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Heat Equations on Vector Bundles - Application to Color Image Regularization

Thomas Batard

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Abstract We use the framework of heat equations associated to generalized Laplacians on vector bundles over Riemannian manifolds in order to regularize color images. We show that most methods devoted to image regularization may be considered in this framework. From a geometric viewpoint, they differ by the metric of the base manifold and the connection of the vector bundle involved. By the regularization operator we propose in this paper, the diffusion process is completely determined by the geometry of the vector bundle. More precisely, the metric of the base manifold determines the anisotropy of the diffusion through the computation of geodesic distances whereas the connection determines the data regularized by the diffusion process through the computation of the parallel transport maps. This regularization operator generalizes the ones based on short-time Beltrami kernel and oriented Gaussian kernels. Then we construct particular connections and metrics involving color information such as luminance and chrominance in order to perform new kinds of regularization.

Keywords Heat equation · Generalized Laplacian · Differential Geometry of Vector Bundle · Riemannian geometry · Regularization · Color Image

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1 Introduction

Geometric Partial Differential Equations find a lot of applications in image processing problems such as segmentation, regularization, inpainting ([1],[7],[12],[13]). The geometric data involved are mostly concepts of Riemannian geometry (curvature, geodesic distance, etc). We refer to [9] for an introduction to differential geometry of manifolds. Riemannian geometry may be viewed as a particular case of differential geometry of vector bundles. Indeed, Riemannian geometry concepts arise from the assignment of the Levi-Cevita connection on the tangent bundle of a manifold equipped with a positive definite metric. More generally, a (differential) geometry on a vector bundle is determined by a connection (or covariant derivation) on this fiber bundle (see e.g. [8],[10] for an introduction to fiber bundles). The use of vector bundles framework to treat image processing problems was already proposed in [2] for edge detection with constraints on multivalued images, and in [3],[4] for image, vectors field and orientations field regularization.

In [4], we have introduced the use of heat equations associated to **generalized Laplacians** in the context of multivalued image regularization. These are equations of the form

$$\frac{\partial u}{\partial t} + Hu = 0, \quad u|_{t=0} = u_0 \quad (1)$$

associated to second-order differential operators H acting on sections of vector bundles E over Riemannian manifolds (X, g) , called generalized Laplacians. As a consequence, it ensures existence and unicity of a kernel $K_t(x, y, H)$ associated to H , called the **heat kernel** of H , generating the solution of (1) by its convolution

with the initial condition u_0 . We have

$$(e^{-tH}u_0)(x) = \int_X K_t(x, y, H)u_0(y) dy$$

where dy stands for the Riemannian density on (X, g) .

The aim of this paper is two-folds.

First, we extend the works in [3],[4] by showing that the framework of heat equations associated to generalized Laplacians appears as a unifying framework for color image regularization, and more generally for multivalued images. Indeed, we show that most of the methods devoted to regularize color images by PDEs can be considered as heat equations associated to generalized Laplacians. More precisely, we show that Divergence-based PDEs [18],[19], Trace-based PDEs [16], Curvature-preserving PDEs [17], and heat equations associated to Laplace-Beltrami operators [14],[15] can be related with generalized Laplacians on a trivial vector bundle of rank 3 over the domain of the image. All these methods differ by the geometric context, i.e. they differ by the Riemannian manifold of the base manifold (the domain of the image) and the connection on the vector bundle. Moreover, using this context of vector bundles we obtain a generalization of Trace-based PDEs by considering the so-called oriented Laplacians as generalized Laplacians.

Then, given a generalized Laplacian H , we propose to regularize images relatively to H using a discretization of operator k_t^0 given by

$$(k_t^0 u_0)(x) = \int_X K_t^0(x, y, H)u_0(y) dy$$

where $K_t^0(x, y, H)$ is the leading term of the asymptotic expansion of the heat kernel $K_t(x, y, H)$ [6].

We show that dealing with the scalar/Beltrami Laplacian, we obtain the operator based on the short-time Beltrami kernel [15], and dealing with oriented Laplacians, we obtain an operator generalizing the operator based on oriented Gaussian kernels [16]. This regularization operator has the property of being completely determined by the geometry of the vector bundle. More precisely, the Riemannian metric of the base manifold determines the anisotropic behaviour of the regularization through the computation of geodesic distances and the connection on the vector bundle determines the data that are regularized through the computation of the parallel transport map. Based on this observation, we propose in this paper new connections and new metrics in order to perform new kinds of regularization. As we deal with color images, we propose new kinds of regularizations using specific color information such as

luminance, chrominance and hue.

The paper is organized as follows. In Sect. 2, we discuss about different methods to regularize images from heat equations of generalized Laplacians. In Sect. 3, we determine the geometric parameters (metric+connection) that make heat equations associated to the Laplace-Beltrami operator, as well as Divergence-based, Trace-based and Curvature-preserving PDEs be heat equations of generalized Laplacians H . Then, we determine the corresponding parallel transport map from which follows the expression of the kernel $K_t^0(x, y, H)$ and the regularization operator k_t^0 . In Sect. 4, we first give a geometric interpretation of luminance, chrominance and hue information on color images, from which we construct connections and metrics in order to perform new kinds of regularizations. At last, Sect. 5 is devoted to experiments. We first compare regularizations induced by the operator k_t^0 related to Laplace-Beltrami operator (short-time Beltrami kernel diffusion), Trace-based PDEs (oriented Gaussian kernel diffusion) and oriented Laplacians. Then we show experiments of the new kinds of regularizations proposed in this paper.

2 Image regularization from heat equations of generalized Laplacians

Generally speaking, for arbitrary base manifold (X, g) and vector bundle E , there is no explicit formula for the heat kernel $K_t(x, y, H)$ of a generalized Laplacian H on E . As a consequence, there is no explicit formula for solutions of the corresponding heat equation. However, in both continuous and discrete settings, solutions may be approximated in different ways.

By the use of heat equations associated with generalized Laplacians in this paper, we are considering a new geometric context for (color) image regularization, i.e. vector bundles over Riemannian manifolds and equipped with connections. Then, besides the construction of a regularization operator induced by such PDEs, we aim at analysing the local behaviour of the regularization process with respect to the geometric data involved, i.e. the Riemannian metric on the base manifold and the connection on the vector bundle.

We first consider two methods based on the computation of approximations of solutions of such PDEs: Euler schemes and convolutions with kernels approximating heat kernels of generalized Laplacians.

2.1 Euler schemes

From the following equality (see [6]),

$$\left\| e^{-tH} u_0 - \sum_{i=0}^k \frac{(-tH)^i}{i!} u_0 \right\|_j = O(t^{k+1})$$

for any j , where $\|\cdot\|_j$ is a norm on derivatives of order j of sections of E , we can use an explicit Euler scheme to approximate in both continuous and discrete settings solutions of generalized heat equations:

$$u_{t+dt} \simeq u_t + dt H u_t$$

It follows that the influence of the geometry on the regularization process is given by the influence of the geometry on the operator H . From the fact that any generalized Laplacian H is of the form

$$H = \Delta^E + F$$

where Δ^E is the connection Laplacian on a vector bundle (E, ∇^E, X, g) and F a zero-order operator [6], we know that the metric g of the base manifold determines the second-order part of H and the connection ∇^E determines its the first-order part.

However, it is not straightforward to interpret the role of the second and first order parts of a generalized Laplacian, and consequently the role of the metric and the connection, in such a regularization process.

2.2 Convolutions with kernels approximating the heat kernel of a generalized Laplacian

There exist kernels $K_t^N(x, y, H)$, for $N \in \mathbb{N}$, approximating the heat kernel of H on normal neighborhoods of the base manifold for small t . It follows that solutions of heat equations associated with generalized Laplacians may be approximated for small t by the convolution of the initial condition with such kernels.

Indeed, for N fixed, the kernel $K_t^N(x, y, H)$ is of the form

$$\left(\frac{1}{4\pi t} \right)^{\frac{m}{2}} e^{-d(x,y)^2/4t} \Psi(d(x,y)^2) \sum_{i=0}^N t^i \Phi_i(x, y, H) J(x, y)^{-\frac{1}{2}}$$

where $\Phi_i(x, y, H) \in \text{End}(E_y, E_x)$, m is the dimension of the base manifold X , and d stands for the geodesic distance on X related to the Levi-Cevita connection on its tangent bundle TX endowed with a Riemannian metric g . Ψ is a function such that the term $\Psi(d(x,y)^2)$ equals

1 if y is inside a normal neighborhood of x and 0 otherwise. Hence we may assume that y is inside a normal neighborhood of x . At last, J are the Jacobians of the coordinates changes from usual coordinates systems to normal coordinates systems.

Then, for $N > m/2$, there exist results about approximations of the heat kernel and solutions of the subsequent heat equation.

1. The kernel $K_t^N(x, y, H)$ is asymptotic to $K_t(x, y, H)$:

$$\left\| \partial_t^k [K_t(x, y, H) - K_t^N(x, y, H)] \right\|_l = O(t^{N-m/2-l/2-k})$$

where $\|\cdot\|_l$ is a norm on sections of class C^l .

2. The operator k_t^N defined by

$$(k_t^N u_0)(x) = \int_X K_t^N(x, y, H) u_0(y) dy \quad (2)$$

satisfies $\lim_{t \rightarrow 0} \|k_t^N u_0 - u_0\|_l = 0$.

The operator (2) determines a regularization operator for the section u_0 .

In this paper, we are concerned by the case $m = 2$ as we deal with images. Hence, the kernel $K_t^N(x, y, H)$ approximates the heat kernel for $N > 1$. However, by the expressions of the maps $\Phi_1(x, y, H)$ and $\Phi_2(x, y, H)$ (see Appendix 1.B), the role of the geometry of the vector bundle in the regularization process induced by the operator k_t^N , for $N > 1$, is hard to understand.

2.3 The regularization operator k_t^0

Motivated by the will to understand the role of the geometry in the regularization process, we deal in this paper with the operator k_t^0 . In this section, we study properties of this operator in both continuous and discrete settings.

The map $\Phi_0(x, y, H)$ is the transport parallel map $\tau(x, y)$ relatively to the connection ∇^E such that $H = \Delta^E + F$ (see Appendix 1.B) along the unique geodesic joining y and x . Indeed, on normal neighborhoods, there is a unique geodesic joining a point to the origin.

Therefore, the kernel $K_t^0(x, y, H)$ is given by

$$\left(\frac{1}{4\pi t} \right)^{m/2} e^{-d(x,y)^2/4t} \Psi(d(x,y)^2) \tau(x, y) J(x, y)^{-1/2}$$

Hence, we have $k_t^0 u_0(x) = (1/4\pi t)^{m/2} \times$

$$\int_X e^{-d(x,y)^2/4t} \Psi(d(x,y)^2) \tau(x,y) u_0(y) J(x,y)^{-1/2} dy \quad (3)$$

and the operator k_t^0 is completely determined by the geometry of the vector bundle, i.e. by the Riemannian metric of the base manifold and the connection on the vector bundle.

Remark 1 Whereas the heat kernel $K_t(x, y, H)$ of a generalized Laplacian H is unique, there is no one to one correspondance between H and the kernel $K_t^0(x, y, H)$. Indeed, $K_t^0(x, y, H)$ only depends on the Riemannian metric g on X and the connection ∇^E on E such that $H = \Delta^E + F$. Then given F_1 a zero-order operator on E and the generalized Laplacian $H_1 = \Delta^E + F_1$, we have

$$K_t^0(x, y, H) = K_t^0(x, y, H_1)$$

The discretization ${}^D k_t^0$ of k_t^0 consists in discrete convolutions with masks. More precisely, the discrete approximation of $k_t^0 u_0$ at a pixel (i, j) is given by the discrete convolution of the map

$$(k, l) \longmapsto \tau((i, j), (k, l)) u_0(k, l)$$

with the mask

$$\begin{cases} Q_t((i, j), (k, l)) = \frac{(1/4\pi t) e^{-d((i,j)-(k,l))^2/4t}}{\sum_{(m,n) \in \mathcal{V}_{(i,j)}} (1/4\pi t) e^{-d((i,j)-(m,n))^2/4t}} \\ \quad \text{if } (k, l) \in \mathcal{V}_{(i,j)} \\ = 0 \quad \text{otherwise} \end{cases}$$

where $\mathcal{V}_{(i,j)}$ is a discrete normal neighborhood of the pixel (i, j) .

More precisely, we have

$${}^D k_t^0 u_0(i, j) = \sum_{(k,l) \in \mathcal{V}_{(i,j)}} Q_t((i, j), (k, l)) \tau((i, j), (k, l)) u_0(k, l) \quad (4)$$

We refer to [3] for details about the construction.

We see that the metric of the base manifold determines the anisotropy of the diffusion through geodesic distances and the connection determines the data regularized through the parallel transport map.

To conclude this Section, we discuss about discrete maximum principle for the regularization process (4).

As we deal with PDEs on vector bundles, we have to associate the maximum principle with a bundle metric. In the following proposition, we give a sufficient condition on the connection ∇^E so that the regularization (4) satisfies the maximum principle for a given bundle metric h .

Proposition 1 *Let h be a metric on a vector bundle (E, X, g) over a Riemannian manifold (X, g) . Let ∇^E be a connection on (E, X, g) compatible with the metric h , and H be a generalized Laplacian on (E, ∇^E, X, g) . Then the discrete regularization operator ${}^D k_t^0$ related to H , defined by*

$${}^D k_t^0 u_0(i, j) = \sum_{(k,l) \in \mathcal{V}_{(i,j)}} Q_t((i, j), (k, l)) \tau((i, j), (k, l)) u_0(k, l)$$

satisfies the maximum principle with respect to h .

3 Heat equations of generalized Laplacians as a unification of many approaches to color image regularization.

In this Section, we show that heat equations associated with generalized Laplacians on vector bundles unify many methods involving PDEs for image regularization. Indeed, both Divergence and Trace-based PDEs, as well as Curvature-preserving PDEs, and PDEs involving Laplace-Beltrami operator and oriented Laplacians can be considered as heat equations associated with generalized Laplacians. We show that they differ by the geometry of the vector bundle involved, that is by the Riemannian metric of the base manifold and by the connection on the vector bundle.

As this article is devoted to color image regularization, we consider functions $I = (I_1, I_2, I_3): \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ in the sequel.

3.1 Scalar/Beltrami Laplacian

Definition 1 The scalar Laplacian on a Riemannian manifold (X, g) is the connection Laplacian Δ^E on a vector bundle of rank 1 over (X, g) , associated to the connection $\nabla^E = d + \omega$ given by $\omega = 0$.

In a local coordinates system on X and frame e_1 of E , the scalar Laplacian is given by

$$\Delta^E(f e_1) = - \sum_{ij} g^{ij} \left(\partial_i \partial_j - \sum_k \Gamma_{ij}^k \partial_k \right) f e_1$$

where Γ_{ij}^k are the symbols of Levi-Cevita connection related to g given in the local coordinates system.

Remark 2 The set of smooth functions $C^\infty(X)$ on a Riemannian manifold (X, g) may be identified with a trivial vector bundle of rank 1 over (X, g) endowed with the connection $\nabla^E = d + \omega$ given by $\omega = 0$. Then if we equip (X, g) of the Levi-Cevita connection related to g , the **Laplace-Beltrami operator** Δ_g defined by

$$\Delta_g(f) = \frac{1}{\sqrt{g}} \sum_{jk} \partial_j (\sqrt{g} g^{jk} \partial_k f)$$

may be identified with minus the scalar Laplacian.

