The number of absorbed individuals in branching Brownian motion with a barrier

Pascal Maillard

To cite this version:

Pascal Maillard. The number of absorbed individuals in branching Brownian motion with a barrier. 2010. hal-00472913v1

HAL Id: hal-00472913
https://hal.archives-ouvertes.fr/hal-00472913v1
Submitted on 13 Apr 2010 (v1), last revised 1 Dec 2011 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
The number of absorbed individuals in branching Brownian motion with a barrier

Pascal Maillard

April 13, 2010

Summary. We study supercritical branching Brownian motion on the real line starting at the origin and with constant drift $c$. At the point $x > 0$, we add an absorbing barrier, i.e. individuals touching the barrier are instantly killed without producing offspring. It is known that there is a critical drift $c_0$, such that this process becomes extinct almost surely if and only if $c \geq c_0$. In this case, if $Z_x$ denotes the number of individuals absorbed at the barrier, we give an equivalent for $P(Z_x = n)$ as $n$ goes to infinity. In the case of a $b$-ary offspring distribution, this answers a conjecture by David Aldous about the total progeny of the process, originally stated for the branching random walk.

Keywords. Branching Brownian motion with absorption, FKPP equation, travelling wave, singularity analysis, Galton–Watson process.


1 Introduction

We define branching Brownian motion as follows. Starting with an initial individual sitting at the origin of the real line, this individual moves according to a 1-dimensional Brownian motion with drift $c$ until an independent exponentially distributed time with rate $\beta$. At that moment it dies and produces $L$ (identical) offspring, $L$ being a random variable in the non-negative integers with $P(L = 1) = 0$. Starting from the position at which its parent has died, each child repeats this process, all independently of one another and of their parent. For a rigorous definition, see for example [8].

We assume that $m = E[L] - 1 \in (0, \infty)$, which means that the process is supercritical. At position $x > 0$, we add an absorbing barrier, i.e. individuals hitting the barrier are instantly killed without producing offspring. Under some moment conditions, this process becomes extinct almost surely if and only if the drift $c \geq c_0 = \sqrt{2\beta m}$. A conjecture of David Aldous ([3]), originally stated for the branching random walk, says that the number $N_x$ of individuals that have lived during the lifetime of the process satisfies $E[N_x] < \infty$ and $E[N_x \log N_x] = \infty$ in the critical speed area ($c = c_0$), and $P(N_x > n) \sim Kn^{-\gamma}$ in the subcritical speed area ($c > c_0$), with some $K > 0, \gamma > 1$. For the branching random walk, the conjecture of the critical case was proven by Addario-Berry and Broutin [1] for general reproduction laws.
satisfying a mild integrability assumption. Aïdékon [2] refined the results for constant $L$ by showing that there are positive constants $\rho, C_1, C_2$, such that for every $x$, we have
\[
\frac{C_1 e^{\rho x}}{n(\log n)^2} \leq P(N_x > n) \leq \frac{C_2 e^{\rho x}}{n(\log n)^2}
\text{ for large } n.
\]
Assuming $L$ constant has the advantage that $N_x$ is directly related to the number $Z_x$ of individuals absorbed at the barrier by $N_x = 1 = (Z_x - 1)(L/(L - 1))$, hence it is possible to study $N_x$ through $Z_x$.

In this sense, Neveu [18] had already proven the critical case conjecture for branching Brownian motion since he showed that the process $Z = (Z_x)_{x \geq 0}$ is actually a continuous-time Galton–Watson process of finite expectation, but not belonging to the $L \log L$ class in the critical case. In this paper, we propose a refinement of this result:

**Theorem 1.1.** Assume $c = c_0$. Assume that $E[L(\log L)^{2+\varepsilon}] < \infty$ for some $\varepsilon > 0$. For $n \geq 2$, let $q_n = \lim_{x \to 0} \frac{1}{2} P(Z_x = n)$. Then for each $x \geq 0$ we have
\[
\sum_{k=n}^{\infty} q_k \sim \frac{c_0}{n(\log n)^2} \quad \text{and} \quad P(Z_x > n) \sim \frac{c_0^2 e^{c_0 x}}{n(\log n)^2}, \quad \text{as } n \to \infty.
\]

The heavy tail of $Z_x$ suggests that its generating function is amenable to singularity analysis in the sense of [10]. This is in fact the case in both the critical and subcritical cases if we impose a stronger condition upon the offspring distribution and leads to the next theorem.

**Theorem 1.2.** Define $f(s) = E[s^L]$ the generating function of the offspring distribution. Assume that $1$ is a regular point of $f$, i.e. that $f$ can be analytically extended in $1$. Let $\delta$ be the span of $L - 1$, i.e. the greatest positive integer, such that $L - 1$ is concentrated on $\delta \mathbb{Z}$. Let $(q_n)_{n \geq 2}$ be defined as in Theorem 1.1. Let $\lambda_c \leq \lambda_c$ be the roots of the equation $\lambda^2 - 2c\lambda + c_0^2 = 0$ and denote by $d = \frac{\lambda_c}{\lambda_c}$ the ratio of the two roots.

- In the critical speed area ($c = c_0$) we have for all $x \geq 0$, as $n \to \infty$,
  \[
  q_{dn+1} \sim \frac{c_0}{d n^2 (\log n)^2} \quad \text{and} \quad P(Z_x = dn + 1) \sim \frac{c_0^2 e^{c_0 x}}{d n^2 (\log n)^2}.
  \]
- In the subcritical speed area ($c > c_0$) there exists a constant $K = K(c, \beta, f) > 0$, such that for all $x \geq 0$, as $n \to \infty$,
  \[
  q_{dn+1} \sim \frac{K}{d n^{d+1}} \quad \text{and} \quad P(Z_x = dn + 1) \sim \frac{e^{\lambda_c x} - e^{\lambda_c x}}{\lambda_c - \lambda_c} \frac{K}{d n^{d+1}}.
  \]

Furthermore, $q_{dn+k} = 0$ for all $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ and $k \in \{2, \ldots, \delta\}$.

The important condition in Theorem 1.2 is that $1$ be a regular point of $f$. For example, the theorem does not apply if $L$ has a power-law tail. Pringsheim’s theorem (see e.g. [5], Band II, p. 289) entails that this condition is in fact equivalent to $f(s) = \sum P(L = n)s^n$ having a radius of convergence greater than one.

The content of the paper is organised as follows: In Section 2 we derive some preliminary results by probabilistic means. Section 3 is devoted to the proof of Theorem 1.1, which draws on a Tauberian theorem and known asymptotics of travelling wave solutions to the FKPP equation. In Section 4 we review results about complex differential equations, singularity analysis of generating functions and continuous-time Galton–Watson processes. Those are needed for the proof of Theorem 1.2, which is done in Section 5.
2 First results by probabilistic methods

The goal of this section is to prove

Proposition 2.1. Assume $c > c_0$ and $E[L^2] < \infty$. Let $\lambda_c < \lambda_c$ be the roots of $\lambda^2 - 2c\lambda + c_0^2$ and denote by $d = \lambda_c/\lambda_c$ their ratio. Then there exists a constant $C = C(x,c,\beta,L) > 0$, such that

$$P(Z_n > n) \geq \frac{C}{n^d} \quad \text{for large } n.$$  

This result is needed to assure that the constant $K$ in Theorem 1.2 is non-zero. It is independent from Section 3 and in particular from Theorem 1.1. Its proof is entirely probabilistic and follows closely [2].

2.1 Notation and preliminary remarks

Our notation borrows from [16]. An individual is an element in the space of Ulam–Harris labels

$$U = \bigcup_{n \in \mathbb{N}_0} \mathbb{N}^n,$$

where $\mathbb{N} = \{1,2,3,\ldots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. This space is endowed with the ordering relations $\preceq$ and $\prec$ defined by

$$u \preceq v \iff \exists w \in U : v = uw \quad \text{and} \quad u \prec v \iff u \preceq v \text{ and } u \neq v.$$  

The space of Galton–Watson trees is the space of subsets $t \subset U$, such that $\emptyset \in t$, $v \in t$ if $v \prec u$ and $u \in t$ and for every $u$ there is a number $L_u \in \mathbb{N}_0$, such that for all $j \in \mathbb{N}$, $uj \in t$ if and only if $j \leq L_u$. Thus, $L_u$ is the number of children of the individual $u$.

Branching Brownian motion is defined on the filtered probability space $(\mathcal{T},F,(F_t),P)$. Here, $\mathcal{T}$ is the space of Galton–Watson trees with each individual $u \in t$ having a mark $(\zeta_u, X_u) \in \mathbb{R}^+ \times C(\mathbb{R}^+, \mathbb{R})$, where $\zeta_u$ is the life length and $X_u(s)$ the position of $u$ at the time $t$ (respectively, the position of its ancestor that was alive at time $t$, or the cemetery symbol $\Delta$ if $u$ dies before the time $t$). The sigma-field $F_t$ contains all the information up to time $t$, and $F = \bigcup_{t \geq 0} F_t$.

Let $y,c \in \mathbb{R}$, $\beta \in \mathbb{R}^+$ and $L$ be some random variable with values in $\mathbb{N}_0$. $P = P_{y,c,\beta,L}$ is the unique probability measure, such that, starting with a single individual at the point $y$,

- each individual moves according to a Brownian motion with drift $c$ until an independent time $\zeta_u$ following an exponential distribution with parameter $\beta$.
- At the time $\zeta_u$ the individual dies and leaves $L_u$ offspring at the position where it has died, with $L_u$ being an independent copy of $L$.
- Each child of $u$ repeats this process, all independently of one another and of the past of the process.

