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Jean-Luc Fouquet, Jean-Marie Vanherpe

▶ **To cite this version:**

Jean-Luc Fouquet, Jean-Marie Vanherpe. Seidel complementation on (P_5, $House$, $Bull$)-free graphs. 2010. hal-00467642

HAL Id: hal-00467642
https://hal.science/hal-00467642

Preprint submitted on 27 Mar 2010

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SEIDEL COMPLEMENTATION ON (P_5, $House$, $Bull$)-FREE GRAPHS

J.L. FOUQUET AND J.M. VANHERPE

Abstract.

1. Seidel complementation

Notation 1.1. *$N(v)$ denotes the neighborhood of the vertex v, $N[v] = N(v) \cup \{v\}$ and $\overline{N(v)} = V - N(v)$.*

Given a graph $G = (V, E)$ and a vertex v of G, the seidel complement of G on v inverses all edges between $N(v)$ and $V - N[v]$. More formaly

Definition 1.2. Let $= (V, E)$ be an undirected graph and let v be a vertex of G. The Seidel complement oat v on G, denoted $G * v$ is defined as follows :
$G = v = (V, E_1 \cup E_2 \cup E_3)$ where $E_1 = \{xy | xy \in E, x \in N[v], y \in N[v]\}$, $E_2 = E \cap \overline{N(v)}^2$ and $E_3 = \{xy | xy \notin E, x \in N(v), y \in \overline{N(v)}\}$.

Remark 1.3. [3]

- *$G * v * v = G$.*
- *If G is a cograph and v is a vertex of G then $G * v$ is a cograph.*
- *G is prime with respect to modular decomposition if and only if $G * v$ is prime with respect to modular decomposition*
- $\overline{G * v} = \overline{G} * v$

2. seidel complementation on (P_5, $\overline{P_5}$,$Bull$)-free graphs

Theorem 2.1. *A graph G is (P_5, $House$, $Bull$)-free if and only if for all vertex v of G, $G * v$ is (P_5, $House$, $Bull$)-free.*

Proof Let v be a vertex of G. Assume that $G * v$ contains an induced subgraph, say H,which is isomorphic to either a P_5 or a $House$ or a $Bull$.

Claim 2.1.1. *The vertex v does not belong to H.*

1991 *Mathematics Subject Classification.* 035 C.

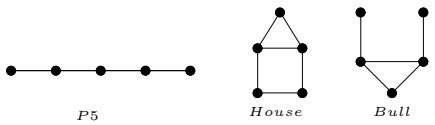


Figure 1. Forbidden configurations for (P_5, $House$, $Bull$)-free graphs.

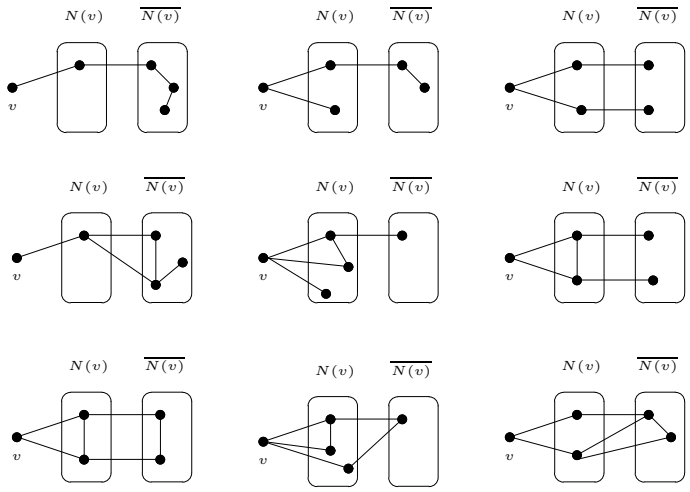


FIGURE 2. The possible cases when v is a vertex of H.

Proof Assume not. Figure 2 describes all cases that can occur whenever v is a vertex of H. It is not difficult to check that, in all cases, $(G * v) * v$ would contain a subgraph isomorphic to P_5 or to $House$ or to $Bull$, a contradiction since $G * v * v = G$ and G is assumed to be (P_5, $House$, $Bull$)-free.

□

In the following we suppose that the vertices of H are in $N(v) \cup \overline{N(v)}$, moreover H has vertices in both sets $N(v)$ and $\overline{N(v)}$, otherwise H would be an induced subgraph of $G * v * v = G$, a contradiction.