In the particular case where the target manifold is $\mathbb{R}^3$ equipped with an Euclidean metric h , the Beltrami framework, introduced by Sochen et al. in [14], yields the following system of PDEs

$$\frac{\partial I_i}{\partial t} = \Delta_g I_i, \quad i = 1, 2, 3 \quad (5)$$

where g is the metric on Ω induced by the metric h , and Δ_g is the **Laplace-Beltrami operator** on (Ω, g) .

In the following proposition, we show that the set $\Delta_g I_i$, $i = 1, 2, 3$ can be viewed as a generalized Laplacian on a vector bundle of rank 3 equipped with the trivial connection. It follows that the system of PDEs (5) can be considered as one generalized heat equation (see Appendix 2.B for the proof).

Proposition 2 *Let g be a Riemannian metric on Ω . Let E be a trivial vector bundle of rank 3 over (Ω, g) of global frame (e_1, e_2, e_3) . Let ∇^E be the connection on E determined by the 1-form $\omega \equiv 0$. Let $I = I_1 e_1 + I_2 e_2 + I_3 e_3 \in \Gamma(E)$. Then the differential operator H of order 2 defined by*

$$H(I) = - \sum_{i=1}^3 \Delta_g I_i e_i \quad (6)$$

is a generalized Laplacian on (E, ∇^E, Ω, g) .

3.2 Oriented Laplacians

Many approaches to color image regularization are based on PDEs of the form

$$\frac{\partial I_i}{\partial t} = \Delta I_i, \quad i = 1, 2, 3 \quad (7)$$

where Δ is an **oriented Laplacian**, i.e. a two-order differential operator acting on functions defined on Ω , of the form

$$\Delta(f) = c_1 d_{\xi, \xi}^2 f + c_2 d_{\eta, \eta}^2 f$$

where c_1, c_2 are two functions and (η, ξ) is a moving frame on Ω (see [16],[17] for an overview).

In the following proposition, we show that the set ΔI_i , $i = 1, 2, 3$ of oriented Laplacians can be viewed as a generalized Laplacian on a well-chosen vector bundle of rank 3 over Ω . It follows that the system of PDEs (7) can be considered as one generalized heat equation.

Proposition 3 *Let (Ω, g) be the Riemannian manifold of dimension 2 with g given by $1/(c_1 c_2 (\xi_1 \eta_2 - \eta_1 \xi_2)^2) \times$*

$$\begin{pmatrix} c_1 \xi_2^2 + c_2 \eta_2^2 & -(c_1 \xi_1 \xi_2 + c_2 \eta_1 \eta_2) \\ -(c_1 \xi_1 \xi_2 + c_2 \eta_1 \eta_2) & c_1 \xi_1^2 + c_2 \eta_1^2 \end{pmatrix}$$

in the cartesian coordinates system (x_1, x_2) on Ω .

Let E be a trivial vector bundle of rank 3 over (Ω, g) , endowed with connection ∇^E given by the symbols Υ_{ij}^k in a global frame (e_1, e_2, e_3) of E and the global frame $(\partial/\partial x_1, \partial/\partial x_2)$ of $T\Omega$, defined by $\Upsilon_{ij}^k = 0$ if $k \neq j$ and

$$\Upsilon_{ij}^k = \begin{cases} \frac{1}{2}(g_{11}a + g_{12}b) & \text{if } i = 1 \text{ and } k = j \\ \frac{1}{2}(g_{12}a + g_{22}b) & \text{if } i = 2 \text{ and } k = j \end{cases}$$

where

$$\begin{aligned} a &= c_1 \frac{\partial \xi_1}{\partial x_1} \xi_1 + c_1 \frac{\partial \xi_1}{\partial x_2} \xi_2 + c_2 \frac{\partial \eta_1}{\partial x_1} \eta_1 + c_2 \frac{\partial \eta_1}{\partial x_2} \eta_2 + 2g^{12} \Gamma_{12}^1 \\ &\quad + g^{11} \Gamma_{11}^1 + g^{22} \Gamma_{22}^1 \\ b &= c_1 \frac{\partial \xi_2}{\partial x_1} \xi_1 + c_1 \frac{\partial \xi_2}{\partial x_2} \xi_2 + c_2 \frac{\partial \eta_2}{\partial x_1} \eta_1 + c_2 \frac{\partial \eta_2}{\partial x_2} \eta_2 + 2g^{12} \Gamma_{12}^2 \\ &\quad + g^{11} \Gamma_{11}^2 + g^{22} \Gamma_{22}^2 \end{aligned}$$

Let $I = I_1 e_1 + I_2 e_2 + I_3 e_3 \in \Gamma(E)$. Then the differential operator H of order 2 defined by

$$H(I) = - \sum_{i=1}^3 \Delta I_i e_i$$

is a generalized Laplacian on (E, ∇^E, Ω, g) .

3.3 Trace-based PDEs

Following the notations of Sect. 3.2, let us assume that there exist a domain $U \subset \Omega$ where the functions c_1, c_2 and components $\xi_1, \xi_2, \eta_1, \eta_2$ are constant. Then PDEs (7) take the following form on U

$$\frac{\partial I_i}{\partial t} = \text{trace}(\mathbf{T} \mathbf{H}_i), \quad i = 1, 2, 3 \quad (8)$$

where $\mathbf{T}$ is the matrix field $c_1 \xi \xi^T + c_2 \eta \eta^T$, and $\mathbf{H}_i$ stands for the Hessian of the function I_i .

Then, assuming that each point $x_0 \in \Omega \cap \mathbb{N}^2$ has such a neighborhood U_{x_0} , and satisfying

$$\bigcup_{x_0 \in \Omega \cap \mathbb{N}^2} U_{x_0} = \Omega$$

we obtain the **Trace-based PDEs** of Tschumperlé et al. [16].

In the following Corollary, we show that Trace-based PDEs may be viewed as sets of local heat equations of generalized Laplacians. Indeed, we show that the set $\text{trace}(\mathbf{TH}_i)|_{U_{x_0}}, i = 1, 2, 3$ can be viewed as a generalized Laplacian on a vector bundle of rank 3 over U_{x_0} , for each $x_0 \in \Omega \cap \mathbb{N}^2$.

Corollary 1 *Let (U_{x_0}, g) be the Riemannian manifold of dimension 2 with g given by $1/(c_1 c_2 (\xi_1 \eta_2 - \eta_1 \xi_2)^2) \times$*

$$\begin{pmatrix} c_1 \xi_2^2 + c_2 \eta_2^2 & -(c_1 \xi_1 \xi_2 + c_2 \eta_1 \eta_2) \\ -(c_1 \xi_1 \xi_2 + c_2 \eta_1 \eta_2) & c_1 \xi_1^2 + c_2 \eta_1^2 \end{pmatrix}$$

in the cartesian coordinates system (x_1, x_2) .

Let E be a trivial vector bundle of rank 3 over (U_{x_0}, g) of global frame (e_1, e_2, e_3) . Let $\nabla^E = d + \omega$ be the connection on E determined by $\omega \equiv 0$. Let $I = I_1 e_1 + I_2 e_2 + I_3 e_3 \in \Gamma(E)$. Then the differential operator H of order 2 defined on U_{x_0} by

$$H(I) = - \sum_{i=1}^3 \text{trace}(\mathbf{TH}_i) e_i$$

is a generalized Laplacian on $(E, \nabla^E, U_{x_0}, g)$.

3.4 Curvature-preserving PDEs

In [17], the author proposes to regularize color images by considering so-called **Curvature-preserving PDEs**

$$\frac{\partial I_i}{\partial t} = \text{trace}(\mathbf{TH}_i) + \frac{2}{\pi} \nabla I_i^T \int_{\alpha=0}^{\pi} \mathbf{J}_{\sqrt{\mathbf{T}} a_\alpha} \sqrt{\mathbf{T}} a_\alpha d\alpha \quad (9)$$

for $i = 1, 2, 3$, where

- the tensor fields $\mathbf{T}$ and $\sqrt{\mathbf{T}}$ are defined by

$$\mathbf{T} = c_1 \xi \xi^T + c_2 \eta \eta^T, \quad \sqrt{\mathbf{T}} = \sqrt{c_1} \xi \xi^T + \sqrt{c_2} \eta \eta^T$$

- $a_\alpha = (\cos \alpha \quad \sin \alpha)^T$

- $\mathbf{J}_{\sqrt{\mathbf{T}} a_\alpha}$ stands for the Jacobian of $\sqrt{\mathbf{T}} a_\alpha$.

- $\mathbf{H}_i$ is the Hessian of the function I_i .

The system of PDEs (9) may be viewed as a heat equation associated to a generalized Laplacian on a well-chosen vector bundle of rank 3 over Ω . This is the purpose of the following proposition.

Proposition 4 *Let (Ω, g) be the Riemannian manifold of dimension 2 with g given by $1/(c_1 c_2 (\xi_1 \eta_2 - \eta_1 \xi_2)^2) \times$*

$$\begin{pmatrix} c_1 \xi_2^2 + c_2 \eta_2^2 & -(c_1 \xi_1 \xi_2 + c_2 \eta_1 \eta_2) \\ -(c_1 \xi_1 \xi_2 + c_2 \eta_1 \eta_2) & c_1 \xi_1^2 + c_2 \eta_1^2 \end{pmatrix}$$

in the cartesian coordinates system (x_1, x_2) on Ω .

Let E be a trivial vector bundle of rank 3 over (Ω, g) with connection ∇^E given by the symbols Υ_{ij}^k in a global frame (e_1, e_2, e_3) of E and frame $(\partial/\partial x_1, \partial/\partial x_2)$ of $T\Omega$, defined by $\Upsilon_{ij}^k = 0$ if $k \neq j$ and

$$\Upsilon_{ij}^k = \begin{cases} \frac{1}{2}(g_{11}a + g_{12}b) & \text{if } i = 1 \text{ and } k = j \\ \frac{1}{2}(g_{12}a + g_{22}b) & \text{if } i = 2 \text{ and } k = j \end{cases}$$

where $a =$

$$\begin{aligned} & \frac{1}{2} \frac{\partial c_1}{\partial x_1} \left(\xi_1^4 + \frac{\sqrt{c_2}}{\sqrt{c_1}} \xi_1^2 \eta_1^2 + \xi_1^2 \xi_2^2 + \frac{\sqrt{c_2}}{\sqrt{c_1}} \xi_1 \xi_2 \eta_1 \eta_2 \right) \\ & + \frac{1}{2} \frac{\partial c_1}{\partial x_2} \left(\xi_1 \xi_2^3 + \frac{\sqrt{c_2}}{\sqrt{c_1}} \xi_1 \xi_2 \eta_2^2 + \xi_1^3 \xi_2 + \frac{\sqrt{c_2}}{\sqrt{c_1}} \xi_1^2 \eta_1 \eta_2 \right) \\ & + \frac{1}{2} \frac{\partial c_2}{\partial x_1} \left(\frac{\sqrt{c_1}}{\sqrt{c_2}} \xi_1^2 \eta_1^2 + \eta_1^4 + \frac{\sqrt{c_1}}{\sqrt{c_2}} \xi_1 \xi_2 \eta_1 \eta_2 + \eta_1^2 \eta_2^2 \right) \\ & + \frac{1}{2} \frac{\partial c_2}{\partial x_2} \left(\frac{\sqrt{c_1}}{\sqrt{c_2}} \xi_2^2 \eta_1 \eta_2 + \eta_1 \eta_2^3 + \frac{\sqrt{c_1}}{\sqrt{c_2}} \eta_1^2 \xi_1 \xi_2 + \eta_1^3 \eta_2 \right) \\ & + \sqrt{c_1 c_2} \left(\frac{\partial \xi_1}{\partial x_2} \xi_2 \eta_2^2 + \frac{\partial \xi_2}{\partial x_2} \xi_1 \eta_2^2 + \frac{\partial \eta_1}{\partial x_2} \xi_2^2 \eta_2 + \frac{\partial \eta_2}{\partial x_2} \xi_2^2 \eta_1 \right) \\ & + 2 \frac{\partial \xi_1}{\partial x_2} \xi_1 \eta_1 \eta_2 + 2 \frac{\partial \eta_1}{\partial x_2} \xi_1 \xi_2 \eta_1 + \frac{\partial \xi_1}{\partial x_1} \xi_2 \eta_1 \eta_2 + \frac{\partial \xi_2}{\partial x_1} \xi_1 \eta_1 \eta_2 \\ & + 2 \frac{\partial \xi_1}{\partial x_1} \xi_1 \eta_1^2 + 2 \frac{\partial \eta_1}{\partial x_1} \xi_1^2 \eta_1 + \frac{\partial \eta_1}{\partial x_1} \xi_1 \xi_2 \eta_2 + \frac{\partial \eta_2}{\partial x_1} \xi_1 \xi_2 \eta_1 \\ & + c_1 \left(2 \frac{\partial \xi_1}{\partial x_1} \xi_1^3 + \frac{\partial \xi_1}{\partial x_2} \xi_2^3 + \frac{\partial \xi_2}{\partial x_1} \xi_1^2 \xi_2 + 2 \frac{\partial \xi_1}{\partial x_2} \xi_1^2 \xi_2 \right. \\ & \left. + \frac{\partial \xi_1}{\partial x_1} \xi_1 \xi_2^2 + \frac{\partial \xi_2}{\partial x_2} \xi_1 \xi_2^2 \right) + c_2 \left(2 \frac{\partial \eta_1}{\partial x_1} \eta_1^3 + 2 \frac{\partial \eta_1}{\partial x_2} \eta_1^2 \eta_2 \right. \\ & \left. + \frac{\partial \eta_1}{\partial x_1} \eta_1 \eta_2^2 + \frac{\partial \eta_2}{\partial x_1} \eta_1^2 \eta_2 + \frac{\partial \eta_1}{\partial x_2} \eta_2^3 + \frac{\partial \eta_2}{\partial x_2} \eta_1 \eta_2^2 \right) \\ & + 2g^{12} \Gamma_{12}^1 + g^{11} \Gamma_{11}^1 + g^{22} \Gamma_{22}^1 \end{aligned}$$

and $b =$

$$\begin{aligned} & \frac{1}{2} \frac{\partial c_1}{\partial x_1} \left(\xi_1^3 \xi_2 + \frac{\sqrt{c_2}}{\sqrt{c_1}} \xi_1 \xi_2 \eta_1^2 + \xi_1 \xi_2^3 + \frac{\sqrt{c_2}}{\sqrt{c_1}} \xi_2^2 \eta_1 \eta_2 \right) \\ & + \frac{1}{2} \frac{\partial c_1}{\partial x_2} \left(\xi_1^2 \xi_2^2 + \frac{\sqrt{c_2}}{\sqrt{c_1}} \xi_1 \xi_2 \eta_1 \eta_2 + \xi_2^4 + \frac{\sqrt{c_2}}{\sqrt{c_1}} \xi_2^2 \eta_2^2 \right) \\ & + \frac{1}{2} \frac{\partial c_2}{\partial x_1} \left(\frac{\sqrt{c_1}}{\sqrt{c_2}} \xi_1^2 \eta_1 \eta_2 + \eta_1^3 \eta_2 + \frac{\sqrt{c_1}}{\sqrt{c_2}} \eta_2^2 \xi_1 \xi_2 + \eta_1 \eta_2^3 \right) \\ & + \frac{1}{2} \frac{\partial c_2}{\partial x_2} \left(\frac{\sqrt{c_1}}{\sqrt{c_2}} \xi_1 \xi_2 \eta_1 \eta_2 + \eta_1^2 \eta_2^2 + \frac{\sqrt{c_1}}{\sqrt{c_2}} \xi_2^2 \eta_2^2 + \eta_2^4 \right) \\ & + \sqrt{c_1 c_2} \left(\frac{\partial \xi_1}{\partial x_1} \xi_2 \eta_1^2 + \frac{\partial \xi_2}{\partial x_1} \xi_1 \eta_1^2 + \frac{\partial \eta_1}{\partial x_1} \xi_1^2 \eta_2 + \frac{\partial \eta_2}{\partial x_1} \xi_1^2 \eta_1 \right) \end{aligned}$$