A common technique in branching processes since [17] is to select a genealogical line of descent from the ancestor $\emptyset$, called the spine and written as $\xi = (\xi_0 = \emptyset, \xi_1, \xi_2, \ldots)$, where $\xi_{n+1}$ is a child of $\xi_n$ for each $n \in \mathbb{N}_0$. This gives the space $\bar{T}$ of marked trees with spine and the sigma-fields $\bar{F}$ and $\bar{F_t}$.

Assume from now on that $m = E[L] - 1 \in (0,\infty)$. Let $N_t$ be the set of those individuals alive at time $t$. Note that every $\bar{F}_t$-measurable function $f(T,\xi)$, $T \in \mathcal{T}$, admits a
representation

\[ f(T, \xi) = \sum_{u \in N_t} f_u(T) \mathbbm{1}_{u \in \xi}, \]

where \( f_u \) are \( \mathcal{F}_t \)-measurable functions for each \( u \in U \). We can therefore define a measure \( \tilde{P} \) on \( (\mathcal{T}, \mathcal{F}, (\mathcal{F}_t)) \) by

\[ \tilde{P} |_{\mathcal{F}_t}(f(T, \xi)) = e^{-\beta m t} P |_{\mathcal{F}_t}\left( \sum_{u \in N_t} f_u(T) \right). \]

It is known ([16]) that this definition is sound and that \( \tilde{P} \) is actually a probability measure with the following properties:

- Under \( \tilde{P} \), the individuals on the spine move according to a Brownian motion with drift \( c \) and die at an accelerated rate \((m + 1)\beta\), independent of the motion.
- When an individual on the spine dies, it leaves a number of offspring at the point where it has died, this number following the size-biased distribution of \( L \). In other words, let \( \tilde{L} \) be a random variable whose law \( \mathcal{L}(\tilde{L}) \) has the density \( L / (m + 1) \) with respect to \( \mathcal{L}(L) \). Then the number of offspring is an independent copy of \( \tilde{L} \).
- Amongst those offspring, the next individual on the spine is chosen uniformly. This individual repeats the behaviour of its parent.
- The other offspring initiate branching Brownian motions according to the law \( P \).

### 2.2 Branching Brownian motion with two barriers

Let \( a, b \in \mathbb{R} \) such that \( y \in (a, b) \). Let \( \tau = \tau_{a,b} \) be the (random) set of those individuals whose paths enter \((-\infty, a] \cup [b, \infty)\) and all of whose ancestors’ paths have stayed inside \((a, b)\). For \( u \in \tau \) we denote by \( \tau(u) \) the first exit time from \((a, b)\) by \( u \)'s path, i.e.

\[ \tau(u) = \inf\{t \geq 0 : X_u(t) \notin (a, b)\} = \min\{t \geq 0 : X_u(t) \in \{a, b\}\}, \]

and set \( \tau(u) = \infty \) for \( u \notin \tau \). \( \tau \) is an (optional) stopping line in the sense of [8].

**Lemma 2.2.** Assume \(|c| > c_0 = \sqrt{2\beta m}\) and define \( \rho = \sqrt{c^2 - c_0^2} \). Let \( Z_{a,b} \) denote the number of individuals leaving the interval \((a, b)\) at the point \( a \), i.e.

\[ Z_{a,b} = \sum_{u \in \tau} \mathbbm{1}_{X_u(\tau) = a}, \]

where \( X_u(\tau) = X_u(\tau(u)) \). Then

\[ E^y[Z_{a,b}] = e^{c(a-y)} \frac{\sinh((b-y)\rho)}{\sinh((b-a)\rho)}. \]

If, furthermore, \( \sigma = E[L(L-1)] < \infty \), then

\[ E^y[Z_{a,b}^2] = \frac{2\beta \sigma e^{c(a-y)}}{\rho \sinh^3((b-a)\rho)} \left[ \sinh((b-y)\rho) \int_a^y e^{c(a-r)} \sinh^2((b-r)\rho) \sinh((r-a)\rho) dr \\
+ \sinh((y-a)\rho) \int_y^b e^{c(a-r)} \sinh^2((b-r)\rho) dr \right] + E^y[Z_{a,b}]. \]
Proof. Define $\xi(\tau) = \xi_i$ if $\xi_i \in \tau$ and write $\tau(\xi) = \tau(\xi(\tau))$ and $X_\xi(\tau) = X_{\xi(\tau)}(\tau)$. Then

$$E^y[Z_{a,b}] = \tilde{P}^y(e^{\beta m \tau(\xi)}, X_\xi(\tau) = a).$$

Let $W^y$ be the law of standard Brownian motion started at $y$ and $(B_t)_{t \geq 0}$ the canonical process. Let $T = T_{a,b}$ be the first exit time from $(a,b)$ of $B_t$. By Girsanov’s theorem, the above quantity is then equal to

$$W^y[e^{c(B_T - y) - \frac{1}{2}(c^2 - c_0^2)T}, B_T = a].$$

Evaluating this expression ([7], p. 212, 3.0.5) gives the first equality.

For $u \in U$, let $\Theta_u$ be the operator that maps a tree in $T$ to its subtree rooted in $u$. Then note that for each $u \in \tau$ we have

$$Z_{a,b} = 1 + \sum_{v \prec u} \sum_{w \in L_v} Z_{a,b} \circ \Theta_w,$$

hence

$$E^y[Z_{a,b}^2] = E^y \left[ \sum_{u \in \tau} \mathbb{1}_{X_u(\tau) = a} Z_{a,b} \right] = E^y[Z_{a,b}] + \tilde{P}^y \left[ e^{\beta m \tau(\xi)} \sum_{v \prec \xi(\tau)} \sum_{w \in L_v \setminus w \neq \xi(\tau)} Z_{a,b} \circ \Theta_w, X_\xi(\tau) = a \right].$$

Conditioning on the path of the spine and the fission times on the spine and using the strong branching property, the second term equals

$$\tilde{P}^y \left[ e^{\beta m \tau(\xi)} \sum_{v \prec \xi(\tau)} E[L_v - 1]E^{X_v(\zeta_v - \tau)}[Z_{a,b}], X_\xi(\tau) = a \right],$$

where $\zeta_v$ denotes the time of death of the individual $v$. But since under $\tilde{P}$ the fission times form a Poisson process of intensity $\beta(m + 1)$ and the reproduction on the spine follows the size-biased law of $L$, both being independent from the motion, this equals

$$W^y \left[ e^{c(B_T - y) - \frac{1}{2}c^2T} \int_0^T \beta \sigma E^{B_t}[Z_{a,b}] dt, B_T = a \right] = \beta \sigma e^{c(a-y)} \int_a^b E^{r}[Z_{a,b}] W^y \left[ e^{-\frac{1}{2}c^2r} L^T_r, B_T = a \right] dr,$$

where $L^T_r$ is the local time of $(B_t)$ at the time $T$ and the point $r$. The last expression can be evaluated explicitly ([7], p. 215, 3.3.8) and gives the desired equality.

Corollary 2.3. Under the assumptions of Lemma 2.2, for each $b > 0$ there are $C^{(1)}_b, C^{(2)}_b > 0$, such that as $a \to -\infty$,

$$E^0[Z_{a,b}] \sim C^{(1)}_b e^{(c + \rho)a}, \quad \text{and} \quad E^0[Z_{a,b}^2] \sim C^{(2)}_b \begin{cases} e^{(c + \rho)a}, & \text{if } c > c_0 \\ e^{2(c + \rho)a}, & \text{if } c < -c_0. \end{cases}$$
2.3 Proof of Proposition 2.1

We now assume \( y = 0 \) and \( \mathbb{E}[L^2] < \infty \). Let \( x > 0 \) and let \( \tau = \tau_x \) be the stopping line of those individuals hitting the point \( x \) for the first time. We set \( Z_x = |\tau_x| \).

**Proof of Proposition 2.1.** Let \( a < 0 \) and \( n \in \mathbb{N} \). Then, by the strong branching property,

\[
P^0(Z_x > n) \geq P^0(Z_x > n \mid Z_{a,x} \geq 1) P^0(Z_{a,x} \geq 1) \geq P^a(Z_x > n) P^0(Z_{a,x} \geq 1).
\]

If \( P^0 \) denotes the law of branching Brownian motion started at the point 0 with drift \(-c\), then

\[
P^0(Z_x > n) = P^0(Z_{a,x} > n) \geq P^0(Z_{a-x,1} > n).
\]

In order to bound this quantity, we choose \( a = a_n \) in such a way that

\[
n = \frac{1}{4} \mathbb{E}^0[Z_{a_n-x,1}]^2 \geq C_1 \quad \text{for large } n.
\]

Furthermore, by Corollary 2.3, \( a_n = -(1/\lambda_c) \log n + O(1) \), which entails that there exists \( C_2 > 0 \), such that

\[
P^0(Z_{a_n,x} \geq 1) \geq \frac{\mathbb{E}^0[Z_{a_n,x}]}{\mathbb{E}^0[Z_{a_n,x}^2]} \geq C_2 \quad \text{for large } n,
\]

again by the Paley–Zygmund inequality. This proves the proposition with \( C = C_1 C_2 \).

\[\square\]

3 Tail asymptotics in the critical case

In Section 3.1 we review the known relation between the process \((Z_x)_{x \geq 0}\) and a differential equation called the FKPP equation. It will be of frequent use during the rest of the paper. Section 3.2 contains the proof of Theorem 1.1 and is independent from Sections 2, 4 and 5.

