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SHAPE AND MATERIAL DESIGN IN PHYSICAL MODELING SOUND SYNTHESIS

Pirouz Djoharian

A.C.R.O.E.
46, avenue Félix Viallet 38000 Grenoble, France
Pirouz.Djoharian@imag.fr

ABSTRACT

This paper deals with the design of physical models by shape and material parameters. To this end, it is shown that acoustic invariants of shape and material are to be found in the reduced frequency and in the frequency-damping relationship respectively. Shape modeling is done within the lumped mass-spring models paradigm. Material parameters are introduced as sound signatures, that are characteristic frequency-damping relationships derived from the theory of linear viscoelasticity. Any conservative model can then be covered with a viscoelastic dress in order to represent an arbitrary shaped material.

1. INTRODUCTION

Among various sound synthesis methods, Physical Modeling has distinguished itself by its ability to produce lifelike realistic sounds with outstanding rich transients. The basic idea is the digital simulation of the prominent sound generation processes existing in various acoustic musical instruments. Thus, Physical Modeling sound synthesis is often understood as a tool for reproducing realistic sounds of existing instruments. However, in an artistic context, the physical model can be seen as an open and challenging framework, allowing musicians to explore their own imagination. The present work attempts to bring to the fore the potential of Physical Modeling to generate abstract but significant synthesis models.

The sound produced by a sound body is affected by many factors including its size and shape, its material, as well as external factors such as the way it is played and the environment. This work is related to the shape and material quality of sound bodies and physical models. Instrument makers have always paid a great attention to the design of instruments’ shape and the choice of materials. Even though, the control of the sound quality of a resonator by means of shape and material features is not an easy task, we want to introduce these parameters into the sound synthesis process. Recent works in Psychoacoustics tend to prove that the auditory system encodes audio information to recognize in some extent shape and material features (Lakatos, 1997). However, it is not clear in which extent sound perceptions of shape and material are independent. Anyhow, in the virtual world, materials may take arbitrary shapes. This will open the way for unusual combination of shape and material, such as strings made of concrete or plates made of human bone, etc.

Shape modeling is based on approximation techniques such as finite difference or finite element methods. In this way, a differential strain operator defined on a continuous region is transformed into a difference operator defined over a discrete grid. The resonator’s shape is then represented by a mass-spring lumped conservative system. Modeling material is based on the theory of linear viscoelasticity. It is shown that in a homogeneous isotropic viscoelastic material, modal frequencies and damping constants satisfy a particular characteristic equation, which is independent of the shape and the boundary conditions. This frequency-damping relationship is what we refer to the sound signature of the material. At the final step, the physical model ends up as a mass-spring-dashpot network. Thus, the modeling techniques presented hereafter, translate shape and material features in terms of mass, spring and dashpot parameters.

The paper is organized as follows. In section 2, an introduction to mechanical properties of materials is briefly reviewed. In section 3, it is shown how shape and material parameters affect vibration properties of mechanical systems. In section 4, a general framework for Physical Modeling, including shape and material design as well as synthesis techniques is presented.

2. MECHANICAL PROPERTIES OF MATERIALS

An acoustical instrument may be divided in two parts: the exciter, e.g. plectrum, hammer, bow, etc., and the resonator, e.g. string, membrane, etc. Even though the exciter is in turn an actual vibrating structure, shape and material features have their greatest importance in the resonators. In most instruments, resonator’s vibrations are of small amplitude. Therefore, the linearity hypothesis is assumed throughout this work.

2.1 Conservative models

Though each material has a particular density and characteristic elastic moduli, free oscillations of conservative models do not show any invariant feature of the material. In fact, inertia and elasticity combine into a single coefficient , which is the speed of sound in the material. However, the shape of a resonator determines its reduced spectrum, i.e. the frequency spectrum up to a scaling factor.
2.1.1 Inertia and elasticity

In homogeneous materials, inertia is defined by a single constant $\mu$, the density of the material. Depending on the dimensionality of the resonator, $\mu$ denotes mass per unit length, unit area or unit volume respectively. Elasticity is the ability to return to an original shape and size when the forces causing the deformation are removed. Deformations may be of various types: elongation, bending, shearing, etc. Elastic property of a material is then expressed in form of a constitutive equation, i.e. an equation that is independent of the geometry of the body and depends only on its material nature. Therefore, a pair of intensive and extensive dependent variables, stress and strain, are introduced. Stress, $\sigma = f/A$, is force per unit area and strain, $\varepsilon = \text{dil}l$, is the fractional and dimensionless change of size (length, volume, angle, etc.). The simplest model of an elastic behavior is the Hooke’s law, which states that the stress is proportional to the strain: $\sigma = k\varepsilon$. The constant $k$ is so defined as the elastic modulus of the material. The typical situation is the case of uniaxial load and the resultant extension of a rod. The ratio $E = \sigma/l = \mu$ is referred to the Young’s modulus of elasticity. Depending on the deformation type, various other elastic moduli are defined: shear, bulk, etc. (Tschoegl, 1989).

