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Motivic decompositions of projective homogeneous varieties and change of coefficients

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Abstract

We prove that under some assumptions on an algebraic group $G$, indecomposable direct summands of the motive of a projective $G$-homogeneous variety with coefficients in $\mathbb{F}_p$ remain indecomposable if the ring of coefficients is any field of characteristic $p$. In particular for any projective $G$-homogeneous variety $X$, the decomposition of the motive of $X$ in a direct sum of indecomposable motives with coefficients in any finite field of characteristic $p$ corresponds to the decomposition of the motive of $X$ with coefficients in $\mathbb{F}_p$. We also construct a counterexample to this result in the case where $G$ is arbitrary.

Résumé

Décompositions motiviques des variétés projectives homogènes et changement des coefficients.

Nous prouvons que sous certaines hypothèses sur un groupe algébrique $G$, tout facteur direct indécomposable du motif associé à une variété projective $G$-homogène à coefficients dans $\mathbb{F}_p$ demeure indécomposable si l’anneau des coefficients est un corps de caractéristique $p$. En particulier pour toute variété projective $G$-homogène $X$, la décomposition du motif de $X$ comme somme directe de motifs indécomposables à coefficients dans tout corps fini de caractéristique $p$ correspond à la décomposition du motif de $X$ à coefficients dans $\mathbb{F}_p$. Nous exhibons de plus un contre-exemple à ce résultat dans le cas où le groupe $G$ est quelconque.

Introduction

Let $F$ be a field, $\Lambda$ be a commutative ring, $CM(F;\Lambda)$ be the category of Grothendieck Chow motives with coefficients in $\Lambda$, $G$ a semi-simple affine algebraic group and $X$ a projective $G$-homogeneous $F$-
variety. The purpose of this note is to study the behaviour of the complete motivic decomposition (in a direct sum of indecomposable motives) of $X \in CM(F; \Lambda)$ when changing the ring of coefficients. In the first part we prove some very elementary results in non-commutative algebra and find sufficient conditions for the tensor product of two connected rings to be connected. In the second part we show that under some assumptions on $G$, indecomposable direct summands of $X$ in $CM(F; \mathbb{F}_p)$ remain indecomposable if the ring of coefficients is any field of characteristic $p$ (Theorem 2.1), since these conditions hold for the reduced endomorphism ring of indecomposable direct summands of $X$. In particular theorem 2.1 implies that the complete decomposition of the motive of $X$ with coefficients in any finite field of characteristic $p$ corresponds to the complete decomposition of the motive of $X$ with coefficients in $\mathbb{F}_p$. Finally we show that theorem 2.1 doesn’t hold for arbitrary $G$ by producing a counterexample.

Let $\Lambda$ be a commutative ring. Given a field $F$, an $F$-variety will be understood as a separated scheme of finite type over $F$. Given such $\Lambda$ and an $F$-variety $X$, we can consider $CH_i(X; \Lambda)$, the Chow group of $i$-dimensional cycles on $X$ modulo rational equivalence with coefficients in $\Lambda$, defined as $CH_i(X) \otimes_\mathbb{Z} \Lambda$. These groups are the first step in the construction of the category $CM(F; \Lambda)$ of Grothendieck Chow motives with coefficients in $\Lambda$. This category is constructed as the pseudo-abelian envelope of the category $CR(F; \Lambda)$ of correspondences with coefficients in $\Lambda$. Our main reference for the construction and the main properties of these categories is [2]. For a field extension $E/F$ and any correspondence $\alpha \in CH(X \times Y; \Lambda)$ we denote by $\alpha_E$ the pull-back of $\alpha$ along the natural morphism $(X \times Y)_E \to X \times Y$. Considering a morphism of commutative rings $\varphi : \Lambda \to \Lambda'$ we define the two following functors. The change of base field functor is the additive functor $res_{E/F} : CM(F; \Lambda) \to CM(E; \Lambda)$ which maps any summand $(X, \pi)[i] \in CM(F; \Lambda)$ to $(X_E, \pi_E)[i]$ and any morphism $\alpha : (X, \pi)[i] \to (Y, \rho)[j]$ to $\alpha_E$. The change of coefficients functor is the additive functor $coeff_{\Lambda'/\Lambda} : CM(F; \Lambda) \to CM(F; \Lambda')$ which maps any summand $(X, \pi)[i]$ to $(X, (id \otimes \varphi)(\pi))[i]$ and any morphism $\alpha : (X, \pi)[i] \to (Y, \rho)[j]$ to $(id \otimes \varphi)(\alpha)$.