3.1 The FKPP equation

As was already observed by Neveu ([18]), the translational invariance of Brownian motion and the strong branching property immediately imply that \( Z = (Z_x)_{x \geq 0} \) is a homogeneous Galton–Watson process. There is therefore an infinitesimal generating function

\[
a(s) = \alpha \left( \sum_{n=0}^{\infty} p_n s^n - s \right), \quad \alpha > 0, \ p_1 = 0,
\]

associated to it ([4], p. 106, [12]). Its probabilistic interpretation is

\[
\alpha = \lim_{x \to 0} \frac{1}{x} P(Z_x \neq 1) \quad \text{and} \quad p_n = \lim_{x \to 0} P(Z_x = n | Z_x \neq 1),
\]

hence \( q_n = \alpha p_n \). Note that with no further conditions on \( c \) and \( L \), \( \sum_n p_n \) need not necessarily be 1.

Neveu showed that \( a(s) \) is intimately linked with the so-called *travelling wave* solutions of a reaction-diffusion equation called the Fisher–Kolmogorov–Petrovskii–Piskounov (FKPP) equation. Although he treated dyadic branching Brownian motion only (i.e. \( L = 2 \) almost surely), it is trivial to extend his result to general offspring distributions.
Fact 3.1. Assume $c \geq c_0$. Define $f(s) = E[s^k]$ and let $q$ be the unique fixed point of $f$ in $[0, 1]$. There exists $t_0 \in \mathbb{R} \cup \{-\infty\}$ and a strictly decreasing smooth function $\psi : (t_0, \infty) \to (q, 1)$ with $\psi(t_0^+) = 1$ and $\psi(\infty) = q$, such that $a = \psi' \circ \psi^{-1}$ on $(q, 1)$, $F_s(s) = E[s^{2k}] = \psi(\psi^{-1}(s) + x)$ and $\psi$ satisfies the following differential equation on $(t_0, \infty)$:

$$\frac{1}{2} \psi'' - c \psi' = \beta(\psi - f \circ \psi).$$

Remark 3.2. In this paper, we are only interested in the case where $Z_x < \infty$ almost surely for every $x \geq 0$. This corresponds to $t_0 = -\infty$ in Fact 3.1. It is known (e.g. [16]) that a sufficient condition for this to happen is

$$c > c_0 \text{ and } E[L \log L] < \infty, \text{ or } c = c_0 \text{ and } E[L(\log L)^{2+\varepsilon}] \text{ for some } \varepsilon > 0. \quad (C)$$

In this case, it is easy to show (see the proof of Lemma 5.1), that $a'(1) \in \{\lambda_c, \overline{\lambda_c}\}$. In particular, $a'(1) = c_0$ if $c = c_0$.

3.2 Proof of Theorem 1.1

We start with the following Abelian-type lemma:

Lemma 3.3. Let $X$ be a random variable concentrated on $\mathbb{N}_0$ and let $\varphi(s) = E[s^X]$ be its generating function. Assume that $E[X(\log X)^\gamma] < \infty$ for some $\gamma > 0$. Then

$$\varphi'(1) - \varphi'(1-s) = O((\log \frac{1}{s})^{-\gamma}), \quad \text{as } s \to 0.$$  

Proof. Let $s_0 > 0$ be such that the function $s \mapsto s(\log \frac{1}{s})^{\gamma}$ is increasing on $[0, s_0]$. Let $s \in (0, s_0)$. Then, with $p_k = P(X = k)$,

$$(\varphi'(1) - \varphi'(1-s))(\log \frac{1}{s})^\gamma = \sum_{k=1}^{\infty} kp_k (1 - (1-s)^{k-1})(\log \frac{1}{s})^\gamma.$$  

If $k > s^{-1}$, then $(1 - (1-s)^{k-1})(\log \frac{1}{s})^\gamma < (\log k)^\gamma$. If $[s_0^{-1}] \leq k < s^{-1}$, then $s(\log \frac{1}{s})^\gamma < k(\log k)^\gamma$ and thus $(1 - (1-s)^{k-1})(\log \frac{1}{s})^\gamma < ks(\log \frac{1}{s})^\gamma \leq (\log k)^\gamma$. Furthermore,

$$\sum_{k=1}^{\lceil s_0^{-1} \rceil} kp_k (1 - (1-s)^{k-1})(\log \frac{1}{s})^\gamma < (s(\log \frac{1}{s})^\gamma) \sum_{k=1}^{\lceil s_0^{-1} \rceil} k^2 p_k < C,$$

for some $C > 0$. Collecting these results, we have

$$(\varphi'(1) - \varphi'(1-s))(\log \frac{1}{s})^\gamma < C + \sum_{k=1}^{\infty} p_k k(\log k)^\gamma \leq C + E[X(\log X)^\gamma] < \infty,$$

for all $s < s_0$, which proves the lemma. \qed

Proof of Theorem 1.1. We have $c = c_0$ by hypothesis. Let $\psi$ be the travelling wave from (3.2). By Remark 3.2, the hypothesis $E[L(\log L)^{2+\varepsilon}] < \infty$ implies that $\psi$ is defined on $\mathbb{R}$. Let $\phi(x) = 1 - \psi(-x)$, such that $a(1-s) = \phi'(\phi^{-1}(s))$ and $u(1-s, x) = 1 - \phi(\phi^{-1}(s) - x)$. It is known from [16] that there exists $K \in (0, \infty)$, such that $\phi(x) \sim Kxe^{-c_0x}$ as $x \to \infty$. Since $a'(1) = c_0$ by Remark 3.2, this entails that $\phi'(x) \sim -c_0Kxe^{-c_0x}$. 

7
Let \( \varphi_1 = \phi' \), \( \varphi_2 = \phi \) and \( g(s) = 2\beta((m + 1)s - (1 - f(1 - s))) \). By Lemma 3.3, \( g(s) = o(s(\log \frac{1}{x})^{-2}) \). Equation (3.2) now implies
\[
\frac{d}{dx} \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = M \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} + g(\varphi_2(x)), \quad \text{with} \quad M = \begin{pmatrix} -2c_0 & -c_0^2 \\ 1 & 1 \end{pmatrix}.
\] (3.3)
The Jordan decomposition of \( M \) is given by
\[
J = A^{-1}MA = \begin{pmatrix} -c_0 & 1 \\ 0 & c_0 \end{pmatrix}, \quad A = \begin{pmatrix} -c_0 & 1 \\ 1 & c_0 \end{pmatrix}.
\] (3.4)
Setting \( \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} = A \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \), we get with \( \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \):
\[
\xi' = J\xi + O(g(\xi_1 + \xi_2)),
\]
which, in integrated form, becomes
\[
\xi(x) = C_0 e^{xJ} + \int_{x_0}^{x} e^{(x-y)J} O(g(\xi_1(y) + \xi_2(y))) dy,
\]
for some constant vector \( C_0 \) and some \( x_0 \in \mathbb{R} \). But \( \xi_i(x) = O(xe^{-c_0x}), \ i = 1, 2 \), hence \( g(\xi_1(x) + \xi_2(x)) = o(e^{-c_0x}/x) \). It follows that \( \varphi_i(x) = C_{i,1}x e^{-c_0x} + C_{i,2}e^{-c_0x} + o(e^{-c_0x}/x) \), \( i = 1, 2 \), and by translating \( \phi \) appropriately along the real line, we can choose it in such a way that for some constant \( C \) we have
\[
\phi(x) = Cxe^{-c_0x} + o(e^{-c_0x}/x), \quad \phi'(x) = -c_0Cxe^{-c_0x} + Ce^{-c_0x} + o(e^{-c_0x}/x).
\]
With these equations and the fact that \( f'(1) - f(1 - s) = o((\log \frac{1}{x})^{-2}) \) by hypothesis and Lemma 3.3, one can show by elementary but tedious calculus that
\[
a''(1 - s) \sim \frac{c_0}{s(\log \frac{1}{x})^2} \quad \text{and} \quad F_x''(1 - s) \sim \frac{c_0^2x e^{-c_0x}}{s(\log \frac{1}{x})^2}, \quad \text{as} \ s \downarrow 0.
\]
By standard Tauberian theorems ([9], XIII.5, Theorem 2), the first equivalence implies that
\[
U(n) = \sum_{k=1}^{n} k^2 \alpha_k \sim \frac{n}{(\log n)^2} \quad \text{as} \ n \to \infty.
\]
By integration by parts, this entails that
\[
\sum_{k=n}^{\infty} \alpha_k = \int_{n}^{\infty} x^{-2}U(dx) \sim c_0 \left( 2 \int_{n}^{\infty} \frac{1}{x^2(\log x)^2} dx - \frac{1}{n(\log n)^2} \right).
\]
But the last integral is equivalent to \( 1/(n(\log n)^2) \) (see e.g. [9], VIII.9, Theorem 1), which proves the first part of the theorem. The second part is proven analogously. \( \square \)
4 Preliminaries for the proof of Theorem 1.2

In the light of Proposition 2.1, one may suggest that under suitable conditions on \( L \) one may extend the proof of Theorem 1.1 to the subcritical case \( c > c_0 \) and prove that \( P(Z_x > n) \sim C'/(n^d) \) for some constant \( C' \). In order to apply the Tauberian theorems, one would then have to establish asymptotics for the \((d + 1)\)-th derivatives of \( a(s) \) and \( F_x(s) \) as \( s \to 1 \). In trying to do this, one quickly sees that the known asymptotics for the travelling wave \((1 - \psi(x)) \sim \text{const} \times e^{-\lambda x} \) as \( x \to -\infty \), see [16]) are not precise enough for this method to work. However, instead of relying on Tauberian theorems, one can analyse the behaviour of the holomorphic function \( a(s) \) near its singular point 1. This method is widely used in combinatorics at least since the seminal paper by Flajolet and Odlyzko [10] and is the basis for our proof of Theorem 1.2. Not only does it work in both the critical and subcritical cases, it even yields asymptotics for the density instead of the tail only. But before turning to the details, we give an overview of the several steps of our proof.

