \delta t \, \|\delta \tilde{e}^{k+1}\|_0^2 \, \leq c \, \sum_{k=0}^{n} \delta t \, \|\delta \tilde{e}^{k+1} - \delta e^{k+1}\|_0^2$$

The first part of the result (control of $\delta \tilde{e}^k$) then follows by lemma 4.1. To obtain the second part, its sufficient to remark that taking $v = 2\, \delta t \, \delta e^{k+1}$ in (17-(ii)) yields:

$$\|\delta e^{k+1}\|_0^2 \, = \, \|\delta \tilde{e}^{k+1}\|_0^2 \, - \, \|\delta e^{k+1} - \delta \tilde{e}^{k+1}\|_0^2 \, \leq \, \|\delta \tilde{e}^{k+1}\|_0^2$$

$\square$

4.2 Analysis for low values of the penalty parameter

The results of this section are gathered in the following theorem:

THEOREM 4.4 The following bounds hold for $1 \leq n \leq N$:

$$\left[\sum_{k=0}^{n} \delta t \, \|e^k\|_0^2\right]^{1/2} + \left[\sum_{k=0}^{n} \delta t \, \|\tilde{e}^k\|_0^2\right]^{1/2} \leq c \; \min\,(\delta t^2, \frac{\delta t^{3/2}}{r^{1/2}})$$

$$\left[\sum_{k=0}^{n} \delta t \, \|\nabla \tilde{e}^k\|_0^2\right]^{1/2} + \left[\sum_{k=0}^{n} \delta t \, \|\epsilon^k\|_0^2\right]^{1/2} \leq c \; \max\,(1, \frac{1}{r^{1/2}}) \; \delta t^{3/2}$$

Proof **First estimate -** Combining (16-(i)) and (16-(ii)) written at the previous time step, one gets:

$$\frac{1}{\delta t}\,(\tilde{e}^{k+1}-\tilde{e}^k,v)+(\nabla\tilde{e}^{k+1},\nabla v)+(\nabla\xi,v)=0\quad\forall v\in V_h$$

where ξ is an element of M_h. Choosing $v=2\,\delta t\,S_h\tilde{e}^{k+1}$ then yields, by the first result of lemma 3.1:

$$|\tilde{e}^{k+1}|_s^2\,+\,|\tilde{e}^{k+1}-\tilde{e}^k|_s^2\,-\,|\tilde{e}^k|_s^2\,+2\delta t\,(\nabla\tilde{e}^{k+1},\nabla S_h\tilde{e}^{k+1})=0$$

Making use of the second result of lemma 3.1 with $\alpha=1/2$ and choosing $w=e^{k+1}$, we then get:

$$|\tilde{e}^{k+1}|_s^2\,+\,|\tilde{e}^{k+1}-\tilde{e}^k|_s^2\,-\,|\tilde{e}^k|_s^2\,+\delta t\,\|\tilde{e}^{k+1}\|_0^2\,\leq c\,\delta t\,\|\tilde{e}^{k+1}-e^{k+1}\|_0^2$$

For $1\leq n\leq N$, summing up these equations from $k=0$ to $k=n$ yields:

$$\sum_{k=0}^{n}\delta t\,\|\tilde{e}^{k+1}\|_0^2\,\leq c\,\sum_{k=0}^{n}\delta t\,\|\tilde{e}^{k+1}-e^{k+1}\|_0^2 \tag{23}$$

Estimates of lemma 4.2 yield the second part of the first bound of the theorem, *i.e.* the control of $\tilde{e}^{k+1}$. The second one is obtained by remarking that, by choosing $v=2\,\delta t\,e^{k+1}$ in equation (16-(ii)), we get:

$$\|e^{k+1}\|_0^2\;=\;\|\tilde{e}^{k+1}\|_0^2\;-\;\|e^{k+1}-\tilde{e}^{k+1}\|_0^2\;\leq\;\|\tilde{e}^{k+1}\|_0^2$$

Second estimate - By lemma 4.1 and the hypothesis on the bilinear form $(\cdot,\cdot)_h$, the following bound holds, $\forall q\in M_h$:

$$|(\nabla\cdot\tilde{e}^{k+1},q)|=|(B_h\tilde{e}^{k+1},q)_h|\leq\;\|B_h\tilde{e}^{k+1}\|_h\;\|q\|_h\;\leq c\;\frac{\delta t^{3/2}}{r^{1/2}}\,\|q\|_h\;\leq c\;\frac{\delta t^{3/2}}{r^{1/2}}\,\|q\|_0$$

Summing equations (16-(i)) and (16-(ii)) and using the fact that, for all v in V_h, $c_h(\tilde{e}^{k+1},v)=(\nabla B_h\tilde{e}^{k+1},v)$, we then obtain from the preceding estimate that $\tilde{e}^{k+1}$ and ϵ^{k+1} obeys the following system:

$$\left\{\begin{array}{ll}(\nabla\tilde{e}^{k+1},\nabla v)+(\nabla\epsilon^{k+1},v)=-\dfrac{1}{\delta t}\,(e^{k+1}-e^k,v) & \forall v\in V_h\\ (\nabla\cdot\tilde{e}^{k+1},q)=(g,q) & \forall q\in M_h\end{array}\right.$$

where $\|g\|_0\;\leq c\;\dfrac{\delta t^{3/2}}{r^{1/2}}$. The results follow from lemma 4.3 by stability of the Stokes problem. □

