A sequence of Albin type continuous martingales with Brownian marginals and scaling

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Dedicated to Lester Dubins (1921-2010) to whom the third author owes a lot.

Summary. Closely inspired by Albin's method which relies ultimately on the duplication formula for the Gamma function, we exploit Gauss' multiplication formula to construct a sequence of continuous martingales with Brownian marginals and scaling.

Key words: Martingales, Brownian marginals

1 Motivation and main results

(1.1) Knowing the law of a "real world" random phenomena, i.e. random process, $(X_t, t \geq 0)$ is often extremely difficult and in most instances, one avails only of the knowledge of the 1-dimensional marginals of $(X_t, t \geq 0)$. However, there may be many different processes with the same given 1-dimensional marginals.

In the present paper, we make explicit a sequence of continuous martingales $(M_m(t), t \geq 0)$ indexed by $m \in \mathbb{N}$ such that for each m,

i) $(M_m(t), t \geq 0)$ enjoys the Brownian scaling property: for any $c > 0$,

$$(M_m(c^2 t), t \geq 0) \overset{(law)}{=} (cM_m(t), t \geq 0)$$

ii) $M_m(1)$ is standard Gaussian.

Note that, combining i) and ii), we get, for any $t > 0$

$$M_m(t) \overset{(law)}{=} B_t,$$

where $(B_t, t \geq 0)$ is a Brownian motion, i.e. M_m admits the same 1-dimensional marginals as Brownian motion.

(1.2) Our main result is the following extension of Albin's construction [1] from $m = 1$ to any integer m.

Theorem 1. *Let $m \in \mathbb{N}$. Then, there exists a continuous martingale $(M_m(t), t \geq 0)$ which enjoys i) and ii) and is defined as follows:*

$$M_m(t) = X_t^{(1)} \dots X_t^{(m+1)} Z_m \tag{1}$$

where $(X_t^{(i)}, t \geq 0)$, for $i = 1, \dots, m+1$, are independent copies of the solution of the SDE

$$dX_t = \frac{1}{m+1} \frac{dB_t}{X_t^m}; \quad X_0 = 0 \tag{2}$$

and, furthermore, Z_m is independent from $(X^{(1)}, \dots, X^{(m+1)})$ and

$$Z_m \overset{(law)}{=} (m+1)^{1/2} \left(\prod_{j=0}^{m-1} \beta\left(\frac{1+2j}{2(m+1)}, \frac{m-j}{m+1}\right) \right)^{\frac{1}{2(m+1)}} \tag{3}$$

where $\beta(a,b)$ denotes a beta variable with parameter (a,b) with density

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} 1_{[0,1]}(x)$$

and the beta variables on the right-hand side of (3) are independent.

Remark: For $m = 1$, $Z_1 = \sqrt{2}\left(\beta(\frac{1}{4}, \frac{1}{2})\right)^{1/4}$ and we recover the distribution of $Y := Z_1$ given by (2) in [1].

(1.3) For the convenience of the reader, we also recall that, if one drops the continuity assumption when searching for martingales $(M(t); t \geq 0)$ satisfying i) and ii), then, the Madan-Yor construction [5] based on the "Azéma-Yor under scaling" method provides such a martingale.
Precisely, starting from a Brownian motion $(B_u, u \geq 0)$ and denoting $S_u = \sup_{s \leq u} B_s$, introduce the family of stopping times

$$\tau_t = \inf\{u, S_u \geq \psi_t(B_u)\}$$

where ψ_t denotes the Hardy-Littlewood function associated with the centered Gaussian distribution μ_t with variance t, i.e.

$$\begin{aligned} \psi_t(x) &= \frac{1}{\mu_t([x, \infty[)} \int_x^\infty y \exp(-\frac{y^2}{2t}) \frac{dy}{\sqrt{2\pi t}} \\ &= \sqrt{t} \exp(-\frac{x^2}{2t}) / \mathcal{N}(x/\sqrt{t}) \end{aligned}$$

where $\mathcal{N}(a) = \int_a^\infty \exp(-\frac{y^2}{2}) dy$. Then, $M_t = B_{\tau_t}$ is a martingale with Brownian marginals.
Another solution has been given by Hamza and Klebaner [4].