1. From (3.2), we derive a first order complex differential equation for \( a(s) \) valid in the unit disk of the complex plane.
2. Using results from Chapters 3 and 4 in [6], we analyse this equation locally around the point 1 of the complex plane to establish asymptotics for \( a(s) \) when \( s \) is near 1.
3. With the help of an equation valid for general continuous-time Galton–Watson processes, we transfer these asymptotics to asymptotics for the generating functions \( F_x(s) = E[s^{Z_x}] \).
4. Finally, we apply results from [10] to turn this analysis into asymptotics for the reproduction probabilities and the density of \( Z_x \).

In the rest of this section, we will define our notation for the complex analytic part of the proof and review some necessary general complex analytic results.

4.1 Notation

In the course of the paper, we will work in the spaces \( \mathbb{C} \) and \( \mathbb{C}^2 \), endowed with the Euclidean topology. An open connected set is called a region, a simply connected region containing a point \( z_0 \) is also called a neighbourhood of \( z_0 \). The closure of a set \( D \) is denoted by \( \overline{D} \), its border by \( \partial D \). The ball of radius \( r \) around \( z_0 \) is denoted by \( \mathbb{D}(z_0, r) = \{ z \in \mathbb{C} : |z - z_0| < r \} \), its closure and border by \( \overline{\mathbb{D}}(z_0, r) \) and \( \partial \mathbb{D}(z_0, r) \), respectively. We further use the abbreviation \( \mathbb{D} = \mathbb{D}(0, 1) \) for the unit disk. For \( 0 < \varphi < \pi \), \( r > 0 \) and \( x \in \mathbb{R} \), we define

\[
G(\varphi, r) = \{ z \in \mathbb{D}(1, r) \setminus \{1\} : |\arg(1-z)| < \pi - \varphi \}, \quad S_+(\varphi, x) = [x, \infty) \times (-\varphi, \varphi),
\]
\[
\Delta(\varphi, r) = \{ z \in \mathbb{D}(0, 1 + r) \setminus \{1\} : |\arg(1-z)| < \pi - \varphi \}, \quad S_-(\varphi, x) = (-\infty, x] \times (-\varphi, \varphi).
\]

Let \( G \) be a region in \( \mathbb{C} \), \( z_0 \in \partial G \) and \( f \) and \( g \) analytic functions in \( G \) with \( g(z) \neq 0 \) for all \( z \in G \). We write

\[
f(z) \in o(g(z)) \iff \lim_{z \to z_0} \frac{|f(z)|}{|g(z)|} = 0,
\]
\[
f(z) \in O(g(z)) \iff \exists K \in \mathbb{C} : f(z) = K g(z) + o(g(z)),
\]
\[
f(z) \sim g(z) \iff f(z) = g(z) + o(g(z)).
\]
4.2 Complex differential equations

In this section, we review some basics about complex differential equations. We start with the fundamental existence and uniqueness theorem, which can be found for example in [6], [13] or [15].

Fact 4.1. Let $G$ be a region in $\mathbb{C}^2$ and $(w_0, z_0)$ a point in $G$. Let $f(w, z) : G \rightarrow \mathbb{C}$ be analytic in $G$, i.e. $f$ is continuous and both partial derivatives exist and are continuous. Then there exists a unique function $w(z)$ analytic in a neighbourhood $U$ of $z_0$, such that

1. $w(z_0) = w_0$,
2. $(w(z), z) \in G$ for all $z \in U$ and
3. $w'(z) = f(w(z), z)$ for all $z \in U$.

In other words, the differential equation $w' = f(w, z)$ with boundary condition $w(z_0) = w_0$ has exactly one solution $w(z)$ which is analytic in $z_0$.

A consequence is the following standard result.

Fact 4.2. Let $H$ be a region in $\mathbb{C}$ and $w(z)$ analytic in $H$. Let $G$ be a region in $\mathbb{C}^2$, such that $(w(z), z) \in G$ for each $z \in H$ and suppose that there exists an analytic function $f(w, z) : G \rightarrow \mathbb{C}$, such that $w'(z) = f(w(z), z)$ for each $z \in H$. Let $z_0 \in \partial H$. Suppose that $w(z)$ is continuous in $z_0$ and that $(w(z_0), z_0) \in G$. Then $z_0$ is a regular point of $w(z)$, i.e. $w(z)$ admits an analytic extension in $z_0$.

4.3 Singularity analysis

We summarise results about the singularity analysis of generating functions. The basic references are [10] and [11], Chapter VI. The results are of two types: those that establish an equivalent for the coefficients of functions that are explicitly known, and those that estimate the coefficients of functions which are dominated by another function. We start with the results of the first type:

Fact 4.3. Let $d \in (1, \infty) \setminus \mathbb{N}$, $k \in \mathbb{N}$, $\gamma \in \mathbb{R}\setminus\{0\}$, $\delta \in \mathbb{R}$ and the functions $f_1$, $f_2$ defined by

$$f_1(z) = (1-z)^d \quad \text{and} \quad f_2(z) = (1-z)^k \left( \log \frac{1}{1-z} \right)^\gamma \left( \log \log \frac{1}{1-z} \right)^\delta,$$

for $z \in \mathbb{C}\setminus[1, +\infty)$. Let $(p_n^{(i)})$ be the coefficients of the Taylor expansion of $f_i$ around the origin, $i = 1, 2$. Then $(p_n^{(i)})$ satisfy the following equivalence relations as $n \rightarrow \infty$:

$$p_n^{(1)} \sim \frac{K_1}{n^{d+1}} \quad \text{and} \quad p_n^{(2)} \sim \frac{K_2 (\log n)^{\gamma-1} (\log \log n)^\delta}{n^{k+1}},$$

for some non-zero constants $K_1 = K_1(d), K_2 = K_2(k, \gamma, \delta)$. We have $K_2(1, -1, 0) = 1$.

Proof. For $f_1$, this is Proposition 1 from [10]. For $f_2$ this is remark 3 at the end of chapter 3 in the same paper. Note that the additional factors $\frac{1}{\delta}$ do not change the nature of the singularities, since $\frac{1}{\delta}$ is analytic in 1. For the last statement, note that $\frac{d}{ds} \Gamma(1-s)(1) = 1$, since $\Gamma$ has a pole of order 1 at $-1$ with residue $-1$.

The results of the second type are contained in the next theorem. It is identical to Corollary 4 in [10]. Note that a potential difficulty here is that it requires analytical extension outside the unit disk.
Fact 4.4. Let $0 < \varphi < \pi/2$, $r > 0$ and $f(z)$ be analytic in $\Delta(\varphi, r)$. Assume that as $z \to 1$,

$$f(z) = o\left((1 - z)^{\alpha}L\left(\frac{1}{1 - z}\right)\right) \quad \text{where } L(u) = (\log u)^\gamma (\log \log u)^\delta$$

for some real constants $\alpha, \gamma, \delta$. Then the coefficients $(p_n)$ of the Taylor expansion of $f$ around 0 satisfy

$$p_n = o\left(\frac{L(n)}{n^{\alpha + 1}}\right) \quad \text{as } n \to \infty.$$

4.4 An equation for continuous-time Galton–Watson processes

The following proposition establishes a relation between the infinitesimal generating function of a Galton–Watson process and its generating function at time $t$. We will use it to transfer asymptotics for $a(s)$ to the generating functions $F_t(s) = E[s^{Y_t}]$. For real $s$, the formulae stated in the proposition are well known, but we will need to use them for complex $s$, which is why we have to include some (complicated) hypotheses to be sure that the functions and integrals appearing in the formulae are well defined.