Remark 6 The second estimate of theorem 4.4 blows up for $r=0$, which is clearly sub-optimal, as the penalty-projection method degenerates in this case into the classical incremental projection method, which is known to be first order convergent for the pressure. This is due to the fact that the techniques used in this proof, which are issued from the analysis of the rotational variant in [14], do not apply to the case $r = 0$. At this point, it is worth to note that a rotational version of the penalty-projection scheme can be defined by just adding $\mu B_h \tilde{u}^{k+1}$ to the pressure increment, where μ stands for the viscosity (here, $\mu = 1$). In other words, the rotational penalty-projection algorithm is obtained by replacing r by $r+\mu$ in (8-(ii)) and leaving (8-(i)) and (8-(iii)) unchanged. For this method, the present analysis yields the same estimates as in theorem 4.4, with r changed to $r+\mu$, and the bound does not blow up anymore for $r = 0$. This scheme has been tested numerically in [16]; roughly speaking, the results for the velocity are left unchanged while, for low values of r, the pressure approximation is significantly improved.

4.3 Analysis for high values of the penalty parameter

We prove in this section the following results.

THEOREM 4.5 For any positive value of the penalty parameter r, the following bounds hold for $1 \le n \le N$:

$$\|e^n\|_0 \,+\, \|\tilde{e}^n\|_0 \,+ \left[\sum_{k=0}^{n} \delta t\, \|\nabla \tilde{e}^k\|_0^2\right]^{1/2} \le c\, \frac{\delta t^{1/2}}{r}$$

$$\left[\sum_{k=0}^{n} \delta t\, \|e^k\|_0^2\right]^{1/2} + \left[\sum_{k=0}^{n} \delta t\, \|\tilde{e}^k\|_0^2\right]^{1/2} \le c\, \frac{\delta t}{r}$$

$$\left[\sum_{k=0}^{n} \delta t\, \|\epsilon^k\|_0^2\right]^{1/2} + \left[\sum_{k=0}^{n} \delta t\, \|\tilde{\epsilon}^k\|_0^2\right]^{1/2} \le c\, \frac{1}{r}$$

The starting point for this part of the analysis is now the system (9). By taking the difference with the variational formulation of the coupled system (5), we obtain a system of equations controlling the splitting errors:

$$\left|\begin{array}{lll}
(i) & \dfrac{1}{\delta t}\,(\tilde{e}^{k+1} - e^k, v) + (\nabla \tilde{e}^{k+1}, \nabla v) + (\nabla \tilde{\epsilon}^{k+1}, v) = 0, & \forall v \in V_h \\[2mm]
(ii) & -(\tilde{e}^{k+1}, \nabla q) + \dfrac{1}{r}\,(\tilde{\epsilon}^{k+1} - \epsilon^k, q)_h = -\dfrac{1}{r}\,(\delta \bar{p}^{k+1}, q)_h, & \forall q \in M_h \\[2mm]
(iii) & \dfrac{1}{\delta t}\,(e^{k+1} - \tilde{e}^{k+1}, v) + (\nabla(\epsilon^{k+1} - \tilde{\epsilon}^{k+1}), v) = 0, & \forall v \in X_h \\[2mm]
(iv) & (e^{k+1}, \nabla q) = 0, & \forall q \in M_h
\end{array}\right. \tag{24}$$

where $\tilde{\epsilon}^{k+1}$ stands for the difference between the intermediate pressure $\tilde{p}^{k+1}$ and the pressure given by the coupled algorithm: $\tilde{\epsilon}^{k+1} = \tilde{p}^{k+1} - \bar{p}^{k+1}$.

We begin by proving the following set of estimates.

LEMMA 4.6 For any positive value of the penalty parameter r, the following bounds hold for $1 \le n \le N$:

$$\|e^n\|_0 + \|\tilde{e}^n\|_0 + \left[\sum_{k=1}^{n} \delta t \, \|\nabla \tilde{e}^k\|_0^2\right]^{1/2} \le c \, \frac{\delta t^{1/2}}{r}$$

$$\left[\sum_{k=1}^{n} \delta t \, \|e^k - \tilde{e}^k\|_0^2\right]^{1/2} + \left[\sum_{k=1}^{n} \delta t \, \|\tilde{e}^k - e^{k-1}\|_0^2\right]^{1/2} \le c \, \frac{\delta t}{r}$$