$$\begin{aligned}
& +2\frac{\partial \xi_2}{\partial x_1} \xi_2 \eta_1 \eta_2 + 2\frac{\partial \eta_2}{\partial x_1} \xi_1 \xi_2 \eta_2 + \frac{\partial \xi_1}{\partial x_2} \xi_2 \eta_1 \eta_2 + \frac{\partial \xi_2}{\partial x_2} \xi_1 \eta_1 \eta_2 \\
& + \frac{\partial \eta_1}{\partial x_2} \xi_1 \xi_2 \eta_2 + \frac{\partial \eta_2}{\partial x_2} \xi_1 \xi_2 \eta_1 + 2\frac{\partial \xi_2}{\partial x_2} \xi_2 \eta_2^2 + 2\frac{\partial \eta_2}{\partial x_2} \xi_2^2 \eta_2 \Big) \\
& + c_1 \left(\frac{\partial \xi_1}{\partial x_1} \xi_1^2 \xi_2 + \frac{\partial \xi_2}{\partial x_1} \xi_1^3 + 2\frac{\partial \xi_2}{\partial x_1} \xi_1 \xi_2^2 + \frac{\partial \xi_1}{\partial x_2} \xi_1 \xi_2^2 \right. \\
& + \frac{\partial \xi_2}{\partial x_2} \xi_1^2 \xi_2 + 2\frac{\partial \xi_2}{\partial x_2} \xi_2^3 + c_2 \left(2\frac{\partial \eta_2}{\partial x_2} \eta_2^3 + 2\frac{\partial \eta_2}{\partial x_1} \eta_1 \eta_2^2 \right. \\
& + \frac{\partial \eta_2}{\partial x_1} \eta_1^3 + \frac{\partial \eta_1}{\partial x_2} \eta_1 \eta_2^2 + \frac{\partial \eta_2}{\partial x_2} \eta_1^2 \eta_2 + \frac{\partial \eta_1}{\partial x_1} \eta_1^2 \eta_2 \Big) \\
& \left. + 2g^{12} \Gamma_{12}^2 + g^{11} \Gamma_{11}^2 + g^{22} \Gamma_{22}^2 \right)
\end{aligned}$$

Let $I = I_1 e_1 + I_2 e_2 + I_3 e_3 \in \Gamma(E)$. Then the differential operator H of order 2 defined by $H(I) =$

$$-\sum_{i=1}^3 \left(\text{trace}(\mathbf{T} \mathbf{H}_i) + \frac{2}{\pi} \nabla I_i^T \int_{\alpha=0}^{\pi} \mathbf{J}_{\sqrt{\mathbf{T} a_\alpha}} \sqrt{\mathbf{T} a_\alpha} d\alpha \right) e_i$$

is a generalized Laplacian on (E, ∇^E, Ω, g) .

Remark 3 The vector bundles involved in Curvature-preserving PDEs and heat equations associated to oriented Laplacians have the same base manifold (Ω, g) , but have different connections ∇^E . Hence, the regularization processes induced by the operator k_t^0 have the same anisotropic behaviour, but the different connections make the data averaged be different.

3.5 Divergence-based PDEs

Divergence-based PDEs (see Weickert [18],[19]) for color image regularization are of the form

$$\frac{\partial I_i}{\partial t} = \text{div}(\mathbf{D} \nabla I_i), \quad i = 1, 2, 3 \quad (10)$$

where $\mathbf{D} = (d_{kl})$ is a field of symmetric definite positive matrices.

In the following proposition, we show that the set of terms $\text{div}(\mathbf{D} \nabla I_i)$, $i = 1, 2, 3$ may be viewed as a generalized Laplacian on a well-chosen vector bundle of rank 3 over Ω . It follows that the system of PDEs (10) can be considered as one generalized heat equation.

Proposition 5 *Let (Ω, g) be the Riemannian manifold of dimension 2 with g given by*

$$\frac{1}{d_{11}d_{22} - (d_{12})^2} \begin{pmatrix} d_{22} & -d_{12} \\ -d_{12} & d_{11} \end{pmatrix}$$

in the cartesian coordinates system (x_1, x_2) on Ω .

Let E be a trivial vector bundle of rank 3 over (Ω, g) , endowed with a connection ∇^E given by the symbols Υ_{ij}^k

in a global frame (e_1, e_2, e_3) of E and the global frame $(\partial/\partial x_1, \partial/\partial x_2)$ of $T\Omega$, defined by $\Upsilon_{ij}^k = 0$ if $k \neq j$ and

$$\Upsilon_{ij}^k = \begin{cases} \frac{1}{2}(g_{11}a + g_{12}b) & \text{if } i = 1 \text{ and } k = j \\ \frac{1}{2}(g_{12}a + g_{22}b) & \text{if } i = 2 \text{ and } k = j \end{cases}$$

where

$$a = \frac{\partial d_{11}}{\partial x_1} + \frac{\partial d_{12}}{\partial x_2} + 2g^{12} \Gamma_{12}^1 + g^{11} \Gamma_{11}^1 + g^{22} \Gamma_{22}^1$$

$$b = \frac{\partial d_{12}}{\partial x_1} + \frac{\partial d_{22}}{\partial x_2} + 2g^{12} \Gamma_{12}^2 + g^{11} \Gamma_{11}^2 + g^{22} \Gamma_{22}^2$$

Let $I = I_1 e_1 + I_2 e_2 + I_3 e_3 \in \Gamma(E)$. Then the differential operator H of order 2 defined by

$$H(I) = -\sum_{i=1}^3 \text{div}(\mathbf{D} \nabla I_i) e_i$$

is a generalized Laplacian on (E, ∇^E, Ω, g) .

Remark 4 Following the notations of Sect. 3.4, let us suppose that $\mathbf{D} = c_1 \xi \xi^T + c_2 \eta \eta^T$, i.e. $\mathbf{D}$ corresponds to the matrix field $\mathbf{T}$. Then, the vector bundle involved in Divergence-based PDEs has the same metric on its base manifold as the vector bundles involved in Curvature-preserving PDEs and heat equations associated to oriented Laplacians. However, all of them have different connections.

3.6 Parallel transport map and kernel $K_t^0(x, y, H)$

The regularization operator k_t^0 related to a generalized Laplacian H corresponds to a convolution with the kernel $K_t^0(x, y, H)$ of H . Following Sect. 2.3, the kernel $K_t^0(x, y, H)$ of a generalized Laplacian H is determined by the geodesic distance related to the metric g , and the transport parallel map related to the connection ∇^E . The aim of this Section is to determine the parallel transport maps related to the methods mentioned above, and the subsequent kernels $K_t^0(x, y, H)$.

Proposition 6 *Let Υ_1, Υ_2 be two functions on Ω . Let ∇^E be the connection on E determined by functions Υ_{ij}^k in the global frame (e_1, e_2, e_3) of E and global frame $(\partial/\partial x_1, \partial/\partial x_2)$ of $T\Omega$, given by*

$$\Upsilon_{ij}^k = \begin{cases} \Upsilon_i & \text{if } k = j \\ 0 & \text{otherwise} \end{cases} \quad (11)$$

Let $Y_0 = \sum_{i=1}^3 Y_0^i e_i(y) \in E_y$, and γ be a C^1 curve on Ω such that $\gamma(0) = y$. Then the parallel transport of Y_0 along γ is $Y(t) =$

$$\sum_{i=1}^3 Y_0^i \exp\left(-\int_0^t \dot{\gamma}_1(s) \Upsilon_1(s) + \dot{\gamma}_2(s) \Upsilon_2(s) ds\right) e_i(\gamma(t)) \quad (12)$$

As all the methods mentioned above have connections of the form (11), the parallel transport maps are all of the form (12), but for different functions Υ_i . It follows the expressions of the corresponding kernels $K_t^0(x, y, H)$.

In what follows, we focus on the kernels $K_t^0(x, y, H)$ related to the extensions of scalar/Beltrami Laplacians, oriented Laplacians, and trace-based PDEs on vector bundles of rank 3.

Corollary 2 (scalar Laplacians) *Under the assumptions and notations of Sect. 3.1, we have*

$$K_t^0(x, y, H) = \begin{cases} \frac{1}{4\pi t} \exp\left(-\frac{d(x, y)^2}{4t}\right) Id(x, y) J(x, y)^{-\frac{1}{2}} \\ \text{if } y \text{ is in a normal neighborhood of } x \\ 0 \text{ otherwise} \end{cases} \quad (13)$$

where $Id(x, y): E_y \longrightarrow E_x$ is the Identity operator on the coordinates.

Remark 5 The kernel (13) is the vector extension of the so-called **short-time Beltrami kernel** [15].

Corollary 3 (Oriented Laplacians) *Under the assumptions and notations of Sect. 3.2, we have*

$$K_t^0(x, y, H) = (1/4\pi t) \exp(-d(x, y)^2/4t) \times$$

$$\begin{cases} \exp\left(-\int_0^t \dot{\gamma}_1(s) \Upsilon_1(s) + \dot{\gamma}_2(s) \Upsilon_2(s) ds\right) Id(x, y) J(x, y)^{-\frac{1}{2}} \\ \text{if } y \text{ is in a normal neighborhood of } x, \text{ where} \\ \Upsilon_1 = \frac{1}{2}(g_{11}a + g_{12}b) \quad \Upsilon_2 = \frac{1}{2}(g_{12}a + g_{22}b) \\ 0 \text{ otherwise} \end{cases}$$

Corollary 4 (Trace-based PDEs) *Under the assumptions and notations of Sect. 3.3, we have*

$$K_t^0(x, y, H) = \frac{1}{4\pi t} \exp\left(-\frac{(y-x)^T \mathbf{T}^{-1} (y-x)}{4t}\right) Id(x, y) \quad (14)$$

$$\forall x \in \Omega, \forall y \in U_x.$$

Remark 6 The kernel $K_t^0(x, y, H)$ in formula (14) corresponds to the heat kernel $K_t(x, y, H)$ of the operator H . It is also the vector extension of the so-called **oriented Gaussian kernel** [16].

Therefore, the kernels $K_t^0(x, y, H)$ of oriented Laplacians may be viewed as generalizations of the oriented Gaussian kernels to mobile frames of components non-locally constant.

Remark 7 As the connection determined by the 1-form $\omega \equiv 0$ is compatible with any metric on E , we deduce that the discrete regularization operator ${}^D k_t^0$ related to the scalar and locally constant oriented Laplacians and their vector extensions satisfy the maximum principle for the metric given by I_3 in the global frame (e_1, e_2, e_3) .

4 New connections and metrics for new kinds of color image regularization

4.1 RGB space embedded into the vector space $\mathbb{R}^3$

In this Section, we show that by embedding the *RGB* color space into the vector space $\mathbb{R}^3$, we can obtain color information like Luminance, Chrominance, Saturation and Hue from linear maps. It follows that we can construct connections, and consequently regularization processes, that take into account these information. This point will be developed in the next Section.

Besides RGB color space we consider in this paper a Hue Saturation Luminance (HSL) color space defined as follows. We set first

$$\begin{pmatrix} Y \\ C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1 & -1/2 & -1/2 \\ 0 & -\sqrt{3}/2 & \sqrt{3}/2 \end{pmatrix} \begin{pmatrix} r \\ g \\ b \end{pmatrix}$$

Then the luminance l , the saturation s and the hue h are respectively given by

$$\begin{aligned} l &= Y \\ s &= \sqrt{C_1^2 + C_2^2} \\ h &= \begin{cases} \arccos(C_2/s) & \text{if } C_2 > 0 \\ 2\pi - \arccos(C_2/s) & \text{otherwise} \end{cases} \end{aligned}$$

Endowing the vector space of basis (e_1, e_2, e_3) that embeds *RGB* with the Euclidean metric, we can decompose any color $\alpha = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3$ by its projection $L(\alpha)$ and its rejection $V(\alpha)$ on the axis generated by the vector $(e_1 + e_2 + e_3)$.

This axis is called the **luminance axis**, it encodes the

luminance information of a color. Its orthogonal 2-plane is called the **chrominance plane**, and encodes the saturation and hue information. A basis is given by the vectors $(2e_1 - e_2 - e_3)$ and $(-e_2 + e_3)$.

Indeed, as it is shown in [2], the luminance of α corresponds to the norm of its projection on the luminance axis. The saturation corresponds to the norm of its projection on the chrominance plane (up to multiplication by the scalar $3/2$), and the hue corresponds to the angle it forms with the projection of the vector e_1 on this plane.

From simple computations, we obtain the behaviour of the luminance, saturation and hue components with respect to scales in RGB space. Let $\lambda \geq 0$, we have

$$l(\lambda \alpha) = \lambda l(\alpha), \quad s(\lambda \alpha) = \lambda s(\alpha), \quad h(\lambda \alpha) = h(\alpha) \quad (15)$$

$$l(L(\alpha) + \lambda V(\alpha)) = l(\alpha), \quad s(L(\alpha) + \lambda V(\alpha)) = \lambda s(\alpha)$$

$$h(L(\alpha) + \lambda V(\alpha)) = h(\alpha) \quad (16)$$

In the following proposition, we give a new geometric interpretation of the hue.

Proposition 7 *Let T be the set of 2-vectors of the form*

$$\{(e_1 + e_2 + e_3) \wedge \alpha, \alpha \in RGB\}$$

where $\wedge$ is the wedge product of the exterior algebra $\bigwedge^* \mathbb{R}^3$ of $\mathbb{R}^3$. Let us equip T of the following equivalence relation

$$B \simeq C \iff B = \lambda C, \quad \text{for } \lambda > 0$$

for $B, C \in T$. Then the set of hues is in bijection with the coset space $T / \simeq$.

Proof see [5].

Geometrically speaking, 2-vectors encode oriented 2-planes. Then, Proposition 7 states that colors of same hue belong to the same 2-plane. It can also be shown that two colors of opposite hue belong to the same 2-plane, but generate oriented 2-planes of different orientation.

4.2 New regularization processes from new connections

In Sect. 2, we related standard methods in color image regularization with some particular connections ∇^E on trivial vector bundles E of rank 3 over Riemannian manifolds (X, g) of dimension 2. In this Section, we determine new connections on such vector bundles, yielding new generalized Laplacians, and consequently new regularization processes of color images. We compute the corresponding parallel transport maps, in order to obtain an expression of the kernels $K_t^0(x, y, H)$ involved in the regularization operator k_t^0 .

Let (e_1, e_2, e_3) be a global frame of E and (x_1, x_2) a coordinates system of X .

