The Marcinkiewicz–Zygmund LLN in Banach Spaces: A Generalized Martingale Approach

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Abstract : A result due to Gut asserts that the Marcinkiewicz-Zygmund strong law of large numbers for real valued random variables is an amart a.s. convergence property. In this paper, a necessary and sufficient condition is given, under which that SLLN is also a quasimartingale. We also study the case of Banach-space valued r.v., by showing that the scalar result remains true when the space is of suitable stable type.
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Suggested running head : On the Marcinkiewicz-Zygmund quasimartingale.

1 Introduction

The speed in the almost sure (a.s.) convergence of sums $(\mathbf{Z}_n)$ associated to triangular arrays of independent random variables (r.v.) has been broadly studied. The classical technique relies on sharp bounds of the tail of the distribution function of these sums (see for instance chapter 9 in Petrov [16], Alt [2] and the references quoted there).
Another way to describe the quality of the convergence of the sequence $(\mathbf{Z}_n)$ is to recognize it as "good" if $(\mathbf{Z}_n)$ has a generalized martingale behaviour. This point of view has only been exploited very scarcely (see for instance chapter 6 in Edgar et Sucheston [3] and the references quoted there, and also Gut [4], Heinkel [8], [9], Hechner [6],...)
In this paper we will consider the Marcinkiewicz-Zygmund strong law of large numbers (SLLN) for Banach space valued r.v. from that point of view. Consider first the scalar case.

Let us begin with recalling the definitions of the two notions of generalized martingales that will be used in the sequel.

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Definition 1 . Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(\mathcal{F}_n)$ be a filtration. We denote by $\mathcal{T}$ the set of bounded stopping times related to $(\mathcal{F}_n)$ that take values in $\mathbb{N}^*$. Let $(\mathbf{Z}_n)$ be a sequence of integrable real r.v. adapted to the filtration $(\mathcal{F}_n)$. Then :

- $(\mathbf{Z}_n)$ is an amart if the net $(\mathbb{E}\mathbf{Z}_\tau)_{\tau \in \mathcal{T}}$ converges.
- $(\mathbf{Z}_n)$ is a quasimartingale if $\sum_{n=1}^{\infty} \mathbb{E}|\mathbb{E}(\mathbf{Z}_{n+1}|\mathcal{F}_n) - \mathbf{Z}_n| < +\infty$.

It is well-known that a quasimartingale is an amart, but that the converse is false in general (see [3] or [4] for further properties of quasimartingales and amarts).

Let us now recall the statement of Marcinkiewicz-Zygmund strong law of large numbers for real r.v. [12] :

Theorem 1 . *Let $(\mathbf{X}_i)$ be a sequence of independent copies of a centered r.v. $\mathbf{X}$ and let $p \in]1, 2[$ be fixed. For every n, denote $\mathbf{S}_n := \mathbf{X}_1 + \cdots + \mathbf{X}_n$. Then the following two properties are equivalent* :

- $\left(\frac{\mathbf{S}_n}{n^{1/p}}\right)$ *converges a.s. to 0.*
- $\mathbb{E}|\mathbf{X}|^p < +\infty$.

Denoting by $\mathcal{F}_n$ the natural filtration $\sigma(\mathbf{X}_1, \ldots, \mathbf{X}_n)$, Allan Gut has precised this law of large numbers as follows (example 4.6 in [5]) :

Theorem 2 . *Let $(\mathbf{X}_i)$ be a sequence of independent copies of a centered r.v. $\mathbf{X}$ and let $p \in]1, 2[$ be fixed. If $\left(\frac{\mathbf{S}_n}{n^{1/p}}\right)$ converges a.s. to 0, then $\left(\frac{\mathbf{S}_n}{n^{1/p}}, \mathcal{F}_n\right)$ is an amart.*

A quasimartingale being a special case of amart, it is natural to wonder if $\left(\frac{\mathbf{S}_n}{n^{1/p}}, \mathcal{F}_n\right)$ is also a quasimartingale, that is if the following holds :

$$\sum_{n \geqslant 1} \frac{1}{n^{1+1/p}} \mathbb{E}|\mathbf{S}_n| < +\infty. \tag{1.1.}$$

The answer to that question is not always positive, as the following example shows :

Example 1 . Let $p \in]1,2[$ be fixed and consider the symmetrically distributed random variable $\mathbf{X}$ which distribution fulfills :

$$\forall t > 0,\ \mathbb{P}(|\mathbf{X}|^p > t) = \mathbf{1}_{[0,e^2]}(t) + \frac{\beta}{t(\ln t)^p(\ln\ln t)}\mathbf{1}_{]e^2,+\infty[}(t)$$

where $\beta := 2^p e^2 \ln 2$.
The r.v. $\mathbf{X}$ being centered, and $|\mathbf{X}|^p$ being integrable, $\left(\frac{\mathbf{S}_n}{n^{1/p}}, \mathcal{F}_n\right)$ is an amart. If (1.1.) would hold, by symmetry of $\mathbf{X}$ it would also be the case of :

$$\sum_{n\geqslant 1} \frac{1}{n^{1+1/p}} \mathbb{E} \sup_{1\leqslant k\leqslant n} |\mathbf{X}_k| < +\infty.$$

As $\mathbb{E}|\mathbf{U}| = \int_0^{+\infty} \mathbb{P}(|\mathbf{U}| > t)dt$, we would then have :

$$\sum_{n\geqslant 2} \frac{1}{n\ln n} \mathbb{P}\left(\sup_{1\leqslant k\leqslant n} |\mathbf{X}_k| > \frac{n^{1/p}}{\ln n}\right) < +\infty. \qquad (1.2.)$$

It is however easy to see that there exists a constant $K > 0$ such that for n large enough, $\mathbb{P}\left(\sup_{1\leqslant k\leqslant n} |\mathbf{X}_k| > \frac{n^{1/p}}{\ln n}\right) \geqslant \frac{K}{\ln\ln n}$, so (1.2.) would imply that the series with general term $\frac{1}{n\ln n(\ln\ln n)}$ converges!
Thus $\left(\frac{\mathbf{S}_n}{n^{1/p}}, \mathcal{F}_n\right)$ is not a quasimartingale.