For a general anisotropic material, the stress-strain relationship is expressed with a larger number of elastic moduli, by means of the tensor algebra. However, the generalized Hooke’s law remains a linear relation between the strain and the stress tensors. All real materials deviate from the linear Hooke’s law in various ways, e.g. non-linearity and time dependence of elastic moduli. Time or frequency dependence of the elastic moduli is known as the viscoelastic properties. Linearity is assumed throughout this work, thus, linear viscoelasticity will be considered as the fundamental behavior of solids.

2.1.2 Reduced spectrum

Let us consider first the simple case of an ideal flexible string of length $l$. Normal mode frequencies $\omega_n$ of transverse vibration of the string are multiples of the fundamental frequency $\omega_0 = c\pi/l$, where $l$ is the length and $c$ the velocity of wave propagation along the string. Therefore, two strings having different lengths $l_1$ and $l_2$ produce the same frequencies provided that $c_1/l_1 = c_2/l_2$. This may be realized for example by tuning the strings, i.e. by stretching them by appropriate tensions $T_1$ and $T_2$ (Fletcher, 1991: 36). This example shows that the frequency spectrum is not an invariant of the material, nor the size. However, all ideal strings have the same reduced spectrum $\omega_n/\omega_0 = n$. The reduced spectrum is indeed invariant under material and size change.

The above example of a string may be generalized to any conservative resonator. The equilibrium equation governing vibrations of a resonator have the following general form:

$$\frac{\partial^2 u}{\partial t^2} + \frac{k}{\mu} Lu = 0$$  \hspace{1cm} (2.1)

where $L$ is a differential operator in space variables, expressing the local strain, $k$ an elastic modulus and $\mu$ the density of the material. For transverse vibrations of flexible strings and membranes, $L = \Delta$, the Laplacian operator. For beams and plates involving bending waves, $L = \Delta^2$, the bi-harmonic operator. The shape and size of the resonator as well as the boundary conditions determine the eigenvalues $\lambda_n$ of $L$ (Courant, 1937: 275). The frequency spectrum is then obtained by

$$\omega_n^2 = \lambda_n c^2$$  \hspace{1cm} (2.2)

where $c^2 = k/\mu$. Therefore, in the conservative case, the material quality appears through a single scaling factor $c$. Resonators made of different materials and homothetical shapes, have then the same reduced spectrum. In other words, the conservative model reflects much more the geometry of the resonator.

2.2 Dissipative models

Loss of the mechanical energy results from numerous external as well as internal factors. External damping consists in transferring energy to other mechanical systems. For instance, in stringed instruments, part of the string vibrational energy is transferred to the soundboard via the bridge. In the same way, a part of the bridge’s energy is in turn supplied to the surrounding air by acoustic radiation. We consider here only internal factors linked to the very nature of the material.

2.2.1 Internal Damping

When a body of material is subjected to a deformation, its microscopic structure experiences local activities. As a rule, the more the microstructure is ordered, the less the deformation produces local activities. These microscopic rearrangements necessarily require a finite time, during which the material properties vary. Thus, constitutive equations must involve the time variable. The linear theory of viscoelasticity considers a material as a causal time invariant linear system, with $\sigma$ and $\varepsilon$ as input-output variables (Tschoegl, 1989). Aging phenomenon is then neglected and temperature assumed to be constant. Considering the strain $\varepsilon$ as the input and $\sigma$ as the output, a constitutive equation has the general form of a convolution

$$\sigma = k_\delta * \varepsilon$$  \hspace{1cm} (2.3)

where $k_\delta$ stands for the impulse response of the material sample. Expressed in the Laplace transform plane, the convolution operation reduces to multiplication by the relaxation $k(s)$, i.e. the Laplace transform of $k(t)$. The step response $k_\delta(t)$, known as relaxation, is the gradual decrease of the stress when the material is held at a constant strain. The harmonic response is expressed by the complex modulus $k^*(\omega) = k(\omega) = k(\omega) + ik^{**}(\omega)$, which is the ratio $\sigma/\varepsilon$ when the material sample experiences harmonic oscillations at frequency $\omega$. The maximum potential energy stored and the amount of energy dissipated in each cycle are respectively proportional to
the storage and loss moduli \( k'(\omega) \) and \( k''(\omega) \) (Lakes, 1999).

The storage modulus \( k' \) is an increasing function of \( \omega \): 
\[
k'(0) = k_0 < k'(\omega) = k_0(\omega) .
\]
In fact, at high rate strain, fewer relaxation processes find enough time to be completed. A relaxation phenomenon requires time but also kinetic energy. Thus, each relaxation process has its best efficiency at an optimal strain rate \( \dot{\epsilon} \). Hence, the plot of \( k'(\omega) \) may exhibit several peaks at various frequencies. Roughly speaking, the frequency axis can be divided into three regions:

1. the rubbery region, where \( k' \) and \( k'' \) have low values and slow variations
2. the transition region, where \( k' \) increases fast and \( k'' \) has one or several peaks.
3. the glassy region, where \( k' \) attains a high stationary value \( k' = k'(\infty) \), and \( k'' \) takes again low values.

The upper limit of the storage modulus is known as the glassy or instantaneous modulus, whereas the lowest value is referred to the equilibrium modulus: \( k_0 = k'(0) \). The ratio \( \chi = k/k_0 \) is a measure of the overall viscoelastic strength. Knowledge of viscoelastic properties of materials is based on measurements. The most easily measurable viscoelastic functions are the step and the harmonic responses.