4.2.1 Generalized Laplacians of parameter α

Definition 2 Let ∇^E be the connection determined by the connection 1-form

$$\omega^E = (dx_1 + dx_2) \otimes \alpha Id$$

for some $\alpha \in C^\infty(X)$, where Id is the identity operator on E . A generalized Laplacian $H = \Delta^E + F$ on (E, ∇^E, X, g) is called a **generalized Laplacian of parameter α** .

Remark 8 The scalar Laplacian is a generalized Laplacian of parameter α , for $\alpha \equiv 0$ and $F \equiv 0$.

Proposition 8 *Let (E, ∇^E, X, g) of Definition 2. Let $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ be a C^1 curve in X such that $\gamma(0) = y$, and $Y_0 = Y_0^1 e_1(y) + Y_0^2 e_2(y) + Y_0^3 e_3(y) \in E_y$. The parallel transport of Y_0 along γ is*

$$Y(t) = \sum_{i=1}^3 \exp \left(- \int_0^t \alpha(s) (\dot{\gamma}_1(s) + \dot{\gamma}_2(s)) ds \right) Y_0^i e_i(\gamma(t))$$

4.2.2 Generalized Laplacians of parameters (h, α, v)

Definition 3 Let h be a definite positive metric on E , and $v \in \Gamma(E)$. Let ∇^E be the connection on E given by the connection 1-form

$$\omega^E = (dx_1 + dx_2) \otimes \alpha P_v^\perp$$

where $\alpha \in C^\infty(X)$, and $P_v^\perp$ is the operator acting on each fiber E_x as the orthogonal projection with respect to h_x on the subspace generated by $v(x)$. A generalized Laplacian $H = \Delta^E + F$ on (E, ∇^E, X, g) is called a **generalized Laplacian of parameters (h, α, v)** .

Proposition 9 Let (E, ∇^E, X, g) of Definition 3. Let $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ be a C^1 curve in X such that $\gamma(0) = y$, and $Y_0 = Y_0^1 e_1(y) + Y_0^2 e_2(y) + Y_0^3 e_3(y) \in E_y$. The parallel transport of Y_0 along γ is $Y(t) = Y^1(t) e_1(\gamma(t)) + Y^2(t) e_2(\gamma(t)) + Y^3(t) e_3(\gamma(t))$ with components satisfying

i. If the coefficients h_{ij} of the metric h and the components (v_1, v_2, v_3) of v are constant in the global frame (e_1, e_2, e_3) . Then $(Y^1(t), Y^2(t), Y^3(t))^T =$

$$P \begin{pmatrix} \exp\left(-\int_0^t \alpha(s)(\dot{\gamma}_1(s) + \dot{\gamma}_2(s))ds\right) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} P^{-1} \begin{pmatrix} Y_0^1 \\ Y_0^2 \\ Y_0^3 \end{pmatrix}$$

where P is a matrix field of change frame from (e_1, e_2, e_3) to (v, w, z) where w and z are sections of constant components in the frame (e_1, e_2, e_3) that are orthogonal to v with respect to the metric h .

ii. If the coefficients h_{ij} of the metric h or the components (v_1, v_2, v_3) of v are non constant in the global frame (e_1, e_2, e_3) . Then $(Y^1(t), Y^2(t), Y^3(t))^T$ is the solution of the differential equation

$$\begin{pmatrix} \frac{\partial Y^1}{\partial t} \\ \frac{\partial Y^2}{\partial t} \\ \frac{\partial Y^3}{\partial t} \end{pmatrix} = -\alpha \frac{(\dot{\gamma}_1 + \dot{\gamma}_2)}{\|v\|^2} B \begin{pmatrix} Y^1 \\ Y^2 \\ Y^3 \end{pmatrix} \quad (17)$$

of initial condition

$$(Y^1(0), Y^2(0), Y^3(0)) = (Y_0^1, Y_0^2, Y_0^3),$$

with $B = (B)_{ij}$ a 3×3 matrix field depending on h and v .

In the case i., the diffusion induced by a generalized Laplacian of parameters (h, α, v) on (E, ∇^E, X, g) may be decomposed into a diffusion induced by a generalized Laplacian of parameter α on the subbundle of rank 1 generated by v and a Beltrami diffusion on the subbundle of rank 2 generated by w and z .

4.2.3 Generalized Laplacians of parameters $(h, \alpha, \mathcal{P}_{v,w})$

Definition 4 Let h be a definite positive metric on E , and $v, w \in \Gamma(E)$. Let ∇^E be the connection on E given by the connection 1-form

$$\omega^E = (dx_1 + dx_2) \otimes \alpha P_{v,w}^\perp$$

where $\alpha \in C^\infty(X)$, and $P_{v,w}^\perp$ is the operator acting on each fiber E_x as the orthogonal projection with respect to h_x on the subspace generated by $v(x)$ and $w(x)$. A generalized Laplacian $H = \Delta^E + F$ on (E, ∇^E, X, g) is called a **generalized Laplacian of parameters $(h, \alpha, P_{v,w})$** .

Proposition 10 Let (E, ∇^E, X, g) of Definition 4. Let $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ be a C^1 curve in X such that $\gamma(0) = y$, and $Y_0 = Y_0^1 e_1(y) + Y_0^2 e_2(y) + Y_0^3 e_3(y) \in E_y$. The parallel transport of Y_0 along γ is $Y(t) = Y^1(t) e_1(\gamma(t)) + Y^2(t) e_2(\gamma(t)) + Y^3(t) e_3(\gamma(t))$ with components satisfying

i. If the coefficients h_{ij} of the metric h and the components (v_1, v_2, v_3) and (w_1, w_2, w_3) of v and w are constant in the global frame (e_1, e_2, e_3) , then

$$\begin{pmatrix} Y^1(t) \\ Y^2(t) \\ Y^3(t) \end{pmatrix} = P C(t) P^{-1} \begin{pmatrix} Y_0^1 \\ Y_0^2 \\ Y_0^3 \end{pmatrix}$$

where $C(t)$ is the matrix

$$\begin{pmatrix} \exp\left(-\int_0^t \alpha(s)A(s)ds\right) & 0 & 0 \\ 0 & \exp\left(-\int_0^t \alpha(s)A(s)ds\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

where P is a matrix field of change frame from (e_1, e_2, e_3) to (v, w, z) for z being a section of constant components in the frame (e_1, e_2, e_3) that is orthogonal to v and w with respect to the metric h , and $A: t \mapsto \dot{\gamma}_1(t) + \dot{\gamma}_2(t)$.

ii. If the coefficients h_{ij} of the metric h or the components (v_1, v_2, v_3) and (w_1, w_2, w_3) of v and w are non constant in the global frame (e_1, e_2, e_3) , then $(Y^1(t), Y^2(t), Y^3(t))$ is the solution of the differential equation

$$\begin{pmatrix} \frac{\partial Y^1}{\partial t} \\ \frac{\partial Y^2}{\partial t} \\ \frac{\partial Y^3}{\partial t} \end{pmatrix} = -\alpha \frac{(\dot{\gamma}_1 + \dot{\gamma}_2)}{\|v\|^2} B \begin{pmatrix} Y^1 \\ Y^2 \\ Y^3 \end{pmatrix} \quad (18)$$

of initial condition

$$(Y^1(0), Y^2(0), Y^3(0)) = (Y_0^1, Y_0^2, Y_0^3),$$

with $B = (B)_{ij}$ a 3×3 matrix field depending on h, v and w .

In the case i , the diffusion induced by a generalized Laplacian of parameters $(h, \alpha, \mathcal{P}_{v,w})$ on (E, ∇^E, X, g) may be decomposed into a diffusion induced by a generalized Laplacian of parameter α on the subbundle of rank 2 generated by v and w and a Beltrami diffusion on the subbundle of rank 1 orthogonal with respect to the metric h .

4.3 Control of the diffusion's speed

In Sect. 2, we showed that the metric of the base manifold determines the anisotropy of the regularization process induced by the operator k_t^0 . In this Section, we construct a family of Riemannian metrics whose aim is to control the speed of the diffusion.

Let $I: (x_1, x_2) \mapsto (I^1(x_1, x_2), I^2(x_1, x_2), I^3(x_1, x_2))$ be a color image on $\Omega \subset \mathbb{R}^2$. The graph φ of I

$$\varphi: (x_1, x_2) \mapsto (x_1, x_2, I^1(x_1, x_2), I^2(x_1, x_2), I^3(x_1, x_2))$$

determines the parametrization of a surface X of dimension 2 embedded in $\mathbb{R}^5$. Then we equip $\mathbb{R}^5$ of a metric

$$h = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}$$

for some positive function λ .

The metric h induces a Riemannian metric g_λ on X of matrix representation

$$\begin{pmatrix} 1 + \lambda(I_{x_1}^1{}^2 + I_{x_1}^2{}^2 + I_{x_1}^3{}^2) & \lambda(I_{x_1}^1 I_{x_2}^1 + I_{x_1}^2 I_{x_2}^2 + I_{x_1}^3 I_{x_2}^3) \\ \lambda(I_{x_1}^1 I_{x_2}^1 + I_{x_1}^2 I_{x_2}^2 + I_{x_1}^3 I_{x_2}^3) & 1 + \lambda(I_{x_2}^1{}^2 + I_{x_2}^2{}^2 + I_{x_2}^3{}^2) \end{pmatrix} \quad (19)$$

in the frame $(\partial/\partial_{x_1}, \partial/\partial_{x_2})$.

This construction makes the couples (X, g_λ) be Riemannian manifolds of dimension 2 of global chart (Ω, φ) .

Given a generalized Laplacian H on $(E, \nabla^E, X, g_\lambda)$ where I is considered as the section

$$I = I^1 e_1 + I^2 e_2 + I^3 e_3,$$

we show in the sequel that λ controls the speed of the regularization $k_t^0 I$ of I related to H .

Indeed, for $(X_1, X_2) \in T_x X$, we have

$$g_{\lambda_x}(X_1, X_2) = (1 + \lambda a)X_1^2 + 2\lambda b X_1 X_2 + (1 + \lambda c)X_2^2$$

for $a = I_{x_1}^1{}^2 + I_{x_1}^2{}^2 + I_{x_1}^3{}^2$, $b = I_{x_1}^1 I_{x_2}^1 + I_{x_1}^2 I_{x_2}^2 + I_{x_1}^3 I_{x_2}^3$ and $c = I_{x_2}^1{}^2 + I_{x_2}^2{}^2 + I_{x_2}^3{}^2$.

Therefore, the more λ is taken high, the more the length of a given curve γ in X is high. In particular, if γ is the unique geodesic joining two points, the geodesic distance between these points increases with λ . It follows that the quantity

$$\left(\frac{1}{4\pi t}\right) e^{-d(x,y)^2/4t}$$

decreases with λ . We deduce that the speed of the diffusion decreases with λ .

In the rest of the Section, we treat the limit cases $\lambda \equiv 0$ and $\lambda \gg 0$.

For $\lambda \equiv 0$, the metric g_0 is Euclidean. It follows that the symbols I_{ij}^k of the Levi-Cevita connection on TX vanish. Then geodesics curves on X correspond to straight lines on the chart Ω and the geodesic distances on X to Euclidean distances on Ω . As a consequence, the kernel $K_t^0(x, y, H)$ related to H is of the form

$$K_t^0(x, y, H) = \left(\frac{1}{4\pi t}\right) e^{-\|\varphi^{-1}(x) - \varphi^{-1}(y)\|^2/4t} \tau(x, y)$$

Therefore, the diffusion $(k_t^0 I)(x)$ of I at a point x consists in an isotropic average around x of the map

$$y \mapsto \tau(x, y) I(y)$$

Remark 9 Let us endow E with the connection $\nabla^E = d + \omega$ with $\omega \equiv 0$, and let H be a generalized Laplacian on (E, ∇^E, X, g_0) (e.g. the vector extension of the Laplace-Beltrami operator of Sect. 3.1). Then, the diffusion $(k_t^0 I)(x)$ of I at a point x consists in an isotropic average around x of the map

$$y \mapsto I^1(y) e_1(x) + I^2(y) e_2(x) + I^3(y) e_3(x)$$

Hence, it corresponds to a Gaussian diffusion on each component $I^i, i = 1, 2, 3$ of the image I .

As we have shown above, taken λ high penalizes the diffusion's speed since it makes the geodesic distances be high. Then, taken λ high enough, numerical approximations of the discrete kernel $K_t^0(x, y, H)$ give

$$K_t^0(x, y, H) = \begin{cases} Id & \text{if } y = x \\ 0 & \text{otherwise} \end{cases}$$

It follows $(k_t^0 I)(x) = I(x)$.

Hence, sufficiently high values of λ annihilate the discrete regularization process.

Finally, we deduce that make λ vary with respect to points on X make the speed of the diffusion vary with respect to pixels on the image. This is a purpose of one of the experiments we present in the next Section.

Remark 10 The speed of the diffusion can also be controlled by the parameter t . Indeed, the more t is high, the more the diffusion is important. However as t is a constant, variations of the parameter t modify the diffusion's speed in a homogeneous way all over the image.

5 Experiments

In this Section, we show experiments about the computation of the discrete regularization operator ${}^Dk_t^0$ related to the kernel $K_t^0(x, y, H)$ of some generalized Laplacians H mentioned above.

5.1 Heat equations of generalized Laplacians as a unification of many approaches to color image regularization

We compare the regularizations based on the vector extensions of the kernels $K_t^0(x, y, H)$ related to scalar Laplacians, locally constant oriented Laplacians and oriented Laplacians we constructed in Sect. 3.6.

For the approach related to scalar Laplacians, the base manifold we consider is the Riemannian manifold (X, g_λ) we constructed in Sect. 4.3 for $\lambda \equiv 0.01$. We construct the vector bundle of rank 3 over (X, g_λ) of total space $E = X \times \mathbb{R}^3$ of global frame (e_1, e_2, e_3) where

$$e_1(x) = (x, (1, 0, 0)), e_2(x) = (x, (0, 1, 0)),$$

$$e_3(x) = (x, (0, 0, 1))$$

We equip E of the connection ∇^E determined by the 1-form $\omega \equiv 0$. Then we consider the initial color image as the section I_0 of the vector bundle $(E, \nabla^E, X, g_\lambda)$ of components (r, g, b) in the global frame (e_1, e_2, e_3) , i.e.

$$I_0(x) = r(x)e_1(x) + g(x)e_2(x) + b(x)e_3(x)$$

which we convolve with the discretization of the kernel $K_t^0(x, y, H)$ related to the vector extension of the scalar Laplacian (see Corollary 2).

Whereas the approach related to scalar Laplacians is completely determined by the metric of the base manifold, the approaches related to oriented Laplacians and locally constant oriented Laplacians are determined by

the choices of functions c_1, c_2 and a moving frame (ξ, η) on the image domain Ω . For the experiments, we take

$$c_1 = \frac{1}{(\lambda^+ + \lambda^- - 1)^{0.5}} \quad c_2 = \frac{1}{(\lambda^+ + \lambda^- - 1)^{0.3}}$$

where λ^+ resp. λ^- is the field of highest resp. lowest eigenvalues of (19) for $\lambda \equiv 0.01$, and (ξ, η) is the corresponding unit eigenvectors field. Then, we consider the Riemannian manifold (Ω, g) where $g = 1/(c_1 c_2 (\xi_1 \eta_2 - \eta_1 \xi_2)^2) \times$

$$\begin{pmatrix} c_1 \xi_2^2 + c_2 \eta_2^2 & -(c_1 \xi_1 \xi_2 + c_2 \eta_1 \eta_2) \\ -(c_1 \xi_1 \xi_2 + c_2 \eta_1 \eta_2) & c_1 \xi_1^2 + c_2 \eta_1^2 \end{pmatrix}$$

in the cartesian coordinates system (x_1, x_2) on Ω .