As situations exist in which $\left(\frac{\mathbf{S}_n}{n^{1/p}}, \mathcal{F}_n\right)$ is obviously a quasimartingale (for instance if $\mathbf{X}$ is square integrable), one can wonder if there exists a regularity condition on the distribution of $\mathbf{X}$, which, added to the integrability of $|\mathbf{X}|^p$ ensures that $\left(\frac{\mathbf{S}_n}{n^{1/p}}, \mathcal{F}_n\right)$ is a quasimartingale. The answer to that question is as follows :

Theorem 3 . *Let $(\mathbf{X}_i)$ be a sequence of independent copies of a centered r.v. $\mathbf{X}$ and let $p \in]1,2[$ be fixed. Then the following two properties are equivalent for the sums $\mathbf{S}_n := \mathbf{X}_1 + \cdots + \mathbf{X}_n$:*

- $\left(\frac{\mathbf{S}_n}{n^{1/p}}, \mathcal{F}_n\right)$ *is a quasimartingale.*
- $\int_0^{+\infty} \mathbb{P}^{1/p}(|\mathbf{X}| > t)dt < +\infty.$

In fact this result is a special case of a much more general statement (theorem 5) that we will state and prove after having introduced some technical tools.

2 Some technical prerequisites

In this section, we will recall the properties of Banach-valued r.v. and prove a lemma concerning power series that will be needed in the sequel. $(\mathbf{B}, \|\cdot\|)$ will be a real separable Banach space, equipped with the Borel σ-algebra $\mathcal{B}$ associated to the norm $\|\cdot\|$. We will consider a r.v. $\mathbf{X}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $(\mathbf{B}, \mathcal{B})$.

2.1 Type of Banach spaces

The generalization of Marcinkiewicz-Zygmund SLLN to a Banach space $\mathbf{B}$ depends on the geometry of $\mathbf{B}$ through the notion of type of a Banach space :

Definition 2 . Let $(\mathbf{B}, \|\cdot\|)$ be a Banach space and $1 \leqslant p < +\infty$. $\mathbf{B}$ is said to be of (Rademacher) type p if there exists a constant $c(p)$ such that for every sequence (ε_i) of independent Rademacher r.v. and every finite sequence (x_i) in $\mathbf{B}$,

$$\left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\|_p \leqslant c(p) \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{1/p}. \tag{2.1.}$$

Remark 1 . Every Banach space is of some type $p \in [1, 2]$. In one sense, the type of a Banach space measures the regularity of the space : every space is of type 1 and if $\mathbf{B}$ is of type p, it is also of type p' for $1 \leqslant p' \leqslant p$. Some Banach spaces, for example the space c_0 of real sequences converging to 0, are of type 1 but of no other type $p' > 1$.

Remark 2 . Let $(\mathbf{B}, \|\cdot\|)$ be a Banach space and $1 \leqslant p \leqslant 2$. If $\mathbf{B}$ is of (Rademacher) type p, there exists a constant $c(p)$ such that for every finite sequence $(\mathbf{X}_i)$ of centered r.v. belonging to L^p,

$$\mathbb{E} \left\| \sum_{i=1}^{n} \mathbf{X}_i \right\|^p \leqslant c(p) \sum_{i=1}^{n} \mathbb{E}\|\mathbf{X}_i\|^p. \tag{2.2.}$$

De Acosta [1] characterized Banach spaces of type p by the Marcinkiewicz-Zygmund SLLN :

Theorem 4 . *Let* $\mathbf{B}$ *be a Banach space and* $1 < p < 2$. *Then the following two properties are equivalent* :

1. $\mathbf{B}$ *is of type* p.
2. *For every r.v.* $\mathbf{X}$ *with values in* $\mathbf{B}$, *if* $(\mathbf{X}_i)_{i\in\mathbb{N}}$ *is a sequence of independent copies of* $\mathbf{X}$, *the following two properties are equivalent* :
   - $\left(\frac{\mathbf{S}_n}{n^{1/p}}\right)$ *converges to 0 a.s.* (*where as above* $\mathbf{S}_n := \mathbf{X}_1 + \cdots + \mathbf{X}_n$).
   - $\mathbb{E}\|\mathbf{X}\|^p < +\infty$ *and* $\mathbb{E}\mathbf{X} = 0$.

We will make that result more precise by showing that the notion of Rademacher type is naturally linked to the "amart" behaviour of $\left\|\frac{\mathbf{S}_n}{n^{1/p}}\right\|$. The "quasimartingale" behaviour is linked to a smaller class of spaces, the Banach spaces of stable type p :
Let $a := (a_k)_{1\leqslant k\leqslant n}$ be a sequence of real numbers and (a_k^*) the non-increasing rearrangement of the sequence $(|a_k|)$.
For a given $q \geqslant 1$, $\|a\|_{q,\infty} := \sup_{1\leqslant k\leqslant n} (k^{1/q}a_k^*)$ is called the weak–ℓ_q norm of the sequence a.
One also defines the Laurent norm $\|\mathbf{X}\|_{p,\infty}$ of a $\mathbf{B}$-valued r.v. $\mathbf{X}$:

$$\|\mathbf{X}\|_{p,\infty} := \left(\sup_{t>0} t^p \mathbb{P}(\|\mathbf{X}\| > t)\right)^{\frac{1}{p}}.$$

Definition 3 . For $1 \leqslant p < 2$, a Banach space $(\mathbf{B}, \|\cdot\|)$ is said to be of stable type p if there is a constant $c(p)$ such that for every sequence (θ_i) of independent standard p–stable r.v. and every finite sequence (x_i) in $\mathbf{B}$,

$$\left\|\sum_i \theta_i x_i\right\|_{p,\infty} \leqslant c(p)\left(\sum_i \|x_i\|^p\right)^{1/p}.$$

Remark 3 . A Banach space of stable type $1 < p < 2$ is also of stable type p' for every $1 \leqslant p' \leqslant p$.
Furthermore, a Banach space $\mathbf{B}$ of stable type $1 \leqslant p < 2$ is also of type p. Conversely, if $\mathbf{B}$ is of type $p > 1$, it is of stable type p' for every $p' < p$.
If $(\mathbf{B}, \|\cdot\|)$ is a Banach space of stable type $p \in]1, 2[$, Maurey–Pisier's theorem [15] asserts that there exists a $q' > p$ such that $\mathbf{B}$ is of stable type q'.