2 Proof of the theorem

Step 1: For $m \in \mathbb{R}$ and $c \in \mathbb{R}$, we consider the stochastic equation:

$$dX_t = c\frac{dB_t}{X_t^m}, \quad X_0 = 0.$$

This equation has a unique weak solution which can be defined as a time-changed Brownian motion

$$(X_t) \overset{(law)}{=} W(\alpha^{(-1)}(t))$$

where W is a Brownian motion starting from 0 and $\alpha^{(-1)}$ is the (continuous) inverse of the increasing process

$$\alpha(t) = \frac{1}{c^2}\int_0^t W_u^{2m} du.$$

We look for $k \in \mathbb{N}$ and c such that $(X_t^{2k}, t \geq 0)$ is a squared Bessel process of some dimension d. It turns out, by application of Itô's formula, that we need to take $k = m + 1$ and $c = \frac{1}{m+1}$. Thus, we find that $(X_t^{2(m+1)}, t \geq 0)$ is a squared Bessel process with dimension $d = k(2k-1)c^2 = \frac{2m+1}{m+1}$.
Note that the law of a BESQ(d) process at time 1 is well known to be that of $2\gamma_{d/2}$, where γ_a denotes a gamma variable with parameter a. Thus, we have:

$$|X_1| \overset{(law)}{=} \left(2\gamma_{\frac{2m+1}{2(m+1)}}\right)^{\frac{1}{2(m+1)}} \tag{4}$$

Step 2: We now discuss the scaling property of the solution of (2). From the scaling property of Brownian motion, it is easily shown that , for any $\lambda > 0$, we get:

$$(X_{\lambda t}, t \geq 0) \overset{(law)}{=} (\lambda^{\alpha} X_t, t \geq 0)$$

with $\alpha = \frac{1}{2(m+1)}$, that is, the process $(X_t, t \geq 0)$ enjoys the scaling property of order $\frac{1}{2(m+1)}$.

Step 3: Consequently, if we multiply $m+1$ independent copies of the process $(X_t, t \geq 0)$ solution of (2), we get a process

$$Y_t = X_t^{(1)} \dots X_t^{(m+1)}$$

which is a martingale and has the scaling property of order $\frac{1}{2}$.

Step 4: Finally, it suffices to find a random variable Z_m independent of the processes $X_t^{(1)}, \dots, X_t^{(m+1)}$ and which satisfies:

$$N \overset{(law)}{=} X_1^{(1)} \dots X_1^{(m+1)} Z_m \tag{5}$$

where N denotes a standard Gaussian variable. Note that the distribution of any of the $X_1^{(i)}$'s is symmetric. We shall take $Z_m \geq 0$; thus, the distribution of Z_m shall be determined by its Mellin transform $\mathcal{M}(s)$. From (5), $\mathcal{M}(s)$ satisfies:

$$\mathbb{E}[(2\gamma_{1/2})^{s/2}] = \left(\mathbb{E}[(2\gamma_{d/2})^{s/2(m+1)}]\right)^{m+1} \mathcal{M}(s)$$

with $d = \frac{2m+1}{m+1}$, that is:

$$2^{s/2}\frac{\Gamma(\frac{1+s}{2})}{\Gamma(\frac{1}{2})} = 2^{s/2}\left(\frac{\Gamma(\frac{d}{2} + \frac{s}{2(m+1)})}{\Gamma(\frac{d}{2})}\right)^{m+1} \mathcal{M}(s)$$

that is precisely:

$$\frac{\Gamma(\frac{1+s}{2})}{\Gamma(\frac{1}{2})} = \left(\frac{\Gamma(\frac{2m+1+s}{2(m+1)})}{\Gamma(\frac{2m+1}{2(m+1)})}\right)^{m+1} \mathcal{M}(s). \tag{6}$$

Now, we recall Gauss multiplication formula ([2], see also [3])

$$\Gamma(kz) = \frac{k^{kz-1/2}}{(2\pi)^{\frac{k-1}{2}}} \prod_{j=0}^{k-1} \Gamma(z + \frac{j}{k}) \tag{7}$$

which we apply with $k = m+1$ and $z = \frac{1+s}{2(m+1)}$. We then obtain, from (7)

$$\frac{\Gamma(\frac{1+s}{2})}{\sqrt{\pi}} = \frac{(m+1)^{s/2}}{(2\pi)^{m/2}} \frac{1}{\sqrt{\pi}} \prod_{j=0}^{m} \Gamma(\frac{1+s+2j}{2(m+1)}) \tag{8}$$