Figure 3 (resp Figure 4, Figure 5) describes all cases that can occur whenever H is isomorphic to a P_5 (resp. a $Bull$, a $House$) and has at most two vertices in $N(v)$. In all cases we get a contradiction with Claim 2.1.1 or a forbidden configuration appears in $G * v * v$. When H has more than 2 vertices in $N(v)$ we get a similar contradiction in considering $\overline{G}$. □

3. SEIDEL COMPLEMENTATION ON THE MODULAR DECOMPOSITION TREE OF (P_5, $\overline{P_5}$,$Bull$)-FREE GRAPHS

3.1. **On (P_5, $\overline{P_5}$)-free graphs.** We shall say that an induced subgraph of a (P_5,$\overline{P_5}$))-free graph is a *buoy*[1] whenever we can find a partition of its vertex set into 5 subsets $A_i, i = 1, \ldots, 5$ (subsript i is to be taken modulo 5, such that A_i and A_{i+1} are joined by every possible edge, while no possible edge are allowed between A_i and A_j when $j \neq i + 1$ *mod*[5], and such that the A_i's are maximal for these properties.

Theorem 3.1. [1, 2] *Let G be a connected (P_5,$\overline{P_5}$)-free graph. If G contains an induced C_5 then every C_5 of G is contained into a buoy, and this buoy is either equal to G or is an homogeneous set of G.*

Corollary 3.2. *Every prime (P_5,$\overline{P_5}$)-free graph is either a C_5 or is C_5-free*

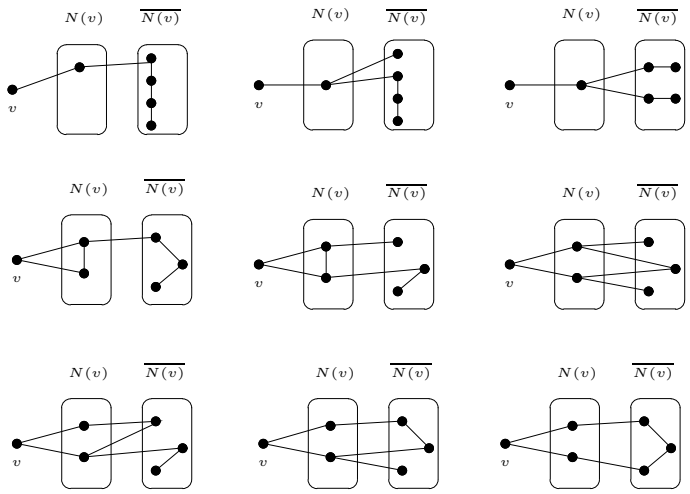


FIGURE 3. Cases : H is a P_5 and has at most 2 vertices in $N(v)$.

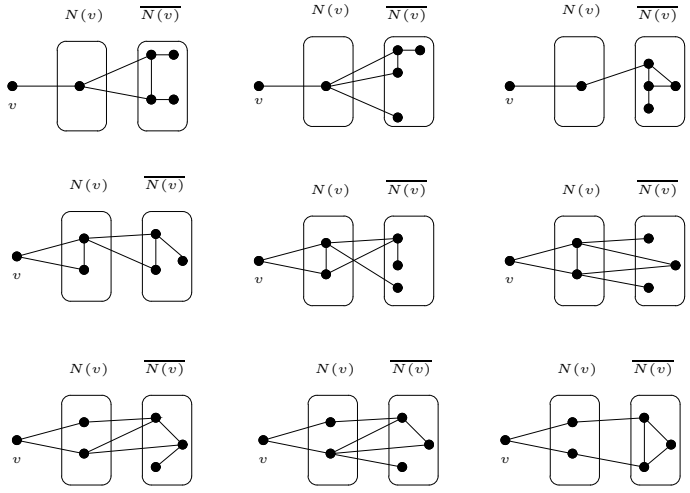


FIGURE 4. Cases : H is a *Bull* and has at most 2 vertices in $N(v)$.

3.2. On (P_5, $\overline{P_5}$, *Bull*)-free graphs.

Theorem 3.3. [1] *Let G be a prime graph. G is a P_5HB-free graph if and only if one of the following conditions is satisfied*

(i) *G is isomorphic to a C_5;*
(ii) *G is bipartite and P_5-free;*
(iii) *$\overline{G}$ is bipartite and P_5-free.*

In a prime P_5-free biparte graph the neighborhoods of two distinct vertices cannot overlap properly, thus :

Proposition 3.4. *Let $G = (V, E)$ be a prime graph of n vertices. G is bipartite and P_5-free iff the following conditions are verified:*

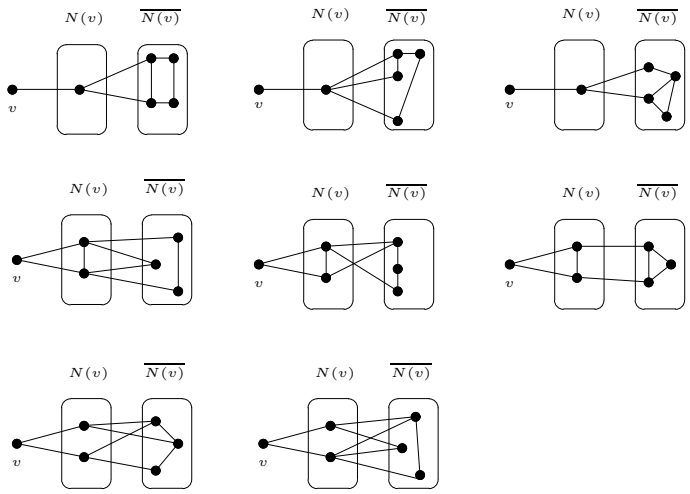


FIGURE 5. Cases : H is a *House* and has at most 2 vertices in $N(v)$.