2.2 Tails of sums of independent r.v. and tails of extremes of individual terms

The first inequality we will need in the proof is inequality (3.3) in [10] :

Proposition 1 . *Let* $(\mathbf{X}_i)$ *be independent symmetric r.v. with values in* $\mathbf{B}$. *Set* $\mathbf{S}_k := \sum_{i=1}^{k} \mathbf{X}_i$. *If* $\mathbf{S}_n$ *converges to* $\mathbf{S}$*, then, for every* $s, t > 0$,

$$\mathbb{P}\left(\|\mathbf{S}\| > 2t + s\right) \leqslant 4\left(\mathbb{P}\left(\|\mathbf{S}\| > t\right)\right)^2 + \mathbb{P}\left(\sup_i \|\mathbf{X}_i\| > s\right).$$

An important corollary of this inequality, that compares the integrability properties of $\sup_n \frac{\|\mathbf{X}_n\|}{a_n}$ and $\sup_n \frac{\|\mathbf{S}_n\|}{a_n}$ is the following one (corollary 3.4 in [10]) :

Corollary 1 . *Let* $(\mathbf{X}_i)$ *be an independent sequence of* $\mathbf{B}$*-valued r.v,* (a_n) *an increasing sequence of positive numbers tending to infinity and* $0 < p < +\infty$. *Then if* $\sup_n \frac{\|\mathbf{S_n}\|}{a_n} < \infty$ *a.s, the following two properties are equivalent* :

i) $\mathbb{E} \sup_n \left(\frac{\|\mathbf{S}_n\|}{a_n}\right)^p < \infty$*;*

ii) $\mathbb{E} \sup_n \left(\frac{\|\mathbf{X}_n\|}{a_n}\right)^p < \infty$.

We will now state and prove a technical lemma, which will be used in the main proof.

2.3 A property of weak–ℓ_p spaces.

Marcus and Pisier [13] showed that if $(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ are independent scalar valued r.v,

$$\forall q \geqslant 1,\ \forall u > 0,\ \mathbb{P}(\|(\mathbf{X}_k)\|_{q,\infty} > u) \leqslant \frac{2e}{u^q} \sup_{t>0} \left(t^q \sum_{k=1}^{n} \mathbb{P}(|\mathbf{X}_k| > t)\right).$$

It can be mentionned that the original Marcus-Pisier inequality involved the constant 262 instead of $2e$! The improved constant is due to Zinn (see [17], lemma 4.11).
Let us derive from the previous inequality the result that will be crucial later.

Lemma 1 . *Let* $\mathbf{B}$ *be a Banach space of stable type* $1 < q < 2$. *Then there exists an universal constant* $c(q) > 0$ *such that for every finite sequence*

$(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ *of independent and integrable* $\mathbf{B}$*-valued r.v with sum* $\mathbf{S}_n := \mathbf{X}_1 + \cdots + \mathbf{X}_n$:

$$\mathbb{E}\,\|\mathbf{S}_n - \mathbb{E}\mathbf{S}_n\| \leqslant c(q)\Delta^{1/q},$$

where

$$\Delta := \Delta(\mathbf{X}_1, \ldots, \mathbf{X}_n) = \sup_{t>0} t^q \sum_{k=1}^{n} \mathbb{P}(\|\mathbf{X}_k\| > t).$$

<u>PROOF :</u>
In the sequel we will denote by C_k positive constants which precise value doesn't matter.
As $\mathbf{B}$ is of stable type $q > 1$, Maurey-Pisier theorem asserts that it is also of stable type q' for a $q' > q$. Therefore $\mathbf{B}$ is of Rademacher type q'.
We consider a sequence $(\varepsilon_k)_{1 \leqslant k \leqslant n}$ of independent Rademacher r.v. defined on $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$, a sequence $(\mathbf{X}_k)_{1 \leqslant k \leqslant n}$ of independent r.v. defined on $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ and a sequence $(\mathbf{X}'_k)_{1 \leqslant k \leqslant n}$ of independent r.v. defined on $(\Omega_3, \mathcal{F}_3, \mathbb{P}_3)$ such that $\mathbf{X}'_k$ and $\mathbf{X}_k$ have the same distribution. Let $\mathbf{S}'_n$ be the sum of the r.v. $\mathbf{X}'_k$: $\mathbf{S}'_n := \mathbf{X}'_1 + \cdots + \mathbf{X}'_n$. We denote by $\mathbf{N}(\mathbf{X}_k)$ the weak ℓ^q-norm of the real n-vector $(\|X_k\|)_{1 \leqslant k \leqslant n}$, that is $\mathbf{N}(\mathbf{X}_k) := \|(\|\mathbf{X}_k\|)_{1 \leqslant k \leqslant n}\|_{q,\infty}$. We will denote by $\mathbb{E}_i$ the expectation on the space Ω_i. Then, applying Fubini's and Jensen's theorems, we obtain, by symmetry of $\mathbf{X}_k - \mathbf{X}'_k$:

$$\begin{aligned}
\mathbb{E}_2\,\|\mathbf{S}_n - \mathbb{E}\mathbf{S}_n\| &\leqslant \mathbb{E}_2\mathbb{E}_3\,\|\mathbf{S}_n - \mathbf{S}'_n\| \\
&\leqslant 2\mathbb{E}_2\mathbb{E}_1 \left\| \sum_{k=1}^{n} \varepsilon_k \mathbf{X}_k \right\| \leqslant 2\mathbb{E}_2 \left(\mathbb{E}_1 \left\| \sum_{k=1}^{n} \varepsilon_k \mathbf{X}_k \right\|^{q'} \right)^{1/q'} \\
&\leqslant C_1 \mathbb{E}_2 \left(\sum_{k=1}^{n} \|\mathbf{X}_k\|^{q'} \right)^{1/q'} \leqslant C_2 \mathbb{E}_2 \mathbf{N}(\mathbf{X}_k) \times \left(\sum_{k=1}^{n} \frac{1}{k^{\frac{q'}{q}}} \right)^{1/q'} \\
&\leqslant C_3 \mathbb{E}\mathbf{N}(\mathbf{X}_k).
\end{aligned}$$

We now use the Marcus-Pisier inequality to obtain :

$$\begin{aligned}
\mathbb{E}\mathbf{N}(\mathbf{X}_k) &= \int_0^{\Delta^{1/q}} \mathbb{P}(\mathbf{N}(\mathbf{X}_k) > t)dt + \int_{\Delta^{1/q}}^{+\infty} \mathbb{P}(\mathbf{N}(\mathbf{X}_k) > t)dt \\
&\leqslant \Delta^{1/q} + \int_{\Delta^{1/q}}^{+\infty} \frac{2e\Delta}{t^q} dt.
\end{aligned}$$

□

We will close this section with a lemma concerning power series that we will use in proof of proposition 3.

2.4 A lemma about power series

Lemma 2 *:*
Let $0 < \alpha < 1$ *be a positive number. Let us define* $\delta_{\alpha,n} := \frac{1}{n^\alpha} - \frac{1}{(n+1)^\alpha}$ *for* $n \geqslant 1$ *and* $G_\alpha(x) := \sum_{n=1}^{+\infty} \delta_{\alpha n}(1-x^n)$ *for* $|x| \leqslant 1$*. Let* (a_i) *be a sequence of non-negative numbers with partial sums* $A_n := a_1 + \cdots + a_n$*. Then we have :*

1. $\sum_{j=1}^{+\infty} j^{-\alpha} a_j = \sum_{n=1}^{+\infty} \delta_{\alpha,n} A_n.$
2. $\frac{1}{(2n)^{\alpha+1}} \leqslant \frac{1}{(n+1)^{\alpha+1}} \leqslant \delta_{\alpha,n} \leqslant \frac{1}{n^{\alpha+1}}, \ \forall n \geqslant 1.$
3. $\exists C_\alpha > 0, \ \forall x \in [0,1], \ (1-x)^\alpha \leqslant G_\alpha(x) \leqslant C_\alpha (1-x)^\alpha.$

<u>PROOF :</u>
The first point is a consequence of the relation $\sum_{n=j}^{+\infty} \delta_{\alpha,n} = \frac{1}{j^\alpha}$ and of the Fubini-Tonelli theorem. The second one is obtained using the mean value theorem.
Let us prove the third one. Using the logarithms, one show that the infinite product $v_\alpha := \prod_{j=1}^{+\infty} \left(1 - \frac{\alpha}{j}\right) e^{\frac{\alpha}{j}}$ converges and that

$$\forall n \geqslant 1, \ 0 < v_\alpha \leqslant \prod_{j=1}^{n} \left(1 - \frac{\alpha}{j}\right) e^{\frac{\alpha}{j}} \leqslant 1 \tag{2.3.}$$

Since $\sum_{j=1}^{n} \frac{1}{j} - 1 \leqslant \sum_{j=2}^{n+1} \frac{1}{j} \leqslant \ln(n+1) = \int_1^{n+1} \frac{1}{x} dx \leqslant \sum_{j=1}^{n} \frac{1}{j}$, one has the following inequalities :

$$\frac{v_\alpha}{e^\alpha (n+1)^\alpha} \leqslant v_\alpha \exp\left(-\sum_{j=1}^{n} \frac{\alpha}{j}\right) \leqslant \prod_{j=1}^{n} \left(1 - \frac{\alpha}{j}\right) \leqslant \exp\left(-\sum_{j=1}^{n} \frac{\alpha}{j}\right) \leqslant \frac{1}{(n+1)^\alpha} \tag{2.4.}$$

As $\frac{1}{(1-x)^{1-\alpha}} = \sum_{i=1}^{+\infty}\left(\prod_{j=1}^{n}\left(1-\frac{\alpha}{j}\right)^{n}\right)x^{n}$, one deduces from (2.4.), denoting $F_\alpha(x) := \sum_{n=0}^{+\infty}\frac{x^n}{(n+1)^\alpha}$ the boundary :

$$\forall 0 \leqslant x < 1,\ \frac{v_\alpha}{e^\alpha}F_\alpha(x) \leqslant (1-x)^{\alpha-1} \leqslant F_\alpha(x). \qquad (2.5.)$$

Applying now the first point with $a_i := (1-x)x^{i-1}$, one obtains $G_\alpha(x) = (1-x)F_\alpha(x)$, which completes the proof of property 3 with $C_\alpha := \frac{e^\alpha}{v_\alpha}$. □

3 The main results

Allan Gut's result (theorem 2) easily extends to Banach spaces of Rademacher type p as follows :

Proposition 2 . *Let* $\mathbf{B}$ *be a Banach space of Rademacher type* $p \in]1,2[$, *and* $(\mathbf{X}_i)$ *be a sequence of independent copies of a centered* $\mathbf{B}$*-valued r.v.* $\mathbf{X}$ *such that* $\mathbb{E}\|\mathbf{X}\|^p < +\infty$. *Then* $\left(\frac{\|\mathbf{S}_n\|}{n^{1/p}}, \mathcal{F}_n\right)$ *is an amart (where* $\mathcal{F}_n := \sigma(\mathbf{X}_1, \ldots, \mathbf{X}_n)$ *).*