$$= (m+1)^{s/2} \prod_{j=0}^{m} \left(\frac{\Gamma(\frac{1+s+2j}{2(m+1)})}{\Gamma(\frac{1+2j}{2(m+1)})}\right) \tag{9}$$

since the two sides of (8) are equal to 1 for $s = 0$. We now plug (9) into (6) and obtain

$$(m+1)^{s/2} \prod_{j=0}^{m} \left(\frac{\Gamma(\frac{1+s+2j}{2(m+1)})}{\Gamma(\frac{1+2j}{2(m+1)})}\right) = \left(\frac{\Gamma(\frac{2m+1+s}{2(m+1)})}{\Gamma(\frac{2m+1}{2(m+1)})}\right)^{m+1} \mathcal{M}(s) \tag{10}$$

We note that for $j = m$, the same term appears on both sides of (10), thus (10) may be written as:

$$(m+1)^{s/2} \prod_{j=0}^{m-1} \left(\frac{\Gamma(\frac{1+s+2j}{2(m+1)})}{\Gamma(\frac{1+2j}{2(m+1)})}\right) = \left(\frac{\Gamma(\frac{2m+1+s}{2(m+1)})}{\Gamma(\frac{2m+1}{2(m+1)})}\right)^{m} \mathcal{M}(s) \tag{11}$$

In terms of independent gamma variables, the left-hand side of (11) equals:

$$(m+1)^{s/2}\mathbb{E}\left[\left(\prod_{j=0}^{m-1}\gamma^{(j)}_{\frac{1+2j}{2(m+1)}}\right)^{\frac{s}{2(m+1)}}\right] \tag{12}$$

whereas the right-hand side of (11) equals:

$$\mathbb{E}\left[\left(\prod_{j=0}^{m-1}\gamma^{(j)}_{\frac{1+2m}{2(m+1)}}\right)^{\frac{s}{2(m+1)}}\right]\mathcal{M}(s) \tag{13}$$

where the $\gamma^{(j)}_{a_j}$ denote independent gamma variables with respective parameters a_j.

Now, from the beta-gamma algebra, we deduce, for any $j \leq m-1$:

$$\gamma^{(j)}_{\frac{1+2j}{2(m+1)}} \overset{(law)}{=} \gamma^{(j)}_{\frac{1+2m}{2(m+1)}}\beta(\frac{1+2j}{2(m+1)}, \frac{m-j}{m+1}).$$

Thus, we obtain, again by comparing (12) and (13):

$$\mathcal{M}(s) = (m+1)^{s/2}\ \mathbb{E}\left[\left(\prod_{j=0}^{m-1}\beta(\frac{1+2j}{2(m+1)}, \frac{m-j}{m+1})\right)^{\frac{s}{2(m+1)}}\right]$$

which entails:

$$\mathbb{E}[Z_m^s] = (m+1)^{s/2}\ \mathbb{E}\left[\left(\prod_{j=0}^{m-1}\beta(\frac{1+2j}{2(m+1)}, \frac{m-j}{m+1})\right)^{\frac{s}{2(m+1)}}\right]$$

that is, equivalently,

$$Z_m \overset{(law)}{=} (m+1)^{1/2}\left(\prod_{j=0}^{m-1}\beta(\frac{1+2j}{2(m+1)}, \frac{m-j}{m+1})\right)^{\frac{1}{2(m+1)}}$$

3 Some remarks about Theorem 1

3.1 A further extension

We tried to extend Theorem 1 by taking a product of independent martingales $X^{(i)}$, solution of (2) with different m_i's. Here are the details of our attempt. We are looking for the existence of a variable Z such that the martingale

$$M(t) = \left(\prod_{j=0}^{p-1} X_t^{(m_j)}\right) Z$$

satisfies the properties i) and ii). Here $p, (m_j)_{0\le j\le p-1}$ are integers and $X^{(m_j)}$ is the solution of the EDS (2) associated to m_j, the martingales being independent for j varying. In order that M enjoys the Brownian scaling property, we need the following relation

$$\sum_{j=0}^{p-1} \frac{1}{m_j+1} = 1. \tag{14}$$

Following the previous computations, see (6), the Mellin transform $\mathcal{M}(s)$ of Z should satisfy

$$\frac{\Gamma(\frac{1+s}{2})}{\Gamma(\frac{1}{2})} = \left(\prod_{j=0}^{p-1} \frac{\Gamma(\frac{2m_j+1+s}{2(m_j+1)})}{\Gamma(\frac{2m_j+1}{2(m_j+1)})}\right) \mathcal{M}(s). \tag{15}$$