We construct the vector bundle of rank 3 over (Ω, g) of total space $E = \Omega \times \mathbb{R}^3$ of global frame (e_1, e_2, e_3) where

$$e_1(x) = (x, (1, 0, 0)), e_2(x) = (x, (0, 1, 0)),$$

$$e_3(x) = (x, (0, 0, 1))$$

Then, we consider the initial color image as the section I_0 of the vector bundle (E, ∇^E, Ω, g) for the corresponding connection ∇^E of components (r, g, b) in the global frame (e_1, e_2, e_3) , i.e.

$$I_0(x) = r(x)e_1(x) + g(x)e_2(x) + b(x)e_3(x)$$

which we convolve with the discretization of the kernel $K_t^0(x, y, H)$ related to the vector extension of the oriented Laplacian (see Corollary 3).

Fig. 2 illustrates different regularizations of the image on Fig. 1 (see <http://www.cgtextures.com>). They are obtained after 10 iterations of the discrete computation of $k_t^0 I_0$, for $t = 0.2$ where the convolutions are done on 5×5 neighborhoods. In the three cases, the diffusions tend to behave as Gaussian diffusions on regions of low color variations and to diffuse in the direction of edges on regions of high color variations. This is the usual behaviour of the Beltrami flow. For the diffusions based on oriented Laplacians and their locally constant restriction, this behaviour comes from the choice of the functions c_1, c_2 and the moving frame (ξ, η) . We see that diffusions based on the scalar and locally constant oriented Laplacians provide similar results. The diffusion based on the oriented Laplacian we have constructed appears to preserve better the geometry of the initial image in the sense that thin edges are more preserved.

Fig. 1 Left: Original Image - Right: Zoom inside the green rectangle.

(a) Laplace-Beltrami operator

(b) oriented Laplacian

(c) locally constant oriented Laplacian

Fig. 2 Regularization of Fig. 1 by convolution with the kernels $K_t^0(x, y, H)$ for different generalized Laplacians H

Fig. 3 Left: Original Image - Right: Zoom inside the blue rectangle.

(a) $\alpha \equiv 0$

(b) $\alpha \equiv 0.2$

(c) $\alpha \equiv 0.3$

Fig. 4 Regularization of Fig. 3 by convolution with the kernels $K_t^0(x, y, H)$ of generalized Laplacians H of parameter α for different values of the parameter α .

5.2 New connections and metrics for new kinds of color image regularization

We compare the regularizations based on the kernels $K_t^0(x, y, H)$ related to generalized Laplacians of parameters α , (α, h, v) and $(\alpha, h, \mathcal{P}_{v,w})$ for different values of the parameters α, h, v and $\mathcal{P}_{v,w}$, and different metrics on the base manifold.

The base manifolds we consider are the Riemannian manifolds (X, g_λ) constructed in Sect. 4.3, for different functions λ . We construct the vector bundle of rank 3 over (X, g_λ) of total space $E = X \times \mathbb{R}^3$ of global frame (e_1, e_2, e_3) where

$$e_1(x) = (x, (1, 0, 0)), e_2(x) = (x, (0, 1, 0)), \\ e_3(x) = (x, (0, 0, 1))$$

Then, we consider the initial color image as the section I_0 of the vector bundle $(E, \nabla^E, X, g_\lambda)$ for the corresponding connection ∇^E of components (r, g, b) in the global frame (e_1, e_2, e_3) , i.e.

$$I_0(x) = r(x)e_1(x) + g(x)e_2(x) + b(x)e_3(x)$$

which we convolve with the discrete kernel $K_t^0(x, y, H)$ related to the generalized Laplacian involved.

On Fig. 4 we compare color image regularizations related to generalized Laplacians of parameter α for different constant functions α : $\alpha \equiv 0, 0.2, 0.3$, and $\lambda \equiv 0.01$. The section I_0 represents the image on Fig. 3, taken from the Berkeley database [11]. We show results after 10 iterations, for $t = 0.2$.

For $\alpha = 0$, it corresponds to the regularization provided by scalar/Beltrami Laplacians (or shortly Beltrami regularization). For $\alpha > 0$, the regularization combines the anisotropic behaviour of the Beltrami regularization and a highlight of some edges. Indeed, the kernel $K_t^0(x, y, H)$ related to such Laplacians is of the form

$$\left(\frac{1}{4\pi t} \right) e^{-d(x,y)^2/4t} \exp \left(-\alpha \int_0^t \dot{\gamma}_1(s) + \dot{\gamma}_2(s) ds \right) Id$$

on normal neighborhoods. Then the anisotropy does not depend on the parameter α since it is entirely determined by the metric g_λ of X . However, the data we regularize depends on α since it is determined by the connection ∇^E . For $\alpha = 0$, the data is I_0 itself whereas for $\alpha > 0$, it is $\exp \left(-\alpha \int_0^t \dot{\gamma}_1(s) + \dot{\gamma}_2(s) ds \right) I_0$. By the exponential map, the diffusion for $\alpha \neq 0$ increases the contrast of the image and consequently highlights some edges, whereas the Beltrami diffusion reduces some edges, as on the green leaves (see images on

right column of Fig. 4).

On Fig. 5 we compare color image regularizations related to generalized Laplacians of parameters (h, α, v) and $(h, \alpha, \mathcal{P}_{v,w})$ for $\alpha = 0.3$, and $\lambda \equiv 0.01$. The section I_0 represents the image on top-left. We show results after 15 iterations for $t = 0.2$.

On top row, we have considered the following three sections of E : $v = e_1 + e_2 + e_3$, $w = -e_2 + e_3$ and $z = 2e_1 - e_2 - e_3$. The metric h on E we consider is given by the matrix I_3 in the frame (e_1, e_2, e_3) . From Sect. 4.1, given a point $x \in X$, the vector $v(x)$ generates the luminance axis in RGB color space and the vectors $w(x)$ and $z(x)$ form a basis of the chrominance plane, denoted by $\mathcal{P}_{w,z}(x)$.

We have computed ${}^D K_t^0 I_0$ related to generalized Laplacians of parameters (h, α, v) (top-center image) and $(h, \alpha, \mathcal{P}_{w,z})$ (top-right image). The metric h makes the luminance vector field v and the chrominance plane field $\mathcal{P}_{w,z}$ be orthogonal, i.e. it makes the triplet (v, w, z) be an orthogonal frame field of E . Hence, from Propositions 9 and 10, we obtain an interpretation of the results. On the top-center image, the diffusion may be decomposed into a diffusion induced by a generalized Laplacian of parameter α on the luminance component of the initial image (top-left), and a Beltrami diffusion on its chrominance component. Conversely on the top-right image, the diffusion may be decomposed into a diffusion induced by a generalized Laplacian of parameter α on the chrominance component of the initial image, and a Beltrami diffusion on its luminance component.

On bottom row, we have considered the following three sections of E : $v_1 = e_1$, $v_2 = e_1 + e_2$ and $v_3 = 96e_1 + 109e_2 + 65e_3$. Given a point $x \in X$, the vector $v(x)$ encodes the red color, the vector $w(x)$ the yellow color and the vector $z(x)$ a color appearing in the background of the initial image.

Let P_i , $i = 1 \dots 3$ be the matrix fields on X given by

$$P_1 = \begin{pmatrix} 1 & 1 & 96 \\ 0 & 1 & 109 \\ 0 & 0 & 65 \end{pmatrix} \quad P_2 = \begin{pmatrix} 1 & 96 & 1 \\ 1 & 109 & 0 \\ 0 & 65 & 0 \end{pmatrix} \quad P_3 = \begin{pmatrix} 96 & 1 & 1 \\ 109 & 0 & 1 \\ 65 & 0 & 0 \end{pmatrix}$$

and h_i be the metrics on E given by the matrix representations $(P_i^T)^{-1} P_i^{-1}$ in the frame (e_1, e_2, e_3) . Such metrics make the sections v_1, v_2 and v_3 be orthonormal.

We have computed ${}^D K_t^0 I_0$ related to generalized Laplacians of parameters (h_1, α, v_1) (bottom-left image), (h_2, α, v_2) (bottom-center image) and (h_3, α, v_3)

Fig. 5 Top left: Original image - Others: Regularizations of the original image by convolutions with the kernels $K_t^0(x, y, H)$ of generalized Laplacians of parameters (h, α, v) and $(h, \alpha, \mathcal{P}_{v,w})$ for different parameters h, v, w .

(bottom-right image). By construction of h_i , all the diffusions have similar behaviour. Indeed, they can be decomposed into a diffusion induced by a generalized Laplacian of parameter α on the v_i component of the initial image, and a Beltrami diffusion on the plane generated by the v_j and v_k components for $j, k \neq i$. It corresponds approximatively to decompose the diffusion into a diffusion induced by a generalized Laplacian of parameter α on petals (bottom-left), inside the petals (bottom-center), background (bottom-right) and a Beltrami diffusion on the other parts.

On Fig. 6 we compare regularizations of the image on Fig. 6(a) induced by the generalized Laplacian of parameter α , for $\alpha \equiv 0$ for different Riemannian metrics g_λ , i.e. for different values of the parameter λ in (19). The regularizations are obtained after 20 iterations for $t = 0.2$. For the Center image, we have taken $\lambda \equiv 0.01$. Then we obtain a Beltrami regularization all over the image. For the Right image, we have taken $\lambda \equiv 5$ where the hue of I_0 is close to the hue given by the angle 110° , and $\lambda \equiv 0.01$ otherwise. As a consequence the regularization does not affect the image where the hue is close to 110° and behaves as a Beltrami regularization elsewhere.

6 Conclusion

We have proposed in this paper a new framework for multivalued image regularization by PDEs, based on

heat equations related to generalized Laplacians on vector bundles. This framework unifies most of the methods devoted to image regularization by PDEs. In this context, we have shown that the different methods differ by the geometry of the vector bundle involved, i.e. by the Riemannian metric of the base manifold and the connection on the vector bundle. Moreover, we have proposed to regularize images by an operator acting as the convolution with some kernel related to the heat kernel of the corresponding generalized Laplacian. By the kernel we have chosen, the diffusion process is entirely determined by the connection on the vector bundle and the Riemannian metric of the base manifold. The connection provides the data that are convolved whereas the metric gives the anisotropic behaviour of the convolution. Based on this observation, we have constructed several connections and metrics to perform several kinds of regularization, with applications on color images. In particular, we have considered connections in such a way that the subsequent diffusion combines a Beltrami diffusion and a highlight of some edges, taking into account luminance and chrominance information of the image. By the metrics we have considered, we have constructed regularizations where the speed of the diffusion varies with respect to pixels of the image. Finally, by this general framework, many other kinds of image regularization may be envisaged by the choice of other connections and metrics. Further work will be devoted to making a link between this local approach and global approaches to image regularization.

Fig. 6 Regularization by convolution with the kernel $K_t^0(x, y, H)$ of a generalized Laplacian of parameter α for $\alpha \equiv 0$, and for different metrics g_λ on the base manifold.

Appendix 1. Heat Equations on Vector Bundles

A. Differential geometry of vector bundles

Definition 5 A **smooth vector bundle of rank n** is a triplet (E, π, X) where X and E are two C^∞ manifolds, and $\pi: E \rightarrow X$ is a surjective map such that the preimage $\pi^{-1}(x)$ of $x \in X$ is endowed with a structure of vector space of dimension n . X is called the **base manifold** and E the **total space** of the vector bundle. The set $\pi^{-1}(x)$ is called the **fiber** over x , and is denoted by E_x .

The vector bundle is said to be **locally trivial** if the following conditions hold: for each $x \in X$, there is a neighborhood U of x and a diffeomorphism $\phi: U \times \mathbb{R}^n \rightarrow \pi^{-1}(U)$ satisfying $\pi \circ \phi(x, f) = x$, and such that the map $\phi_x: \mathbb{R}^n \rightarrow E_x$ is a vector space isomorphism. The couple (U, ϕ) is called a **local trivialization**.

The vector bundle is said to be **trivial** if there exists a diffeomorphism $\Phi: X \times \mathbb{R}^n \rightarrow E$ satisfying $\pi \circ \Phi(x, f) = x$, and such that the map $\Phi_x: \mathbb{R}^n \rightarrow E_x$ is a vector space isomorphism.

Example 1 Let X be a C^∞ manifold of dimension m . The disjoint union of tangent spaces $TX := \bigsqcup T_x X$ for $x \in X$, is the total space of a vector bundle (TX, π, X) of rank m called the **tangent bundle** of X .

Definition 6 A metric h on a vector bundle is the assignment of a scalar product h_x on each fiber $\pi^{-1}(x)$.

Example 2 A Riemannian metric on a manifold is a positive definite metric on its tangent bundle.

Definition 7 A **section** of a smooth vector bundle (E, π, X) is a differentiable map $S: X \rightarrow E$ such that $\pi \circ S = Id_X$.

Let $(f_1, \dots, f_n)$ be a basis of $\mathbb{R}^n$. In a local trivialization (U, ϕ) of (E, π, X) , any section may be written

$$S(x) = \sum_{i=1}^n s_i(x) \phi(x, f_i)$$

for some functions $s_i \in C^\infty(U)$.

The set $\{\phi(\cdot, f_1), \dots, \phi(\cdot, f_n)\}$ is called a **local frame** of (E, π, X) .

The set of sections of (E, π, X) is denoted by $\Gamma(E)$.

Example 3 Tangent vector fields on X are the sections of the tangent bundle (TX, π, X) .

Definition 8 A **connection** on (E, π, X) is a map $\nabla^E: \Gamma(TX) \times \Gamma(E) \rightarrow \Gamma(E)$ satisfying the following axioms:

$$\begin{aligned} \nabla_{f_u + g_v}^E Y &= f \nabla_u^E Y + g \nabla_v^E Y \\ \nabla_u^E f Y &= (d_u f) Y + f \nabla_u^E Y \end{aligned}$$

for $f, g \in C^\infty(X)$, $u, v \in \Gamma(TX)$ and $Y \in \Gamma(E)$.

Hence, a connection on (E, π, X) may be written as $d + \omega$ for some $\omega \in \Gamma(T^*X \otimes \text{End}(E))$.

In local frames $(e_1, \dots, e_n)$ of E and $(u_1, \dots, u_m)$ of TX , a connection is determined by $n^2 \times m$ functions Υ_{ij}^k such that

$$\nabla_{u_i}^E e_j = \sum_{k=1}^n \Upsilon_{ij}^k e_k$$

Example 4 The **Levi-Cevita** connection is the connection on the tangent bundle of a Riemannian manifold (X, g) determined by the m^3 functions

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl} (\partial_j g_{li} + \partial_i g_{lj} - \partial_l g_{ij})$$

with respect to the local frame $(\partial/\partial x_1, \dots, \partial/\partial x_m)$ of TX given by a local coordinates system $(x_1, \dots, x_m)$ on X .

Definition 9 Let E be a vector bundle equipped with a metric h . A connection ∇^E on E is **compatible with the metric** h if it satisfies

$$u \cdot h(S, S') = h(\nabla_u^E S, S') + h(S, \nabla_u^E S')$$

for any $u \in \Gamma(TX)$, and $S, S' \in \Gamma(E)$, where $\cdot$ stands for the derivative of functions.