PROOF :
This straightforward consequence of theorem 4 has perhaps been noticed before. By lack of a suitable reference, we give an elementary proof. Consider the sequence of inequalities :

$$\begin{aligned}\mathbb{E}\sup_{k\geqslant 1}\frac{\|\mathbf{X}_k\|}{k^{1/p}} &\leqslant 1 + \int_1^{+\infty}\mathbb{P}\left(\sup_{k\geqslant 1}\frac{\|\mathbf{X}_k\|}{k^{1/p}} > t\right)dt \leqslant 1 + \int_1^{+\infty}\sum_{k=1}^{+\infty}\mathbb{P}\left(\frac{\|\mathbf{X}\|^p}{t^p} > k\right)dt \\ &\leqslant 1 + \int_1^{+\infty}\frac{\mathbb{E}\|\mathbf{X}\|^p}{t^p} < +\infty.\end{aligned}$$

As $\left(\frac{\mathbf{S}_n}{n^{1/p}}\right)$ converges a.s. to 0, corollary 1 implies that $\mathbb{E}\sup_{n\geqslant 1}\frac{\|\mathbf{S}_n\|}{n^{1/p}} < +\infty$. Finally, the inequality $\sup_{\substack{\tau\in\mathcal{T}\\ \tau\geqslant N}}\mathbb{E}\frac{\|\mathbf{S}_\tau\|}{\tau^{1/p}} \leqslant \mathbb{E}\sup_{n\geqslant N}\frac{\|\mathbf{S}_n\|}{n^{1/p}}$ implies that the sequence $\left(\frac{\|\mathbf{S}_n\|}{n^{1/p}}\right)$ is an amart by the bounded convergence theorem. □

The general version of theorem 3 is as follows :

Theorem 5 . *Let* $\mathbf{B}$ *be a Banach space of stable type* $p \in]1, 2[$, *and* $\mathbf{X}$ *a centered* $\mathbf{B}$*-valued r.v.. The following two properties are equivalent :*

1. $\left(\frac{\mathbf{S}_n}{n^{1/p}}, \mathcal{F}_n\right)$ *is a quasimartingale.*
2. $\int_0^{+\infty} \mathbb{P}^{1/p}(\|\mathbf{X}\| > t)dt < +\infty.$ $(\star)$

Remark 4 . An easy computation shows that $(\star)$ implies the integrability of $\|\mathbf{X}\|^p$. This fact will be used in the proof of proposition 3.

The hypothesis $(\star)$ appears to be a regularity property of the quantiles of the r.v. $\|\mathbf{X}\|$. Remember that for all $n \in \mathbb{N}$, the quantiles u_n of order $\left(1 - \frac{1}{n}\right)$ of $\|\mathbf{X}\|$ are defined as follows :

$$u_n := \inf\left\{x \,\middle|\, \mathbb{P}(\|\mathbf{X}\| \leqslant x) > 1 - \frac{1}{n}\right\} = \inf\left\{x \,\middle|\, \mathbb{P}(\|\mathbf{X}\| > x) < \frac{1}{n}\right\}.$$

Proposition 3 . *The following three properties are equivalent :*

(*a*) $\int_0^{+\infty} \mathbb{P}^{1/p}(\|\mathbf{X}\| > t)dt < +\infty.$

(*b*) $\sum_{n\geqslant 1} \frac{u_n}{n^{1+1/p}} < +\infty$

(*c*) $\sum_{n\geqslant 1} \frac{1}{n^{1+1/p}} \mathbb{E} \sup_{1\leqslant k\leqslant n} \|\mathbf{X}_k\| < +\infty$

<u>PROOF :</u>
In the sequel, we will denote by $f(t)$ the tail of the distribution of $\|\mathbf{X}\|$:

$$f(t) := \mathbb{P}(\|\mathbf{X}\| > t).$$

We also define $u_0 := 0$, and denote $\alpha := 1/p$.
Let us show that (a) and (b) are equivalent. For every $j \in \mathbb{N}^*$, one defines $t_j := u_{j+1} - u_j$. Note that (u_n) is an increasing sequence, with $\sup_{n\geqslant 1} u_n = \|\mathbf{X}\|_\infty$ and that $\forall t \geqslant \|\mathbf{X}\|_\infty,\ f(t) = 0$. Therefore (t_j) is a sequence of non-negative numbers. First write

$$u_1 + \sum_{j\geqslant 1} \int_{u_j}^{u_{j+1}} f^\alpha(t)dt \geqslant \int_0^{+\infty} f^\alpha(t)dt \geqslant \sum_{j\geqslant 1} \int_{u_j}^{u_{j+1}} f^\alpha(t)dt.$$

Therefore property (a) is equivalent to the convergence of the series with general terms $\int_{u_j}^{u_{j+1}} f^\alpha(t)dt$.
As $\forall t \in]u_j, u_{j+1}[,\ \frac{1}{j+1} \leqslant f(t) \leqslant \frac{1}{j}$, one gets

$$\frac{t_j}{2^\alpha j^\alpha} \leqslant \frac{t_j}{(j+1)^\alpha} \leqslant \int_{u_j}^{u_{j+1}} f^\alpha(t)dt \leqslant \frac{t_j}{j^\alpha}.$$

Hence, property (a) is equivalent to the convergence of the series with general term $\frac{t_j}{j^\alpha}$.
Applying lemma2 with $a_j := t_j$, one obtains

$$\sum_{j=1}^{+\infty} \frac{t_j}{j^\alpha} = \sum_{n=1}^{+\infty} \delta_{\alpha,n}(u_{n+1} - u_1).$$

