Strongly singular MASAs and mixing actions in finite von Neumann algebras

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Abstract

Let Γ be a countable group and let Γ_0 be an infinite abelian subgroup of Γ. We prove that if the pair (Γ, Γ_0) satisfies some combinatorial condition called (SS), then the abelian subalgebra $A = L(\Gamma_0)$ is a singular MASA in $M = L(\Gamma)$ which satisfies a weakly mixing condition. If moreover it satisfies a stronger condition called (ST), then it provides a singular MASA with a strictly stronger mixing property. We describe families of examples of both types coming from free products, HNN extentions and semidirect products, and in particular we exhibit examples of singular MASA's that satisfy the weak mixing condition but not the strong mixing one.

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1 Introduction

If M is a von Neumann algebra, if A is a maximal abelian von Neumann subalgebra of M (henceforth called a MASA), let $\mathcal{N}_M(A)$ be the *normaliser* of A in M: it is the subgroup of the unitary group $U(M)$ of all elements u such that $uAu^* = A$. Then A is *singular* in M if $\mathcal{N}_M(A)$ is as small as possible, namely, if $\mathcal{N}_M(A) = U(A)$. Until recently, it was quite difficult in general to exhibit singular MASA's in von Neumann algebras, though S. Popa proved among others in [11] that all separable type II_1 factors admit singular MASA's.

Example. This example is due to F. Radulescu [12]. Let $L(F_N)$ be the factor associated to the non abelian free group on N generators $X_1, \ldots, X_N$ and let A be the abelian von Neumann subalgebra generated by $X_1 + \ldots + X_N + X_1^{-1} + \ldots + X_N^{-1}$. Then A is a singular MASA in $L(F_N)$. A is called the *radial* or *Laplacian* subalgebra because its elements coincide with convolution operators by functions that depend only on the length of the elements of F_N.

Recently, T. Bildea generalized Radulescu's example in [2] by using the notion of asymptotic homomorphism (see below): Let G be either F_N or a free product $G_1 \star \ldots \star G_m$ of $m \geq 3$ groups, each finite of order $p \geq 2$ with $m \geq p$. Then the radial subalgebra is a singular MASA in $L(G)^{\bar{\otimes}_k}$.

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Motivated by S. Popa's articles [11] and [10], the authors of [14] and [17] introduced sufficient conditions on an abelian von Neumann subalgebra A of a finite von Neumann algebra M that imply that A is even a strongly singular MASA in M. This means that A satisfies the apparently stronger condition: for all $u \in U(M)$, one has

$$\sup_{x \in M, \|x\| \leq 1} \|E_A(x) - E_{uAu^*}(x)\|_2 \geq \|u - E_A(u)\|_2.$$

In fact, it was proved by A. Sinclair, R. Smith, S. White and A. Wiggins in [19] that all singular MASA's are strongly singular. Nevertheless, it is sometimes easier to prove directly strong singularity.

Proposition 1.1 *([14]) Suppose that the pair $A \subset M$ satisfies the following condition (WM): $\forall x, y \in M$ and $\forall \varepsilon > 0$, there exists $v \in U(A)$ such that*

$$\|E_A(xvy) - E_A(x)vE_A(y)\|_2 \leq \varepsilon.$$

Then A is a strongly singular MASA in M.

Let us reproduce an adaptation of the proof of Lemma 2.1 of [14] for convenience:

Proof. Fix $u \in U(M)$ and $\varepsilon > 0$, and take $x = u^*$, $y = u$. There exists $v \in U(A)$ such that

$$\|E_A(u^*vu) - E_A(u^*)vE_A(u)\|_2 = \|E_A(v^*u^*vu) - E_A(u^*)E_A(u)\|_2 \leq \varepsilon.$$

(Commutativity of A is crucial here.) Hence, we get :

$$\begin{aligned}
\|E_A - E_{uAu^*}\|_{\infty,2}^2 &\geq \|v - uE_A(u^*vu)u^*\|_2^2 \\
&= \|u^*vu - E_A(u^*vu)\|_2^2 \\
&= 1 - \|E_A(u^*vu)\|_2^2 \\
&\geq 1 - (\|E_A(u^*)vE_A(u)\|_2 + \varepsilon)^2 \\
&\geq 1 - (\|E_A(u)\|_2 + \varepsilon)^2 \\
&= \|u - E_A(u)\|_2^2 - 2\varepsilon\|E_A(u)\|_2 - \varepsilon^2.
\end{aligned}$$

As ε is arbitrary, we get the desired inequality. ☐

Earlier, in [17], A. Sinclair and R. Smith used a stronger condition (that we call *condition (AH)* here) in order to get singular MASA's:

Given $v \in U(A)$, the conditional expectation E_A is an *asymptotic homomorphism with respect to v* if

$$(AH) \qquad \lim_{|k| \to \infty} \|E_A(xv^k y) - E_A(x)v^k E_A(y)\|_2 = 0$$

for all $x, y \in M$.

Both conditions (WM) and (AH) remind one of mixing properties of group actions, because of the following equality (since A abelian):

$$\|E_A(vxv^*y) - E_A(x)E_A(y)\|_2 = \|E_A(xv^*y) - E_A(x)v^*E_A(y)\|_2$$

$\forall x, y \in M, \ \forall v \in U(A)$.

Thus our point of view here is the following: every subgroup G of $U(A)$ acts by inner automorphisms on M by conjugation

$$\sigma_v(x) = vxv^* \quad \forall v \in G, \ \forall x \in M,$$

and as will be seen, condition (WM) is in some sense equivalent to a weakly mixing action of G, and condition (AH) is equivalent to a strongly mixing action of the cyclic subgroup of $U(A)$ generated by the distinguished unitary v.

Section 2 is devoted to weakly mixing actions of subgroups $G \subset U(A)$, to crossed products by weakly mixing actions of countable groups, and to pairs (Γ, Γ_0) where Γ_0 is an abelian subgroup of the countable group Γ which provide pairs $A = L(\Gamma_0) \subset M = L(\Gamma)$ with A weakly mixing in M (see Definition 2.1). It turns out that the weak mixing property is completely determined by a combinatorial property of the pair (Γ, Γ_0) called *condition (SS)*, already introduced in [14] to provide examples of strongly singular MASA's.

Section 3 is devoted to a (strictly) stronger condition (called *condition (ST)*) on pairs (Γ, Γ_0) as above which is related to strongly mixing actions of groups.

Section 4 contains various families of examples (free products with amalgamation, HNN extentions, semidirect products), and some of them prove that condition (ST) is strictly stronger than condition (SS) so that they provide two distinct levels of "mixing MASA's": the weak ones and the strong ones.

We will see that many pairs (Γ, Γ_0) satisfy a strictly stronger condition than condition (ST), based on malnormal subgroups: Γ_0 is said to be a *malnormal subgroup* of Γ if for every $g \in \Gamma \setminus \Gamma_0$, one has $g\Gamma_0 g^{-1} \cap \Gamma_0 = \{1\}$. Such pairs have been considered first in the pioneering article [10] in order to control normalizers (in particular relative commutants) of $L(\Gamma_0)$ and of its diffuse subalgebras, and, as a byproduct, to produce singular MASA's $L(\Gamma_0)$. They were also used in [13], [17] to provide more examples of (strongly) singular MASA's in type II_1 factors that fit Popa's criteria of Proposition 4.1 in [10].

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2 Weak mixing

In the rest of the article, M denotes a finite von Neumann algebra, and τ is some normal, faithful, finite, normalised trace on M (henceforth simply called a *trace* on M). It defines a scalar product on M: $\langle a, b\rangle = \tau(b^*a) = \tau(ab^*)$, and the corresponding completion is the Hilbert space $L^2(M, \tau)$ on which M acts by left multiplication extending the analogous operation on M. As usual, we denote by $\|\cdot\|_2$ the corresponding Hilbert norm. When the trace τ must be specified, we write $\|\cdot\|_{2,\tau}$. We denote also by $M_\star$ the predual of M, i.e. the Banach space of all normal linear functionals on M. We will always assume for convenience that $M_\star$ is separable, or equivalently, that $L^2(M, \tau)$ is a separable Hilbert space. Recall that, for every $\varphi \in M_\star$, there exist $\xi, \eta \in L^2(M, \tau)$ such that $\varphi(x) = \langle x\xi, \eta\rangle$ for all $x \in M$.