2.2.2 Lumped viscoelastic models

By combining pure springs and pure dashpots in series-parallel assemblies, one obtains a wide variety of viscoelastic behaviors. Relaxance of such model is a rational fraction \( k(s) = P(s)/Q(s) \) and equation (2.3) may be replaced by a constant coefficient differential equation. The simplest combinations involve a pair of spring and dashpot with stiffness and viscosity constants \( k \) and \( z \). Series and parallel mountings are respectively known as the Maxwell and the Kelvin models (Fig. 2.1).

The relaxation function of a Maxwell unit is a decaying exponential \( k(t) = \exp(-zt) \) where \( t = z/k \) (resp. \( \tau = z/k \)) is the relaxation time (resp. frequency). By adding a pure spring \( k_0 \) in parallel, one obtains the Zener model, which is the simplest model of linear viscoelastic solid. By assembling \( n \) Maxwell units with spring and dashpot constants \( k_i \) and \( z_i \), one obtains an \( n \)-order Wiechert model (Tschoegl, 1989). The relaxation and the relaxation function of a Wiechert model are given below:

\[
k(s) = k_0 + \sum_{i=1}^{n} \frac{k_i z_i}{s + k_i + z_i} = \sum_{i=1}^{n} \frac{k_i z_i}{s + k_i + z_i}
\]

\[
k(s) = k_0 + \sum_{i=1}^{n} k_i e^{-\frac{z_i s}{k_i}}
\]

where \( \zeta = k/z \). If the relaxation frequencies \( \zeta_i \) are spread enough, the loss modulus \( k''(\omega) \) has \( n \) peaks located at \( \zeta_i \).

Every lumped parameter model may be represented as a Wiechert model. The decomposition is obtained by partial fraction expansion of the relaxance \( k(s) = P(s)/Q(s) \). In real materials, many relaxation processes are responsible for viscoelastic behavior (Lakes, 1999: 289). Each relaxation process corresponds to a particular peak of the loss modulus \( k''(\omega) \). Wiechert models with \( n = 4 \) to 10 elements can represent linear viscoelastic properties of solids with a good approximation. Relaxation frequencies \( \zeta_i \) and related weighting coefficients \( k_i \) may be calculated by model fitting methods so that \( \sum_{i} k_i \exp(-\zeta_i s) \) approaches a measured relaxation function (Tschoegl, 1989).

![Figure 2.1 Lumped viscoelastic models: Kelvin (a), Maxwell (b), Zener (c) and a 2-order Wiechert mounting (d).](image)

We define the dimensionless moduli (denoted by underlined characters), by normalizing with respect to \( k_e \). For example, the normalized relaxance \( \tilde{k}(s) \), is defined by

\[
\tilde{k}(s) = \frac{k(s)}{k_e} \quad \text{(2.6)}
\]

2.2.3 Continuous relaxation spectrum

Wiechert models having a large number of Maxwell units are generalized in models having an infinite number of relaxation peaks. The distribution of relaxation frequencies is given by a weighting function \( H(\zeta) \), known as the relaxation frequency spectrum. The relaxance of an infinite order Wiechert model is then a generalization of (2.4) to an infinite sum

\[
k(s) = k_0 - \int_{0}^{\infty} \frac{H(\zeta)}{s + \zeta} d\zeta
\]

Some particular spectra allow analytical expression of the response functions. The simplest example is the wedge spectrum model

\[
H(\zeta) = \begin{cases} 
\frac{k_0}{\zeta} & \text{if } \zeta_1 < \zeta < \zeta_2 \\
0 & \text{otherwise}
\end{cases}
\]

where \( \zeta_0, k_0 \) and \( \theta \) are parameters. As a particular case, for \( \theta = 0 \), one obtains the box spectrum. In this case, the loss modulus has a peak at \( \zeta_{\text{eq}} \). Wiechert models in comparison to the wedge spectrum models for \( \theta = 0 \) (box) and \( \theta = 0.5 \). All of the four viscoelastic models considered have the same overall viscoelastic strength \( \chi = 2 \). Note that two or several models may be used in conjunction to model the entire relaxation spectrum. Each model contributes then in an additive manner to the various response functions.
an overdamped (resp. underdamped) solution. It can be shown that this property may be extended to box and wedge spectrum oscillators as well (Djoharian, 2000). The case of a continuous spectrum resulting from superposition of several boxes and/or wedges is similar to a Wiechert oscillator having several discrete peaks. For reasons that become apparent hereafter, we are particularly interested in the relation between $\alpha$ and $\omega_0$. Some general features of the $\alpha(\omega_0)$ function are outlined below:

- The damping constant $\alpha$ has an upper bound $\alpha_{\max}$.
- In the rubbery region, $\alpha^2 \approx \omega_0^2/\zeta_i$ and $\alpha \propto \omega_0^2$.
- In the glassy region, $\omega \approx \omega_0$ and $\alpha \propto \omega_0$.