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1. On the tensor product of connected rings

Recall that a ring $A$ is connected if there are no idempotents in $A$ besides 0 and 1.

**Proposition 1.1** Let $A$ be a finite and connected ring. Then any element $a$ in $A$ is either nilpotent or invertible. The set $\mathcal{N}$ of nilpotent elements in $A$ is a two-sided and nilpotent ideal.

In order to prove Proposition 1.1 we will need the following elementary lemma.

**Lemma 1.2** Let $A$ be a finite ring. An appropriate power of any element $a$ of $A$ is idempotent.

**Proof.** For any $a \in A$, the set $\{a^n, \ n \in \mathbb{N}\}$ is finite, hence there is a couple $(p, k) \in \mathbb{N}^2$ (with $k$ non-zero) such that $a^p = a^{p+k}$. The sequence $(a^n)_{n \geq p}$ is $k$-periodic and for example if $s$ is the lowest integer such that $p < sk$, $a^{sk}$ is idempotent.

Proof of Proposition 1.1. For any $a \in A$, an appropriate power of $a$ is an idempotent by lemma 1.2. Since $A$ is connected, this power is either 0 or 1, that is to say $a$ is either nilpotent or invertible.

We now show that the set $\mathcal{N}$ of nilpotent elements in $A$ is a two-sided ideal. First if $a$ is nilpotent in $A$, then for any $b$ in $A$, $ab$ and $ba$ are not invertible, hence $ab$ and $ba$ belong to $\mathcal{N}$.

It remains to show that the sum of two nilpotent elements in $A$ is nilpotent. Setting $\nu$ for the number of nilpotent elements in $A$, we claim that for any sequence $a_1, \ldots, a_\nu$ in $\mathcal{N}$, $a_1 \ldots a_\nu = 0$. Indeed if $a_{\nu+1}$ is any nilpotent in $A$ the finite sequence $\Pi_1 = a_1, \Pi_2 = a_1 a_2, \ldots, \Pi_{\nu+1} = a_1 a_2 \ldots a_{\nu+1}$ consists of nilpotents and by the pigeon-hole principle $\Pi_k = \Pi_s$ for some $k$ and $s$ satisfying $1 \leq k < s \leq \nu + 1$. Therefore $\Pi_s = \Pi_k a_{k+1} \ldots a_s = \Pi_k$ which implies that $\Pi_k (1 - a_{k+1} \ldots a_s) = 0$ and $\Pi_k = 0$ since $1 - a_{k+1} \ldots a_s$ is
invertible. With this in hand it is clear that for any \( a \) and \( b \) in \( \mathcal{N} \), \((a + b)^\nu = 0 \). Furthermore \( \mathcal{N}^\nu = 0 \) and \( \mathcal{N} \) is nilpotent.

**Corollary 1.3** Let \( A \) be a finite and connected \( F_p \)-algebra endowed with a ring morphism \( \varphi : A \to F_p \). Then the set \( \mathcal{N} \) of nilpotent elements in \( A \) is precisely \( \text{ker}(\varphi) \). Furthermore for any connected \( F_p \)-algebra \( E \), \( A \otimes_{F_p} E \) is connected.

Proof. For any \( a \in \mathcal{N} \) and \( n \in \mathbb{N}^* \) such that \( a^n = 0 \), \( 0 = \varphi(a^n) = \varphi(a)^n \), hence \( a \) lies in the kernel of \( \varphi \). On the other hand if \( \varphi(a) = 0 \), \( a \) is not invertible thus \( a \) is nilpotent and \( \mathcal{N} = \text{ker}(\varphi) \). Since \( \mathcal{N} \) is nilpotent, \( \mathcal{N} \otimes E \) is also nilpotent. The sequence

\[
0 \to \mathcal{N} \otimes E \to A \otimes E \xrightarrow{\psi} E \to 0
\]

is exact and we want to show that any idempotent \( P \in A \otimes \mathbb{F}_p \) is either 0 or 1. Since \( E \) is connected, \( \psi(P) \) is either 0 or 1. We may replace \( P \) by \( 1 - P \) and so assume that \( P \) lies in the kernel of \( \psi \), which implies that the idempotent \( P \) is nilpotent, hence \( P = 0 \).