We recall (see (9)) the Gauss multiplication formula

$$\frac{\Gamma(\frac{1+s}{2})}{\sqrt{\pi}} = p^{s/2} \prod_{j=0}^{p-1} \left(\frac{\Gamma(\frac{1+s+2j}{2p})}{\Gamma(\frac{1+2j}{2p})}\right) \tag{16}$$

To find $\mathcal{M}(s)$ from (15), (16), we give some probabilistic interpretation:

$$\frac{\Gamma(\frac{1+s+2j}{2p})}{\Gamma(\frac{1+2j}{2p})} = \mathbb{E}[\gamma_{(1+2j)/2p}^{s/2p}]$$

whereas

$$\frac{\Gamma(\frac{2m_j+1+s}{2(m_j+1)})}{\Gamma(\frac{2m_j+1}{2(m_j+1)})} = \mathbb{E}[\gamma_{(1+2m_j)/2(m_j+1)}^{s/2(m_j+1)}].$$

Thus, we would like to factorize

$$\gamma_{(1+2j)/2p}^{1/2p} \stackrel{(law)}{=} \gamma_{(1+2m_j)/2(m_j+1)}^{1/2(m_j+1)} z_{m_j,p}^{(j)} \tag{17}$$

for some variable $z_{m_j,p}^{(j)}$ to conclude that

$$Z = p^{1/2} \prod_{j=0}^{p-1} z_{m_j,p}^{(j)}.$$

It remains to find under which condition the identity (17) may be fulfilled. We write

$$\gamma_{(1+2j)/2p} \stackrel{(law)}{=} \gamma_{(1+2m_j)/2(m_j+1)}^{p/(m_j+1)} (z_{m_j,p}^{(j)})^{2p}. \tag{18}$$

Now, if $\frac{1+2j}{2p} < \frac{1+2m_j}{2(m_j+1)}$, we may apply the beta-gamma algebra to obtain

$$\gamma_{(1+2j)/2p} \overset{(law)}{=} \gamma_{(1+2m_j)/2(m_j+1)}\beta(\frac{1+2j}{2p}, \frac{1+2m_j}{2(m_j+1)} - \frac{1+2j}{2p})$$

but in (18), we need to have on the right-hand side $\gamma^{p/(m_j+1)}_{(1+2m_j)/2(m_j+1)}$ instead of $\gamma_{(1+2m_j)/2(m_j+1)}$.
However, it is known that

$$\gamma_a \overset{(law)}{=} \gamma_a^c \gamma_{a,c}$$

for some variable $\gamma_{a,c}$ independent of γ_a for any $c \in (0, 1]$. This follows from the self-decomposable character of $\ln(\gamma_a)$. Thus, we seem to need $\frac{p}{m_j+1} \leq 1$. But, this condition is not compatible with (14) unless $m_j = m = p - 1$.

3.2 Asymptotic study

We study the behavior of the product $X_1^{(1)} \dots X_1^{(m+1)}$, resp. Z_m, appearing in the right-hand side of the equality in law (5), when $m \longrightarrow \infty$. Recall from (4) that

$$|X_1| \overset{(law)}{=} \left(2\gamma_{\frac{2m+1}{2(m+1)}}\right)^{\frac{1}{2(m+1)}}.$$

We are thus led to consider the product

$$\Theta^{(p)}_{a,b,c} = \left(\prod_{i=1}^{p} \gamma^{(i)}_{a-b/p}\right)^{c/p}$$

where in our set up of Theorem 1, $p = m + 1$, $a = 1$, $b = c = 1/2$.

$$\begin{aligned}
\mathbb{E}[(\Theta^{(p)}_{a,b,c})^s] &= \prod_{i=1}^{p} \mathbb{E}[\left(\gamma^{(i)}_{a-b/p}\right)^{cs/p}] \\
&= \left(\frac{\Gamma(a - \frac{b}{p} + \frac{cs}{p})}{\Gamma(a - \frac{b}{p})}\right)^p \\
&= \exp[p(\ln(\Gamma(a + \frac{cs - b}{p})) - \ln(\Gamma(a - \frac{b}{p})))] \\
&\longrightarrow \exp(\frac{\Gamma'(a)}{\Gamma(a)} cs).
\end{aligned}$$