Let Γ be a countable group and let Γ_0 be an abelian subgroup of Γ. Denote by $L(\Gamma)$ (respectively $L(\Gamma_0)$) the von Neumann algebra generated by the left regular representation λ of Γ (respectively Γ_0). Recall that $\lambda : \Gamma \to U(\ell^2(\Gamma))$ is defined by $(\lambda(g)\xi)(h) = \xi(g^{-1}h)$ for all $g, h \in \Gamma$ and $\xi \in \ell^2(\Gamma)$. It extends linearly to the group algebra $\mathbb{C}\Gamma$, and $L(\Gamma)$ is the weak-operator closure of $\lambda(\mathbb{C}\Gamma) =: L_f(\Gamma)$. The normal functional $\tau(x) = \langle x\delta_1, \delta_1\rangle$ is a faithful trace on $L(\Gamma)$. For $x \in L(\Gamma)$, denote by $\sum_{g\in\Gamma} x(g)\lambda(g)$ its "Fourier expansion": $x(g) = \tau(x\lambda(g^{-1}))$ for every $g \in \Gamma$, and the series $\sum_g x(g)\lambda(g)$ converges to x in the

$\|\cdot\|_2$-sense so that $\sum_{g\in\Gamma}|x(g)|^2 = \|x\|_2^2$.

Let $1 \in B$ be a unital von Neumann subalgebra of the von Neumann algebra M gifted with some trace τ as above and let E_B be the τ-preserving conditional expectation of M onto B. Then E_B is characterised by the following two conditions: $E_B(x) \in B$ for all $x \in M$ and $\tau(E_B(x)b) = \tau(xb)$ for all $x \in M$ and all $b \in B$. It enjoys the well-known properties:

(1) $E_B(b_1 x b_2) = b_1 E_B(x) b_2$ for all $x \in M$ and all $b_1, b_2 \in B$;

(2) $\tau \circ E_B = \tau$.

(3) If $M = L(\Gamma)$ is the von Neumann algebra associated to the countable group Γ, if H is a subgroup of Γ, if $B = L(H)$ and if $x \in M$, then $E_B(x) = \sum_{h\in H} x(h)\lambda(h)$.

Let also α be a τ-preserving action of Γ on M. Recall that it is *weakly mixing* if, for every finite set $F \subset M$ and for every $\varepsilon > 0$, there exists $g \in \Gamma$ such that

$$|\tau(\alpha_g(a)b) - \tau(a)\tau(b)| < \varepsilon \quad \forall a, b \in F.$$

In [8], S. Popa introduced a relative version of weakly mixing actions:
If $1 \in A \subset M$ is a von Neumann subalgebra such that $\alpha_g(A) = A\ \forall g \in \Gamma$, the action α is called *weakly mixing relative to* A if, for every finite set $F \subset M \ominus A := \{x \in M \ :\ E_A(x) = 0\}$, for every $\varepsilon > 0$, one can find $g \in \Gamma$ such that

$$\|E_A(x^*\alpha_g(y))\|_2 \leq \varepsilon \quad \forall x, y \in F.$$

Here we consider a (countable) subgroup G of the unitary group $U(A)$ which acts on M by conjugation:

$$\sigma_v(x) = vxv^* \quad \forall v \in G,\ \forall x \in M.$$

This allows us to introduce the following definition:

Definition 2.1 *The abelian von Neumann subalgebra A is* **weakly mixing in** *M if there exists a subgroup G of $U(A)$ such that the corresponding action by conjugation is weakly mixing relative to A in Popa's sense.*

Notice that, A being abelian, it is equivalent to asking that, for every finite set $F \subset M$ and every $\varepsilon > 0$, there exists $v \in G$ such that

$$\|E_A(xvy) - E_A(x)vE_A(y)\|_2 \leq \varepsilon \quad \forall x, y \in F.$$

Weakly mixing actions provide singular MASA's, as was already kwown to many people; see for instance [7], [9] and [14]. More precisely, one has:

Proposition 2.2 *Let Γ_0 be an abelian group which acts on a finite von Neumann algebra B and which preserves a trace τ, then the abelian von Neumann subalgebra $A = L(\Gamma_0)$ of the crossed product $M = B \rtimes \Gamma_0$ is weakly mixing in M if and only if the action of Γ_0 is.*

Sketch of proof. Observe first that, for $x = \sum_\gamma x(\gamma)\lambda(\gamma) \in M$, the projection of x onto A is given by $E_A(x) = \sum_\gamma \tau(x(\gamma))\lambda(\gamma)$.

Suppose that the action α of Γ_0 is weakly mixing. It suffices to prove that, for all finite sets $E \subset \Gamma_0$ and $F \subset B$ and for every $\varepsilon > 0$, there exists $\gamma \in \Gamma_0$ such that

$$\|E_A(a\lambda(g)\lambda(\gamma)b\lambda(h)) - E_A(a\lambda(g))\lambda(\gamma)E_A(b\lambda(h))\|_2 < \varepsilon$$

for all $a, b \in F$ and all $g, h \in E$. But, $E_A(a\lambda(g)) = \tau(a)\lambda(g)$, for all $a \in F$ and $g \in \Gamma$, and $E_A(a\lambda(g)\lambda(\gamma)b\lambda(h)) = \tau(a\alpha_\gamma(\alpha_g(b)))\lambda(g\gamma h)$ which suffices to get the conclusion.

Conversely, if G is a subgroup of $U(A)$ whose action is weakly mixing relative to A, let F be as above and let $\varepsilon > 0$. There exists a unitary $u \in G$ such that

$$\sum_{a,b\in F} \|E_A(aub) - E_A(a)uE_A(b)\|_2^2 < \frac{\varepsilon^2}{2}.$$

As above, since $u = \sum_\gamma u(\gamma)\lambda(\gamma)$ with $u(\gamma) \in \mathbb{C}$ for every γ, one has

$$E_A(aub) = \sum_\gamma u(\gamma)\tau(a\alpha_\gamma(b))\lambda(\gamma)$$

and

$$E_A(a)uE_A(b) = \sum_\gamma u(\gamma)\tau(a)\tau(b)\lambda(\gamma).$$

This implies that

$$\sum_{a,b\in F}\sum_\gamma |u(\gamma)|^2|\tau(a\alpha_\gamma(b)) - \tau(a)\tau(b)|^2 < \frac{\varepsilon^2}{2}.$$

As $\sum_\gamma |u(\gamma)|^2 = 1$, this implies the existence of $\gamma \in \Gamma_0$ such that

$$\sum_{a,b\in F} |\tau(a\alpha_\gamma(b)) - \tau(a)\tau(b)|^2 < \varepsilon^2.$$

$\square$

When $M = L(\Gamma)$ and $A = L(\Gamma_0)$, where Γ_0 is an abelian subgroup of Γ, it turns out that weak mixing of A is equivalent to a combinatorial property of the pair of groups (Γ, Γ_0) as it appears in [14]:

Proposition 2.3 *For a pair (Γ, Γ_0) as above, the following two conditions are equivalent:*

(1) (SS) For every finite subset $C \subset \Gamma \setminus \Gamma_0$, there exists $\gamma \in \Gamma_0$ such that $g\gamma h \notin \Gamma_0$ for all $g, h \in C$;

(2) (WM) $A = L(\Gamma_0)$ is weakly mixing in $M = L(\Gamma)$.

Proof. If (Γ, Γ_0) satisfies condition (SS), take $G = \lambda(\Gamma_0)$. Let F be a finite subset of $L(\Gamma)$ such that $E_A(x) = 0$ for every $x \in F$. By density, one assumes that F is contained in $L_f(\Gamma)$. We intend to prove that there exists $\gamma \in \Gamma_0$ such that $E_A(x\lambda(\gamma)y) = 0$ for all $x, y \in F$. By the above assumptions, there exists a finite set $C \subset \Gamma \setminus \Gamma_0$ such that every $x \in F$ has support in C. Fix x and y in F and choose $\gamma \in \Gamma_0$ as in condition (SS) with respect to C. Then

$$\begin{aligned} E_A(x\lambda(\gamma)y) &= E_A\left(\sum_{g,h\in C} x(g)y(h)\lambda(g\gamma h)\right) \\ &= \sum_{g,h\in C, g\gamma h\in\Gamma_0} x(g)y(h)\lambda(g\gamma h) = 0 \end{aligned}$$

since the set $\{(g,h) \in C \times C \ : \ g\gamma h \in \Gamma_0\}$ is empty. This shows that A is weakly mixing in M.