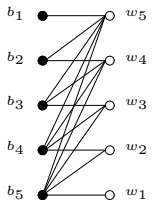


FIGURE 6. A Prime P_5-free bipartite graph.

(i) *There exists a partition of* $V(G)$ *into two stable sets* $B = \{b_1, b_2, \dots b_{\frac{n}{2}}\}$ *and* $W = \{w_1, w_2, \dots w_{\frac{n}{2}}\}$.
(ii) *The neighbors of* b_i $(i = 1 \dots \frac{n}{2})$ *are precisely* $w_1, \dots w_{\frac{n}{2}-i+1}$.

3.3. **Seidel complementation of a Prime P_5-free bipartite graph.**

Proposition 3.5. *Let* $G = (B \cup W, E)$ *be a prime bipartite* P_5*-free graph such that* $B = \{b_1, \dots b_{\frac{n}{2}}\}$ *and* $W = \{w_1, \dots, w_{\frac{n}{2}}\}$*, then* $G * b_i$ *is a prime* P_5*-free bipartite graph together with the bipartition :*

$B' = \{b_{i-1}, b_{i-2}, \dots b_i, w_1, w_2, \dots, w_{\frac{n}{2}}\}, W' = \{b_{i+1}, \dots, b_{\frac{n}{2}}, w_{\frac{n}{2}}, w_{\frac{n}{2}-1}, \dots w_{\frac{n}{2}-i+1}\}$

Proof We have $N(b_i) = \{w_{\frac{n}{2}}, w_{\frac{n}{2}-1}, \dots w_{\frac{n}{2}-i+1}\}$ and $\overline{N(b_i)} = \{b_1, b_2, \dots, b_{i-1}, w_{\frac{n}{2}-i}, \dots, w_1\}$. It must be pointed out that the subgraphs induced by the sets $S_1 = \{w_{\frac{n}{2}}, w_{\frac{n}{2}-1}, \dots w_{\frac{n}{2}-i+1}, b_1, b_2, \dots, b_{i-1}\}$ and $S_2 = \{b_{i+1}, \dots, b_{\frac{n}{2}}, w_{\frac{n}{2}-i}, \dots w_1\}$ are prime P_5-free bipartite graphs as well as $S_3 = \{w_{\frac{n}{2}}, w_{\frac{n}{2}-1}, \dots w_{\frac{n}{2}-i+1}, b_{i+1}, \dots, b_{\frac{n}{2}}\}$ induces a complete bipartite graph. The result follows when considering the Seidel complementation on b_i (see Figure 7). □

Corollary 3.6. *For a prime* P_5*-free bipartite graph (or its complement) Seidel complementation at any vertex can be performed in constant time.*

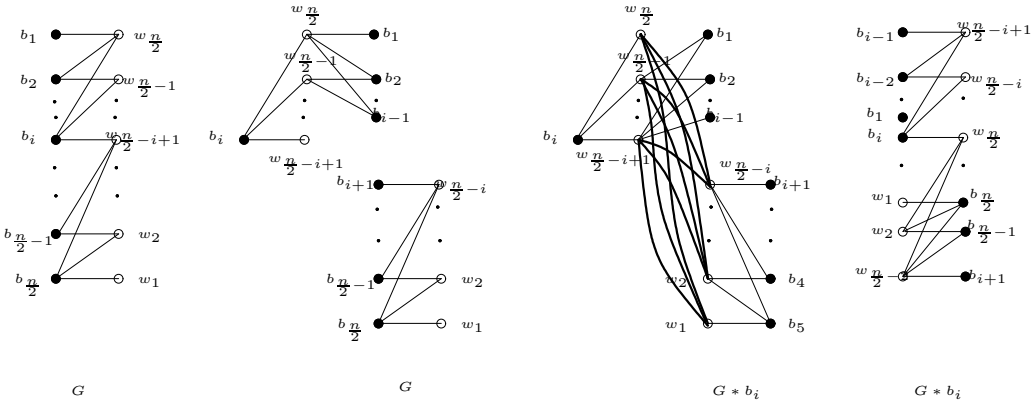


FIGURE 7. Seidel complementation on b_2.

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L.I.F.O., FACULTÉ DES SCIENCES, B.P. 6759, UNIVERSITÉ D'ORLÉANS, 45067 ORLÉANS CEDEX 2, FR