To be specific, for a Wiechert oscillator having $n$ relaxation peaks sorted as $\zeta_1 < \ldots < \zeta_n$, if $\omega_0 \ll \zeta_i$, then $\alpha^2 \approx \omega_0^2 = k/m$ and

$$\alpha = \frac{k}{\omega_0^2} \frac{\alpha_0^2}{\zeta_i} \quad (3.3)$$

If the relaxation peaks $\zeta_i$ are spread enough, it can be shown that, in the transition region, $\alpha$ grows by stages, namely plateau regions delimited by $\zeta_i$s. Beyond the highest peak $\zeta_n$, $\omega = \omega_0$ and $\alpha$ approaches its upper bound

$$\alpha_{\max} = \frac{k}{\omega_0^2} \frac{\alpha_0^2}{\zeta_n} \quad (3.4)$$

### 3.2 Multiple degree of freedom

Consider first the case of a finite degree of freedom system, i.e. a network of $p$ little masses interconnected by viscoelastic links. The viscoelastic properties of the various links are expressed by the relaxance matrix $[K(s)]$. The equilibrium equation in the transform plane is

$$[M]s^2X(s) + [K(s)]X(s) = F(s) \quad (3.5)$$

where $[M]$ is the diagonal matrix of masses, $X(s)$ and $F(s)$ the Laplace transform of the displacement and external force vectors. The system is said to be homogeneous, if $[K(s)]$ may be factored into $[K(s)] = \hat{K}(s)[K]$, where $[K]$ is a real symmetric matrix. For homogeneous systems, classical modal analysis is applicable to the underlying conservative mass-spring system defined by $[M] \text{ and } [K]$. In the coordinate system $y$, defined by the mode shape vectors, equation (3.5) is converted into a set of $n$ uncoupled equations

$$[s^2 + \omega_j^2 \delta_j(s)]y_j(s) = f_j(s) \quad (3.6)$$

For every mode, (3.6) is the equilibrium equation of a single degree of freedom (SDOF) oscillator, called the modal oscillator. It follows that a homogeneous viscoelastic oscillator may be decomposed into SDOF oscillators defined by (3.6). The mode shape vectors are identical to those of the underlying conservative system. Damped frequencies and damping constants of the $j$-th mode are obtained from the characteristic equation

$$s^2 + 2\zeta_j \alpha_j \omega_j + \omega_j^2 = 0 \quad (3.7)$$

where $\zeta_j$ and $\alpha_j$ are the damping ratio and damping constant of the $j$-th mode, respectively.

### 3.3 Interconnection of oscillators

Examples of the interconnected response include the transmission of a ground motion past a viscoelastic oscillator through coupling via a ground link. As discussed in Djoharian (2000), the viscoelastic oscillator may be decomposed into SDOF oscillator, with characteristic equation

$$s^2 + 2\zeta_1 \alpha_1 \omega_1 + \omega_1^2 = 0 \quad (3.8)$$

where $\zeta_1$ and $\alpha_1$ are the damping ratio and damping constant of the oscillator. The damping constant $\alpha_1$ has an upper bound $\alpha_{\max}^1$. In the rubbery region, $\alpha_1^2 \approx \omega_1^2/\zeta_1$ and $\alpha_1 \propto \omega_1^2$. In the glassy region, $\omega_1 \approx \omega_1$ and $\alpha_1 \propto \omega_1$. To be specific, for a Wiechert oscillator having $n$ relaxation peaks sorted as $\zeta_1 < \ldots < \zeta_n$, if $\omega_1 \ll \zeta_i$, then $\alpha_1^2 \approx \omega_1^2 = k/m$ and

$$\alpha_1 = \frac{k}{\omega_1^2} \frac{\alpha_0^2}{\zeta_1} \quad (3.9)$$

If the relaxation peaks $\zeta_i$ are spread enough, it can be shown that, in the transition region, $\alpha_1$ grows by stages, namely plateau regions delimited by $\zeta_i$s. Beyond the highest peak $\zeta_n$, $\omega_1 = \omega_1$ and $\alpha_1$ approaches its upper bound

$$\alpha_{\max}^1 = \frac{k}{\omega_1^2} \frac{\alpha_0^2}{\zeta_n} \quad (3.10)$$

Figure 2.2 Plots of the normalized relaxation (top), storage (middle) and loss (bottom) moduli for a Zener, Wiechert, Box and Wedge spectrum ($\theta = 0.5$) models.

### 3.4 VIBRATING SYSTEMS

We consider now vibrational properties of viscoelastic oscillators. According to the nature of the viscoelastic model used, we will refer to Zener, Wiechert, box spectrum, ... oscillators.

#### 3.1 Single degree of freedom

Let us consider a single mass $m$, connected to the ground by a viscoelastic model defined by the relaxance $k(s)$. Introducing the (glassy) natural frequency $\omega_0$ as $\omega_0^2 = k/m$, the characteristic equation of free oscillations, expressed in the transform plane is

$$s^2 + \omega_0^2 k(s) = 0 \quad (3.1)$$

Solving (3.1) for complex roots $s = -\alpha \pm i \omega_0$ determine the damped frequency and the damping constant $\omega_0$ and $\alpha$. Apart from the case of the Zener oscillator, analytical solutions are hopeless. However, for a lightly damped oscillator, namely $\chi \approx 1$, we have $\omega = \omega_0$ and the following approximation for $\alpha$ can be stated

$$\alpha = \frac{k}{\omega_0^2} \frac{\omega_0^2}{\alpha_0} \quad (3.2)$$

A simple examination of pole-zero locations in equation (3.1) shows that an $n$-order Wiechert oscillator has at least $n$ overdamped free solutions $\exp(-\alpha t)$ and at most, a single underdamped solution $\exp(-\alpha t)\exp(i\omega t)$. For instance, the Zener oscillator has at least (resp. most)
\[ s^2 + \omega_0^2 \mu \delta(s) = 0 \]  
\hspace{2cm} (3.7) 