Now using the second point of lemma 2, one gets :

$$\frac{1}{2^{\alpha+1}} \sum_{n=1}^{+\infty} \frac{u_{n+1} - u_1}{n^{\alpha+1}} \leqslant \sum_{n=1}^{+\infty} \delta_{1/pn}(u_{n+1} - u_1) \leqslant \sum_{n=1}^{+\infty} \frac{u_{n+1} - u_1}{n^{\alpha+1}}.$$

This shows the equivalence of properties (a) and (b).
Let us show the equivalence between (a) and (c). First notice that

$$\mathbb{E} \sup_{1\leqslant k\leqslant n} \|\mathbf{X}_k\| = \int_0^{+\infty} \mathbb{P}(\sup_{1\leqslant k\leqslant n} \|\mathbf{X}_k\| > t)dt = \int_0^{+\infty} \left(1 - (1 - f(t))^n\right) dt.$$

According to third point of lemma 2,

$$\exists C_\alpha > 0,\ \forall t \geqslant 0,\ f^\alpha(t) \leqslant G_\alpha(1-f(t)) = \sum_{n=1}^{+\infty} \delta_{\alpha n} \left(1 - (1 - f(t))^n\right) \leqslant C_\alpha f^\alpha(t).$$

Integrating this inequality and using Fubini-Tonelli theorem, one gets

$$\int_0^{+\infty} f^\alpha(t)dt \leqslant \sum_{n=1}^{+\infty} \delta_{\alpha n} \mathbb{E} \sup_{1\leqslant k\leqslant n} \|\mathbf{X}_k\| \leqslant C_\alpha \int_0^{+\infty} f^\alpha(t)dt.$$

The second point of lemma 2 gives the equivalence between (a) and (c).

$\square$

4 Proof of theorem 5

Let us show that 1 implies 2.
Suppose that $\left(\frac{\mathbf{S}_n}{n^{1/p}}\right)$ is a quasimartingale – that is $\sum \mathbb{E}\left\|\mathbb{E}\left(\frac{\mathbf{S}_{n+1}}{(n+1)^{1/p}}\middle|\mathcal{F}_n\right) - \frac{\mathbf{S}_n}{n^{1/p}}\right\| < +\infty$ or equivalently $\sum \frac{\mathbb{E}\|\mathbf{S}_n\|}{n^{1+1/p}} < +\infty$. Taking the same notations as in the proof of lemma 1, applying Jensen's and Levy's inequalities, one obtains :

$$\begin{aligned}\mathbb{E}\sup\|\mathbf{S}_k\| =&\mathbb{E}_2\sup\|\mathbf{S}_k - \mathbb{E}(\mathbf{S}'_k)\| \leqslant \mathbb{E}_2\mathbb{E}_3\sup\|\mathbf{S}_k - \mathbf{S}'_k\| \\ &\leqslant 2\mathbb{E}_1\mathbb{E}_2\|\mathbf{S}_n - \mathbf{S}'_n\| \leqslant 4\mathbb{E}\|\mathbf{S}_n\|.\end{aligned}$$

So by proposition 3, properties 1 and 2 are equivalent.

Let us show that 2 implies 1.
As $\mathcal{B}$ is of stable type p, according to Maurey-Pisier's theorem it is also of stable type q for some $q > p$ and so of Rademacher type q.
Now, for every $n \in \mathbb{N}^*$, and every $k = 1, \ldots, n$, one considers the following truncated r.v. :

$$\mathbf{U}_k := \mathbf{U}_{n,k} := \mathbf{X}_k \mathbf{1}_{(\|\mathbf{X}_k\| > u_n)} \quad \text{and} \quad \mathbf{V}_k := \mathbf{V}_{n,k} := \mathbf{X}_k \mathbf{1}_{(\|\mathbf{X}_k\| \leqslant u_n)}$$

To these truncated r.v. associate the sums :

$$\mathbf{T}_n^{(1)} := \sum_{k=1}^n \mathbf{U}_k \quad \text{and} \quad \mathbf{T}_n^{(2)} := \sum_{k=1}^n \mathbf{V}_k$$

If for every $j = 1{,}2$ the series with general term $\mathbb{E}\|\mathbf{T}_n^{(j)} - \mathbb{E}\mathbf{T}_n^{(j)}\|/n^{1+1/p}$ converges, then $\left(\frac{\mathbf{S}_n}{n^{1/p}}, \mathcal{F}_n\right)$ clearly will be a quasimartingale.
So let us check that these two series converge.

Lemma 3 .

$$\sum_{n\geqslant 1} \frac{1}{n^{1+1/p}} \mathbb{E}\|\mathbf{T}_n^{(1)} - \mathbb{E}\mathbf{T}_n^{(1)}\| < +\infty.$$

<u>PROOF :</u>
We will first prove that

$$\sum_{n\geqslant 1} \frac{1}{n^{1+1/p}} \mathbb{E}\|\mathbf{T}_n^{(1)}\| < +\infty. \tag{4.1.}$$

First notice the following elementary inequality

$$\begin{aligned}\mathbb{E}\|\mathbf{T}_n^{(1)}\| &\leqslant n\mathbb{E}\|\mathbf{X}\|\mathbf{1}_{(\|\mathbf{X}\|>u_n)}\\ &= nu_n\mathbb{P}(\|\mathbf{X}\|>u_n)+n\int_{u_n}^{+\infty}f(t)dt \leqslant u_n+n\int_{u_n}^{+\infty}f(t)dt.\end{aligned}$$