Proof Choosing $v = 2\,\delta t\,\tilde{e}^{k+1}$ in (24-(i)), we get for $k = 0, \ldots, N-1$:

$$\|\tilde{e}^{k+1}\|_0^2 + \|\tilde{e}^{k+1} - e^k\|_0^2 - \|e^k\|_0^2 + 2\delta t \, \|\nabla \tilde{e}^{k+1}\|_0^2 + 2\delta t \, (\tilde{e}^{k+1}, \nabla \tilde{\epsilon}^{k+1}) = 0 \tag{25}$$

Taking $q = 2\,\delta t\,\tilde{\epsilon}^{k+1}$ in (24-(ii)) yields:

$$\begin{aligned} -2\delta t \, (\tilde{e}^{k+1}, \nabla \tilde{\epsilon}^{k+1}) + \frac{\delta t}{r} \left[\|\tilde{\epsilon}^{k+1}\|_h^2 + \|\tilde{\epsilon}^{k+1} - \epsilon^k\|_h^2 - \|\epsilon^k\|_h^2 \right] \\ = -\frac{2\delta t}{r} \, (\delta \bar{p}^{k+1}, \tilde{\epsilon}^{k+1})_h \end{aligned} \tag{26}$$

Then, taking $v = 2\,\delta t\,e^{k+1}$ in (24-(iii)), one obtains with (24-(iv)):

$$\|e^{k+1}\|_0^2 + \|e^{k+1} - \tilde{e}^{k+1}\|_0^2 - \|\tilde{e}^{k+1}\|_0^2 = 0 \tag{27}$$

Finally, let $\xi \in M_h$ be defined by:

$$(\nabla \xi, \nabla q) = (\epsilon^{k+1}, q)_h \qquad \forall q \in M_h$$

Choosing $v = 2\,\nabla \xi$ in (24-(iii)) yields:

$$\begin{aligned} &\frac{2}{\delta t} \, (e^{k+1} - \tilde{e}^{k+1}, \nabla \xi) + 2\,(\nabla(\epsilon^{k+1} - \tilde{\epsilon}^{k+1}), \nabla \xi) \\ &\qquad = \frac{2}{\delta t} \, (e^{k+1} - \tilde{e}^{k+1}, \nabla \xi) + 2\,(\epsilon^{k+1} - \tilde{\epsilon}^{k+1}, \epsilon^{k+1})_h = 0 \end{aligned}$$

and thus:

$$\frac{\delta t}{r} \left[\|\epsilon^{k+1}\|_h^2 + \|\epsilon^{k+1} - \tilde{\epsilon}^{k+1}\|_h^2 - \|\tilde{\epsilon}^{k+1}\|_h^2 \right] = -\frac{2}{r} \, (e^{k+1} - \tilde{e}^{k+1}, \nabla \xi)$$

By equation (16-(ii)), the right hand side of these latter equation reads:

$$-\frac{2}{r} \, (e^{k+1} - \tilde{e}^{k+1}, \nabla \xi) = \frac{2\delta t}{r} \, (\nabla \phi, \nabla \xi)$$

where ϕ is the pressure increment defined in the proof of lemma 4.2, which is known from inequalities (21) and (22) and the Poincaré-Friedrichs inequality to satisfy:

$$\|\phi\|_h \le c \; \delta t \quad , \qquad \|\phi\|_h \le c \, \frac{\delta t^{1/2}}{r^{1/2}}$$

By the definition of ξ, we then get:

$$-\frac{2}{r} \, (e^{k+1} - \tilde{e}^{k+1}, \nabla \xi) = \frac{2\delta t}{r} \, (\epsilon^{k+1}, \phi)_h$$

and, finally:

$$\frac{\delta t}{r}\left[\,\|\epsilon^{k+1}\|_h^2 \,+\, \|\epsilon^{k+1}-\tilde\epsilon^{k+1}\|_h^2 \,-\, \|\tilde\epsilon^{k+1}\|_h^2\,\right] = \frac{2\delta t}{r}\,(\epsilon^{k+1},\phi)_h \tag{28}$$

Summing up the four equations (25)-(28), we have:

$$\begin{aligned}
&\|e^{k+1}\|_0^2 \,+\, \|e^{k+1}-\tilde e^{k+1}\|_0^2 \,+\, \|\tilde e^{k+1}-e^k\|_0^2 \,-\, \|e^k\|_0^2 \,+2\delta t\,\|\nabla\tilde e^{k+1}\|_0^2\\
&\qquad +\frac{\delta t}{r}\left[\,\|\epsilon^{k+1}\|_h^2 \,+\, \|\epsilon^{k+1}-\tilde\epsilon^{k+1}\|_h^2 \,+\, \|\tilde\epsilon^{k+1}-\epsilon^k\|_h^2 \,-\, \|\epsilon^k\|_h^2\,\right]\\
&\qquad\qquad = -\frac{2\delta t}{r}\,(\delta\bar p^{k+1},\tilde\epsilon^{k+1})_h + \frac{2\delta t}{r}\,(\epsilon^{k+1},\phi)_h
\end{aligned}$$