The decomposition into normal modes holds for distributed systems as well. The homogeneity condition above must be replaced by isotropy, in which case, the equilibrium equation for zero external force becomes 
\[ \mu \delta u(x,s) = \delta(s) Lu(x,s) \]  
\hspace{2cm} (3.8) 

where \( \mu \) is the density of the material, \( \delta(s) \) the normalized relaxance and \( L \) a (self-adjoint) linear differential operator containing derivatives with respect to space coordinates (Courant, 1953). The eigenfunctions are identical to those of the underlying conservative system. Complex frequencies \( s = -\alpha \pm i \omega \) are solutions of the dispersion equation (3.7), where \( \alpha \) is an eigenvalue of \( L \).

### 3.3 Sound signature of a material

As previously stated, a homogeneous / isotropic multi-degree of freedom oscillator may be decomposed into its normal modes. All these modes are SDOF oscillators with the same normalized relaxance and \( L \). A (self-adjoint) linear differential operator containing derivatives with respect to space coordinates (Courant, 1953). The eigenfunctions are identical to those of the underlying conservative system. Complex frequencies \( s = -\alpha \pm i \omega \) are solutions of the dispersion equation (3.7), where \( \alpha \) is an eigenvalue of \( L \).

Being a microscopic scaled phenomenon, most of the relaxation processes are not much affected by the macroscopic shape of the material sample. However, it should be noted that some relaxation phenomena are size dependent. For instance, thermoelastic relaxation involving heat transfer between extended and compressed regions, is size dependent (Lakes, 1999). Figure 3.1 shows some examples of sound signatures corresponding to simple viscoelastic models.

### 4. Physical Modeling

As stated above, shape and material modeling may be done in two independent steps: shape modeling by designing a conservative model, and material modeling by wearing this skeleton with a viscoelastic dress to represent a particular material.

#### 4.1 Shape Modeling

According to the simulation technique used, synthesis models may take different forms. Here, shape models are created using the mass-spring model paradigm.

##### 4.1.1 Spatial design

A conservative resonator is characterized by physical data such as density and elasticity as well as geometrical data: the dimensionality (line, surface, etc.) and the boundary conditions (fixed, free, etc.). All the shape information is contained in the differential operator \( L \), its spatial region of definition and the corresponding boundary conditions. To get rid of the time variable in (3.8), we may replace it by the eigenvalue equation (or the reduced wave equation)

\[ kL(u) = \alpha^2 \mu u \]  
\hspace{2cm} (4.1) 

where \( k \) is an elastic modulus and \( \mu \) the density. The above equation has to be approximated by a discrete model, expressed in matrix form

\[ [K] [U] = \alpha_0^2 [M] [U] \]  
\hspace{2cm} (4.2) 

where \([K]\) and \([M]\) are the stiffness and mass matrices of a conservative network.

The first step is to partition the continuous domain of the resonator, in small regions: intervals for one dimensional and rectilinear or curvilinear polygons for 2-dimensional regions. Now, the two main methods for deriving the matrices \([K]\) and \([M]\) are finite difference (FDM) and finite element (FEM) methods.

Finite difference methods are based on the approximation of the differential operator. To be specific let us consider the case of a one dimensional flexible resonator involving \( L = \partial^2 \partial x \) (i.e. a string, a thin bar or an air column). In the case of a string, \( k \) is the force applied at the two endpoints and \( \mu \) the linear density (Fletcher, 1989: 36). To each node \( O \), with left and right neighbors \( W \) and \( E \), one can assign a mesh region \( R_0 \) as the interval bounded by the middle of each mesh. At the node \( O \), equation (4.2) may be approximated by

\[ k \left[ \frac{u_W + u_E}{h_E} - \left( \frac{1}{h_W} + \frac{1}{h_E} \right) u_O \right] = \mu \left( \frac{h_W + h_E}{2} \right) u_O \]  
\hspace{2cm} (4.3) 

A little mass \( m_0 \) equal to the mass of \( R_0 \) is associated to the node \( O \). To each edge is associated a spring with a stiffness constant proportional to the inverse of the edge length (Fig. 4.1).
For two-dimensional flexible resonators (membranes), usual mesh shapes are triangular or rectangular polygons. As in the one dimensional case, to each node $O$, one can assign a mesh region $R_O$, defined as the polygon bounded by the perpendicular bisectors of the edges containing $O$. For instance, for triangular meshes, the mesh region is the polygon defined by the circumcenters of all triangles containing $O$ (Fig. 4.2). The difference equation at $O$ may be derived by integrating (4.1) over $R_O$ (Varga, 2000: 183)

$$k \int \int_{R_O} \Delta u dxdy = \lambda \int \int_{R_O} u dxdy$$

(4.4)