Proposition 4.5. Let $(Y_t)_{t \geq 0}$ be a continuous-time Galton–Watson process starting at 1. Let $a(s)$ be its infinitesimal generating function and $F_t(s) = E[s^{Y_t}]$. Let $a'(1) = \lambda \in (0, \infty)$. Suppose that $a(s)$ has an analytic extension into some simply connected region $D_a$. Let $Z_a = \{s \in D_a : a(s) = 0\}$. Suppose further that $F_t(s)$ has an analytic extension into some simply connected region $D_F$. Let there be a simply connected region $D \subset G \cap D_F$ with $F_t(D) \subset G$ and $D \cap \mathbb{D} = \emptyset$. Then the following equations hold for all $s \in D$:

$$\int_s^{F_t(s)} \frac{1}{a(r)} \, dr = t, \quad (4.1)$$

and

$$1 - F_t(s) = e^{\lambda t} (1 - s) \exp\left(\int_{F_t(s)}^{s} f^*(r) \, dr\right), \quad (4.2)$$

where $f^*(s)$ is defined for all $s \in D_a \setminus Z_a$ as

$$f^*(s) = \frac{a'(1)}{a(s)} + \frac{1}{1 - s}. \quad (4.3)$$

Proof. We note first that $\frac{1}{a(s)}$ and $f^*$ are analytic in the simply connected region $G$, hence the integrals are univalent analytic functions in $D$. Equation (4.1) now follows readily from Kolmogorov’s backward equation for $s \in \mathbb{D} \cap D$ (see [4], p. 106). The law of the permanence of functional equations (see for example [5], Band I, p. 203) then implies that (4.1) holds for every $s \in D$. This proves the first equation. Since $G$ is simply connected and does not contain the point $1 \in Z_a$, the primitive $\int \frac{ds}{1 - s} = - \log(1 - s)$ is univalent on $G$, which immediately gives the second equation. \qed

Corollary 4.6. If 1 is a regular point of $a(s)$, then it is a regular point for $F_t(s)$ for every $t \geq 0$.  

11
Proof. Define $G = \{ s \in \mathbb{D} : \text{Re} s > q \}$. Then $G \cap Z_a = \emptyset$, since $q$ is the only zero of $a$ in $\mathbb{D}$ (every probability generating function $g$ with $g'(1) > 1$ has exactly one fixed point $q$ in $\mathbb{D}$; this can easily be seen by applying Schwarz's lemma to $\tau^{-1} \circ g \circ \tau$, where $\tau$ is the Möbius transform that maps 0 onto $q$). Let $s_1 \in (q, 1)$ be such that $F_t(s) \in G$ for every $s \in H = \{ s \in \mathbb{D} : \text{Re} s > s_1 \}$. We can then apply Proposition 4.5 to conclude that (4.2) holds for every $s \in H$.

Since $a(s)$ is analytic in a neighbourhood $U$ of 1 by hypothesis, it is easy to show that $f^*$ is analytic in $U$ as well. Thus, $f^*$ has a primitive $F^*$ in $H \cup U$. We define the function $g(s) = (1 - s) \exp(F^*(s))$ on $H \cup U$. Since $g'(1) = -\exp(F^*(1)) \neq 0$, there exists an inverse $g^{-1}$ of $g$ in a neighbourhood $U_1$ of $g(1) = 0$. Let $U_2 \subset U$ be a neighbourhood of 1, such that $e^{\text{Im} g(s)} \in U_1$ for every $s \in U_2$. Define the analytic function $\tilde{F}_t(s) = g^{-1}(e^{\text{Im} g(s)})$ for $s \in U_2$. Then by (4.2), we have $F_t(s) = \tilde{F}_t(s)$ for every $s \in H \cap U_2$, hence $\tilde{F}_t$ is an analytic extension of $F_t$ in 1.

Corollary 4.7. Suppose that $a(s)$ has an analytic extension into $G(\varphi_0, r_0)$ for some $0 < \varphi_0 < \pi$ and $r_0 > 0$. Then for every $\varphi_0 < \varphi < \pi$ there exists $r > 0$, such that $F_t(s)$ can be extended into $G(\varphi, r)$, mapping $G(\varphi, r)$ into $G(\varphi_0, r_0)$.

Proof. Suppose w.l.o.g. that $a(s) \neq 0$ in $G(\varphi_0, r_0)$. Let $A(s)$ be a primitive of $(a(s))^{-1}$ on $G(\varphi_0, r_0)$. Since $a'(0) = \lambda > 0$, it is easy to show that there is $r_1 \leq r_0$, such that $A(s)$ is injective on $G(\varphi_0, r_1)$. Fix $\varphi \in (\varphi_0, \pi)$ and define $\varphi_1 = (\varphi_0 + \varphi)/2$. Since $A(s) \sim \lambda^{-1} \log(1-s)$ as $s \to 1$, there exists $R \in \mathbb{R}$, such that $A(G(\varphi_0, r_1))$ covers the strip $S = S(R, \pi - \varphi_1)$, hence $A^{-1}$ exists on $S$. Furthermore, there is an $r > 0$, such that $A(s) + t \in S$ for every $s \in G(\varphi, r)$. We can thus define the function $\tilde{F}_t(s) = A^{-1}(t + A(s))$ on $G(\varphi, r)$. By Proposition 4.5, $\tilde{F}_t(s) = F_t(s)$ on $G(\varphi, r) \cap \mathbb{D}$, hence $\tilde{F}_t$ is an analytic extension of $F_t$, mapping $G(\varphi, r)$ into $G(\varphi_0, r_0)$ by definition.

5 Proof of Theorem 1.2

We turn back to the branching Brownian motion and to our Galton–Watson process $Z = (Z_x)_{x \geq 0}$ of the number of individuals absorbed at the point $x$. Throughout this section, we place ourselves under the hypotheses of Theorem 1.2, i.e. we assume that $c \geq c_0 = \sqrt{2\beta m}$ and that the radius of convergence of $f(s) = E[s^d]$ is greater than 1. The equation $\lambda^2 - 2c\lambda + c_0^2 = 0$ then has the solutions $\lambda_c = c - \sqrt{c^2 - c_0^2}$ and $\lambda_\infty = c + \sqrt{c^2 - c_0^2}$, hence $\lambda_c = \lambda_\infty = c_0$ if $c = c_0$ and $\lambda_c < c_0 < \lambda_\infty$ otherwise. The ratio $d = \lambda_c/\lambda_\infty$ is therefore greater than or equal to one, according to whether $c > c_0$ or $c < c_0$, respectively.

We recall from section 3.1 that there is an infinitesimal generating function $a(s)$ associated to $Z$, which will be analysed thoroughly throughout this section. As a first step, we derive a first-order complex differential equation for $a(s)$:

Lemma 5.1. On $\overline{\mathbb{D}}$, we have:

$$a'(s)a(s) = 2ca(s) + 2\beta(s - f(s)). \quad (5.1)$$

Proof. On $(q, 1)$, we have by Fact 3.1

$$a' = \frac{\psi'' \circ \psi^{-1}}{\psi' \circ \psi^{-1}} = 2c + 2\beta \frac{\psi' \circ \psi^{-1} - f \circ \psi \circ \psi^{-1}}{\psi' \circ \psi^{-1}} = 2c + 2\beta \frac{\text{Id} - f}{a},$$
whence (5.1) follows. This implies that the equation holds on \( \mathbb{D} \) as well and if \( a(s) \) and \( a'(s) \) converge for each \( s \in \partial \mathbb{D} \), Abel’s limit theorem entails that the equation is satisfied on \( \overline{\mathbb{D}} \). Since the coefficients of \( a + \alpha s \) and \( a' + \alpha \) are real and non-negative, it is enough to verify the convergence in 1. But since condition \((C)\) in Remark 3.2 is verified by hypothesis, \( a(1) = 0 \) and since the coefficients of \( a \) are positive,

\[
a'(1) = \lim_{s \to 1} a'(s) = 2c + 2\beta \lim_{s \to 1} \frac{s - f(s)}{a(s)} = 2c - \frac{2\beta m}{a'(1)},
\]

which implies \( a'(1) < \infty \).

The next lemma assures that we can ignore certain degenerate cases appearing in the course of the analysis of \((5.1)\). It is the analytic interpretation of the probabilistic results in Section 2.

**Lemma 5.2.** 1 is a singular point of \( a(s) \). If \( c = c_0 \), then \( a''(1) = +\infty \).

**Proof.** If \( c = c_0 \), the second assertion follows from Theorem 1.1 or from Neveu’s result that \((Z_s)\) does not belong to the \( L \log L \) class (see the remark before Theorem 1.1). This implies that the radius of convergence of the power series \( a(s) \) is 1, and Pringsheim’s theorem now implies that 1 is a singular point of \( a(s) \). If \( c > c_0 \), Proposition 2.1 implies that \( E[s^{2c}] = \infty \) for every \( s > 1 \), whence 1 is a singular point of the generating function \( F_x(s) \) by Pringsheim’s theorem. By Corollary 4.6, it follows that 1 is a singular point of \( a(s) \) as well.

The next theorem is the core of the proof of Theorem 1.2.

**Theorem 5.3.** Let \( \lambda_c, \overline{\lambda_c} \) and \( d \) be defined as in Theorem 1.2. Then for every angle \( \varphi \in (0, \pi) \) there exists \( r > 0 \), such that \( a(s) \) possesses an analytical extension (denoted by \( a(s) \) as well) into \( G(\varphi, r) \). Moreover, as \( s \to 1 \) in \( G(\varphi, r) \), the following holds.