Example 5 For a Riemannian metric g on TX , the corresponding Levi-Cevita connection is compatible with g .

Definition 10 Let ∇^E be a connection on a vector bundle (E, π, X) , and γ a C^1 curve in X such that $\gamma(0) = y$. The **parallel transport** of $Y_0 \in E_y$ along the curve γ is the section $Y(t)$ that is solution of the following differential equation

$$\begin{cases} \nabla_{\dot{\gamma}}^E Y(t) = 0 \\ Y(0) = Y_0 \end{cases}$$

A section Y along a curve γ is **parallel** if $\nabla_{\dot{\gamma}}^E Y(t) = 0$ for each t .

Example 6 Let (X, g) be a Riemannian manifold endowed with the Levi-Cevita connection ∇ on its tangent bundle. A **geodesic curve** on X is a C^1 curve γ whose tangent vector field $\dot{\gamma}$ is parallel along γ , i.e. $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$.

B. Heat Kernels of Generalized Laplacians

We refer to [6] for more details on this part.

Definition 11 Let E be a vector bundle over a Riemannian manifold (X, g) , endowed with a connection ∇^E . Let ∇ be the Levi-Cevita connection of (TX, g) . To any pair of tangent vector fields V and W on X , we associate an invariant second derivative $\nabla_{V,W}^2: \Gamma(E) \rightarrow \Gamma(E)$ by setting

$$\nabla_{V,W}^2 \varphi \equiv \nabla_V^E \nabla_W^E \varphi - \nabla_{\nabla_V W}^E \varphi$$

Then the **connection Laplacian** $\Delta^E: \Gamma(E) \rightarrow \Gamma(E)$ is defined by

$$\Delta^E \varphi = -\text{trace}(\nabla_{\cdot}^2 \varphi)$$

where trace denotes the contraction with the metric g .

In particular, if e_i is a local orthonormal frame of TX , the operator Δ^E is given by

$$\Delta^E = - \sum_i \left(\nabla_{e_i}^E \nabla_{e_i}^E - \nabla_{\nabla_{e_i} e_i}^E \right)$$

With respect to the local frame $\partial_i := \partial/\partial x_i$ defined by a local coordinates system on X , we have

$$\Delta^E = - \sum_{ij} g^{ij} \left(\nabla_{\partial_i}^E \nabla_{\partial_j}^E - \sum_k \Gamma_{ij}^k \nabla_{\partial_k}^E \right)$$

where the symbols Γ_{ij}^k are defined by $\nabla_{\partial_i} \partial_j = \sum_k \Gamma_{ij}^k \partial_k$.

Definition 12 Let E be a vector bundle over a Riemannian manifold (X, g) . A **Generalized Laplacian** on E is a second-order differential operator $H: \Gamma(E) \rightarrow \Gamma(E)$ that may be written

$$H = \Delta^E + F$$

for some connection ∇^E and a zero-order operator F .

In particular, any connection Laplacian is a generalized Laplacian. To precise the geometric data associated with a generalized H , we write sometimes that H is a generalized Laplacian on (E, ∇^E, X, g) .

To any generalized Laplacian H on a vector bundle E over a compact manifold X , one may associate an operator $e^{-tH}: \Gamma(E) \rightarrow \Gamma(E)$, for $t > 0$, with the property that $u(x) = e^{-tH} u_0(x)$ satisfies the **heat equation**

$$\frac{\partial u}{\partial t} + Hu = 0, \quad u|_{t=0} = u_0$$

We shall define e^{-tH} as an integral operator of the form

$$(e^{-tH} u_0)(x) = \int_X K_t(x, y, H) u_0(y) dy$$

where $K_t(x, y, H): E_y \longrightarrow E_x$ is a linear map depending smoothly on x, y and t . It is called the **heat kernel of H** .

In the following theorem, we summarize some results on approximations of the heat kernel and solutions of the heat equation.

Theorem 1 *Let $x \in X$. We denote by $\mathbf{y}^i$ the normal coordinates of a point y in the injectivity radius of X at x , ∂_i the corresponding partial derivatives, and by $g_{ij}(\mathbf{y})$ the scalar product of ∂_i and ∂_j at $\mathbf{y}$. Moreover, we define*

$$J(x, y) = \det(g_{ij}(\mathbf{y}))^{1/2} \quad \text{for } y = \exp_x(\mathbf{y})$$

Let ϵ chosen smaller than the injectivity radius of X . Let $\Psi: \mathbb{R}_+ \longrightarrow [0, 1]$ be a smooth function such that $\Psi(s) = 1$ if $s < \epsilon^2/4$ and $\Psi(s) = 0$ if $s > \epsilon^2$.

Let $\tau(x, y): E_y \longrightarrow E_x$ be the parallel transport along the unique geodesic curve joining x and y , and $d(x, y)$ its length.

Let $K_t^N(x, y, H)$ be the kernel defined by

$$\left(\frac{1}{4\pi t}\right)^{\frac{m}{2}} e^{-d(x, y)^2/4t} \Psi(d(x, y)^2) \sum_{i=0}^N t^i \Phi_i(x, y, H) J(x, y)^{-\frac{1}{2}}$$

where the sections Φ_i are given by $\Phi_0(x, y, H) = \tau(x, y)$ and $\tau(x, y)^{-1} \Phi_i(x, y, H) =$

$$- \int_0^1 s^{i-1} \tau(x_s, y)^{-1} (B_x \cdot \Phi_{i-1})(x_s, y, H) ds$$

B_x is the operator $J^{1/2} \circ H_x \circ J^{-1/2}$ where H_x is the operator H with respect to the first variable.

1. For every $N > m/2$, the kernel $K_t^N(x, y, H)$ is asymptotic to $K_t(x, y, H)$:

$$\left\| \partial_t^k [K_t(x, y, H) - K_t^N(x, y, H)] \right\|_l = O(t^{N-m/2-l/2-k})$$

where $\|\cdot\|_l$ is a norm on C^l sections.

2. Let us denote by k_t^N the operator defined by

$$(k_t^N u_0)(x) = \int_X K_t^N(x, y, H) u_0(y) dy$$

Then for every N , $\lim_{t \rightarrow 0} \|k_t^N u_0 - u_0\|_l = 0$.

3. We have the following estimate:

$$\left\| e^{-tH} u_0 - \sum_{i=0}^k \frac{(-tH)^i}{i!} u_0 \right\|_j = O(t^{k+1})$$

which justifies the notation e^{-tH} .

Appendix 2. Proofs

A. Discrete maximum principle for the operator k_t^0

Proof (of Proposition 1) We first show that for a connection compatible with the metric, the parallel transport preserves the scalar product.

Let $S, S' \in \Gamma(E)$, and γ a C^1 curve in X . By definition of a connection compatible with a metric, we have

$$\dot{\gamma} \cdot h(S, S') = h(\nabla_{\dot{\gamma}}^E S, S') + h(S, \nabla_{\dot{\gamma}}^E S') \quad (20)$$

Let us suppose that S, S' are parallel along γ , i.e. $\nabla_{\dot{\gamma}}^E S = \nabla_{\dot{\gamma}}^E S' = 0$. Then, (20) gives $\dot{\gamma} \cdot h(S, S') = 0$. As a consequence, the scalar product of S and S' is invariant along γ . We deduce that the parallel transport preserves the scalar product. Denoting by $\|\cdot\|_h$ the norm with respect to h , it follows that

$$\begin{aligned} \|\nabla_{\dot{\gamma}}^E u_0(i, j)\|_h &\leq \sum_{(k, l) \in \mathcal{V}(i, j)} Q_t((i, j), (k, l)) \|u_0(k, l)\|_h \\ &\leq \max_{(k, l) \in \mathcal{V}(i, j)} \|u_0(k, l)\|_h \sum_{(k, l) \in \mathcal{V}(i, j)} Q_t((i, j), (k, l)) \\ &\leq \max_{(k, l) \in \mathcal{V}(i, j)} \|u_0(k, l)\|_h \end{aligned}$$

since $\sum_{(k, l) \in \mathcal{V}(i, j)} Q_t((i, j), (k, l)) = 1$. Hence

$$\max_{t \in \mathbb{R}^+} \max_{(i, j) \in \Omega \cap \mathbb{N}^2} \|\nabla_{\dot{\gamma}}^E u_0(i, j)\|_h = \max_{(i, j) \in \Omega \cap \mathbb{N}^2} \|u_0(i, j)\|_h$$

and the regularization operator $\nabla_{\dot{\gamma}}^E k_t^0$ satisfies the maximum principle. $\square$

B. Scalar Laplacian/Laplace-Beltrami operator

Proof (of Proposition 2) Following the notations of Proposition 2, the connection Laplacian on (E, ∇^E, Ω, g) is of the form

$$\Delta^E(I) = -(\Delta_g I_1) e_1 - (\Delta_g I_2) e_2 - (\Delta_g I_3) e_3$$

Hence, the operator (6) corresponds to Δ^E , and is therefore the generalized Laplacian on

(E, ∇^E, Ω, g) associated with the zero-order operator $F \equiv 0$.

C. Oriented Laplacian

Let $\Omega \subset \mathbb{R}^2$ be the domain of an image. Let f be a smooth function defined on Ω . We consider the differential operator of order 2

$$\Delta(f) = -c_1 d_{\xi, \xi}^2 f - c_2 d_{\eta, \eta}^2 f \quad (21)$$

where (ξ, η) is a moving frame, and c_1, c_2 are two strictly positive functions on Ω .

In the following proposition, we show that Δ may be viewed as a generalized Laplacian on a vector bundle of rank 1 over Ω .

Proposition 11 *Let (Ω, g) be the Riemannian manifold of dimension 2 with g given by $1/(c_1 c_2 (\xi_1 \eta_2 - \eta_1 \xi_2)^2) \times$*

$$\begin{pmatrix} c_1 \xi_2^2 + c_2 \eta_2^2 & -(c_1 \xi_1 \xi_2 + c_2 \eta_1 \eta_2) \\ -(c_1 \xi_1 \xi_2 + c_2 \eta_1 \eta_2) & c_1 \xi_1^2 + c_2 \eta_1^2 \end{pmatrix}$$

in the cartesian coordinates system (x_1, x_2) on Ω .

Let E be a trivial vector bundle of rank 1 over (Ω, g) of global frame e_1 , endowed with the connection ∇^E defined by $\nabla_{\partial_{x_1}}^E e_1 = \Upsilon_1 e_1$ and $\nabla_{\partial_{x_2}}^E e_1 = \Upsilon_2 e_1$ where

$$\Upsilon_1 = \frac{1}{2}(g_{11}a + g_{12}b) \quad \Upsilon_2 = \frac{1}{2}(g_{12}a + g_{22}b)$$

and

$$\begin{aligned} a &= c_1 \frac{\partial \xi_1}{\partial x_1} \xi_1 + c_1 \frac{\partial \xi_1}{\partial x_2} \xi_2 + c_2 \frac{\partial \eta_1}{\partial x_1} \eta_1 + c_2 \frac{\partial \eta_1}{\partial x_2} \eta_2 + 2g^{12} \Gamma_{12}^1 \\ &\quad + g^{11} \Gamma_{11}^1 + g^{22} \Gamma_{22}^1 \\ b &= c_1 \frac{\partial \xi_2}{\partial x_1} \xi_1 + c_1 \frac{\partial \xi_2}{\partial x_2} \xi_2 + c_2 \frac{\partial \eta_2}{\partial x_1} \eta_1 + c_2 \frac{\partial \eta_2}{\partial x_2} \eta_2 + 2g^{12} \Gamma_{12}^2 \\ &\quad + g^{11} \Gamma_{11}^2 + g^{22} \Gamma_{22}^2 \end{aligned}$$

Let $\tilde{f} = f e_1 \in \Gamma(E)$. The differential operator H of order 2 defined by

$$H(\tilde{f}) = \Delta(f) e_1 \quad (22)$$

may be written as $H = \Delta^E + F$ for some zero-order operator F .

Proof Developping (22) we obtain $H(\tilde{f}) =$

$$\begin{aligned} &\left[-\left(c_1 \xi_1^2 + c_2 \eta_1^2\right) \frac{\partial^2 f}{\partial x_1^2} - \left(2c_1 \xi_1 \xi_2 + 2c_2 \eta_1 \eta_2\right) \frac{\partial^2 f}{\partial x_1 \partial x_2} \right. \\ &\quad \left. - \left(c_1 \xi_2^2 + c_2 \eta_2^2\right) \frac{\partial^2 f}{\partial x_2^2} \right. \\ &\quad \left. + \frac{\partial f}{\partial x_1} \left(-c_1 \xi_1 \frac{\partial \xi_1}{\partial x_1} - c_1 \xi_2 \frac{\partial \xi_1}{\partial x_2} - c_2 \eta_1 \frac{\partial \eta_1}{\partial x_1} - c_2 \eta_2 \frac{\partial \eta_1}{\partial x_2} \right) \right. \\ &\quad \left. + \frac{\partial f}{\partial x_2} \left(-c_1 \xi_1 \frac{\partial \xi_2}{\partial x_1} - c_1 \xi_2 \frac{\partial \xi_2}{\partial x_2} - c_2 \eta_1 \frac{\partial \eta_2}{\partial x_1} - c_2 \eta_2 \frac{\partial \eta_2}{\partial x_2} \right) \right] e_1 \end{aligned}$$

Then we search (g, ∇^E, F) such that $H = \Delta^E + F$ on (E, ∇^E, Ω, g) . By definition of connection Laplacians

(see Appendix 1.B), we have

$$\begin{aligned} \Delta^E(\tilde{f}) &= -g^{11} \left(\nabla_{\partial_{x_1}}^E \nabla_{\partial_{x_1}}^E \tilde{f} - \Gamma_{11}^1 \nabla_{\partial_{x_1}}^E \tilde{f} - \Gamma_{11}^2 \nabla_{\partial_{x_2}}^E \tilde{f} \right) \\ &\quad - g^{12} \left(\nabla_{\partial_{x_1}}^E \nabla_{\partial_{x_2}}^E \tilde{f} - \Gamma_{12}^1 \nabla_{\partial_{x_1}}^E \tilde{f} - \Gamma_{12}^2 \nabla_{\partial_{x_2}}^E \tilde{f} \right) \\ &\quad - g^{21} \left(\nabla_{\partial_{x_2}}^E \nabla_{\partial_{x_1}}^E \tilde{f} - \Gamma_{21}^1 \nabla_{\partial_{x_1}}^E \tilde{f} - \Gamma_{21}^2 \nabla_{\partial_{x_2}}^E \tilde{f} \right) \\ &\quad - g^{22} \left(\nabla_{\partial_{x_2}}^E \nabla_{\partial_{x_2}}^E \tilde{f} - \Gamma_{22}^1 \nabla_{\partial_{x_1}}^E \tilde{f} - \Gamma_{22}^2 \nabla_{\partial_{x_2}}^E \tilde{f} \right) \end{aligned}$$

A connection ∇^E on a vector bundle of rank 1 being defined by $\nabla_{\partial_{x_1}}^E e_1 = \Upsilon_1 e_1$ and $\nabla_{\partial_{x_2}}^E e_1 = \Upsilon_2 e_1$ for some functions Υ_1 and Υ_2 , we obtain $\Delta^E(\tilde{f}) =$

$$\begin{aligned} &\left(-g^{11} \frac{\partial^2 f}{\partial x_1^2} - g^{12} \frac{\partial^2 f}{\partial x_1 \partial x_2} - g^{21} \frac{\partial^2 f}{\partial x_1 \partial x_2} - g^{22} \frac{\partial^2 f}{\partial x_2^2} \right) e_1 \\ &\quad + \left(\left(-g^{12} \Upsilon_2 + g^{12} \Gamma_{12}^1 - g^{21} \Upsilon_2 + g^{21} \Gamma_{21}^1 - 2g^{11} \Upsilon_1 + g^{11} \Gamma_{11}^1 \right. \right. \\ &\quad \left. \left. + g^{22} \Gamma_{22}^1 \right) \frac{\partial f}{\partial x_1} + \left(-g^{12} \Upsilon_1 + g^{12} \Gamma_{12}^2 - g^{21} \Upsilon_1 + g^{21} \Gamma_{21}^2 + g^{11} \Gamma_{11}^2 \right. \right. \\ &\quad \left. \left. - 2g^{22} \Upsilon_2 + g^{22} \Gamma_{22}^2 \right) \frac{\partial f}{\partial x_2} \right) e_1 \\ &\quad + f \left(-g^{11} \frac{\partial \Upsilon_1}{\partial x_1} - g^{11} (\Upsilon_1)^2 + g^{11} \Gamma_{11}^1 \Upsilon_1 + g^{11} \Gamma_{11}^2 \Upsilon_2 - g^{12} \frac{\partial \Upsilon_2}{\partial x_1} \right. \\ &\quad \left. - g^{12} \Upsilon_1 \Upsilon_2 + g^{12} \Gamma_{12}^1 \Upsilon_1 + g^{12} \Gamma_{12}^2 \Upsilon_2 - g^{21} \frac{\partial \Upsilon_1}{\partial x_2} - g^{21} \Upsilon_1 \Upsilon_2 + \right. \\ &\quad \left. g^{21} \Gamma_{21}^1 \Upsilon_1 \right. \\ &\quad \left. + g^{21} \Gamma_{21}^2 \Upsilon_2 g^{22} \frac{\partial \Upsilon_2}{\partial x_2} - g^{22} (\Upsilon_2)^2 + g^{22} \Gamma_{22}^1 \Upsilon_1 + g^{22} \Gamma_{22}^2 \Upsilon_2 \right) e_1 \end{aligned}$$

We see that the second-order part determines the metric g , the first-order part determines the connection ∇^E and the zero-order part determines the zero-order operator F .