The right hand integral may be approximated by $A_O u_O$ where $A_O$ is the area of $R_O$. The left hand may be transformed into a boundary integral over $\partial R_O$, using the Green’s formula (Courant, 1953: 280)

$$k \int_{\partial R_O} \frac{\partial u}{\partial n} ds = \lambda \mu A_O$$

(4.5)

where $n$ is the outer normal vector to $\partial R_O$. Now, every side of $\partial R_O$ bisects some particular edge $OP$ (Fig. 4.2). The left hand integral in (4.5) is then divided in two integrals, each approximated by central difference

$$k \int_{\partial R_O} \frac{\partial u}{\partial n} ds = k \Omega \frac{u_P - u_O}{OP}$$

(4.6)

It follows that to every edge $OP$ is associated a spring with a stiffness constant proportional to the ratio of the lengths of $\Omega P$ and $OP$. If an edge belongs to two meshes, both contributions must be added. Note that for boundary nodes, prescribed values of $u$ or $\partial u/\partial n$ must be taken into account. Stiffness and mass constants corresponding to triangular meshes are given below (Fig. 4.2)

$$\begin{align*}
{k_{OP} = k (\cot Q + \cot R)} \\
{m_O = \mu \sum_{Q} \frac{\cot Q}{8} OP^2}
\end{align*}$$

(4.7)

Figure 4.1 Approximation of a continuous flexible string by a mass-spring network.

The above method may be applied to any surface and any (rectilinear or curvilinear) meshes. Note that rectangular meshes are equivalent to two triangular meshes in which the hypotenuse spring vanishes ($\cot Q = \cot R = \cot \pi/2 = 0$).

Finite element methods use continuous domains, but the approximate solutions are found in form of piecewise polynomial functions, characterized by their values at control nodes. For linear elements, FEM produces stiffness matrix identical to FDM above. However, the mass matrix produced by usual FEM such as Ritz, Galerkin, etc., is non-diagonal: the discrete system generated has inertial interconnections. For simulation systems such as CORDIS-ANIMA which do not support inertial coupling, mass lumping techniques may be used (Reddy, 1993: 232).

For two-dimensional flexible resonators (membranes), usual mesh shapes are triangular or rectangular polygons. As in the one dimensional case, to each node $O$, one can assign a mesh region $R_O$, defined as the polygon bounded by the perpendicular bisectors of the edges containing $O$. For instance, for triangular meshes, the mesh region is the polygon defined by the circumcenters of all triangles containing $O$ (Fig. 4.2). The difference equation at $O$ may be derived by integrating (4.1) over $R_O$ (Varga, 2000: 183)

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The right hand integral may be approximated by $A_O u_O$ where $A_O$ is the area of $R_O$. The left hand may be transformed into a boundary integral over $\partial R_O$, using the Green’s formula (Courant, 1953: 280)

$$k \int_{\partial R_O} \frac{\partial u}{\partial n} ds = \lambda \mu A_O$$

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where $n$ is the outer normal vector to $\partial R_O$. Now, every side of $\partial R_O$ bisects some particular edge $OP$ (Fig. 4.2). The left hand integral in (4.5) is then divided in two integrals, each approximated by central difference

$$k \int_{\partial R_O} \frac{\partial u}{\partial n} ds = k \Omega \frac{u_P - u_O}{OP}$$

(4.6)

It follows that to every edge $OP$ is associated a spring with a stiffness constant proportional to the ratio of the lengths of $\Omega P$ and $OP$. If an edge belongs to two meshes, both contributions must be added. Note that for boundary nodes, prescribed values of $u$ or $\partial u/\partial n$ must be taken into account. Stiffness and mass constants corresponding to triangular meshes are given below (Fig. 4.2)

$$\begin{align*}
{k_{OP} = k (\cot Q + \cot R)} \\
{m_O = \mu \sum_{Q} \frac{\cot Q}{8} OP^2}
\end{align*}$$

(4.7)

Figure 4.1 Approximation of a continuous flexible string by a mass-spring network.

The above method may be applied to any surface and any (rectilinear or curvilinear) meshes. Note that rectangular meshes are equivalent to two triangular meshes in which the hypotenuse spring vanishes ($\cot Q = \cot R = \cot \pi/2 = 0$).

Finite element methods use continuous domains, but the approximate solutions are found in form of piecewise polynomial functions, characterized by their values at control nodes. For linear elements, FEM produces stiffness matrix identical to FDM above. However, the mass matrix produced by usual FEM such as Ritz, Galerkin, etc., is non-diagonal: the discrete system generated has inertial interconnections. For simulation systems such as CORDIS-ANIMA which do not support inertial coupling, mass lumping techniques may be used (Reddy, 1993: 232).