Conversely, if A satisfies the latter condition in M, choose a subgroup G of $U(A)$ such that its action on M is weakly mixing relative to A and let $C \subset \Gamma \setminus \Gamma_0$ be a finite set. Take $F = \lambda(C)$, and observe that $E_A(\lambda(g)) = 0$ for every $g \in C$. Finally, choose any $0 < \varepsilon < 1/2$. Then there exists $v \in G$ such that

$$\sum_{g,h\in C} \|E_A(\lambda(g)v\lambda(h)\|_2^2 < \varepsilon^2.$$

We have $v = \sum_{\gamma\in\Gamma_0} v(\gamma)\lambda(\gamma)$ and $\|v\|_2^2 = 1 = \sum_\gamma |v(\gamma)|^2$. Moreover,

$$\begin{aligned} E_A(\lambda(g)v\lambda(h)) &= \sum_{\gamma\in\Gamma_0} v(\gamma)E_A(\lambda(g\gamma h)) \\ &= \sum_{\gamma\in\Gamma_0, g\gamma h\in\Gamma_0} v(\gamma)\lambda(g\gamma h). \end{aligned}$$

Thus,

$$\|E_A(\lambda(g)v\lambda(h)\|_2^2 = \sum_{\gamma\in\Gamma_0, g\gamma h\in\Gamma_0} |v(\gamma)|^2.$$

Assume that for every $\gamma \in \Gamma_0$ one can find g_γ and h_γ in C such that $g_\gamma\gamma h_\gamma \in \Gamma_0$. Then we would get

$$\begin{aligned} 1 = \sum_{\gamma\in\Gamma_0} |v(\gamma)|^2 &= \sum_{\gamma\in\Gamma_0, g_\gamma\gamma h_\gamma\in\Gamma_0} |v(\gamma)|^2 \\ &\leq \sum_{g,h\in C} \sum_{\gamma\in\Gamma_0, g\gamma h\in\Gamma_0} |v(\gamma)|^2 < \varepsilon^2 \end{aligned}$$

which is a contradiction. □

Before ending the present section, let us recall examples of pairs (Γ, Γ_0) that satisfy condition (SS). They are taken from [14]. Let Γ be a group of isometries of some metric space (X, d) and let Γ_0 be an abelian subgroup of Γ. Assume that there is a Γ_0-invariant subset Y of X such that

(C1) there exists a compact set $K \subset Y$ such that $\Gamma_0 K = Y$;

(C2) if $Y \subset_\delta g_1Y \cup g_2Y \cup \ \ldots \ \cup g_nY$ for some g_j's in Γ, and some $\delta > 0$, then there exists j such that $g_j \in \Gamma_0$. (Recall that for subsets P, Q of X and $\delta > 0$, $P \subset_\delta Q$ means that $d(p, Q) \leq \delta$ for every $p \in P$.)

Then it is proved in Proposition 4.2 of [14] that the pair (Γ, Γ_0) satisfies condition (SS).

Now let G be a semisimple Lie group with no centre and no compact factors. Let Γ be a torsion free cocompact lattice in G. Then Γ acts freely on the symmetric space $X = G/K$, where K is a maximal compact subgroup of G, and the quotient manifold $M = \Gamma\backslash X$ has universal covering space X. In particular, Γ is the fundamental group $\pi(M)$ of M. Let r be the rank of X and let $T^r \subset M$ be a totally geodesic embedding of a flat r-torus in M, so that the inclusion $i : T^r \to M$ induces an injective homomorphism $i_* : \pi(T^r) \to \pi(M)$. Thus $\Gamma_0 = i_*\pi(T^r) \cong \mathbb{Z}^r$ is an abelian subgroup of Γ in a natural way. Then it is proved in Theorem 4.9 of [14] that the pair (Γ, Γ_0) satisfies conditions (C1) and (C2) above, hence that it satisfies condition (SS) as well. In the same vein, the authors also get examples of pairs coming from groups acting cocompactly on locally finite euclidean buildings.

3 Strong mixing

Let F be the Thompson's group; it admits the following presentation:

$$F = \langle x_0,\ x_1,\ \ldots \mid x_i^{-1} x_n x_i = x_{n+1},\ 0 \leq i < n \rangle.$$

Let Γ_0 be the subgroup generated by x_0. In [6], the first named author proved that the pair (F, Γ_0) satisfies a stronger property than condition (SS) that was called *condition (ST)* and which is described as follows:

Definition 3.1 *Let (Γ, Γ_0) be a pair as above. Then we say that it satisfies **condition (ST)** if, for every finite subset $C \subset \Gamma \setminus \Gamma_0$ there exists a finite subset $E \subset \Gamma_0$ such that $gg_0h \notin \Gamma_0$ for all $g_0 \in \Gamma_0 \setminus E$ and all $g, h \in C$.*

Remark. Observe that, taking finite unions of exceptional sets E of Γ_0 if necessary, condition (ST) is equivalent to:

For all $g, h \in \Gamma \setminus \Gamma_0$, there exists a finite subset E of Γ_0 such that $g\gamma h \notin \Gamma_0$ for all $\gamma \in \Gamma_0 \setminus E$.

Since the subset $\Gamma \setminus \Gamma_0$ is stable under the mapping $g \mapsto g^{-1}$, when Γ_0 is an infinite cyclic group generated by some element t, condition (ST) is still equivalent to:

For all $g, h \in \Gamma \setminus \Gamma_0$, there exists a positive integer N such that, for every $|k| > N$, one has $gt^k h \notin \Gamma_0$.

For future use in the present section, for every subset S of a group Γ we put $S^* = S \setminus \{1\}$.

We observe in the next section that condition (ST) is strictly stronger than condition (SS); examples are borrowed from Section 5 of [18]. As it is the case for condition (SS), it turns out that condition (ST) is completely characterized by the pair of von Neumann algebras $L(\Gamma_0) \subset L(\Gamma)$.

To prove that, we need some facts and definitions. Let M and τ be as above. Let us say that a subset S of M is *τ-orthonormal* if $\tau(xy^*) = \delta_{x,y}$ for all $x, y \in S$. We will need a weaker notion which is independent of the chosen trace τ.

Proposition 3.2 *Let M be a finite von Neumann algebra, let τ be a finite trace on M as above and let S be an infinite subset of the unitary group $U(M)$. The following conditions are equivalent:*

(1) for every $\varphi \in M_\star$ and for every $\varepsilon > 0$, there exists a finite subset F of S such that $|\varphi(u)| \leq \varepsilon$ for all $u \in S \setminus F$;

(2) for every $x \in M$ and for every $\varepsilon > 0$, there exists a finite set $F \subset S$ such that $|\tau(ux)| \leq \varepsilon$ for all $u \in S \setminus F$;

(2') for any trace τ' on M, for every $x \in M$ and for every $\varepsilon > 0$, there exists a finite set $F \subset S$ such that $|\tau'(ux)| \leq \varepsilon$ for all $u \in S \setminus F$;

(3) for every τ-orthonormal finite set $\{x_1, \ldots, x_N\} \subset M$ and for every $\varepsilon > 0$ there exists a finite set $F \subset S$ such that

$$\sup\{|\tau(ux^*)|\ ;\ x \in \mathrm{span}\{x_1, \ldots, x_N\},\ \|x\|_2 \leq 1\} \leq \varepsilon \quad \forall u \in S \setminus F;$$

(3') for every trace τ' on M, for every τ'-orthonormal finite set $\{x_1, \ldots, x_N\} \subset M$ and for every $\varepsilon > 0$ there exists a finite set $F \subset S$ such that

$$\sup\{|\tau'(ux^*)| \ ; \ x \in \mathrm{span}\{x_1, \ldots, x_N\}, \ \|x\|_{2,\tau'} \leq 1\} \leq \varepsilon \quad \forall u \in S \setminus F;$$

In particular, if $S \subset U(M)$ satisfies the above conditions, if θ is a $*$-isomorphism of M onto some von Neumann algebra N, then $\theta(S) \subset U(N)$ satisfies the same conditions.