- If \( d = 1 \), then

\[
a(1 - s) = -c_0s + c_0 \frac{s}{\log \frac{1}{s}} + c_0s \log \log \frac{1}{s} + O\left( \frac{s}{\log \frac{1}{s}} \right), \tag{5.2}
\]

- If \( d > 1 \), then there is a \( K = K(c, \beta, f) \in \mathbb{C} \setminus \{0\} \) and a polynomial \( h(s) = \sum_{n=2}^{[d]} c_n s^n \), such that

\[
\begin{align*}
&\text{if } d \notin \mathbb{N} : \quad a(1 - s) = -\lambda_c s + h(s) + Ks^d + o(s^d), \tag{5.3} \\
&\text{if } d \in \mathbb{N} : \quad a(1 - s) = -\lambda_c s + h(s) + Ks^d \log s + o(s^d). \tag{5.4}
\end{align*}
\]

Before turning to the proof, we explain its basic ideas in the case \( c > c_0 \). Starting point is the differential equation \((5.1)\). We will see that the behaviour of \( a(s) \) around the point 1 will be closely related to the behaviour of the solutions to an equation of the form

\[
\frac{db}{ds} = d \cdot \frac{b + \text{terms in } b \text{ and } s \text{ of order } \geq 2}{s + \text{terms in } b \text{ and } s \text{ of order } \geq 2},
\]

around the point 0. If we remove the higher order terms from this equation, the family of solutions is given by \( b(s) = Ks^d \) for \( K \in \mathbb{C} \), hence we can hope that our solution behaves
asymptotically as $Ks^d$ for some $K \in \mathbb{C}$. This is actually the case if $d \notin \mathbb{N}$, but if $d$ is a natural number, it will behave like a solution to

$$\frac{db}{ds} = d \cdot \frac{b + Cs^d}{s},$$

where $C$ is some non-zero constant. But the family of solutions to this equation is given by $b(s) = Cs^d(\log s + K)$ for $K \in \mathbb{C}$, which is where the additional logarithmic factor comes into play.

At first sight, this dichotomy might seem strange, but it becomes evident if one remembers that we expect the coefficients of $a(s)$ (i.e. the infinitesimal reproduction probabilities) to behave like $1/n^d$. In light of Fact 4.3, a logarithmic factor must therefore appear if $d$ is a natural number, otherwise $a(s)$ would be analytic in $1$, in which case its coefficients would decrease at least exponentially.

**Proof of Theorem 5.3.** During this proof, we need to successively shrink the radius $r$ to ensure analyticity of the functions we are studying. These radii will be denoted by $r_1 \geq r_2 \geq r_3 \geq \ldots$, in order of appearance.

We set $b(s) = a(1 - s)$. Then

$$-b'(s)b(s) = 2cb(s) + 2\beta(1 - s - f(1 - s)) \quad \text{on } \overline{D}(1,1).$$

Since $f$ is regular in $1$ by hypothesis, $f(1) = 1$ and $f'(1) = E[L] = m + 1$, there exists $0 < r_1 < 1 - q$ and a function $h_1$ analytic on $D(0, r_1)$ with $h(0) = h'(0) = 0$, such that $f(1 - s) = 1 - (m + 1)s + h_1(s)$ for $s \in D(0, r_1)$.

As a first step, we analyse the above equation for real non-negative $s$. Since $b(s) \neq 0$ on $(0, r_1)$, we can divide both sides by $b(s)$ to obtain

$$\frac{db}{ds} = -2cb - c^2b + 2\beta h_1(s) \quad \text{on } (0, r_1). \quad (5.5)$$

We introduce the parameter $dt = \frac{ds}{b}$, living on $[t_0, \infty)$ for some $t_0 \in \mathbb{R}$, to obtain the system

$$\frac{db}{dt} = -2cb - c^2b + 2\beta h_1(s), \quad \frac{ds}{dt} = b,$$

which, in matrix form, becomes

$$\frac{d}{dt} \begin{pmatrix} b \\ s \end{pmatrix} = M \begin{pmatrix} b \\ s \end{pmatrix} + \begin{pmatrix} 2\beta h_1 \\ 0 \end{pmatrix}, \quad M = \begin{pmatrix} -2c & -c^2 \\ 1 & 0 \end{pmatrix}. \quad (5.6)$$

Note that this is nothing else than (3.3). Thus, the Jordan decomposition of $M$ is given by (3.4), if $c = c_0$, and

$$A^{-1}MA = \begin{pmatrix} -\lambda_c & 0 \\ 0 & -\lambda_c \end{pmatrix}, \quad A = \begin{pmatrix} -\lambda_c & -\lambda_c \\ 1 & 1 \end{pmatrix}, \quad \text{if } c > c_0. \quad (5.7)$$

Setting $\begin{pmatrix} b \\ s \end{pmatrix} = A \begin{pmatrix} B \\ S \end{pmatrix}$ transforms (5.6) into

$$\frac{dB}{dt} = -\lambda_c B + P(B, S), \quad \frac{dB}{dt} = -\lambda_c S + Q(B, S), \quad \text{if } c > c_0 \text{ and } (5.8)$$

$$\frac{dS}{dt} = -c_0 S + D(B, S), \quad \frac{dS}{dt} = -c_0 S + E(B, S), \quad \text{if } c = c_0. \quad (5.9)$$
where $P$, $Q$, $D$ and $E$ are of the form

$$
P(w, z) = \sum_{i,j} p_{ij} w^i z^j, \quad Q(w, z) = \sum_{i,j} q_{ij} w^i z^j,$$

canonical, with $\sum_{i,j} = \sum_{i+j \geq 2}$

(5.10)

and all are analytic in a neighbourhood $(\mathbb{D}(0, r_2))^2$ of the point $(0, 0)$.

In Chapter 4 of [6], Bieberbach now tells us how to get rid of the non-linear terms. Let us first treat the subcritical case $(c > c_0)$. Then $\lambda_c > \lambda_c$. He proves in this case that it is possible to find $u(w, z)$ and $v(w, z)$, analytic in $(\mathbb{D}(0, r_3))^2$ and of the form

$$
u(w, z) = w + \sum_{i,j} u_{ij} w^i z^j, \quad \nu(w, z) = z + \sum_{i,j} v_{ij} w^i z^j,$$

(5.11)

such that, when passing to $b = u(B, S)$ and $s = v(B, S)$, the equation (5.8) becomes, if $d \notin \mathbb{N}$:

$$\frac{db}{dt} = -\lambda_c b, \quad \frac{ds}{dt} = -\lambda_c s,$$

(5.12)

and if $d = n \in \mathbb{N}\{1\}$: either (5.12) or

$$\frac{db}{dt} = -\lambda_c b + C s^n, \quad \frac{ds}{dt} = -\lambda_c s, \quad C \in \mathbb{C}\{0\}.$$

(5.13)

The general solution of (5.12) is

$$b(t) = \text{const} \times e^{-\lambda_c t}, \quad s(t) = \text{const} \times e^{-\lambda_c t},$$

(5.14)

and the solutions of (5.13) are

$$b(t) = Ce^t e^{-\bar{\lambda}_c t} + \text{const} \times e^{-\bar{\lambda}_c t}, \quad s(t) = C e^{-\bar{\lambda}_c t}, \quad C \neq 0,$$

(5.15)

or

$$b(t) = \text{const} \times e^{-\bar{\lambda}_c t}, \quad s(t) = 0,$$

(5.16)

where const denotes any complex number. Thus, $b$ and $s$ must be of this form. They therefore possess a unique analytical extension onto $\mathbb{C}$, defined by (5.14), (5.15) or (5.16) for all $t \in \mathbb{C}$.

Let us now work the way back to $b(s)$. First of all, we note that by (5.11) and the implicit function theorem (see for example [14], Theorem 5.1.2), there are $u^*, v^*$, such that

$$\begin{pmatrix} B \\ S \end{pmatrix} = \begin{pmatrix} u^*(b, s) \\ v^*(b, s) \end{pmatrix} = \begin{pmatrix} b + \sum_{i,j} u_{ij} b^i s^j \\ s + \sum_{i,j} v_{ij} b^i s^j \end{pmatrix},$$

(5.17)

and $u^*$, $v^*$ are analytic on $(\mathbb{D}(0, r_4))^2$. Furthermore, we have $b = -\lambda_c s + (\lambda_c - \lambda_c) B$, so we have to define $B$ in dependence on $s$. In the degenerate cases where $b = 0$ or $s = 0$ it is easy to see that $B(s)$ is analytic in $0$ and therefore also $b(s)$, which is impossible because of Lemma 5.2. We can therefore assume $b \neq 0$ and $s \neq 0$.

If $d \notin \mathbb{N}$, (5.14) and (5.17) now imply as $\text{Re} t \rightarrow \infty$, with $n = |d|:

$$s = ce^{-\lambda_c t} + c_2 e^{-2\lambda_c t} + \ldots + c_n e^{-n\lambda_c t} + c_d e^{-d\lambda_c t} + o(e^{-d\lambda_c t}),$$

(5.18)

and

$$s' = -(c/\lambda_c) e^{-\lambda_c t} + O(e^{-\min(d,2)\lambda_c t}),$$

(5.19)

where $c$ and $c_d$ are nonzero (remember that $\lambda_c = d\lambda_c$). Fix $0 < \varphi < \pi$. From (5.18) and (5.19) it is easy to show that there is a $t_1 \in \mathbb{R}$, such that $s(t)$ maps $S_+(t_1, (\pi - \varphi/2)/\lambda_c)$
injectively onto \( G = \{ s \in \mathbb{D}(0, r) : |\arg s| < \pi - \varphi \} \). Taking into account that, because of (5.14) and (5.17),
\[
B = c_2' e^{-2\lambda s t} + \ldots + c_n' e^{-n\lambda s t} + c_d' e^{-d\lambda s t} + o(e^{-d\lambda s t}) \quad \text{as } \Re t \to \infty,
\]
with \( c_d' \) nonzero, it follows that
\[
B(s) = c_2'' s^2 + \ldots + c_n'' s^n + \frac{c_d''}{c_d^2} s^d + o(s^d) \quad \text{as } s \to 0,
\]
which proves (5.3).