We decompose the last term of this equation to obtain:

$$\begin{aligned}
&\|e^{k+1}\|_0^2 \,+\, \|e^{k+1}-\tilde e^{k+1}\|_0^2 \,+\, \|\tilde e^{k+1}-e^k\|_0^2 \,-\, \|e^k\|_0^2 \,+2\delta t\,\|\nabla\tilde e^{k+1}\|_0^2\\
&\qquad +\frac{\delta t}{r}\left[\,\|\epsilon^{k+1}\|_h^2 \,+\, \|\epsilon^{k+1}-\tilde\epsilon^{k+1}\|_h^2 \,+\, \|\tilde\epsilon^{k+1}-\epsilon^k\|_h^2 \,-\, \|\epsilon^k\|_h^2\,\right]\\
&\qquad\qquad = \underbrace{\frac{2\delta t}{r}\,(-\delta\bar p^{k+1}+\phi,\tilde\epsilon^{k+1})_h}_{T_1} + \underbrace{\frac{2\delta t}{r}\,(\epsilon^{k+1}-\tilde\epsilon^{k+1},\phi)_h}_{T_2}
\end{aligned} \tag{29}$$

The following of the proof makes use of an idea issued from the penalty methods analysis [24, section 5.2]. By lemma 3.2 and taking benefit from the fact that both $\|\delta\bar p^{k+1}\|_0$ and $\|\phi\|_h$ are known to be bounded by $c\,\delta t$, we choose $w \in V_h$ such that:

$$(\nabla\cdot w, q) = (-\delta\bar p^{k+1}+\phi, q)_h \quad \forall q \in M_h, \qquad \|w\|_1 \,\le c\,\delta t$$

By equation (24-(i)), we then have:

$$T_1 = \frac{2\delta t}{r}\,(\nabla\cdot w,\tilde\epsilon^{k+1}) = -\frac{2\delta t}{r}\,(w,\nabla\tilde\epsilon^{k+1}) = \frac{2\delta t}{r}\left[\frac{1}{\delta t}(\tilde e^{k+1}-e^k,w) + (\nabla\tilde e^{k+1},\nabla w)\right]$$

and thus:

$$\begin{aligned}
|T_1| \;&\le \frac12\,\|\tilde e^{k+1}-e^k\|_0^2 \,+\frac{2}{r^2}\,\|w\|_0^2 \,+\delta t\,\|\nabla\tilde e^{k+1}\|_0^2 \,+\frac{\delta t}{r^2}\,\|\nabla w\|_0^2\\
&\le \frac12\,\|\tilde e^{k+1}-e^k\|_0^2 \,+\delta t\,\|\nabla\tilde e^{k+1}\|_0^2 \,+c\,\frac{\delta t^2}{r^2}
\end{aligned}$$

In addition, using the fact that $\|\phi\|_h^2$ is known to be bounded by $c\,\delta t/r$:

$$|T_2| \le \frac{\delta t}{2r}\,\|\epsilon^{k+1}-\tilde\epsilon^{k+1}\|_h^2 \,+\frac{2\delta t}{r}\,\|\phi\|_h^2 \,\le \frac{\delta t}{2r}\,\|\epsilon^{k+1}-\tilde\epsilon^{k+1}\|_h^2 \,+c\,\frac{\delta t^2}{r^2}$$

Returning to equation (29) and absorbing terms, we then obtain:

$$\begin{aligned}
&\|e^{k+1}\|_0^2 \,+\, \|e^{k+1}-\tilde e^{k+1}\|_0^2 \,+\frac12\,\|\tilde e^{k+1}-e^k\|_0^2 \,-\, \|e^k\|_0^2 \,+\delta t\,\|\nabla\tilde e^{k+1}\|_0^2\\
&\quad +\frac{\delta t}{r}\left[\,\|\epsilon^{k+1}\|_h^2 \,+\frac12\,\|\epsilon^{k+1}-\tilde\epsilon^{k+1}\|_h^2 \,+\, \|\tilde\epsilon^{k+1}-\epsilon^k\|_h^2 \,-\, \|\epsilon^k\|_h^2\,\right] \le c\frac{\delta t^2}{r^2}
\end{aligned}$$

The proof of the lemma then follows by summing up this inequality written for $k = 1$ up to n and using the fact that both $e^0 = 0$ and $\epsilon^0 = 0$. $\square$

Then we can prove the following stronger estimate for the velocity.