2. Application to motivic decompositions of projective homogeneous varieties

For any semi-simple affine algebraic group \( G \), the full subcategory of \( CM(F; \Lambda) \) whose objects are finite direct sums of twists of direct summands of the motives of projective \( G \)-homogeneous \( F \)-varieties will be denoted \( CM_G(F; \Lambda) \). We now use corollary 1.3 to study how motivic decompositions in \( CM_G(F; \Lambda) \) behave when extending the ring of coefficients. A pseudo-abelian category \( C \) satisfies the Krull-Schmidt principle if the monoid \((\mathfrak{C}, \oplus)\) is free, where \( \mathfrak{C} \) denotes the set of the isomorphism classes of objects of \( C \).

In the sequel \( \Lambda \) will be a connected ring and \( X \) an \( F \)-variety. A field extension \( E/F \) is a splitting field of \( X \) if the \( E \)-motive \( X_E \) is isomorphic to a finite direct sum of twists of Tate motives. The \( F \)-variety \( X \) is geometrically split if \( X \) splits over an extension of \( F \), and \( X \) satisfies the nilpotence principle, if for any field extension \( E/F \) the kernel of the morphism \( \text{res}_{E/F} : \text{End}(M(X)) \to \text{End}(M(X_E)) \) consists of nilpotents. Any projective homogeneous variety (under the action of a semi-simple affine algebraic group) is geometrically split and satisfies the nilpotence principle (see [1]), therefore if \( \Lambda \) is finite the Krull-Schmidt principle holds for \( CM_G(F; \Lambda) \) by [5, Corollary 3.6], and we can serenely deal with motivic decompositions in \( CM_G(F; \Lambda) \).

Let \( G \) be a semi-simple affine adjoint algebraic group over \( F \) and \( p \) a prime. The absolute Galois group \( \text{Gal}(F_{\text{sep}}/F) \) acts on the Dynkin diagram of \( G \) and we say that \( G \) is of inner type if this action is trivial.

By [1] the subfield \( F_G \) of \( F_{\text{sep}} \) corresponding to the kernel of this action is a finite Galois extension of \( F \), and we will say that \( G \) is \( p \)-inner if \( [F_G : F] \) is a power of \( p \). We now state the main result.

**Theorem 2.1** Let \( G \) be a semi-simple affine adjoint \( p \)-inner algebraic group and \( M \in CM_G(F; \mathbb{F}_p) \). Then for any field \( L \) of characteristic \( p \), \( M \) is indecomposable if and only if \( \text{coeff}_{FL/L}(M) \) is indecomposable. If \( X \) is geometrically split the image of any correspondence \( \alpha \in CH_{\text{dim}(X)}(X \times X; \Lambda) \) by the change of base field functor \( \text{res}_{E/F} \to \text{End}(M(X_E)) \) consists of nilpotents. The reduced endomorphism ring of any direct summand \((X, \pi)\) of \( M \) is defined as \( \text{End}_{CM_G(F; \Lambda)}((X, \pi)) \) and denoted by \( \overline{\text{End}}((X, \pi)) \).

Let \( X \) be a complete and irreducible \( F \)-variety. The pull-back of the natural morphism \( \text{Spec}(F(X)) \times X \to X \times X \) gives rise to \( \text{mult} : CH_{\text{dim}(X)}(X \times X; \Lambda) \to CH_0(X_F(X); \Lambda) \to \Lambda \) (where the second map is the usual degree morphism). For any correspondence \( \alpha \in CH_{\text{dim}(X)}(X \times X; \Lambda) \), \( \text{mult}(\alpha) \) is called the multiplicity of \( \alpha \) and we say that a direct summand \((X, \pi)\) given by a projector \( \pi \in CH_{\text{dim}(X)}(X \times X; \Lambda) \) is upper if \( \text{mult}(\pi) = 1 \). If \((X, \pi)\) is an upper direct summand of a complete and irreducible \( F \)-variety, the multiplicity \( \text{mult} : \text{End}_{CM_G(F; \Lambda)}((X, \pi)) \to \Lambda \) is a morphism of rings by [4, Corollary 1.7].