In the case \( d = n \in \mathbb{N}\setminus\{1\} \), assume that \( b \) and \( s \) are of the form (5.14), which entails \( b = s^n \). Again, this implies that \( b(s) \) is analytic in a neighbourhood of 0, contradicting Lemma 5.2.

Therefore, \( b \) and \( s \) must be of the form (5.15). Similar arguments as above then lead to
\[
B = c_2'' s^2 + \ldots + c_n'' s^n - K s^n \log s + o(s^n) \quad \text{as } \Re t \to \infty,
\]
for some non-zero constant \( K \), proving (5.4).

We turn to the case \( d = 1 \). In what follows, we denote by \( \varepsilon_1, \varepsilon_2, \ldots \) some positive constants that are as small as necessary. We have \( B = -b + (1 - c_0) s \), which is a strictly increasing function of \( s \) on \((0, \varepsilon_1)\), since \( \frac{dB}{ds}(0) = -c_0 \). But \( s \) is a decreasing function of \( t \) for large \( t \), thus \( B \) admits an inverse \( t = t(B) \) on \((0, \varepsilon_2)\). Setting \( \widetilde{B} = -c_0 B \), (5.9) then entails
\[
\frac{dS}{dB} = \frac{S + \tilde{E}(\tilde{B}, S)}{B + S + D(B, S)} \quad \text{on } (-\varepsilon_2, 0),
\]
where \( \tilde{E} \) and \( \tilde{D} \) are of the form (5.10). Now it is possible to show (see Lemma 5.5 below) that there exists \( h(z) = \sum_{k \geq 2} h_k z^k \) analytic in 0, such that \( \tilde{S} = S - h(\tilde{B}) \) verifies
\[
\frac{d\tilde{S}}{dB} = \frac{\tilde{S}}{B + \tilde{S} + F(\tilde{B}, \tilde{S})} \quad \text{on } (-\varepsilon_3, 0),
\]
where \( F \) is of the form (5.10). Now, \( \tilde{S} = b + c_0 s + O(s^2) \), hence \( \frac{d\tilde{S}}{ds}(0) = 0 \). But by Lemma 5.2, \( \frac{d\tilde{S}}{ds}(0) = +\infty \), hence \( \tilde{S} \) is a strictly increasing function of \( s \) on \((0, \varepsilon_4)\). It therefore has an inverse \( t = t(\tilde{S}) \) on \((0, \varepsilon_5)\) and
\[
\frac{d\tilde{B}}{dS} = \frac{\tilde{B} + \tilde{S} + F(\tilde{B}, \tilde{S})}{\tilde{S}} \quad \text{on } (0, \varepsilon_5).
\]

According to [6], \( \S \), there exists now an analytic function
\[
u(w, z) = w + \sum_{i,j \geq 0} u_{i,2j} w^i z^j,
\]
such that \( b = u(\tilde{B}, \tilde{S}) \) and \( s = \tilde{S} \) satisfy the equation
\[
\frac{db}{ds} = \frac{b + s}{s} \quad \text{on } (0, \varepsilon_6),
\]
whose solution verifies $b(s) = s(\log s + c)$ for some $c \in \mathbb{C}$ and can thus be extended onto the whole slit plane $\mathbb{C}\setminus(-\infty,0]$. Introducing the parameter $r = \log s \in \mathbb{R} \times (-\pi,\pi)$ gives $b = re^r + ce^r$ and $s = e^r$.

As above, the implicit function theorem entails that there exists an analytic function $v(w,z)$ of the form (5.25), such that $\widetilde{B} = v(b,s)$. We get, as $\text{Re}r \to -\infty$,

$$s = -\frac{1}{c_0}\widetilde{B} + h(\widetilde{B}) + s = -\frac{1}{c_0}re^r + O(e^r),$$

and

$$s'(r) = -\frac{1}{c_0}re^r + O(e^r).$$

Again, we can show that for every $0 < \varphi < \pi$ there exists $r_1 \in \mathbb{R}$, such that the map $s(r)$ is injective on the strip $S_{-}(r_1, \pi - \varphi/2)$ and covers the region $G = \{s \in \mathbb{D}(0,\rho) : |\text{arg} s| < \pi - \varphi\}$ for some $\rho > 0$. Consequently, the inverse $r = r(s)$ exists on $G$. Elementary calculus shows that

$$r = \log(c_0s) - \log \log \frac{1}{s} + \frac{\log \log \frac{1}{s}}{\log \frac{1}{s}} + O\left(\frac{1}{\log \frac{1}{s}}\right) \quad \text{as} \ s \to 0.$$ 

Since $b = -c_0s + S$ and $S = e^r + O(r^2e^{2r})$ as $\text{Re}r \to -\infty$, this immediately implies (5.2).

**Remark 5.4.** The reason why we cannot explicit the constant $K$ in Theorem 5.3 is that we are analysing (5.1) only locally around the point 1. Since the solution of (5.1) with boundary conditions $a(q) = a(1) = 0$ is unique (this follows from the uniqueness of the travelling wave solutions to the FKPP equation), a global analysis of this equation should be able to exhibit the value of $K$. But it is probably easier to refine the probabilistic arguments of Section 2, which already give a lower bound that can be easily made explicit.

The following lemma was needed in the proof of Theorem 5.3.

**Lemma 5.5.** Let $w(z)$ satisfy the following equation on $(0,\varepsilon)$ or $(-\varepsilon,0)$ for some $\varepsilon > 0$:

$$\frac{dw}{dz} = \frac{w + P(w,z)}{z + w + Q(w,z)}, \quad w(0) = 0, \quad (5.27)$$

with $P,Q$ of the form (5.10). Then there exists $h(z) = \sum_{k \geq 2} h_k z^k$ analytic in $0$, such that $\tilde{w} = w - h(z)$ satisfies

$$\frac{d\tilde{w}}{dz} = \frac{\tilde{w}}{z + \tilde{w} + P^*(w,z)}, \quad \tilde{w}(0) = 0, \quad (5.28)$$

with $P^*$ of the form (5.10).

**Proof.** We first show that (5.27) has an analytic solution $w_0(z)$ with $w_0(z) = w_0'(z) = 0$. We choose the ansatz $w_0 = z \cdot w_1$ and get

$$w_1' = \frac{-w_1^2 + zP_1(w_1,z) - zw_1Q_1(w_1,z)}{z(1 + w_1 + zQ_1(w_1,z))}, \quad (5.29)$$

where $P_1(w,z) = P(zw,z)/z^2$ and $Q_1(w,z) = Q(zw,z)/z^2$ are analytic in $(0,0)$. Writing $1/(1 + w_1 + zQ_1(w_1,z))$ as a power series in $w_1$ and $z$ and multiplying the numerator in (5.29) by it, we get

$$w_1' = \frac{-w_1^2 + zR(w_1,z)}{z}, \quad \text{with} \ R(w,z) = \sum_{i,j \geq 0} r_{ij} w^i z^j. \quad (5.30)$$

17
Setting further $w_1 = z(w_2 + r_0)$ with $w_2(0) = 0$ gives
\[ w'_2 = -\frac{w_2 + zR_1(w_2, z)}{z}, \quad R_1(w, z) \text{ analytic in } (0, 0). \quad (5.31) \]

We can now continue as in [6], p. 64 to conclude that the coefficients of $w_2$ can be determined from (5.31) and form a convergent power series in a neighbourhood of $(0, 0)$. We then continue as in [6], p. 65 by showing that $\tilde{w} = w - w_0$ satisfies (5.28).

The asymptotics established in Theorem 5.3 for the infinitesimal generating function can now be readily transferred to the generating functions $F_x(s)$.

**Corollary 5.6.** Let $\lambda_c, \overline{\lambda}_c$ and $d$ be defined as in Theorem 1.2. Set $F_x(s) = E[Z^x | s \in \mathbb{D}]$. Then for every $x \geq 0$ and angle $0 < \varphi < \pi$ there exists $r > 0$, such that $F_x$ possesses an analytical extension (denoted by $F_x$ as well) into $G(\varphi, r)$. Furthermore, for every $x \geq 0$ the following holds as $s \to 1$ in $G(\varphi, r)$.

- If $d = 1$, then
\[
F_x(1 - s) = 1 - c_0 s + c_0^2 x e^{c_0 x} \left( \frac{s}{\log \frac{1}{s}} + \frac{s \log \log \frac{1}{s}}{(\log \frac{1}{s})^2} \right) + O \left( \frac{s}{(\log \frac{1}{s})^2} \right). \quad (5.32)
\]

- If $d > 1$, then there is a polynomial $h_x(s) = \sum_{n=2}^{d} c_n s^n$, such that

\[
\begin{aligned}
&\text{if } d \notin \mathbb{N} : \quad F_x(1 - s) = 1 - \lambda c s + h_x(s) + K_x d s^d + o(s^d), \\
&\text{if } d \in \mathbb{N} : \quad F_x(1 - s) = 1 - \lambda c s + h_x(s) + K_x d s^d \log s + o(s^d), 
\end{aligned} \quad (5.33)\]

where $K_x = K(\overline{\lambda}_c - \lambda_c)$, with $K$ being the constant from Theorem 5.3.