LEMMA 4.7 For any positive value of the penalty parameter r, we have for $1 \leq n \leq N$:

$$\left[\sum_{k=0}^{n} \delta t \, \|e^k\|_0^2\right]^{1/2} + \left[\sum_{k=0}^{n} \delta t \, \|\tilde{e}^k\|_0^2\right]^{1/2} \leq c \, \frac{\delta t}{r}$$

Proof The proof follows basically the same ideas as for lemma 4.3 and the first estimate of theorem 4.4. We start from the sum of equation (24-(i)) and equation (24-(iii)) written at the previous time step, which gives for $1 \leq k \leq N-1$:

$$\frac{1}{\delta t} \, (\tilde{e}^{k+1} - \tilde{e}^k, v) + (\nabla \tilde{e}^{k+1}, \nabla v) + (\nabla \xi, v) = 0 \qquad \forall v \in V_h$$

where ξ is an element of M_h. In addition, by defining $\tilde{e}^0 = 0$, this equation is also valid for $k = 0$, as, because $e^0 = 0$, it is exactly the same relation that (24-(i)) taken at $k = 0$. Choosing $v = 2 \, \delta t \, S_h \tilde{e}^{k+1}$ in this equation yields, by the first result of lemma 3.1:

$$|\tilde{e}^{k+1}|_s^2 + |\tilde{e}^{k+1} - \tilde{e}^k|_s^2 - |\tilde{e}^k|_s^2 + 2\delta t \, (\nabla \tilde{e}^{k+1}, \nabla S_h \tilde{e}^{k+1}) = 0$$

Making use of the second result of lemma 3.1 with $\alpha = 1/2$ and choosing $w = e^{k+1}$ then yields:

$$|\tilde{e}^{k+1}|_s^2 + |\tilde{e}^{k+1} - \tilde{e}^k|_s^2 - |\tilde{e}^k|_s^2 + \delta t \, \|\tilde{e}^{k+1}\|_0^2 \leq c \, \delta t \, \|\tilde{e}^{k+1} - e^{k+1}\|_0^2$$

The result then follows by summing up this inequality from $k = 0$ to $k = n$, using lemma 4.6 and, finally, remarking that, for any k, $\|e^{k+1}\|_0^2 \leq \|\tilde{e}^{k+1}\|_0^2$. $\square$

We finish by giving a bound for the pressure splitting error.

LEMMA 4.8 For any positive value of the penalty parameter r, we have for $1 \leq n \leq N$:

$$\left[\sum_{k=0}^{n} \delta t \, \|\epsilon^k\|_0^2\right]^{1/2} + \left[\sum_{k=0}^{n} \delta t \, \|\tilde{\epsilon}^k\|_0^2\right]^{1/2} \leq c \, \frac{1}{r}$$

Proof From equation (24-(i)), we infer thanks to the *inf-sup* stability of the discretization, for $0 \leq k \leq N-1$:

$$\|\tilde{\epsilon}^k\|_0 \leq c \left[\frac{1}{\delta t} \, \|\tilde{e}^{k+1} - e^k\|_0 + \|\nabla \tilde{e}^{k+1}\|_0\right]$$

Squaring this relation and summing over the time steps thus yields, since $\tilde{\epsilon}^0 = 0$:

$$\sum_{k=0}^{n} \delta t \, \|\tilde{\epsilon}^k\|_0^2 \leq c \left[\frac{1}{\delta t^2} \sum_{k=1}^{n} \delta t \, \|\tilde{e}^k - e^{k-1}\|_0^2 + \sum_{k=1}^{n} \delta t \, \|\nabla \tilde{e}^k\|_0^2\right]$$

which, by lemma 4.6, yields the second part of the desired inequality. Still from the *inf-sup* stability of the discretization, we obtain from equation (24-(iii)), for $0 \leq k \leq N-1$:

$$\|\epsilon^{k+1} - \tilde{\epsilon}^{k+1}\|_0 \;\leq c\; \frac{1}{\delta t}\; \|e^{k+1} - \tilde{e}^{k+1}\|_0$$

which, squaring and summing over the time steps yields:

$$\sum_{k=0}^{n} \delta t\; \|\epsilon^k - \tilde{\epsilon}^k\|_0^2 \;\leq c\; \frac{1}{\delta t^2} \sum_{k=1}^{n} \delta t\; \|e^k - \tilde{e}^k\|_0^2$$

Invoking once again lemma 4.6 and the triangle inequality concludes the proof. □

Gathering the results of the three previous lemmas, the proof of theorem 4.5 is complete.

5 Numerical tests

The aim of this section is to check the validity of the theoretical analysis against a practical test case for which an analytic solution can be exhibited. This solution is built as follows. We choose a stream function and a geometrical domain such that homogeneous Dirichlet conditions holds:

$$\psi = \frac{1}{4\pi}\;\left[\,\sin(2\pi x)\sin(2\pi y)\,\right]^2 \exp(-t) \quad , \qquad \Omega =]0,1[\times]0,1[\quad , \qquad u = \begin{bmatrix} \dfrac{\partial \psi}{\partial y} \\ -\dfrac{\partial \psi}{\partial x} \end{bmatrix}$$

then we pick an arbitrary pressure in $\mathrm{L}^2_0(\Omega)$:

$$p = \sin(2\pi x)\sin(2\pi y)\exp(-t)$$

and the right-hand side f is computed in order that the equations of the Stokes problem (1) be satisfied for (u, p).

The spatial approximation is performed with the so-called Taylor-Hood ($i.e.$$\mathrm{P}_2$–$\mathrm{P}_1$) element, with a mesh obtained by cutting (along the diagonals) in four simplices each square of the 20×20 regular grid; note however that, since we consider the splitting error and not the difference between a numerical solution and the analytic one, results are almost unsensitive to the spatial discretization.