By identification, we have

$$g^{-1} = \begin{pmatrix} c_1 \xi_1^2 + c_2 \eta_1^2 & c_1 \xi_1 \xi_2 + c_2 \eta_1 \eta_2 \\ c_1 \xi_1 \xi_2 + c_2 \eta_1 \eta_2 & c_1 \xi_2^2 + c_2 \eta_2^2 \end{pmatrix}$$

in the frame $(\partial/\partial x_1, \partial/\partial x_2)$ from which follows that g is given by $1/(c_1 c_2 (\xi_1 \eta_2 - \eta_1 \xi_2)^2) \times$

$$\begin{pmatrix} c_1 \xi_2^2 + c_2 \eta_2^2 & -(c_1 \xi_1 \xi_2 + c_2 \eta_1 \eta_2) \\ -(c_1 \xi_1 \xi_2 + c_2 \eta_1 \eta_2) & c_1 \xi_1^2 + c_2 \eta_1^2 \end{pmatrix}$$

in the frame $(\partial/\partial x_1, \partial/\partial x_2)$.

Identifying first order parts we obtain two equalities, where left terms have been simplified using symmetries of coefficients g_{ij} and symbols Γ_{ij}^k

$$-2g^{11} \Upsilon_1 - 2g^{12} \Upsilon_2 + 2g^{12} \Gamma_{12}^1 + g^{11} \Gamma_{11}^1 + g^{22} \Gamma_{22}^1 = -c_1 \xi_1 \frac{\partial \xi_1}{\partial x_1} -$$

$$\begin{aligned}
& c_1 \xi_2 \frac{\partial \xi_1}{\partial x_2} - c_2 \eta_1 \frac{\partial \eta_1}{\partial x_1} - c_2 \eta_2 \frac{\partial \eta_1}{\partial x_2} \\
& - 2g^{12} \Upsilon_1 - 2g^{22} \Upsilon_2 + 2g^{12} \Gamma_{12}^2 + g^{11} \Gamma_{11}^2 + g^{22} \Gamma_{22}^2 = -c_1 \xi_1 \frac{\partial \xi_2}{\partial x_1} - \\
& c_1 \xi_2 \frac{\partial \xi_2}{\partial x_2} - c_2 \eta_1 \frac{\partial \eta_2}{\partial x_1} - c_2 \eta_2 \frac{\partial \eta_2}{\partial x_2}
\end{aligned}$$

We put

$$\begin{aligned}
a &= c_1 \frac{\partial \xi_1}{\partial x_1} \xi_1 + c_1 \frac{\partial \xi_1}{\partial x_2} \xi_2 + c_2 \frac{\partial \eta_1}{\partial x_1} \eta_1 + c_2 \frac{\partial \eta_1}{\partial x_2} \eta_2 + 2g^{12} \Gamma_{12}^1 \\
& \quad + g^{11} \Gamma_{11}^1 + g^{22} \Gamma_{22}^1 \\
b &= c_1 \frac{\partial \xi_2}{\partial x_1} \xi_1 + c_1 \frac{\partial \xi_2}{\partial x_2} \xi_2 + c_2 \frac{\partial \eta_2}{\partial x_1} \eta_1 + c_2 \frac{\partial \eta_2}{\partial x_2} \eta_2 \\
& \quad + 2g^{12} \Gamma_{12}^2 + g^{11} \Gamma_{11}^2 + g^{22} \Gamma_{22}^2
\end{aligned}$$

Then we obtain the system

$$\begin{cases} -2g^{11} \Upsilon_1 - 2g^{12} \Upsilon_2 = -a \\ -2g^{12} \Upsilon_1 - 2g^{22} \Upsilon_2 = -b \end{cases}$$

whose solution is

$$\begin{pmatrix} \Upsilon_1 \\ \Upsilon_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

At last, it is immediat that

$$\begin{aligned}
F &= \left(g^{11} \frac{\partial \Upsilon_1}{\partial x_1} + g^{11} (\Upsilon_1)^2 - g^{11} \Gamma_{11}^1 \Upsilon_1 - g^{11} \Gamma_{11}^2 \Upsilon_2 + g^{12} \frac{\partial \Upsilon_2}{\partial x_1} \right. \\
& \quad + g^{12} \frac{\partial \Upsilon_1}{\partial x_2} + 2g^{12} \Upsilon_1 \Upsilon_2 - 2g^{12} \Gamma_{12}^1 \Upsilon_1 - 2g^{12} \Gamma_{12}^2 \Upsilon_2 + g^{22} \frac{\partial \Upsilon_2}{\partial x_2} \\
& \quad \left. + g^{22} (\Upsilon_2)^2 - g^{22} \Gamma_{22}^1 \Upsilon_1 - g^{22} \Gamma_{22}^2 \Upsilon_2 \right) Id
\end{aligned}$$

where Id is the Identity operator on E . $\square$

Then, Proposition 3 appears as a Corollary of Proposition 11 in which H is extended to a generalized Laplacian on a vector bundle of rank 3.

Proof (of Proposition 3) Following the notations of Proposition 3, we have $H = \Delta^E + F$ where F is the zero order operator on E defined by

$$\begin{aligned}
& \left(g^{11} \frac{\partial \Upsilon_1}{\partial x_1} + g^{11} (\Upsilon_1)^2 - g^{11} \Gamma_{11}^1 \Upsilon_1 - g^{11} \Gamma_{11}^2 \Upsilon_2 + g^{12} \frac{\partial \Upsilon_2}{\partial x_1} \right. \\
& \quad + g^{12} \frac{\partial \Upsilon_1}{\partial x_2} + 2g^{12} \Upsilon_1 \Upsilon_2 - 2g^{12} \Gamma_{12}^1 \Upsilon_1 - 2g^{12} \Gamma_{12}^2 \Upsilon_2 + g^{22} \frac{\partial \Upsilon_2}{\partial x_2} \\
& \quad \left. + g^{22} (\Upsilon_2)^2 - g^{22} \Gamma_{22}^1 \Upsilon_1 - g^{22} \Gamma_{22}^2 \Upsilon_2 \right) Id
\end{aligned}$$

where Id is the identity operator on E . $\square$

D. Curvature-preserving PDEs

Let Ω be the domain of an image. Let f be a smooth function defined on Ω . We consider the differential op-

erator of order 2

$$f \longmapsto \text{trace}(\mathbf{TH}) + \frac{2}{\pi} \nabla f^T \int_{\alpha=0}^{\pi} \mathbf{J}_{\sqrt{\mathbf{T}} a_\alpha} \sqrt{\mathbf{T}} a_\alpha d\alpha \quad (23)$$

where

- the tensor fields $\mathbf{T}$ and $\sqrt{\mathbf{T}}$ are defined by

$$\mathbf{T} = c_1 \xi \xi^T + c_2 \eta \eta^T, \quad \sqrt{\mathbf{T}} = \sqrt{c_1} \xi \xi^T + \sqrt{c_2} \eta \eta^T$$

- $a_\alpha = (\cos \alpha \quad \sin \alpha)^T$

- $\mathbf{J}_{\sqrt{\mathbf{T}} a_\alpha}$ stands for the Jacobian of $\sqrt{\mathbf{T}} a_\alpha$.

- $\mathbf{H}$ is the Hessian of the function f .

In the following proposition, we show that (23) may be viewed as a generalized Laplacian on a vector bundle of rank 1 over Ω .

Proposition 12 *Let (Ω, g) be the Riemannian manifold of dimension 2 with g given by $1/(c_1 c_2 (\xi_1 \eta_2 - \eta_1 \xi_2)^2) \times$*

$$\begin{pmatrix} c_1 \xi_2^2 + c_2 \eta_2^2 & -(c_1 \xi_1 \xi_2 + c_2 \eta_1 \eta_2) \\ -(c_1 \xi_1 \xi_2 + c_2 \eta_1 \eta_2) & c_1 \xi_1^2 + c_2 \eta_1^2 \end{pmatrix}$$

in the cartesian coordinates system (x_1, x_2) on Ω .

Let E be a trivial vector bundle of rank 1 over (Ω, g) of global frame e_1 , endowed with the connection ∇^E defined by $\nabla_{\partial_{x_1}}^E e_1 = \Upsilon_1 e_1$ and $\nabla_{\partial_{x_2}}^E e_1 = \Upsilon_2 e_1$ where

$$\Upsilon_1 = \frac{1}{2}(g_{11}a + g_{12}b) \quad \Upsilon_2 = \frac{1}{2}(g_{12}a + g_{22}b)$$

where $a =$

$$\begin{aligned}
& \frac{1}{2} \frac{\partial c_1}{\partial x_1} \left(\xi_1^4 + \frac{\sqrt{c_2}}{\sqrt{c_1}} \xi_1^2 \eta_1^2 + \xi_1^2 \xi_2^2 + \frac{\sqrt{c_2}}{\sqrt{c_1}} \xi_1 \xi_2 \eta_1 \eta_2 \right) \\
& + \frac{1}{2} \frac{\partial c_1}{\partial x_2} \left(\xi_1 \xi_2^3 + \frac{\sqrt{c_2}}{\sqrt{c_1}} \xi_1 \xi_2 \eta_2^2 + \xi_1^3 \xi_2 + \frac{\sqrt{c_2}}{\sqrt{c_1}} \xi_1^2 \eta_1 \eta_2 \right) \\
& + \frac{1}{2} \frac{\partial c_2}{\partial x_1} \left(\frac{\sqrt{c_1}}{\sqrt{c_2}} \xi_1^2 \eta_1^2 + \eta_1^4 + \frac{\sqrt{c_1}}{\sqrt{c_2}} \xi_1 \xi_2 \eta_1 \eta_2 + \eta_1^2 \eta_2^2 \right) \\
& + \frac{1}{2} \frac{\partial c_2}{\partial x_2} \left(\frac{\sqrt{c_1}}{\sqrt{c_2}} \xi_2^2 \eta_1 \eta_2 + \eta_1 \eta_2^3 + \frac{\sqrt{c_1}}{\sqrt{c_2}} \eta_1^2 \xi_1 \xi_2 + \eta_1^3 \eta_2 \right) \\
& + \sqrt{c_1 c_2} \left(\frac{\partial \xi_1}{\partial x_2} \xi_2 \eta_2^2 + \frac{\partial \xi_2}{\partial x_2} \xi_1 \eta_2^2 + \frac{\partial \eta_1}{\partial x_2} \xi_2^2 \eta_2 + \frac{\partial \eta_2}{\partial x_2} \xi_2^2 \eta_1 \right. \\
& + 2 \frac{\partial \xi_1}{\partial x_2} \xi_1 \eta_1 \eta_2 + 2 \frac{\partial \eta_1}{\partial x_2} \xi_1 \xi_2 \eta_1 + \frac{\partial \xi_1}{\partial x_1} \xi_2 \eta_1 \eta_2 + \frac{\partial \xi_2}{\partial x_1} \xi_1 \eta_1 \eta_2 \\
& + 2 \frac{\partial \xi_1}{\partial x_1} \xi_1 \eta_1^2 + 2 \frac{\partial \eta_1}{\partial x_1} \xi_1^2 \eta_1 + \frac{\partial \eta_1}{\partial x_1} \xi_1 \xi_2 \eta_2 + \frac{\partial \eta_2}{\partial x_1} \xi_1 \xi_2 \eta_1 \left. \right) \\
& + c_1 \left(2 \frac{\partial \xi_1}{\partial x_1} \xi_1^3 + \frac{\partial \xi_1}{\partial x_2} \xi_2^3 + \frac{\partial \xi_2}{\partial x_1} \xi_1^2 \xi_2 + 2 \frac{\partial \xi_1}{\partial x_2} \xi_1^2 \xi_2 \right. \\
& + \frac{\partial \xi_1}{\partial x_1} \xi_1 \xi_2^2 + \frac{\partial \xi_2}{\partial x_2} \xi_1 \xi_2^2 \left. \right) + c_2 \left(2 \frac{\partial \eta_1}{\partial x_1} \eta_1^3 + 2 \frac{\partial \eta_1}{\partial x_2} \eta_1^2 \eta_2 \right. \\
& + \frac{\partial \eta_1}{\partial x_1} \eta_1 \eta_2^2 + \frac{\partial \eta_2}{\partial x_1} \eta_1^2 \eta_2 + \frac{\partial \eta_1}{\partial x_2} \eta_2^3 + \frac{\partial \eta_2}{\partial x_2} \eta_1 \eta_2^2 \left. \right) \\
& + 2g^{12} \Gamma_{12}^1 + g^{11} \Gamma_{11}^1 + g^{22} \Gamma_{22}^1
\end{aligned}$$