Proof. (1) $\Rightarrow$ (2′) $\Rightarrow$ (2) and (3′) $\Rightarrow$ (3) are trivial.
(2) $\Rightarrow$ (3′): If τ' is a trace on M, if $\{x_1, \ldots, x_N\} \subset M$ is τ'-orthonormal and if $\varepsilon > 0$ is fixed, there exists $h \in M$ such that

$$\|\tau' - \tau(h\cdot)\| \leq \frac{\varepsilon}{2\sqrt{N} \cdot \max \|x_j\|}.$$

Furthermore, one can find a finite set $F \subset S$ such that

$$|\tau(ux_j^*h)| \leq \frac{\varepsilon}{2\sqrt{N}} \quad \forall u \in S \setminus F \quad \text{and} \quad \forall j = 1, \ldots, N.$$

This implies that

$$|\tau'(ux_j^*)| \leq \frac{\varepsilon}{\sqrt{N}} \quad \forall u \in S \setminus F \quad \text{and} \quad \forall j = 1, \ldots, N.$$

Let $x \in \mathrm{span}\{x_1, \ldots, x_N\}$, $\|x\|_{2,\tau'} \leq 1$. Let us write $x = \sum_{j=1}^N \xi_j x_j$, where $\xi_j = \tau'(xx_j^*)$, and $\sum_{j=1}^N |\xi_j|^2 = \|x\|_{2,\tau'}^2 \leq 1$ since the x_j's are τ'-orthonormal. Hence we get, for $u \in S \setminus F$:

$$|\tau'(ux^*)| = |\sum_{j=1}^N \overline{\xi_j} \tau'(ux_j^*)| \leq \left(\sum_{j=1}^N |\xi_j|^2\right)^{1/2} \left(\sum_{j=1}^N |\tau'(ux_j^*)|^2\right)^{1/2} \leq \varepsilon$$

uniformly on the set $\{x \in \mathrm{span}\{x_1, \ldots, x_N\} \ ; \|x\|_{2,\tau'} \leq 1\}$.
(3) $\Rightarrow$ (1): Let $\varphi \in M_\star$ and $\varepsilon > 0$. We choose $x \in M$ such that $\|\varphi - \tau(\cdot x)\| \leq \varepsilon/2$. Applying condition (3) to the singleton set $\{x/\|x\|_2\}$ as orthonormal set, we find a finite subset F of S such that $|\tau(ux)| \leq \varepsilon/2$ for every $u \in S \setminus F$. Hence we get $|\varphi(u)| \leq \varepsilon$ for all $u \in S \setminus F$.

The last statement follows readily from condition (1). □

Definition 3.3 *Let M be a finite von Neumann algebra gifted with some fixed finite trace τ. We say that an infinite subset $S \subset U(M)$ is* ***almost orthonormal*** *if it satisfies the equivalent conditions in Proposition 3.2.*

Remarks. (1) Since M has separable predual, an almost orthonormal subset S of $U(M)$ is necessarily countable. Indeed, let $\{x_n \ ; \ n \geq 1\}$ be a $\|\cdot\|_2$-dense countable subset of the unit ball of M with respect to the operator norm. For $n \geq 1$, put

$$S_n = \{u \in S \ ; \ \max_{1 \leq j \leq n} |\tau(ux_j^*)| \geq \frac{1}{n}\}.$$

Then each S_n is finite, $S_n \subset S_{n+1}$ for every n and $S = \bigcup_n S_n$. Thus, if $S = (u_n)_{n \geq 1}$ is a sequence of unitary elements, then S is almost orthonormal if and only if u_n tends weakly to zero. In particular, every diffuse von Neumann algebra contains almost orthonormal sequences of unitaries.

(2) The reason why we choose the above definition comes from the fact that if S is almost orthonormal in M, then for every $u \in S$, for every $\varepsilon > 0$, there exists a finite set $F \subset S$ such that $|\tau(v^*u)| < \varepsilon$ for all $v \in S \setminus F$. A typical example of an almost orthonormal subset in a finite von Neumann algebra is a τ-orthonormal subset S of $U(M)$ for *some* trace τ on M: indeed, for every $x \in M$, the series $\sum_{u\in S} |\tau(xu^*)|^2$ converges. For instance, let $v \in U(M)$ be such that $\tau(v^k) = 0$ and for all integers $k \in \mathbb{Z} \setminus \{0\}$. Then the subgroup generated by v is almost orthonormal. As another example, let Γ be a countable group and let Γ_1 be an infinite subgroup of Γ. Set $G = \lambda(\Gamma_1)$. Then G is almost orthonormal in $M = L(\Gamma)$. Indeed, if $x \in L(\Gamma)$, then $\tau(\lambda(g)x) = \tau(x\lambda(g)) = x(g^{-1})$ obviously tends to 0 as g tends to infinity of Γ_1.

We come now to the main definition of our article. To motivate it, recall that if Γ is a (countable) group and if α is a τ-preserving action of Γ on M, then it is called *strongly mixing* if, for every finite set $F \subset M$ and for every $\varepsilon > 0$, there exists a finite set $E \subset \Gamma$ such that

$$|\tau(\alpha_g(a)b) - \tau(a)\tau(b)| < \varepsilon$$

for all $a, b \in F$ and all $g \notin E$.

Definition 3.4 *Let M and τ be as above and let A be an abelian, unital von Neumann subalgebra of M. Let G be a subgroup of $U(A)$. We say that the action of G is* ***strongly mixing relative to*** *A if, for all $x, y \in M$, one has:*

$$\lim_{u\to\infty, u\in G} \|E_A(uxu^{-1}y) - E_A(x)E_A(y)\|_2 = 0.$$

Furthermore, we say that A itself is ***strongly mixing*** *in M if, for every almost orthonormal infinite subgroup G of $U(A)$, the action of G by inner automorphisms on M is strongly mixing relative to A.*

Remark. The above property is independent of the trace τ and it is a conjugacy invariant. Indeed, E_A is the unique conditional expectation onto A and almost orthonormality is independent of the chosen trace.

We present now our main result.

Theorem 3.5 *Let Γ be an infinite group and let Γ_0 be an infinite abelian subgroup of Γ. Let $M = L(\Gamma)$ and $A = L(\Gamma_0)$ be as above. Then the following conditions are equivalent:*

(1) the action of Γ_0 by inner automorphisms on M is strongly mixing relative to A;

(2) for every finite subset $C \subset \Gamma \setminus \Gamma_0$ there exists a finite subset $E \subset \Gamma_0$ such that $gg_0h \notin \Gamma_0$ for all $g_0 \in \Gamma_0 \setminus E$ and all $g, h \in C$; namely, the pair (Γ, Γ_0) satisfies condition (ST);

(3) for every almost orthonormal infinite subset $S \subset U(A)$, for all $x, y \in M$ and for every $\varepsilon > 0$, there exists a finite subset $F \subset S$ such that

$$\|E_A(uxu^*y) - E_A(x)E_A(y)\|_2 < \varepsilon \quad \forall u \in S \setminus F;$$

(4) A is strongly mixing in M.

Proof. Trivially, (3) $\Rightarrow$ (4) $\Rightarrow$ (1).
(1) $\Rightarrow$ (2): If C is as in (2), set

$$x = \sum_{g \in C} \lambda(g).$$

Thus $E_A(x) = 0$, and there exists a finite subset $E \subset \Gamma_0$ such that

$$\|E_A(\lambda(g_0^{-1})x\lambda(g_0)x)\|_2 = \|E_A(x\lambda(g_0)x)\|_2 < 1 \quad \forall g_0 \in \Gamma_0 \setminus E.$$