Figures 1, 2, 3 and 4 show the L^2 norm of the splitting error of respectively the velocity then the pressure obtained at $t = 1$, as a function of the time step and the penalty parameter. For the velocity splitting error, we observe as expected a second-order convergence with respect to the time step at $r = 0$, and, at large values of r, a convergence as $\delta t/r$. The pressure splitting error convergence is the same as the velocity one, $i.e.$ better than expected: δt^2 (instead of $\delta t^{3/2}$) at $r = 0$ and $\delta t/r$ for large values r. This "over-optimal" pressure convergence is frequently encountered, and, at least at $r = 0$, it is clear that this behaviour cannot be hoped in any case. Several explanations have been proposed to explain it. In particular, it

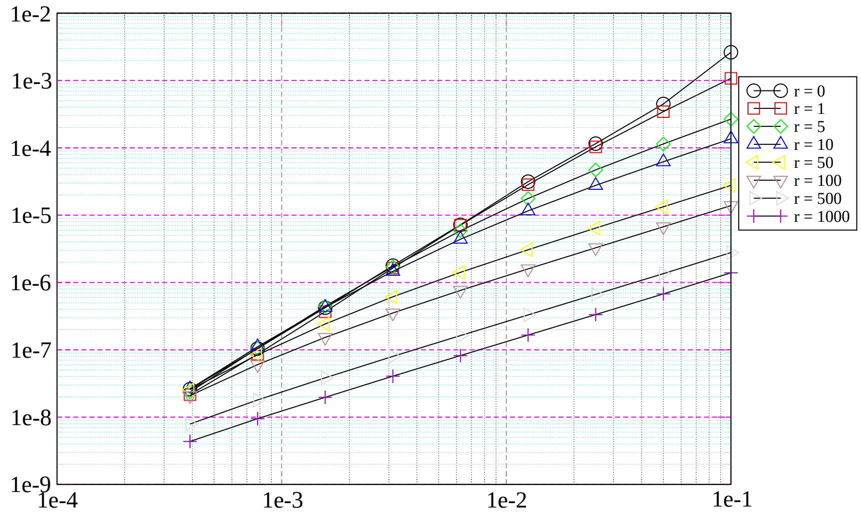

Figure 1: L^2 norm of $\tilde{e}$ at $t = 1$, as a function of the time step.

may be argued that, in finite dimension, all the norms are equivalent and a numerical experiment with a fixed meshing cannot discriminate between a convergence in L^2 norm and a faster convergence in a weaker norm; one will find an example of such a result in [11], for the pressure obtained with the standard incremental projection method (*i.e.* the method recovered here for $r = 0$). Note also that the pressure convergence rate has been observed to be strongly dependent on the regularity of the domain. A discussion on this topic, together with some reports of numerical experiments, can be found in [14, section 5.3] and references therein.

6 Conclusion

The general conclusion which can be drawn from this study is that, from the point of view of convergence properties, the proposed penalty-projection scheme builds a bridge between the rotational projection scheme [14] and the penalty scheme described in [24], which, to the best of our knowledge, belongs to the most advanced methods in their class, *i.e.* the projection and penalty methods, respectively. Indeed, for low value of the penalty parameter r, splitting error estimates of the so-called rotational projection scheme are recovered, *i.e.* second and $3/2$ order convergence for the velocity and the pressure, respectively; for high values of the penalty parameter, we obtain the $\delta t/r$ behaviour for the velocity splitting error known for the penalty scheme.

An extensive computational study of this method for Navier-Stokes equations, with Dirichlet and open boundary conditions, can be found in [16]; these results are coherent with the present ones. In addition, we observe in [16] that, for second-order in time discretizations, the splitting errors become dominant for usual projection schemes, at time steps affordable in practical applications; the cancelling of this

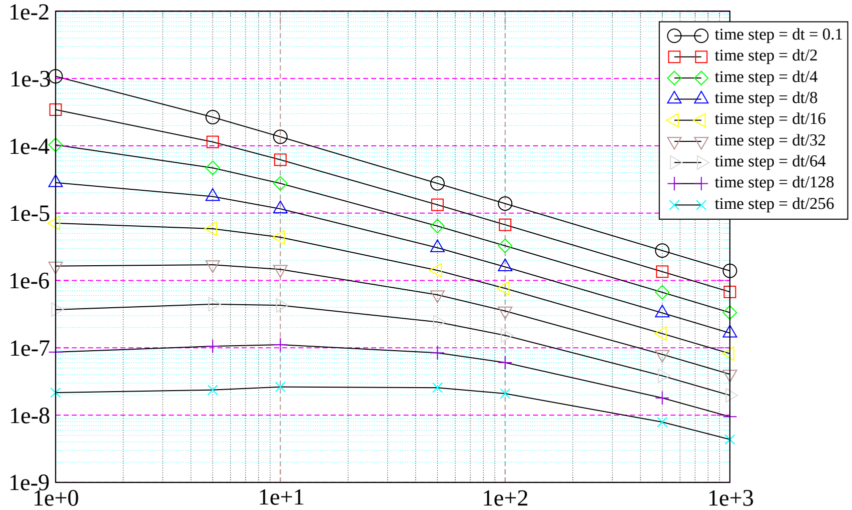

Figure 2: L^2 norm of $\tilde{e}$ at $t = 1$, as a function of the penalty parameter.