For two-dimensional flexible resonators (membranes), usual mesh shapes are triangular or rectangular polygons. As in the one dimensional case, to each node $O$, one can assign a mesh region $R_O$, defined as the polygon bounded by the perpendicular bisectors of the edges containing $O$. For instance, for triangular meshes, the mesh region is the polygon defined by the circumcenters of all triangles containing $O$ (Fig. 4.2). The difference equation at $O$ may be derived by integrating (4.1) over $R_O$ (Varga, 2000: 183)

$$k \int \int_{R_O} \Delta u dxdy = \lambda \int \int_{R_O} u dxdy$$

(4.4)

The right hand integral may be approximated by $A_O u_O$ where $A_O$ is the area of $R_O$. The left hand may be transformed into a boundary integral over $\partial R_O$, using the Green’s formula (Courant, 1953: 280)

$$k \int_{\partial R_O} \frac{\partial u}{\partial n} ds = \lambda \mu A_O$$

(4.5)

where $n$ is the outer normal vector to $\partial R_O$. Now, every side of $\partial R_O$ bisects some particular edge $OP$ (Fig. 4.2). The left hand integral in (4.5) is then divided in two integrals, each approximated by central difference

$$k \int_{\partial R_O} \frac{\partial u}{\partial n} ds = k \Omega \frac{u_P - u_O}{OP}$$

(4.6)

It follows that to every edge $OP$ is associated a spring with a stiffness constant proportional to the ratio of the lengths of $\Omega P$ and $OP$. If an edge belongs to two meshes, both contributions must be added. Note that for boundary nodes, prescribed values of $u$ or $\partial u/\partial n$ must be taken into account. Stiffness and mass constants corresponding to triangular meshes are given below (Fig. 4.2)

$$\begin{align*}
{k_{OP} = k (\cot Q + \cot R)} \\
{m_O = \mu \sum_{Q} \frac{\cot Q}{8} OP^2}
\end{align*}$$

(4.7)

Figure 4.1 Approximation of a continuous flexible string by a mass-spring network.

The above method may be applied to any surface and any (rectilinear or curvilinear) meshes. Note that rectangular meshes are equivalent to two triangular meshes in which the hypotenuse spring vanishes ($\cot Q = \cot R = \cot \pi/2 = 0$).

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Figure 4.2 Mass-spring network resulting from finite difference approximation over triangular meshes.

The above method may be applied to any surface and any (rectilinear or curvilinear) meshes. Note that rectangular meshes are equivalent to two triangular meshes in which the hypotenuse spring vanishes ($\cot Q = \cot R = \cot \pi/2 = 0$).

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Figure 4.3 Approximation of planar (a, b) and non planar drums (c, d). In each case, the reduced frequency and the mode shape of the second mode are displayed.

The above constructions may be extended to bars and plates involving bending waves. We give here only the mass-spring model corresponding to a bar (Fig. 4.4).

Figure 4.4 Finite difference scheme corresponding to a bar.

4.1.2 Topological Design

Some geometrical constructions may be extended to abstract operations on mass-spring networks. The most
prominent example is the product of two networks (Djoharian, 1993: 62). The basic idea is the separation of variables in equation (4.1) (Courant, 1953: 287). It can be applied if the variables are separated in the expression of \( L \) and in the boundary conditions as well. Then, eigenfunctions may be found in form of a product, i.e. \( u(x,y) = f(x)g(y) \). Then the corresponding mode satisfies
\[
\omega^2 = \omega_1^2 + \omega_2^2
\]
where \( \omega_1 \) and \( \omega_2 \) are frequencies of the modes of the separated equations. This is obviously the case for the rectangular and cylindrical membranes. In matrix form, the product operation is expressed with the Kronecker tensor product of matrices (Lancaster, 1969)
\[
[M] = [M_1] \otimes [M_2] \quad [K] = [M_1] \otimes [K_1] \otimes [M_2] \quad (4.9)
\]
The product operation defined by (4.9) may be extended to any two networks, regardless of any geometrical interpretations. The topological product may be generalized in \( \mathcal{L} \)-product and twisted product (Djoharian, 1993: 63). The \( \mathcal{L} \)-product of two networks A and B is obtained by substituting a copy of B to each node of A and replacing every spring \( k \) with a liaison \( k\mathcal{L} \). The product is twisted if in the substitution process, the network B may be subjected to some permutations \( \xi \) (Fig. 4.5). These abstract topological constructions may have some acoustical pertinence. Indeed, it can be shown that if the liaison \( \mathcal{L} \) is orthogonal and if the permutation \( \xi \) leaves B unchanged (i.e. B is \( \xi \)-symmetric) then the spectrum of the product is obtained from the spectra of A and B by relations similar to (4.8) (Djoharian, 1993: 63). This is a way to model exotic surfaces such as the Möbius strip (Fig. 4.5.b) or the Klein bottle (Barr, 1964).

![Figure 4.5 Product (a) and a twisted product (b)](image)

### 4.1.3 Spectral Design

As in the instrument making, in the physical modeling, there is an interplay between the design of shape and spectral considerations. Controlling the interaction between shape and spectrum features is a difficult task. However, with mass-spring networks one has more freedom to design models having a particular spectrum. For instance, given a spectrum \( \omega_1, \ldots, \omega_n \), one can define a homogeneous periodic mass-spring network having \( 2n \) unitary masses and spring constants defined as the discrete Fourier transform of \( \{\omega_1^2, \ldots, \omega_n^2\} \). The resulting network has the desired spectrum, but the mode shapes are determined by the Fourier matrix. To control relative amplitudes, the Lanczos algorithm produces a one-dimensional network having a prescribed spectrum and relative weights as well (Gladwell, 1986).