But

$$E_A(x\lambda(g_0)x) = E_A\left(\sum_{g,h \in C} \lambda(gg_0h)\right) = \sum_{g,h \in C, gg_0h \in \Gamma_0} \lambda(gg_0h),$$

hence $\|E_A(x\lambda(g_0)x)\|_2^2 = |\{(g,h) \in C \times C \ ; \ gg_0h \in \Gamma_0\}| < 1$ for all $g_0 \notin E$, which implies that $gg_0h \notin \Gamma_0$ for $g_0 \notin E$ and for all $g, h \in C$.
(2) $\Rightarrow$ (3): Let S be an almost orthonormal infinite subset of $U(A)$, let $x, y \in M$ and fix $\varepsilon > 0$. Using A-bilinearity of E_A, we can assume that $E_A(x) = E_A(y) = 0$, and we have to prove that one can find a finite set $F \subset S$ such that $\|E_A(uxu^*y)\|_2 < \varepsilon$ for all $u \in S \setminus F$. To begin with, let us assume furthermore that x and y have finite support, and let us write $x = \sum_{g \in C} x(g)\lambda(g)$ and $y = \sum_{h \in C} y(h)\lambda(h)$ with $C \subset \Gamma \setminus \Gamma_0$ finite. Let $E \subset \Gamma_0$ be as in (2) with respect to the finite set C of $\Gamma \setminus \Gamma_0$, namely $gg_0h \notin \Gamma_0$ for all $g, h \in C$, and $g_0 \notin E$. We claim then that $E_A(x\lambda(g_0^{-1})y) = 0$ if $g_0 \in \Gamma_0 \setminus E^{-1}$. Indeed, if $g_0 \in \Gamma_0 \setminus E^{-1}$, we have:

$$\begin{aligned} E_A(x\lambda(g_0^{-1})y) &= E_A\left(\sum_{g,h \in C} x(g)y(h)\lambda(gg_0^{-1}h)\right) \\ &= \sum_{g,h \in C, gg_0^{-1}h \in \Gamma_0} x(g)y(h)\lambda(gg_0^{-1}h) = 0 \end{aligned}$$

because $g_0^{-1} \in \Gamma_0 \setminus E$.
Choosing $\lambda(E) \subset A$ as a τ-orthonormal system, there exists a finite subset F of S such that, if $u \in S \setminus F$:

$$\sup\{|\tau(uz^*)| \ ; \ z \in \mathrm{span}(\lambda(E)), \ \|z\|_2 \leq 1\} < \frac{\varepsilon^2}{|E|\|x\|\|y\|}.$$

Thus, for fixed $u \in S \setminus F$, take $z := \sum_{g_0 \in E} u(g_0)\lambda(g_0)$, so that $z \in \mathrm{span}(\lambda(E))$, $\|z\|_2 \leq 1$ and

$$\sum_{g_0 \in E} |u(g_0)|^2 = \tau(uz^*) < \frac{\varepsilon^2}{|E|\|x\|\|y\|}.$$

Then

$$\begin{aligned} \|E_A(uxu^*y)\|_2 &= \|uE_A(xu^*y)\|_2 = \|E_A(xu^*y)\|_2 \\ &\leq \sum_{g_0 \in E} |u(g_0)|\|E_A(x\lambda(g_0^{-1})y)\|_2 < \varepsilon, \end{aligned}$$

using Cauchy-Schwarz Inequality.
Finally, if $x, y \in M$ are such that $E_A(x) = E_A(y) = 0$, if $\varepsilon > 0$, let $x', y' \in L_f(\Gamma)$ be such that $\|x'\| \leq \|x\|$, $\|y'\| \leq \|y\|$, $E_A(x') = E_A(y') = 0$ and

$$\|x' - x\|_2, \|y' - y\|_2 < \frac{\varepsilon}{3 \cdot \max(\|x\|, \|y\|)}.$$

Take a finite subset $F \subset S$ such that $\|E_A(ux'u^*y')\|_2 < \varepsilon/3$ for every $u \in S \setminus F$. Then, if $u \in S \setminus F$,

$$\|E_A(uxu^*y)\|_2 < \|y\|\|x - x'\|_2 + \|x\|\|y - y'\|_2 + \frac{\varepsilon}{3} < \varepsilon.$$

This ends the proof of Theorem 3.5. $\square$

As in the case of weak mixing, one has for crossed products:

Proposition 3.6 *Let Γ_0 be an abelian group which acts on a finite von Neumann algebra B and which preserves a trace τ, then the abelian von Neumann subalgebra $A = L(\Gamma_0)$ of the crossed product $M = B \rtimes \Gamma_0$ is strongly mixing in M if and only if the action of Γ_0 is.*

In fact, Proposition 3.6 and Theorem 4.2 of [15] prove that weakly mixing MASA's are not strongly mixing in general:

Let Γ_0 be an infinite abelian group and let α be a measure-preserving, free, weakly mixing but not strongly mixing action on some standard probability space $(X, \mathcal{B}, \mu)$ as in Theorem 4.2 of [15]. Set $B = L^\infty(X, \mathcal{B}, \mu)$ and let M be the corresponding crossed product II$_1$-factor. Then the abelian subalgebra $A = L(\Gamma_0)$ is a weakly mixing MASA in M, but it is not strongly mixing.

Typical examples of strongly mixing actions are given by (generalized) Bernoulli shift actions: Consider a finite von Neumann algebra $B \neq \mathbb{C}$ gifted with some trace τ_B, let Γ_0 be an infinite abelian group that acts *properly* on a countable set X : for every finite set $Y \subset X$, the set $\{g \in \Gamma_0 \ ; \ g(Y) \cap Y \neq \emptyset\}$ is finite. Let $(N, \tau) = \bigotimes_{x \in X}(B, \tau_B)$ be the associated infinite tensor product. Then the corresponding Bernoulli shift action is the action σ of Γ_0 on N given by

$$\sigma_g(\otimes_{x\in X} b_x) = \otimes_{x\in X} b_{gx}$$

for every $\otimes_x b_x \in N$ such that $b_x = 1$ for all but finitely many x's. Then it is easy to see that properness of the action implies that σ is a strongly mixing action. The classical case corresponds to the simply transitive action by left translations on Γ_0.

Let A be a diffuse von Neumann subalgebra of M and let Q be a finite von Neumann algebra. It is proved in Theorem 2.3 of [4] that the normalisers $\mathcal{N}_M(A)$ and $\mathcal{N}_{M\star Q}(A)$ are equal. In particular, if A is a singular MASA in M then it is also a singular MASA in the free product $M \star Q$. Then, using the same arguments as in the proofs of Lemma 2.2, Theorem 2.3 and Corollary 2.4 of the above mentioned article, we obtain:

Proposition 3.7 *Let M and Q be finite von Neumann algebras and let A be a strongly mixing MASA in M. Then A is also strongly mixing in the free product von Neumann algebra $M \star Q$.*

4 Examples

From now on, we consider pairs (Γ, Γ_0) where Γ_0 is an abelian subgroup of Γ. We will give examples of families of pairs that satisfy condition (ST) on the one hand, and of pairs that satisfy condition (SS) but not (ST) on the other hand.

4.1 Some free products examples

It was noted in [6] that if Γ is the free group F_N of rank $N \geq 2$ on free generators $a_1, \ldots, a_N$ and if Γ_0 is the subgroup generated by some fixed a_i, then the pair (F_N, Γ_0) satisfies condition (ST). See also Corollary 3.4 of [17]. The next result extends the above case to some amalgamated products.

Proposition 4.1 *Let $\Gamma = \Gamma_0 \star_Z \Gamma_1$ be an amalgamated product where Γ_0 is an infinite abelian group, Z is a finite subgroup of Γ_0 and of Γ_1. If $\Gamma_1 \neq Z$ then the pair (Γ, Γ_0) satisfies condition (ST).*

Proof. Let R and S be sets of representatives for the left cosets of Z in Γ_0 and Γ_1 respectively, such that $1 \in R$ and $1 \in S$. Recall that every element $g \in \Gamma$ has a unique normal form

$$g = r_1 s_1 \ldots r_l s_l z$$

with $r_i \in R$, $s_i \in S$ for every i, such that only r_1 or s_1 can be equal to 1, and $z \in Z$. Notice that g does not belong to Γ_0 if and only if $s_1 \neq 1$. Let $g, h \in \Gamma \setminus \Gamma_0$. It suffices to find a finite subset $E \subset \Gamma_0$ such that, for every $\gamma \in \Gamma_0 \setminus E$, $g\gamma h \notin \Gamma_0$. Let us write $g = r_1 s_1 \ldots r_l s_l z$ with $r_i \in R$, $s_i \in S$ and $z \in Z$, and $h = u_1 v_1 \ldots u_k v_k w$ with $u_j \in R$, $v_j \in S$ and $w \in Z$. If $\gamma = rt$ with $r \in R$ and $t \in Z$, then

$$\begin{aligned} g\gamma h &= r_1 s_1 \ldots r_l s_l z r t u_1 v_1 \ldots u_k v_k w \\ &= r_1 s_1 \ldots r_l s_l (r u_1) v_1' z_1' \ldots u_k v_k w \end{aligned}$$

with $v_1' \in S$ and $z_1' \in Z$ such that $v_1' z_1' = (zt) v_1$ is the decomposition of $(zt)v_1$ in the partition $\Gamma_1 = \coprod_{\sigma \in S} \sigma Z$. Observe that $v_1' \neq 1$ because $v_1 \neq 1$. Continuing in the same way, we move elements of Z to the right as most as possible and we get finally

$$g\gamma h = r_1 s_1 \ldots r_l s_l (r u_1) v_1' \ldots u_k v_k' w'$$

with every $v_j' \in S$ and $w' \in Z$. If $s_l \neq 1$ one can take $E = Z \cup u_1^{-1} Z$, and if $s_l = 1$ one can take $E = Z \cup (r_l u_1)^{-1} Z$. $\square$