error obtained at high values of the penalty parameter r then leads to a drastic gain of accuracy. In addition, moderate values of r are generally sufficient to obtain a significant increase of precision, and any non-zero value of r suppresses the pressure boundary layers and avoids the loss of convergence observed for the standard incremental projection method (but not for the rotational projection scheme presented in [14]) in the case of open boundary conditions. Unfortunately, the introduction of the penalty term in the momentum balance equation has unpleasant effects which are not shared by other projection methods: first, it couples the block of equations related to each velocity component (which, however, are already coupled if the viscosity is not constant, since, in this case, the general form for the divergence of the stress tensor must be used); second, at large values for r, the algebraic system associated to the prediction step becomes severely ill-conditionned. To solve this latter problem, multi-grid algorithms are currently under study at IRSN. However, in today practical applications, it is thus particularly interesting to use a scheme in which the choice for the value of r is left open for the user: this is exactly what the penalty-projection scheme allows.

References

[1] C. Amrouche and V. Girault. On the existence and regularity of the solution of Stokes problem in arbitrary dimension. *Proc. Japan Acad., Série A*, 67:171–175, 1991.

[2] D.L. Brown, R. Cortez, and M.L. Minion. Accurate projection methods for the incompressible Navier-Stokes equations. *Journal of Computational Physics*, 168:464–499, 2001.

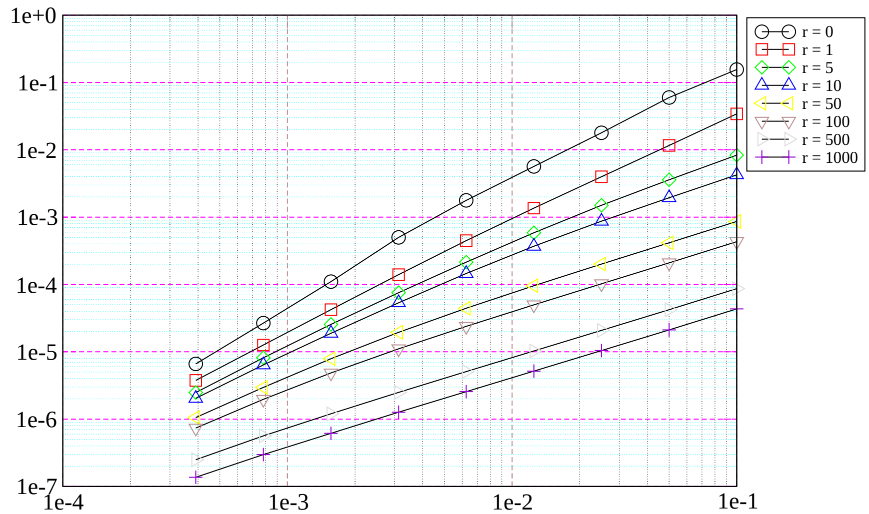

Figure 3: L^2 norm of ϵ at $t = 1$, as a function of the time step.

[3] J.-P. Caltagirone and J. Breil. Sur une méthode de projection vectorielle pour la résolution des équations de Navier-Stokes. *Comptes-Rendus de l'Académie des Sciences, Paris – Série II*, 327:1179–1184, 1999.

[4] L. Cattabriga. Su un problema al contorno relativo al sistemo di equazioni di Stokes. *Rend. Sem. Mat. Univ. Padova*, 31:308–340, 1961.

[5] A.J. Chorin. Numerical solution of the Navier-Stokes equations. *Mathematics of Computation*, 22:745–762, 1968.

[6] P.G. Ciarlet. Handbook of numerical analysis volume II : Finite elements methods – Basic error estimates for elliptic problems. In P. Ciarlet and J.L. Lions, editors, *Handbook of Numerical Analysis, Volume II*, pages 17–351. North Holland, 1991.

[7] M. Fortin and R. Glowinski. *Méthodes de Lagrangien Augmenté*. Dunod, Paris, 1982.

[8] V. Girault and P.-A. Raviart. *Finite Element Methods for Navier-Stokes Equations. Theory and Algorithms*, volume 5 of *Springer Series in Computational Mathematics*. Springer-Verlag, 1986.

[9] K. Goda. A multistep technique with implicit difference schemes for calculating two- or three-dimensional cavity flows. *Journal of Computational Physics*, 30:76–95, 1979.

[10] J.-L. Guermond. Some implementations of projection methods for Navier-Stokes equations. *Mathematical Modelling and Numerical Analysis*, 30(5):637–667, 1996.