**Proof.** Let $0 < \varphi_0 < \varphi$ and let $r_0$ be such that $a(s)$ can be analytically extended onto $G(\varphi_0, r_0)$. Such an $r_0$ exists by the virtue of Theorem 5.3. Corollary 4.7 now implies that there exists $r > 0$, such that $F_x(s)$ can be analytically extended onto $G(\varphi, r)$ and maps $G(\varphi, r)$ into $G(\varphi_0, r_0)$. Hence, the functions

\[
w(s) = 1 - F_x(1 - s) \quad \text{and} \quad I(s) = \int_s^{w(s)} f^*(1 - r) \, dr,
\]

where $f^*(s)$ is defined as in (4.3), are univalent analytic functions in

\[
D = 1 - G(\varphi, r) = \{s \in \mathbb{D}(0, r) \setminus \{0\} : |\arg s| < \pi - \varphi\}.
\]

In what follows, we always assume that $s \in D$.

We treat the case $d > 1$ first. By Theorem 5.3, $a(1 - s) = -\lambda c s + s^2 h_0(s) + \sigma(s) + o(s^d)$, where $h_0(s)$ is a polynomial in $s$ and $\sigma(s) = K s^d$ or $\sigma(s) = K s^d \log s$, according to whether $d \notin \mathbb{N}$ or $d \in \mathbb{N}$, respectively. It follows that for some polynomial $h_1(s)$ with $h_1(0) = 0$,

\[
I(s) = [h_1(r)]_s^{w(s)} - \int_s^{w(s)} \frac{\sigma(r)}{\lambda_c r^2} \, dr + o(s^{d-1}). \quad (5.35)
\]

By Proposition 4.5, we further have

\[
w(s) = s e^{\lambda_c x} \exp(I(s)) = s e^{\lambda_c x} (1 + I(s) + O(I(s)^2)) \quad \text{as } s \to 0. \quad (5.36)
\]
Repeated application of (5.36) shows that \( w(s) = P(s) + o(s^{d-1}) \), where \( P \) is a polynomial of degree \([d - 1]\) with \( P(0) = 0 \). Straightforward calculus now shows that for some \( C \in \mathbb{C} \),

\[
\int_{s}^{w(s)} \frac{\sigma(r)}{\lambda e^{r^2}} \, dr = \frac{K_x}{e^{\lambda x}} \frac{\sigma(s)}{s} + C s^{d-1} + o(s^{d-1}),
\]

yielding (5.33) and (5.34).

In the critical case \( d = 1 \), Theorem 5.3 tells us that

\[
f^*(1 - s) = -\frac{1}{s} \left( \frac{1}{\log \frac{1}{s}} + \frac{\log \log \frac{1}{s}}{(\log \frac{1}{s})^2} + O \left( \frac{1}{(\log \frac{1}{s})^2} \right) \right).
\]

Write \( \lambda = \lambda_c = c_0 \). For our first approximation of \( w(s) \), we note that

\[
I(s) \sim -\int_{s}^{w(s)} \frac{1}{r \log \frac{1}{r}} \, dr \sim -\frac{\lambda x}{\log \frac{1}{s}} \text{ as } s \to 0,
\]

hence, by the virtue of (5.36),

\[
w(s) = se^{\lambda x} \left( 1 - \frac{\lambda x}{\log \frac{1}{s}} + o \left( \frac{1}{\log \frac{1}{s}} \right) \right).
\]

To obtain a finer approximation, we decompose \( I(s) \) into

\[
I(s) = \int_{s}^{w(s)} f^*(1 - r) \, dr + \int_{se^{\lambda x}}^{w(s)} f^*(1 - r) \, dr =: I_1(s) + I_2(s).
\]

We then have

\[
I_1(s) = -\frac{\lambda x}{\log \frac{1}{s}} - \frac{\lambda x}{\log \frac{1}{s}} \log \frac{1}{s} + O \left( \frac{1}{(\log \frac{1}{s})^2} \right),
\]

and, because of (5.39),

\[
-I_2(s) \sim \int_{se^{\lambda x}}^{w(s)} \frac{1}{r \log \frac{1}{r}} \, dr \sim \frac{\lambda x}{(\log \frac{1}{s})^2}.
\]

This finishes the proof. \( \square \)

**Proof of Theorem 1.2.** Most of the work has been done in Theorem 5.3 and Corollary 5.6. In order to apply the methods from singularity analysis reviewed in Section 4.3, it remains to check the necessary analyticity conditions and to treat the case where the law of \( L \) has a span of 2 or more.

We first turn to the infinitesimal probabilities. It is clear that if \((p_k)_{k \in \mathbb{N}_0}\) represents the infinitesimal reproduction law of \((Z_x)\), the span of \((p_{k+1})_{k \in \mathbb{N}_0}\) is the same as the span \( \delta \) of \( L - 1 \), since the process starts with one individual and the number of individuals increases by \( l - 1 \) when an individual gives birth to \( l \) children.

Let us first treat the case \( \delta = 1 \). Denote \( h(s) = \sum_{k \in \mathbb{N}_0} p_k s^k \). Let \( s_0 \in \partial \mathbb{D} \setminus \{1\} \). Since \( \delta = 1 \), we have \( |h(s_0)| < 1 = |s_0| \), hence \( a_0 := a(s_0) \neq 0 \). We want to show that \( a(s) \) can be analytically extended in \( s_0 \). For this purpose, write the differential equation (5.1) in the form

\[
a' = \frac{2ca + 2\beta(s - f(s))}{a} =: g(a, s).
\]

19
Since the radius of convergence of $f$ is greater than 1 by hypothesis, $g$ is analytic in $(a_0, s_0)$. Fact 4.2 then shows that $s_0$ is a regular point of $a$. Since $s_0 \in \partial \mathbb{D} \setminus \{1\}$ was chosen arbitrarily, this means that every point of $\partial \mathbb{D} \setminus \{1\}$ is a regular point of $a$.

Now let $0 < \varphi < \pi/2$ and $r > 0$ from Theorem 5.3, i.e. $a$ can be analytically extended onto $G(\varphi, r)$ and satisfies one of the asymptotic relations (5.2), (5.3) or (5.4), according to whether $d = 1$, $d \notin \mathbb{N}$ or $d \in \mathbb{N} \setminus \{1\}$, respectively. Since every point of $\partial \mathbb{D} \setminus \{1\}$ is regular, it follows by elementary arguments that $a$ can actually be extended onto $\Delta(\varphi, r_0)$ and satisfies one of the asymptotic relations (5.2), (5.3) and (5.4) immediately prove the first half of Theorem 1.2 (note our definition of $O(\cdot)$ in Section 4.1).

If $\delta \geq 2$, then $p_{k+\delta n} = 0$ for all $k \in \{2, \ldots , n\}$ and $a(s)/s$ is $\delta$-periodic. We can then show as above that $a$ has exactly $\delta$ singularities on $\partial \mathbb{D}$ located on $1, e^{i2\pi/\delta}, \ldots, e^{i2(\delta - 1)\pi/\delta}$. If we set

$$h(s) = \alpha\left(\sum_n p_{1+\delta n} s^n - 1\right),$$

then $a(s) = s \cdot h(s^\delta)$, and so the only singularity on $\partial \mathbb{D}$ of $h$ is the point 1. Furthermore, if $q(s)$ denotes a version of $\sqrt[\delta]{s}$ which is analytic in a neighbourhood $U$ of 1 and which satisfies $q(1) = 1$, then $h(s) = a(q(s))/q(s)$ near 1. But since $q'(1) = 1/\delta$, we have

$$h(1 - s) = a(1 - \left(\frac{1}{\delta} s + c_2 s^2 + c_3 s^3 + \cdots\right)(1 + c'_1 s + c'_2 s + \cdots),$$

for some constants $c_n, c'_n$, and so equations (5.2), (5.3) and (5.4) transfer to $h$ with the coefficient of the main singular term divided by $\delta^d$. We can therefore use Facts 4.3 and 4.4 for $h$ to conclude.

The proof of the statements for the probabilities $P(Z_x = n)$ is completely analogous, drawing on Corollary 5.6 instead. The only thing that remains to show is that for each $x \geq 0$, $F_x$ is analytic in every point $s_0 \in \partial \mathbb{D}$ (we assume $\delta = 1$, the case $\delta \geq 2$ is handled as above). To this end, we deduce from Kolmogorov’s forward and backward equations (see e.g. [4], p. 106 or [12], p. 99) the equation

$$F'_x(s) = \frac{a(F_x(s))}{a(s)} \text{ on } \mathbb{D}.$$

Now, since $a(s)$ is analytic and non-zero in $s_0$, the function $f(w,s) = a(w)/a(s)$ is analytic in $(a(s_0), s_0)$, hence we can apply Fact 4.2 to conclude that $F_x$ is analytic in $s_0$ as well. This finishes the proof of the theorem.

**Acknowledgements**

The author is grateful to Elie Aïdékon for stimulating discussions and to Louis Boutet de Monvel for useful comments on the work.

**References**