Proposition 3 implies that the series with general term $\frac{u_n}{n^{1+1/p}}$ converges. It remains to check the convergence of the series with general term $\frac{1}{n^{1/p}}\int_{u_n}^{+\infty}f(t)dt$. Writing

$$\sum_{n=1}^{+\infty}n^{-1/p}\int_{u_n}^{+\infty}f(t)dt=\sum_{n=1}^{+\infty}n^{-1/p}\sum_{j=n}^{+\infty}\int_{u_j}^{u_{j+1}}f(t)dt$$

and exchanging the two summations, one gets

$$\sum_{j=1}^{+\infty}\left(\int_{u_j}^{u_{j+1}}f(t)dt\right)\sum_{n=1}^{j}\frac{1}{n^{1/p}}\leqslant C_4\sum_{j\geqslant 1}\left(\int_{u_j}^{u_{j+1}}f^{1/p}(t)dt\right)\frac{j^{1-1/p}}{j^{1-1/p}}\leqslant C_4\int_0^{+\infty}f^{1/p}(t)dt.$$

This completes the proof of relation (4.1.).
As $\mathbb{E}\|\mathbf{T}_n^{(1)}-\mathbb{E}\mathbf{T}_n^{(1)}\|\leqslant 2\mathbb{E}\|\mathbf{T}_n^{(1)}\|$, the series with general term $\mathbb{E}\|\mathbf{T}_n^{(1)}-\mathbb{E}\mathbf{T}_n^{(1)}\|$ converges. □

According to lemma 1, the convergence of the series with general term $\mathbb{E}\|\mathbf{T}_n^{(2)}-\mathbb{E}\mathbf{T}_n^{(2)}\|/n^{1+1/p}$ will be completed if we prove the following lemma :

Lemma 4 . *The series with general term $\Delta_n^{1/q}/n^{1+1/p}$ converges, where $\Delta_n:=\Delta(\mathbf{V}_{n,1},\ldots,\mathbf{V}_{n,n})$.*

<u>Proof :</u>
To begin with, one will show that the following inequality holds :

$$\Delta_n\leqslant n\left(\int_0^{u_n}f^{1/q}dt\right)^q. \qquad (4.2.)$$

Let us denote for simplicity $\mathbf{V}_1:=\mathbf{V}_{n,1}$ and observe the following inequality :

$$\sup_{x>0}(x^q\mathbb{P}(\|\mathbf{V}_1\|>x)) = \sup_{x\in[0,u_n]}(x^q\mathbb{P}(\|\mathbf{V}_1\|>x))\leqslant \sup_{x\in[0,u_n]}x^qf(x)$$

As the function f is decreasing,

$$\forall x\geqslant 0,\ x^qf(x)\leqslant\left(\int_0^x f^{1/q}(t)dt\right)^q,$$

therefore

$$\forall x > 0,\ x^q \mathbb{P}(\|\mathbf{V}_1\| > x) \leqslant \left(\int_0^{u_n} f^{1/q}(t)dt\right)^q$$

which ends the proof of relation (4.2.).
To conclude the proof of lemma 4, it remains to check that the series with general term $n^{\left(\frac{1}{q}-\frac{1}{p}-1\right)} \int_0^{u_n} f^{1/q}(x)dx$ is convergent.
Observing that

$$J := \sum_{n\geqslant 1} n^{\left(\frac{1}{q}-\frac{1}{p}-1\right)} \int_0^{u_n} f^{1/q}(x)dx \leqslant \sum_{n\geqslant 1} n^{\left(\frac{1}{q}-\frac{1}{p}-1\right)} \sum_{0\leqslant j\leqslant n} \int_{u_j}^{u_{j+1}} f^{1/q}(x)dx,$$

and exchanging one more time the summations in n and j, one obtains

$$J \leqslant C_5 \sum_{j\geqslant 1} j^{\left(\frac{1}{q}-\frac{1}{p}\right)} \left(\int_{u_j}^{u_{j+1}} f^{1/q}(x)dx\right) + u_1 \leqslant C_6 \left(\int_0^{\infty} f^{1/p}(x)dx + u_1\right) < +\infty.$$

This concludes the proof of lemma 4 and also the one of theorem 5.

□

5 Is it possible to weaken the stable type hypothesis in theorem 5?

Let $p \in]1,2[$. According to proposition 2 it is natural to wonder whether the stable type hypothesis in theorem 5 could be improved to a (Rademacher) type hypothesis.
The LLN in stable type spaces has some very specific aspects (see for instance [14] and paragraph 9.3 in [11]) which suggest that it is not possible to improve this stable type hypothesis. The quasimartingale behaviour in the Kolmogorov setting SLLN considered in [7] goes in the same direction.
As $1 < p < 2$, ℓ_p is of Rademacher type p but not of stable type p. We will construct in this section an ℓ_p–valued r.v. $\mathbf{X}$ (for $1 < p < 2$), which fulfills the regularity condition $(\star)$ and such that $\left(\frac{\mathbf{S}_n}{n^{1/p}}, \mathcal{F}_n\right)$ is not a quasimartingale.
Let us denote by (e_k) the canonical basis of ℓ_p.
Let $(\xi_n)_{n\geqslant 2}$ be a sequence of independent random variables having a Pareto distribution of parameters $(1,p)$. So for every n, ξ_n has the density :

$$f(x) := \frac{p}{x^{p+1}} \mathbf{1}_{[1,+\infty[}$$

with respect to the Lebesgue measure.
Let $(\varepsilon_n)_{n\geqslant 2}$ be a sequence of independent Rademacher r.v., independent of the sequence (ξ_n), and consider the random variable :

$$\mathbf{X} := \sum_{n=2}^{+\infty} \frac{1}{n^{1/p}(\ln n)^{\frac{p+1}{p}-\frac{p-1}{2p}}} \varepsilon_n \xi_n \mathbf{1}_{\{\xi_n \leqslant n^{1/p}\}} e_n.$$

Let us show that $\mathbf{X}$ is a random variable with values in ℓ_p :

$$\mathbb{E}\|\mathbf{X}\|_p^p = \sum_{n=2}^{+\infty} \frac{1}{n(\ln n)^{p+1-\frac{p-1}{2}}} \mathbb{E}|\xi_n \mathbf{1}_{\{\xi_n \leqslant n^{1/p}\}}|^p \leqslant C_7 \sum_{n=2}^{+\infty} \frac{1}{n(\ln n)^{\frac{p+3}{2}}} \ln n < +\infty.$$