### 4.2 Material Modeling

As stated in §3.3, shape invariant material properties appear through some \( \alpha(\omega) \) relationship, reflecting the viscoelastic properties of materials. Strictly speaking, this is valid for isotropic materials only. However, fibered materials may be modeled in some extent as products.

#### 4.2.1 Isotropic materials

Modeling an isotropic material can be realized by two different methods: (A) by the definition of the material viscoelastic model, via the relaxance \( k(s) \) (or any other viscoelastic function) and, (B) : by explicit definition of the sound signature in the form of \( \alpha(\omega) \) function. The first method is the full description of the viscoelastic behavior of the material. All vibrational properties of the body of material are so modeled. The B method ignores the overdamped components of the transient responses. This method is well suited to modal synthesis programs, where the physical model is designed by direct access to spectral parameters (cf. §4.3.2).

All sound signatures can be realized by Kelvin viscoelastic oscillators. The dashpot interconnection is represented by a matrix \( [Z] \) which is entirely defined by its spectrum and its eigenvectors that are identical to those of the underlying conservative system. It should be noted that the proportional viscosity, i.e. \( [Z] = \tau[K] \), generates a sound signature \( \alpha(\omega) = \tau\alpha^2 \). According to §3.1, this may model sound signature of high damping materials at low frequencies.

#### 4.2.2 Anisotropic Materials

Acoustic properties of an anisotropic material are connected to the geometry of the body. However, for a fibered material the sound signature \( \alpha(\omega) \) can be obtained from the sound signature of each of its components. Moreover, the product operation presented in §4.1.2 may be extended to viscoelastic networks. Equation (4.8) is then replaced by the characteristic equation
\[
s^2 + \alpha_1^2 k_1 (s) + \alpha_2^2 k_2 (s) = 0 \quad (4.10)
\]
However, it should be noted that this applies only to problems of the product type. Plates and complex geometries are then excluded.

### 4.3 Sound synthesis

Lumped viscoelastic oscillators are governed by constant coefficient differential equations. Thus, numerical simulation in time domain based on difference methods can be used. For continuous spectrum oscillators, frequency domain simulation is more relevant. We discuss here only time domain simulation.

#### 4.3.1 State space simulation

The special feature of a viscoelastic oscillator is the high order of the differential equation as well as the
derivatives of the input. For a single degree of freedom the general equation has the following form
\[ a_n x^{(n+2)} + \cdots + a_2 x = b_0 f^{(n)} + \cdots + b_1 f \]
where \( n \) is the number of the relaxation peaks.

By standard state variable techniques, e.g. by modal realization, equation (4.11) can be reduced to a first order equation (Takahashi, 1973). The first order equation can be simulated then in various ways: standard finite difference techniques such as Euler, Runge-Kutta, etc., by impulse invariant digital filters, or by a discrete state space simulation (Hildebrand, 1968; Oppenheim, 1989).

For multiple degree of freedom oscillators, state variable techniques are feasible. However, finite difference simulation of a high order Wiechert oscillator is a heavy task. Especially as, for stability reasons, the sampling frequency must be chosen greater than the highest relaxation peak. An alternative method is to use Kelvin oscillators with a suitable choice of viscosity matrix \([Z]\) (cf. §4.2.1). Differential equation governing oscillation of a Kelvin oscillator is
\[ [M] \ddot{X} + [Z] \dot{X} + [K] X = F \]  
(4.12)

Various finite difference schemes as well as digital filter techniques may be used. For instance, CORDIS-ANIMA system uses a two step finite difference scheme with backward derivative and centered acceleration (Florens, 1991).

4.3.2 Modal synthesis

Modal synthesis enables direct control of damping parameters. Thus, material sound signatures can be used in a direct and explicit way. Classical modal synthesis (Adrien, 1991; Florens, 1991) is equivalent to the Kelvin oscillator approach above, since it neglects overdamped solutions. For homogeneous oscillators, an alternative modal synthesis may be used. According to §3.2, a homogeneous oscillator can be decomposed into high order viscoelastic real modes. The modal synthesis algorithm is then similar to the classical modal synthesis. But, here each mode corresponds to a high order differential equation:
\[ a_n y^{(n+2)} + \cdots + a_2 y = b_0 f^{(n)} + \cdots + b_1 f \]  
(4.13)

Each modal equation can be simulated by the same techniques as before, i.e. finite differences or IIR digital filters.

5. CONCLUSION

A general framework for designing shape and material in Physical Modeling sound synthesis has been presented. It is shown that acoustic invariants of shape and material has to be found in the reduced frequency spectrum and in the frequency-damping constant relationship.

Shape modeling based on the discretization of spatial domains as well as abstract topological constructions of mass-spring models have been outlined. Material modeling uses the linear theory of viscoelasticity. It is shown how rheological laws of viscoelasticity generate sound signatures, i.e. frequency-damping relationships. This enables us to organize shape and material modeling in two independent steps: 1) modeling of geometric data by mass-spring models; 2) wearing this skeleton by a viscoelastic dress to represent a particular material. According to the synthesis method, the second step can be achieved by the choice of a viscoelastic rheological law or by the direct specification of a sound signature.

6. REFERENCES