**Proposition 2.1** Let \( G \) be a semi-simple affine algebraic group and \( M = (X, \pi) \in CM(F; \mathbb{F}_p) \) the upper direct summand of the motive of an irreducible and projective \( G \)-homogeneous \( F \)-variety. Then for any
field \( L \) of characteristic \( p \), \( M \) is indecomposable if and only if \( \text{coeff}_{\mathbb{F}_p}(M) \) is indecomposable.

Proof. Since the change of coefficients functor is additive and maps any non-zero projector to a non-zero projector, it is clear that if \( \text{coeff}_{\mathbb{F}_p}(M) \) is indecomposable, \( M \) is also indecomposable. Considering a splitting field \( E \) of \( X \), the reduced endomorphism ring \( \overline{\text{End}}(M) := \pi \circ \overline{\text{End}}(X) \circ \pi \) is connected since \( M \) is indecomposable and finite. Corollary 1.3, with \( A = \overline{\text{End}}(M) \), \( E = L \) and \( \varphi = \text{mult} \) implies that \( \overline{\text{End}}(M) \otimes L = \overline{\text{End}}(\text{coeff}_{\mathbb{F}_p}(M)) \) is connected, therefore by the nilpotence principle \( \overline{\text{End}}(\text{coeff}_{\mathbb{F}_p}(M)) \) is also connected, that is to say \( \text{coeff}_{\mathbb{F}_p}(M) \) is indecomposable.

Proof of Theorem 2.1. Recall that \( G \) is a semi-simple affine adjoint \( p \)-inner algebraic group and consider a projective \( G \)-homogeneous \( F \)-variety \( X \). By [6, Theorem 1.1], any indecomposable direct summand \( M \) of \( X \) is a twist of the upper summand of the motive of an irreducible and projective \( G \)-homogeneous \( F \)-variety \( Y \), thus we can apply proposition 2.1 to each indecomposable direct summand of \( X \).

Remark: If \( \Lambda \) is a finite, commutative and connected ring, complete motivic decompositions in \( \text{CM}(F; \Lambda) \) remain complete when the coefficients are extended to the residue field of \( \Lambda \) by [7, Corollary 2.6], hence the study of motivic decompositions in \( \text{CM}(F; \Lambda) \), where \( \Lambda \) is any finite connected ring whose residue field is of characteristic \( p \), is reduced to the study motivic decompositions in \( \text{CM}(F; \mathbb{F}_p) \).

We now produce a counterexample to Theorem 2.1 in the case where the algebraic group \( G \) doesn’t satisfy the needed assumptions. Let \( L/F \) be a Galois extension of degree 3. By [1, Section 7], the endomorphism ring \( \overline{\text{End}}(M(\text{Spec}(L))) \) of the motive associated with the \( F \)-variety \( \text{Spec}(L) \) with coefficients in \( F_2 \) is the \( \mathbb{F}_2 \)-algebra of \( \text{Gal}(L/F) \), i.e. \( \mathbb{F}_2[X]/(X^3-1) \cong \mathbb{F}_2 \times \mathbb{F}_4 \), hence \( M(\text{Spec}(L)) = M \oplus N \), with \( \text{End}(N) = \mathbb{F}_4 \) and both \( M \) and \( N \) are indecomposable. Now \( \text{End}(\text{res}_{\mathbb{F}_4/\mathbb{F}_2}(N)) = \mathbb{F}_4 \otimes \mathbb{F}_4 \)  is not connected since \( 1 \otimes \alpha + \alpha \otimes 1 \) is a non-trivial idempotent for any \( \alpha \in \mathbb{F}_4 \setminus \mathbb{F}_2 \), hence \( \text{res}_{\mathbb{F}_4/\mathbb{F}_2}(N) \) is decomposable.

Consider the \( \text{(PGL}_2)_L \)-homogeneous \( L \)-variety \( \mathbb{P}_1^L \). The Weil restriction \( \mathcal{R}(\mathbb{P}_1^L) \) is a projective homogeneous \( F \)-variety under the action of the Weil restriction of \( (\text{PGL}_2)_L \), and the minimal extension such that \( \mathcal{R}(\text{(PGL}_2)_L) \) is of inner type is \( L \). By [3, Example 4.8], the motive with coefficients in \( F_2 \) of \( \mathcal{R}(\mathbb{P}_1^L) \) contains two twists of \( \text{Spec}(L) \) as direct summands, therefore at least two indecomposable direct summands of \( \mathcal{R}(\mathbb{P}_1^L) \) split off over \( \mathbb{F}_4 \).

References