Finally, we get from Theorem 3.5 and Proposition 3.7:

Proposition 4.2 *Let Γ be an infinite group, let Γ_0 be an infinite abelian subgroup of Γ such that the pair (Γ, Γ_0) satisfies condition (ST) and let G be any countable group. Then the pair $(\Gamma \star G, \Gamma_0)$ satisfies also condition (ST).*

4.2 The case of malnormal subgroups

In [10], S. Popa introduced a condition on pairs (Γ, Γ_0) in order to obtain orthogonal pairs of von Neumann subalgebras; it was also used later in [17] to get asymptotic homomorphism conditional expectations. We say that Γ_0 is a **malnormal** subgroup of Γ if it satisfies the following condition:

$(\star)$ *For every $g \in \Gamma \setminus \Gamma_0$, one has $g \Gamma_0 g^{-1} \cap \Gamma_0 = \{1\}$.*

Then we observe that condition $(\star)$ implies condition (ST). Indeed, for $g, h \in \Gamma$, set $E(g,h) = \{\gamma \in \Gamma_0 \; ; \; g\gamma h \in \Gamma_0\} = g^{-1}\Gamma_0 h^{-1} \cap \Gamma_0$. If (Γ, Γ_0) satisfies $(\star)$, and if $g, h \in \Gamma \setminus \Gamma_0$, then $E(g,h)$ contains at most one element (see the proof of Lemma 3.1 of [17]). In turn, condition (ST) means exactly that $E(g,h)$ is finite for all $g, h \in \Gamma \setminus \Gamma_0$. Thus, if Γ_0 is torsion free, then conditions $(\star)$ and (ST) are equivalent because, in this case, if $g \in \Gamma \setminus \Gamma_0$, the finite set $E(g^{-1}, g) = g\Gamma_0 g^{-1} \cap \Gamma_0$ is a finite subgroup of Γ_0, hence is trivial.

However, condition $(\star)$ is strictly stronger than condition (ST) in general: let $\Gamma = \Gamma_0 \star_Z \Gamma_1$ be as in Proposition 4.1 above, and assume further that $Z \neq \{1\}$ and that there exists $g \in \Gamma_1 \setminus Z$ such that $zg = gz$ for every $z \in Z$. Then $g\Gamma_0 g^{-1} \cap \Gamma_0 \supset Z$, and (Γ, Γ_0) does not satisfy $(\star)$. Observe that, in this case, Γ is not necessarily an ICC group. However, replacing it by a non trivial free product group $\Gamma \star G$, we get a pair $(\Gamma \star G, \Gamma_0)$ satisfying condition (ST) by Proposition 3.7 but not $(\star)$, and $\Gamma \star G$ is an ICC group.

It is known that in some classes of groups, maximal abelian subgroups are malnormal. This is e.g. the case in hyperbolic groups [5] or in groups acting freely on Λ-trees [1]. We present here explicitely a sufficient condition to get malnormal subgroups in groups acting on trees:

Proposition 4.3 *Let Γ be a group acting on a tree T without inversion. Let $t \in \Gamma$ and set $\Gamma_0 = \langle t \rangle$. Assume that:*

(1) there exist neither $u \in \Gamma$ nor $n \in \mathbb{Z} \setminus \{\pm 1\}$ such that $t = u^n$;

(2) the induced automorphism of T (again denoted by t) is hyperbolic;

(3) the subgroup $\bigcap_{v \in \mathrm{axis}(t)} \mathrm{Stab}(v)$ is trivial;

(4) no element of Γ induces a reflection of $\mathrm{axis}(t)$.

Then Γ_0 is a malnormal subgroup of Γ, and in particular the pair (Γ, Γ_0) satisfies condition (ST).

Before proving this, we introduce the necessary terminology about actions on trees. If a group G acts without inversion on a tree X, it is well-known [16, Chap I.6.4] that an element $g \in G$ is either *elliptic*, that is it fixes a vertex, or *hyperbolic*, that is g preserves an infinite geodesic (called *axis*) on which it acts by a non-trivial translation. It is easy to see that a hyperbolic element g has a unique axis. It will be denoted $\mathrm{axis}(g)$.

Proof of Proposition 4.3. Given an element $g \in \Gamma$ such that $g\Gamma_0 g^{-1} \cap \Gamma_0 \neq \{1\}$, we have to prove that $g \in \Gamma_0$. First, we write $gt^k g^{-1} = t^{k'}$ for some $k, k' \in \mathbb{Z}^*$. The elements t^k and $t^{k'}$ having $\mathrm{axis}(t)$ as axis, g has to preserve $\mathrm{axis}(t)$. Then, by hypotheses (3) and (4), g is an hyperbolic element and $\mathrm{axis}(g) = \mathrm{axis}(t)$. Let us denote by $\ell(\gamma)$ the translation length of an element $\gamma \in \Gamma$. By Bézout's Theorem, there exists an element $s = g^m t^n$ with $m, n \in \mathbb{Z}$ such that $\ell(s)$ is the greatest common divisor of $\ell(g)$ and $\ell(t)$. Then, there exist $\delta, \varepsilon \in \{\pm 1\}$ such that $gs^{\delta\ell(g)/\ell(s)}$ and $ts^{\varepsilon\ell(t)/\ell(s)}$ fix every vertex of $\mathrm{axis}(t)$. By hypothesis (3), we get $g = s^{-\delta\ell(g)/\ell(s)}$ and $t = s^{-\varepsilon\ell(t)/\ell(s)}$. Then (1) gives $t = s^{\pm 1}$, so that $g = t^{\pm\ell(g)/\ell(t)}$. As desired, g is an element of Γ_0. $\square$

As it will be seen below, wide families of HNN extentions satisfy hypotheses of Proposition 4.3. Thus, let $\Gamma = HNN(\Lambda, H, K, \phi)$ be an HNN extension where H, K are subgroups of Λ and, as usual, where $\phi : H \to K$ is an isomorphism. Denote by t the stable letter such that $t^{-1}ht = \phi(h)$ for all $h \in H$, and by Γ_0 the subgroup generated by t. Recall that a sequence $g_0, t^{\varepsilon_1}, \ldots, t^{\varepsilon_n}, g_n$, $(n \geq 0)$ is *reduced* if $g_i \in \Lambda$ and $\varepsilon_i = \pm 1$ for every i, and if there is no subsequence t^{-1}, g_i, t with $g_i \in H$ or t, g_i, t^{-1} with $g_i \in K$. As is well known, if the sequence $g_0, t^{\varepsilon_1}, \ldots, t^{\varepsilon_n}, g_n$ is reduced and if $n \geq 1$ then the corresponding element $g = g_0 t^{\varepsilon_1} \cdots t^{\varepsilon_n} g_n \in \Gamma$ is non trivial (Britton's Lemma). We also say that such an element is in *reduced form*. Furthermore, if $g = g_0 t^{\varepsilon_1} \cdots t^{\varepsilon_n} g_n$ and $h = h_0 t^{\delta_1} \cdots t^{\delta_m} g_m$ are in reduced form and if $g = h$, then $n = m$ and $\varepsilon_i = \delta_i$ for every i. Hence the *length* $\ell(g)$ of $g = g_0 t^{\varepsilon_1} \cdots t^{\varepsilon_n} g_n$ (in reduced form) is the integer n. Finally, recall from [20] that, for every positive integer j, $\mathrm{Dom}(\phi^j)$ is defined by $\mathrm{Dom}(\phi) = H$ for $j = 1$ and, by induction, $\mathrm{Dom}(\phi^j) = \phi^{-1}(\mathrm{Dom}(\phi^{j-1}) \cap K) \subset H$ for $j \geq 2$.

If $\Gamma = HNN(\Lambda, H, K, \phi)$, its *Bass-Serre tree* has Γ/Λ as set of vertices and Γ/H as set of oriented edges. The origin of the edge γH is $\gamma\Lambda$ and its terminal vertex is $\gamma t\Lambda$. Chapter I.5 in [16], and Theorem 12 in particular, ensures that it is a tree. It is obvious that the Bass-Serre tree is endowed with an orientation-preserving Γ-action.