Now let us show that $\mathbf{X}$ fulfills the condition $\int_0^{+\infty} \mathbb{P}^{1/p}(\|\mathbf{X}\| > t)dt < +\infty$.
The r.v. $\mathbf{X}$ can be considered as a sum of independent symmetrically distributed r.v. which are all of norm smaller than one. So by proposition 1, $\mathbb{P}(\|\mathbf{X}\| > 3n) \leqslant 4(\mathbb{P}(\|\mathbf{X}\|^p > n^p))^2 \leqslant 4\frac{(\mathbb{E}\|\mathbf{X}\|^p)^2}{n^{2p}}$.
As $\int_0^{+\infty} \mathbb{P}^{1/p}(\|\mathbf{X}\| > t)dt \leqslant 3 + \sum_{n=1}^{+\infty} \mathbb{P}^{1/p}(\|\mathbf{X}\| > 3n)$, $(\star)$ therefore holds.
Let $(\mathbf{X}_j)$ be a sequence of independent copies of $\mathbf{X}$. We denote, for every j,

$$\mathbf{X}_j := \sum_{n=2}^{+\infty} \frac{1}{n^{1/p}(\ln n)^{\frac{p+1}{p}-\frac{p-1}{2p}}} \varepsilon_n^j \xi_n^j \mathbf{1}_{\{\xi_n \leqslant n^{1/p}\}} e_n.$$

We also denote, for every n, $\mathbf{S}_n := \mathbf{X}_1 + \cdots + \mathbf{X}_n$.
Let $\omega \in \Omega$ be chosen. Define :

$$v_k(\omega) := \frac{1}{k^{\frac{p-1}{p}}(\ln k)^{\frac{p-1}{p}+\frac{p-1}{2p}}} sgn\left(\sum_{j=1}^{n} \xi_k^j \varepsilon_k^j \mathbf{1}_{\{\xi_k^j \leqslant k^{1/p}\}}(\omega)\right),$$

with $sgn(0) = 1$. The sequence $(v_k(\omega))$ is an element of $\ell_{\frac{p}{p-1}}$. Denoting $v(\omega) := (cv_k(\omega))$ where c is chosen such that $\|v\|_{\frac{p}{p-1}} = 1$ and setting

$$\begin{array}{rccc} \psi : & \ell_p & \longrightarrow & \mathbb{R} \\ & (x_i) & \longmapsto & c\sum v_i(\omega)x_i \end{array}$$

one gets :

$$\frac{\|\mathbf{S}_n(\omega)\|_p}{n^{1/p}} \geqslant \frac{|\psi(\mathbf{S}_n)(\omega)|}{n^{1/p}} \geqslant \sum_{k=2}^{+\infty} \frac{1}{k(\ln k)^2} \left| \sum_{j=1}^{n} \frac{\xi_k^j \varepsilon_k^j \mathbf{1}_{\{\xi_k^j \leqslant k^{1/p}\}}(\omega)}{n^{1/p}} \right|.$$

Therefore, taking the expectation, one obtains :

$$\begin{aligned}
\mathbb{E}\frac{\|\mathbf{S}_n\|}{n^{1/p}} &\geqslant \sum_{k=2}^{+\infty} \frac{1}{k(\ln k)^2}\mathbb{E}\left|\sum_{j=1}^{n} \frac{\xi_k^j \varepsilon_k^j \mathbf{1}_{\{\xi_k^j \leqslant k^{1/p}\}}}{n^{1/p}}\right| \\
&\geqslant C_8 \sum_{k=2}^{+\infty} \frac{1}{k(\ln k)^2}\mathbb{E} \sup_{1\leqslant j\leqslant n} \frac{\xi_k^j \mathbf{1}_{\{\xi_k^j \leqslant k^{1/p}\}}}{n^{1/p}} \\
&\geqslant C_9 \sum_{k=4^p n}^{+\infty} \frac{1}{k(\ln k)^2} \int_0^{+\infty} \mathbb{P}(\sup_{1\leqslant j\leqslant n} \xi_k^j \mathbf{1}_{\{\xi_k^j \leqslant k^{1/p}\}} > n^{1/p} t) dt
\end{aligned}$$

For $k \geqslant 4^p n$, a calculus similar to one that has already been done earlier shows that

$$\begin{aligned}
\mathbb{E}\frac{\|\mathbf{S}_n\|}{n^{1/p}} &\geqslant C_{10} \sum_{k=4^p n}^{+\infty} \frac{1}{k(\ln k)^2} \int_0^{+\infty} \mathbb{P}(\sup_{1\leqslant j\leqslant n} \xi_k^j \mathbf{1}_{\{\xi_k^j \leqslant k^{1/p}\}} > n^{1/p} t) dt \\
&\geqslant C_{11} \sum_{k=4^p n}^{+\infty} \frac{1}{k(\ln k)^2} \int_2^{3} n\mathbb{P}(n^{1/p} t < \xi_k^1 \leqslant k^{1/p}) dt \\
&\geqslant C_{12} \sum_{k=4^p n}^{+\infty} \frac{1}{k(\ln k)^2} \int_2^{3} \left(\frac{1}{t^p} - \frac{1}{4^p}\right) dt \geqslant C_{13} \sum_{k=4^p n}^{+\infty} \frac{1}{k(\ln k)^2} dt \geqslant C_{14} \frac{1}{\ln n}
\end{aligned}$$

So

$$\sum_{n=2}^{+\infty} \frac{\mathbb{E}\|\mathbf{S}_n\|}{n^{1+\frac{1}{p}}} \geqslant \sum_{n=2}^{+\infty} \frac{1}{n(\ln n)} = +\infty$$

and therefore $\left(\frac{\mathbf{S}_n}{n^{1/p}}, \mathcal{F}_n\right)$ is not a quasimartingale.

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