Thus, it follows that

$$\Theta^{(p)}_{a,b,c} \xrightarrow[p\to\infty]{\mathbb{P}} \exp(\frac{\Gamma'(a)}{\Gamma(a)} c),$$

implying that

$$|X_1^{(1)} \dots X_1^{(m+1)}| \xrightarrow[p\to\infty]{\mathbb{P}} \exp(-\gamma/2) \tag{19}$$

and

$$\exp(-\gamma/2)Z_m \xrightarrow[m\to\infty]{(law)} |N|. \tag{20}$$

where $\gamma = -\Gamma'(1)$ is the Euler constant.
We now look for a central limit theorem for $\Theta^{(p)}_{a,b,c}$. We consider the limiting distribution of

$$\sqrt{p}\left\{\frac{c}{p}\sum_{i=1}^{p}\ln(\gamma^{(i)}_{a-b/p}) - c\frac{\Gamma'(a)}{\Gamma(a)}\right\}.$$

$$\begin{aligned}
&\mathbb{E}\left(\exp\left[cs\sqrt{p}\left\{\frac{1}{p}\sum_{i=1}^{p}\ln(\gamma^{(i)}_{a-b/p}) - \frac{\Gamma'(a)}{\Gamma(a)}\right\}\right]\right)\\
&= \mathbb{E}\left[\prod_{i=1}^{p}\left(\gamma^{(i)}_{a-b/p}\right)^{cs/\sqrt{p}}\right]\exp(-cs\sqrt{p}\frac{\Gamma'(a)}{\Gamma(a)})\\
&= \mathbb{E}\left[\left(\gamma^{(i)}_{a-b/p}\right)^{cs/\sqrt{p}}\right]^{p}\exp(-cs\sqrt{p}\frac{\Gamma'(a)}{\Gamma(a)})\\
&= \left(\frac{\Gamma(a-\frac{b}{p}+\frac{cs}{\sqrt{p}})}{\Gamma(a-\frac{b}{p})}\right)^{p}\exp(-cs\sqrt{p}\frac{\Gamma'(a)}{\Gamma(a)})\\
&= \exp[p(\ln(\Gamma(a-\frac{b}{p}+\frac{cs}{\sqrt{p}})) - \ln(\Gamma(a-\frac{b}{p}))) - cs\sqrt{p}\frac{\Gamma'(a)}{\Gamma(a)}]\\
&= \exp(\frac{c^2s^2}{2}(\ln(\Gamma))''(a) + O(m^{-1/2}))
\end{aligned}$$

We thus obtain that

$$\sqrt{p}\left\{\frac{c}{m}\sum_{i=1}^{m}\ln(\gamma^{(i)}_{a-b/m}) - c\frac{\Gamma'(a)}{\Gamma(a)}\right\} \xrightarrow{(law)} N(0,\sigma^2) \tag{21}$$

where $N(0,\sigma^2)$ denotes a centered Gaussian variable with variance:

$$\sigma^2 = c^2(\ln(\Gamma))''(a) = c^2\left[\frac{\Gamma''(a)}{\Gamma(a)} - \left(\frac{\Gamma'(a)}{\Gamma(a)}\right)^2\right].$$

or, equivalently

$$\left(\Theta^{(p)}_{a,b,c}\exp(\frac{\Gamma'(a)}{\Gamma(a)}c)\right)^{\sqrt{p}} \xrightarrow[p\to\infty]{(law)} \exp(N(0,c^2(\ln(\Gamma))''(a))). \tag{22}$$

References

1. Albin, J. M. P. A continuous non-Brownian motion martingale with Brownian motion marginal distributions. *Statist. Probab. Lett.* **78** (2008), no. 6, 682–686.

2. Andrews, G.; Askey, R. and Roy, R. *Special functions*. Encyclopedia of Mathematics and its Applications, 71. Cambridge University Press, Cambridge, 1999.
3. Chaumont, L. and Yor, M. *Exercises in probability. A guided tour from measure theory to random processes, via conditioning*. Cambridge Series in Statistical and Probabilistic Mathematics, 13. Cambridge University Press, Cambridge, 2003.
4. Hamza, K. and Klebaner, F. A family of non-Gaussian martingales with Gaussian marginals. *J. Appl. Math. Stoch. Anal.* (2007), Art. ID 92723, 19 pp.
5. Madan, D. and Yor, M. Making Markov martingales meet marginals: with explicit constructions. *Bernoulli* **8** (2002), no. 4, 509–536.