Corollary 4.4 *Suppose that for each $\lambda \in \Lambda^*$, there exists $j > 0$ such that $\lambda \notin \mathrm{Dom}(\phi^j)$. Then Γ_0 is a malnormal subgroup of Γ.*

Proof. Let T be the Bass-Serre tree of the HNN-extension. We check the hypotheses of Proposition 4.3. Since (1), (2) and (4) are obvious, we prove (3). We have $\bigcap_{v \in \mathrm{axis}(t)} \mathrm{Stab}(v) = \bigcap_{k \in \mathbb{Z}} \Lambda_k$, where $\Lambda_k = t^{-k}\Lambda t^k$. Assume by contradiction that the intersection contains a non trivial element λ. By hypothesis, there exists $j \in \mathbb{N}^*$, such that $\lambda \notin \mathrm{Dom}(\phi^j)$ and we may assume j to be minimal for this property. This means that $t^{1-j}\lambda t^{j-1} = \phi^{j-1}(\lambda) \in \Lambda \setminus H$, and $t^{-j}\lambda t^j \notin \Lambda$ by Britton's Lemma. We get $\lambda \notin \Lambda_{-j}$, a contradiction. This proves (3). $\square$

As it will be recalled in the first example below, Thompson's group F is an HNN extention, and it satisfies condition (ST) with respect to the subgroup generated by x_0, by Lemma 3.2 of [6].

Examples. (1) For every integer $k \geq 1$, denote by F_k the subgroup of F generated by $x_k, x_{k+1}, \ldots$, and denote by σ the "shift map" defined by $\sigma(x_n) = x_{n+1}$, for $n \geq 0$. Its restriction to F_k is an isomorphism onto F_{k+1}, and in particular, the inverse map $\phi : F_2 \to F_1$ is an isomorphism which satisfies $\phi(x) = x_0 x x_0^{-1}$ for every $x \in F_2$. As in Proposition 1.7 of [3], it is evident that F is the HNN extension $HNN(F_1, F_2, F_1, \phi)$ with $t = x_0^{-1}$ as stable letter. With these choices, F satisfies the hypotheses of Proposition 4.1, and this proves that the pair (F, Γ_0) satisfies condition (ST).
(2) Let m and n be non-zero integers. The associated *Baumslag-Solitar group* is the group which has the following presentation:

$$BS(m, n) = \langle a, b \mid ab^m a^{-1} b^{-n} \rangle.$$

Since $a^{-1}b^n a = b^m$, $BS(m, n)$ is an HNN extension $HNN(\mathbb{Z}, n\mathbb{Z}, m\mathbb{Z}, \phi)$ where $\phi(nk) = mk$ for every integer k. Assume first that $|n| > |m|$ and denote by Γ_0 the subgroup generated by a. Then it is easy to check that the pair $(\Gamma_0, BS(m, n))$ satisfies the condition in Proposition 4.1. Thus, it satisfies also condition (ST). If $|m| > |n|$, replacing a by a^{-1} (which does not change Γ_0), one gets the same conclusion. Thus, all Baumslag-Solitar groups $BS(m, n)$ with $|m| \neq |n|$ satisfy condition (ST) with respect to the subgroup generated by a. Observe that the latter class is precisely the class of Baumslag-Solitar groups that are ICC ([20]).

We turn to free products. The *Bass-Serre tree* of $\Gamma = A * B$ has $(\Gamma/A) \sqcup (\Gamma/B)$ as set of vertices and Γ as set of oriented edges. The origin of the edge γ is γA and its terminal vertex is γB. Again, Chapter I.5 in [16] ensures that it is a tree. Here is another consequence of Proposition 4.3:

Corollary 4.5 *Let $\Gamma = A * B$ be a free product such that $|A| \geq 2$ and B contains an element b of order at least 3. Let a be a non trivial element of A and $\Gamma_0 = \langle ab \rangle$. Then Γ_0 is malnormal in Γ.*

Proof. Let T be the Bass-Serre tree of the free product. Again, we check the hypotheses of Proposition 4.3, with $t = ab$. Since (1) and (2) are obvious, we focus on (3) and (4). The axis of t has the following structure.

$$\ldots \leftarrow b^{-1}a^{-1}b^{-1}A \rightarrow b^{-1}a^{-1}B \leftarrow b^{-1}A \rightarrow B \leftarrow A \rightarrow aB \leftarrow abA \rightarrow \ldots$$

To prove (3), we remark that the vertices A and abA are on axis(t) and that $\mathrm{Stab}(A) \cap \mathrm{Stab}(abA) = A \cap abAb^{-1}a^{-1} = \{1\}$.
To prove (4), assume by contradiction that there exists an elliptic element $g \in \Gamma$ which induces a reflection on axis(t). Up to conjugating g by a power of t, we may assume that the fixed point of g on axis(t) is A or B. If g fixes A, we have $gb^{-1}A = abA$, so that $g \in A$ and $bg^{-1}ab \in A$. This is a contradiction since $|b| \geq 3$. Now, if g fixes B, we have $gabA = b^{-1}a^{-1}b^{-1}A$, so that $g \in B$ and $babgab \in A$. This implies $g = b^{-1}$, $a^2 = 1$, and $b^2 = 1$, which is again a contradiction since $|b| \geq 3$. $\square$

4.3 Some semidirect products

Let H be a discrete group, let Γ_0 be an infinite abelian group and let $\alpha : \Gamma_0 \to \mathrm{Aut}(H)$ be an action of Γ_0 on H. Then the semi-direct product $\Gamma = H \rtimes_\alpha \Gamma_0$ is the direct product set $H \times \Gamma_0$ gifted with the multiplication

$$(h, \gamma)(h', \gamma') = (h\alpha_\gamma(h'), \gamma\gamma') \quad \forall (h, \gamma), (h', \gamma') \in \Gamma.$$

The action α lifts from H to the von Neumann algebra $L(H)$, and $L(\Gamma) = L(H) \rtimes \Gamma_0$ is a crossed product von Neumann algebra in a natural way. In Theorem 2.2 of [14], the authors consider a sufficient condition on the action α on H which ensures that $L(\Gamma_0) \subset L(\Gamma)$ is a strongly singular MASA in $L(\Gamma)$ and that $L(\Gamma)$ is a type II_1 factor. In fact, it turns out that their condition implies that $L(\Gamma_0)$ is strongly mixing in $L(\Gamma)$, as we prove here:

Proposition 4.6 *Let H and Γ_0 be infinite discrete groups, Γ_0 being abelian, let α be an action of Γ_0 on H and let $\Gamma = H \rtimes_\alpha \Gamma_0$. If, for each $\gamma \in \Gamma_0^*$, the only fixed point of α_γ is 1_H, then:*

(1) Γ is an ICC group;

(2) the pair (Γ, Γ_0) satisfies condition (ST);

(3) the action of Γ_0 on $L(H)$ is strongly mixing.

In particular, $L(\Gamma)$ is a type II_1 factor and $L(\Gamma_0)$ is strongly mixing in $L(\Gamma)$.

Proof. Statement (1) is proved in [14].
Thus it remains to prove (2) and (3).
To do that, we claim first that if α is as above, then the triple (Γ_0, H, α) satisfies the following condition whose proof is inspired by that of Theorem 2.2 of [14]:

For every finite subset F of H^*, there exists a finite set E in Γ_0 such that $\alpha_\gamma(F) \cap F = \emptyset$ for all $\gamma \in \Gamma_0 \setminus E$.

Indeed, if F is fixed, set $I(F) = \{\gamma \in \Gamma_0 \ ; \ \alpha_\gamma(F) \cap F \neq \emptyset\}$. If $I(F)$ would be infinite for some F, set for $f \in F$:

$$S_f = \{\gamma \in \Gamma_0 \ ; \ \alpha_\gamma(f) \in F\},$$

so that $I(F) = \bigcup_{f \in F} S_f$, and S_f would be infinite for at least one $f \in F$. There would exist then distinct elements γ_1 and γ_2 of Γ_0 such that $\alpha_{\gamma_1}(f) = \alpha_{\gamma_2}(f)$, since F is finite. This is impossible because $\alpha_{\gamma_1^{-1}\gamma_2}$ cannot have any fixed point in $F \subset H^*$.