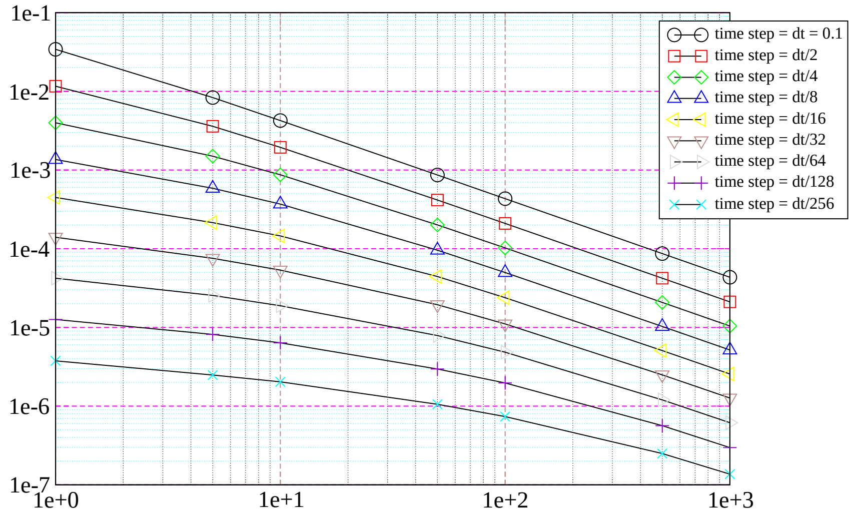

Figure 4: L^2 norm of ϵ at $t = 1$, as a function of the penalty parameter.

[11] J.-L. Guermond. Un résultat de convergence d'ordre deux en temps pour l'approximation des équations de Navier-Stokes par une technique de projection incrémentale. *Mathematical Modelling and Numerical Analysis*, 33(1):169–189, 1999.

[12] J.-L. Guermond and L. Quartapelle. On the approximation of the unsteady Navier-Stokes equations by finite element projection methods. *Numerische Mathematik*, 80:207–238, 1998.

[13] J.L. Guermond, P. Minev, and J. Shen. Error analysis of pressure-correction schemes for the time-dependent Stokes equations with open boundary conditions. *SIAM journal on Numerical Analysis*, 43(1):239–258, 2005.

[14] J.L. Guermond and J. Shen. On the error estimates for the rotational pressure-correction projection methods. *Mathematics of Computation*, 73(248):1719–1737, 2004.

[15] J.G. Heywood and R. Rannacher. Finite element approximation of the nonstationary Navier-Stokes problem. I. Regularity of solutions and second-order error estimates for spatial discretization. *SIAM Journal on Numerical Analysis*, 19(2):275–311, 1982.

[16] M. Jobelin, C. Lapuerta, J.-C. Latché, Ph. Angot, and B. Piar. A finite element penalty-projection method for incompressible flows. *Journal of Computational Physics*, 217:502–518, 2006.

[17] M. Jobelin, B. Piar, Ph. Angot, and J.-C. Latché. Une méthode de pénalité-projection pour les écoulements dilatables. *to appear in the Revue Européenne de Mécanique Numérique*, 2008.

[18] A. Prohl. *Projection and Quasi-Compressibility Methods for Solving the Incompressible Navier-Stokes Equations*. Teubner, 1997.

[19] A. Quarteroni, F. Saleri, and A. Veneziani. Analysis of the Yosida method for the incompressible Navier-Stokes equations. *Journal de Mathématiques Pures et Appliquées*, 78:473–503, 1999.

[20] A. Quarteroni and A. Valli. *Numerical Approximation of Partial Differential Equations*. Springer, 1994.

[21] J. Shen. On error estimates of projection methods for Navier-Stokes equations: First-order schemes. *SIAM Journal on Numerical Analysis*, 29(1):57–77, 1992.

[22] J. Shen. On error estimates of some higher order projection and penalty-projection methods for Navier-Stokes equations. *Numerische Mathematik*, 62:49–73, 1992.

[23] J. Shen. Remarks on the pressure estimates for the projection methods. *Numerische Mathematik*, 67:513–520, 1994.

[24] J. Shen. On error estimates of the penalty method for unsteady Navier-Stokes equations. *SIAM Journal on Numerical Analysis*, 32(2):386–403, 1995.

[25] J. Shen. On error estimates of projection methods for Navier-Stokes equations: Second-order schemes. *Mathematics of Computation*, 65(215):1039–1065, 1996.

[26] R. Temam. Une méthode d'approximation de la solution des équations de Navier-Stokes. *Bull. Soc. Math. France*, 98:115–152, 1968.

[27] R. Temam. Sur l'approximation de la solution des équations de Navier-Stokes par la méthode des pas fractionnaires II. *Arch. Rat. Mech. Anal.*, 33:377–385, 1969.

[28] L.J.P. Timmermans, P.D. Minev, and F.N. Van de Vosse. An approximate projection scheme for incompressible flow using spectral elements. *International Journal for Numerical Methods in Fluids*, 22:673–688, 1996.