Let us prove (2). Fix some finite set $C \subset \Gamma \setminus \Gamma_0$. Without loss of generality, we assume that $C = C_1 \times C_2$ with $C_1 \subset H^*$ and $C_2 \subset \Gamma_0$ finite. Take $F = C_1 \cup C_1^{-1} \subset H^*$ in the condition above and let $E_1 \subset \Gamma_0$ be a finite set such that $\alpha_\gamma(F) \cap F = \emptyset$ for all $\gamma \in \Gamma_0 \setminus E_1$. Put $E = \bigcup_{\gamma \in C_2} \gamma^{-1} E_1$, which is finite. Then it is easy to check that for all $(h, \gamma), (h', \gamma') \in C$ and for every $g \in \Gamma_0 \setminus E$, one has

$$(h, \gamma)(1, g)(h', \gamma') = (h\alpha_{\gamma g}(h'), \gamma g \gamma') \notin \{1_H\} \times \Gamma_0.$$

Finally, in order to prove (3), it suffices to see that, if $a, b \in L_f(H)$, then there exists a finite subset E of Γ_0 such that

$$\tau(\alpha_\gamma(a)b) = \tau(a)\tau(b) \quad \forall \gamma \notin E.$$

Thus, let $S \subset \Gamma_0$ be a finite subset such that $a = \sum_{\gamma \in S} a(\gamma)\lambda(\gamma)$ and $b = \sum_{\gamma \in S} b(\gamma)\lambda(\gamma)$. Choose $E \subset \Gamma_0$ finite such that $\alpha_\gamma(S^* \cup (S^*)^{-1}) \cap (S^* \cup (S^*)^{-1}) = \emptyset$ for every $\gamma \in \Gamma_0 \setminus E$. Then, if $\gamma \notin E$, we have

$$\tau(\alpha_\gamma(a)b) = \sum_{h_1, h_2 \in S} a(h_1)b(h_2)\tau(\lambda(\alpha_\gamma(h_1)h_2)) = \tau(a)\tau(b)$$

since $\alpha_\gamma(h_1)h_2 \neq 1$ for $h_1, h_2 \in S^*$. $\square$

Example. Let $d \geq 2$ be an integer and let $g \in GL(d, \mathbb{Z})$. Then it defines a natural action of $\Gamma_0 = \mathbb{Z}$ on $H = \mathbb{Z}^d$ which has no non trivial fixed point if and only if the list of eigenvalues of g contains no root of unity. (See for instance Example 2.5 of [20].)

We give below an application of Proposition 4.6 to some HNN extensions (a wider class than in Corollary 4.4); as it follows from [16, Chap I.1.4, Prop 5], if $\Gamma = HNN(\Lambda, H, K, \phi)$ with stable letter t such that $t^{-1}ht = \phi(h)$ for all $h \in H$, if $\Gamma_0 = \langle t \rangle$ as above and if σ_t denotes the quotient map $\Gamma \to \Gamma_0$, then Γ is a semidirect product group $N \rtimes \Gamma_0$ where $N = \ker(\sigma_t)$ is a direct limit, or *amalgam*, of the system

$$\ldots \Lambda_{-1} \nwarrow K_{-1} = H_0 \nearrow \Lambda_0 \nwarrow K_0 = H_1 \nearrow \Lambda_1 \ldots$$

where $\Lambda_i = t^{-i}\Lambda t^i$, $H_i = t^{-i}Ht^i$ and $K_i = t^{-i}Kt^i$ for all $i \in \mathbb{Z}$. The conjugation operation $n \mapsto t^{-1}nt$ corresponds then to a shift to the right direction.

Corollary 4.7 *Suppose that for all $j \in \mathbb{N}^*$, the homomorphism ϕ^j has no non trivial fixed point, that is, for all $h \in H$, $\phi^j(h) = h$ implies $h = 1$. Then the following hold:*

1. *the group Γ is ICC;*
2. *the pair (Γ, Γ_0) satisfies condition (ST);*
3. *the algebra $L(\Gamma)$ is a type II_1 factor, in which $L(\Gamma_0)$ is strongly mixing.*

Proof. It suffices to prove that, for any $k \in \mathbb{Z}^*$ and any $n \in N^*$, one has $t^{-k}nt^k \neq n$. Assume by contradiction that $t^{-k}nt^k = n$ for some $k \in \mathbb{Z}^*$, and some $n \in N^*$. Up to replacing k by $-k$, we assume that $k \in \mathbb{N}^*$. Then there exists $s \in \mathbb{N}$ such that n is in the subgroup of N generated by $\Lambda_{-s}, \ldots, \Lambda_0, \ldots, \Lambda_s$. Let now ℓ be a multiple of k such that $\ell - s > s + k$. The element $t^{-\ell}nt^\ell$ is in the subgroup of N generated by $\Lambda_{\ell-s}, \ldots, \Lambda_\ell, \ldots, \Lambda_{\ell+s}$. Then we consider the subgroups N_l generated by $\ldots, \Lambda_{s-1}, \Lambda_s$, and N_r generated by $\Lambda_{s+1}, \Lambda_{s+2}, \ldots$, and, since $\ell - s > s$, we have $N = N_l *_{K_s = H_{s+1}} N_r$ with $n \in N_l$ and $t^{-\ell}nt^\ell \in N_r$. Since ℓ is a multiple of k, we have $t^{-\ell}nt^\ell = n$ and this element is in $K_s = H_{s+1}$. By similar arguments (shifting the "cutting index" in the construction of N_l, N_r) we obtain that $n \in K_{s+1} = H_{s+2}, \ldots, n \in K_{\ell-s-1} = H_{\ell-s}$. Hence, the elements $n' := t^{\ell-s}nt^{s-\ell}$, $t^{-1}n't = t^{\ell-s-1}nt^{s-\ell+1}$, $\ldots$, $t^{-\ell+2s+1}n't^{\ell-2s-1} = t^{s+1}nt^{-s-1}$ are in $H_0 = H$. Thus, since $\ell - s > k$, we have $n', t^{-1}n't, \ldots, t^{-k}n't^k \in H$. Consquently, $\phi^k(n')$ exists, and $\phi^k(n') = t^{-k}n't^k = t^{\ell-s-k}nt^{k+s-\ell} = t^{\ell-s}n't^{s-\ell} = n'$. On the other hand, ϕ^k has no non trivial fixed point. This is a contradiction. ☐

4.4 Final examples and remarks

Next, let us look at examples where Γ_0 is not cyclic (inspired by Sinclair and Smith, [18]): let $\mathbb{Q}$ be the additive group of rational numbers and denote by $\mathbb{Q}^\times$ the multiplicative group of nonzero rational numbers.

For each positive integer n, set

$$F_n = \{\frac{p}{q} \cdot 2^{kn} \ ; \ p, q \in \mathbb{Z}_{\text{odd}}, \ k \in \mathbb{Z}\} \subset \mathbb{Q}^\times$$

and

$$F_\infty = \{\frac{p}{q} \ ; \ p, q \in \mathbb{Z}_{\text{odd}}\} \subset \mathbb{Q}^\times.$$

Next, for $n \in \mathbb{N} \cup \{\infty\}$, set

$$\Gamma(n) = \left\{ \begin{pmatrix} f & x \\ 0 & 1 \end{pmatrix} \ ; \ f \in F_n, \ x \in \mathbb{Q} \right\}$$

and let $\Gamma_0(n)$ be the subgroup of diagonal elements of $\Gamma(n)$. $\Gamma(n)$ is an ICC, amenable group. Then the pair $(\Gamma(n), \Gamma(n)_0)$ satisfies condition (ST) for every n.

However, if we consider larger matrices, the corresponding pairs of groups do not satisfy condition (ST). Let us fix two positive integers m and n, and set

$$\Gamma(m,n) = \left\{ \begin{pmatrix} 1 & x & y \\ 0 & f_1 & 0 \\ 0 & 0 & f_2 \end{pmatrix} \ ; \ f_1 \in F_m, \ f_2 \in F_n, \ x, y \in \mathbb{Q} \right\}$$

and let $\Gamma_0(m,n)$ be the corresponding diagonal subgroup. Then we have:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & f & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which belongs to Γ_0 for all $f \in F_m$. Thus the pair $(\Gamma(m,n), \Gamma(m,n)_0)$ does not satisfy condition (ST), though it satisfies condition (SS).

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