and $b =$

$$\begin{aligned}
& \frac{1}{2} \frac{\partial c_1}{\partial x_1} \left(\xi_1^3 \xi_2 + \frac{\sqrt{c_2}}{\sqrt{c_1}} \xi_1 \xi_2 \eta_1^2 + \xi_1 \xi_2^3 + \frac{\sqrt{c_2}}{\sqrt{c_1}} \xi_2^2 \eta_1 \eta_2 \right) \\
& + \frac{1}{2} \frac{\partial c_1}{\partial x_2} \left(\xi_1^2 \xi_2^2 + \frac{\sqrt{c_2}}{\sqrt{c_1}} \xi_1 \xi_2 \eta_1 \eta_2 + \xi_2^4 + \frac{\sqrt{c_2}}{\sqrt{c_1}} \xi_2^2 \eta_2^2 \right) \\
& + \frac{1}{2} \frac{\partial c_2}{\partial x_1} \left(\frac{\sqrt{c_1}}{\sqrt{c_2}} \xi_1^2 \eta_1 \eta_2 + \eta_1^3 \eta_2 + \frac{\sqrt{c_1}}{\sqrt{c_2}} \eta_2^2 \xi_1 \xi_2 + \eta_1 \eta_2^3 \right) \\
& + \frac{1}{2} \frac{\partial c_2}{\partial x_2} \left(\frac{\sqrt{c_1}}{\sqrt{c_2}} \xi_1 \xi_2 \eta_1 \eta_2 + \eta_1^2 \eta_2^2 + \frac{\sqrt{c_1}}{\sqrt{c_2}} \xi_2^2 \eta_2^2 + \eta_2^4 \right) \\
& + \sqrt{c_1 c_2} \left(\frac{\partial \xi_1}{\partial x_1} \xi_2 \eta_1^2 + \frac{\partial \xi_2}{\partial x_1} \xi_1 \eta_1^2 + \frac{\partial \eta_1}{\partial x_1} \xi_1^2 \eta_2 + \frac{\partial \eta_2}{\partial x_1} \xi_1^2 \eta_1 \right. \\
& + 2 \frac{\partial \xi_2}{\partial x_1} \xi_2 \eta_1 \eta_2 + 2 \frac{\partial \eta_2}{\partial x_1} \xi_1 \xi_2 \eta_2 + \frac{\partial \xi_1}{\partial x_2} \xi_2 \eta_1 \eta_2 + \frac{\partial \xi_2}{\partial x_2} \xi_1 \eta_1 \eta_2 \\
& \left. + \frac{\partial \eta_1}{\partial x_2} \xi_1 \xi_2 \eta_2 + \frac{\partial \eta_2}{\partial x_2} \xi_1 \xi_2 \eta_1 + 2 \frac{\partial \xi_2}{\partial x_2} \xi_2 \eta_2^2 + 2 \frac{\partial \eta_2}{\partial x_2} \xi_2^2 \eta_2 \right) \\
& + c_1 \left(\frac{\partial \xi_1}{\partial x_1} \xi_1^2 \xi_2 + \frac{\partial \xi_2}{\partial x_1} \xi_1^3 + 2 \frac{\partial \xi_2}{\partial x_1} \xi_1 \xi_2^2 + \frac{\partial \xi_1}{\partial x_2} \xi_1 \xi_2^2 \right. \\
& + \frac{\partial \xi_2}{\partial x_2} \xi_1^2 \xi_2 + 2 \frac{\partial \xi_2}{\partial x_2} \xi_2^3 + c_2 \left(2 \frac{\partial \eta_2}{\partial x_2} \eta_2^3 + 2 \frac{\partial \eta_2}{\partial x_1} \eta_1 \eta_2^2 \right. \\
& \left. + \frac{\partial \eta_2}{\partial x_1} \eta_1^3 + \frac{\partial \eta_1}{\partial x_2} \eta_1 \eta_2^2 + \frac{\partial \eta_2}{\partial x_2} \eta_1^2 \eta_2 + \frac{\partial \eta_1}{\partial x_1} \eta_1^2 \eta_2 \right) \\
& + 2g^{12} \Gamma_{12}^2 + g^{11} \Gamma_{11}^2 + g^{22} \Gamma_{22}^2
\end{aligned}$$

Let $\tilde{f} = f e_1 \in \Gamma(E)$. Then the differential operator H of order 2 defined by $H(\tilde{f}) =$

$$- \left(\text{trace}(\mathbf{TH}) + \frac{2}{\pi} \nabla f^T \int_{\alpha=0}^{\pi} \mathbf{J}_{\sqrt{\mathbf{T}a_\alpha}} \sqrt{\mathbf{T}a_\alpha} d\alpha \right) e_1 \quad (24)$$

may be written as $H = \Delta^E + F$ for some zero-order operator F .

Proof The second part of (24) is given by the term $\text{trace}(\mathbf{TH})$, and determines the metric g on Ω . It corresponds to the second-order part of the generalized Laplacian (22). We deduce that the metric is given by $1/(c_1 c_2 (\xi_1 \eta_2 - \eta_1 \xi_2)^2) \times$

$$\begin{pmatrix} c_1 \xi_2^2 + c_2 \eta_2^2 & -(c_1 \xi_1 \xi_2 + c_2 \eta_1 \eta_2) \\ -(c_1 \xi_1 \xi_2 + c_2 \eta_1 \eta_2) & c_1 \xi_1^2 + c_2 \eta_1^2 \end{pmatrix}$$

in the coordinates system (x_1, x_2) .

The first-order part is given by the term

$$\frac{2}{\pi} \nabla f^T \int_{\alpha=0}^{\pi} \mathbf{J}_{\sqrt{\mathbf{T}a_\alpha}} \sqrt{\mathbf{T}a_\alpha} d\alpha \quad (25)$$

and determines the connection ∇^E on E .

Computing (25), we obtain

$$\frac{\partial f}{\partial x_1} (a - 2g^{12} \Gamma_{12}^1 - g^{11} \Gamma_{11}^1 - g^{22} \Gamma_{22}^1)$$

$$+ \frac{\partial f}{\partial x_2} (b - 2g^{12} \Gamma_{12}^1 - g^{11} \Gamma_{11}^1 - g^{22} \Gamma_{22}^1)$$

Under the identification with the first-order of Δ^E , we obtain the system

$$\begin{cases} -2g^{11} \Upsilon_1 - 2g^{12} \Upsilon_2 = -a \\ -2g^{12} \Upsilon_1 - 2g^{22} \Upsilon_2 = -b \end{cases}$$

whose solution is

$$\begin{pmatrix} \Upsilon_1 \\ \Upsilon_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

At last, we deduce that the zero-order operator F such that $H = \Delta^E + F$ is given by

$$\begin{aligned}
F = & \left(g^{11} \frac{\partial \Upsilon_1}{\partial x_1} + g^{11} (\Upsilon_1)^2 - g^{11} \Gamma_{11}^1 \Upsilon_1 - g^{11} \Gamma_{11}^2 \Upsilon_2 + g^{12} \frac{\partial \Upsilon_2}{\partial x_1} \right. \\
& + g^{12} \frac{\partial \Upsilon_1}{\partial x_2} + 2g^{12} \Upsilon_1 \Upsilon_2 - 2g^{12} \Gamma_{12}^1 \Upsilon_1 - 2g^{12} \Gamma_{12}^2 \Upsilon_2 + g^{22} \frac{\partial \Upsilon_2}{\partial x_2} \\
& \left. + g^{22} (\Upsilon_2)^2 - g^{22} \Gamma_{22}^1 \Upsilon_1 - g^{22} \Gamma_{22}^2 \Upsilon_2 \right) Id
\end{aligned}$$

where Id is the Identity operator on E . $\square$

Then Proposition 4 appears as a Corollary of Proposition 12, dealing with the extension of H to a generalized Laplacian on a vector bundle of rank 3.

Proof (of Proposition 4) Following the notations of Proposition 3, we have $H = \Delta^E + F$ where F is the zero order operator on E defined by

$$\begin{aligned}
& \left(g^{11} \frac{\partial \Upsilon_1}{\partial x_1} + g^{11} (\Upsilon_1)^2 - g^{11} \Gamma_{11}^1 \Upsilon_1 - g^{11} \Gamma_{11}^2 \Upsilon_2 + g^{12} \frac{\partial \Upsilon_2}{\partial x_1} \right. \\
& + g^{12} \frac{\partial \Upsilon_1}{\partial x_2} + 2g^{12} \Upsilon_1 \Upsilon_2 - 2g^{12} \Gamma_{12}^1 \Upsilon_1 - 2g^{12} \Gamma_{12}^2 \Upsilon_2 + g^{22} \frac{\partial \Upsilon_2}{\partial x_2} \\
& \left. + g^{22} (\Upsilon_2)^2 - g^{22} \Gamma_{22}^1 \Upsilon_1 - g^{22} \Gamma_{22}^2 \Upsilon_2 \right) Id
\end{aligned}$$

where Id is the identity operator on E . $\square$

E. Divergence-based PDEs

Let Ω be the domain of an image. Let f be a smooth function defined on Ω . We consider the differential operator of order 2

$$f \mapsto \text{div}(D \nabla f) \quad (26)$$

where $D = (d_{ij})$ is a field of symmetric definite positives matrices on Ω , and div resp. ∇ stands for the Euclidean divergence resp. gradient operator.

In the following proposition, we show that (26) may be viewed as a generalized Laplacian on a vector bundle of rank 1 over Ω .

Proposition 13 Let (Ω, g) be the Riemannian manifold of dimension 2 with g given by

$$\frac{1}{d_{11}d_{22} - (d_{12})^2} \begin{pmatrix} d_{22} & -d_{12} \\ -d_{12} & d_{11} \end{pmatrix}$$

in the cartesian coordinates system (x_1, x_2) on Ω .

Let E be a trivial vector bundle of rank 1 over (Ω, g) of global frame e_1 , endowed with the connection ∇^E defined by $\nabla_{\partial_{x_1}}^E e_1 = \Upsilon_1 e_1$ and $\nabla_{\partial_{x_2}}^E e_1 = \Upsilon_2 e_1$ where

$$\Upsilon_1 = \frac{1}{2}(g_{11}a + g_{12}b) \quad \Upsilon_2 = \frac{1}{2}(g_{12}a + g_{22}b)$$

and

$$\begin{aligned} a &= \frac{\partial d_{11}}{\partial x_1} + \frac{\partial d_{12}}{\partial x_2} + 2g^{12} \Gamma_{12}^1 + g^{11} \Gamma_{11}^1 + g^{22} \Gamma_{22}^1 \\ b &= \frac{\partial d_{12}}{\partial x_1} + \frac{\partial d_{22}}{\partial x_2} + 2g^{12} \Gamma_{12}^2 + g^{11} \Gamma_{11}^2 + g^{22} \Gamma_{22}^2 \end{aligned}$$

Let $\tilde{f} = f e_1 \in \Gamma(E)$. The differential operator H defined by

$$H(\tilde{f}) = -\text{div}(D\nabla f) e_1 \quad (27)$$

may be written as $H = \Delta^E + F$ for some zero-order operator F .

Proof The proof follows the same arguments as the proof of Propositions 11 and 12. Indeed, we developp (27) and we obtain

$$\begin{aligned} & \left[d_{11} \frac{\partial^2 f}{\partial x_1^2} + 2d_{12} \frac{\partial^2 f}{\partial x_1 \partial x_2} + d_{22} \frac{\partial^2 f}{\partial x_2^2} + \frac{\partial f}{\partial x_1} \left(\frac{\partial d_{11}}{\partial x_1} + \frac{\partial d_{12}}{\partial x_2} \right) \right. \\ & \left. + \frac{\partial f}{\partial x_2} \left(\frac{\partial d_{12}}{\partial x_1} + \frac{\partial d_{22}}{\partial x_2} \right) \right] e_1 \end{aligned}$$

From the second order part, we obtain

$$g^{-1} = \begin{pmatrix} d_{11} & d_{12} \\ d_{12} & d_{22} \end{pmatrix}$$

in the coordinates system (x_1, x_2) .

Then, from the first order part, we have

$$\begin{aligned} -2g^{11}\Upsilon_1 - 2g^{12}\Upsilon_2 + 2g^{12}\Gamma_{12}^1 + g^{11}\Gamma_{11}^1 + g^{22}\Gamma_{22}^1 &= \frac{\partial d_{11}}{\partial x_1} + \frac{\partial d_{12}}{\partial x_2} \\ -2g^{12}\Upsilon_1 - 2g^{22}\Upsilon_2 + 2g^{12}\Gamma_{12}^2 + g^{11}\Gamma_{11}^2 + g^{22}\Gamma_{22}^2 &= \frac{\partial d_{12}}{\partial x_1} + \frac{\partial d_{22}}{\partial x_2} \end{aligned}$$

We put

$$a = -\frac{\partial d_{11}}{\partial x_1} - \frac{\partial d_{12}}{\partial x_2} + 2g^{12}\Gamma_{12}^1 + g^{11}\Gamma_{11}^1 + g^{22}\Gamma_{22}^1$$

$$b = -\frac{\partial d_{12}}{\partial x_1} - \frac{\partial d_{22}}{\partial x_2} + 2g^{12}\Gamma_{12}^2 + g^{11}\Gamma_{11}^2 + g^{22}\Gamma_{22}^2$$

Then we obtain the system

$$\begin{cases} -2g^{11}\Upsilon_1 - 2g^{12}\Upsilon_2 = -a \\ -2g^{12}\Upsilon_1 - 2g^{22}\Upsilon_2 = -b \end{cases}$$

whose solution is

$$\begin{pmatrix} \Upsilon_1 \\ \Upsilon_2 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} g_{11} & g_{12} \\ g_{12} & g_{22} \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

At last, as there is no zero-order part in (27), we deduce that the zero-order operator F such that $H = \Delta^E + F$ is

$$\begin{aligned} & \left(g^{11} \frac{\partial \Upsilon_1}{\partial x_1} + g^{11}(\Upsilon_1)^2 - g^{11}\Gamma_{11}^1\Upsilon_1 - g^{11}\Gamma_{11}^2\Upsilon_2 + g^{12} \frac{\partial \Upsilon_2}{\partial x_1} \right. \\ & + 2g^{12}\Upsilon_1\Upsilon_2 - 2g^{12}\Gamma_{12}^1\Upsilon_1 - 2g^{12}\Gamma_{12}^2\Upsilon_2 + g^{21} \frac{\partial \Upsilon_1}{\partial x_2} \\ & \left. - g^{22} \frac{\partial \Upsilon_2}{\partial x_2} + g^{22}(\Upsilon_2)^2 - g^{22}\Gamma_{22}^1\Upsilon_1 - g^{22}\Gamma_{22}^2\Upsilon_2 \right) Id \end{aligned}$$

where Id is the Identity operator on E .

Proof (of Proposition 5) Following the notations of Proposition 5, we have $H = \Delta^E + F$ where F is the zero order operator on E defined by

$$\begin{aligned} & \left(g^{11} \frac{\partial \Upsilon_1}{\partial x_1} + g^{11}(\Upsilon_1)^2 - g^{11}\Gamma_{11}^1\Upsilon_1 - g^{11}\Gamma_{11}^2\Upsilon_2 + g^{12} \frac{\partial \Upsilon_2}{\partial x_1} \right. \\ & + 2g^{12}\Upsilon_1\Upsilon_2 - 2g^{12}\Gamma_{12}^1\Upsilon_1 - 2g^{12}\Gamma_{12}^2\Upsilon_2 + g^{21} \frac{\partial \Upsilon_1}{\partial x_2} \\ & \left. - g^{22} \frac{\partial \Upsilon_2}{\partial x_2} + g^{22}(\Upsilon_2)^2 - g^{22}\Gamma_{22}^1\Upsilon_1 - g^{22}\Gamma_{22}^2\Upsilon_2 \right) Id \end{aligned}$$

where Id is the identity operator on E . $\square$

F. Parallel transport map and the kernel $K_t^0(x, y, H)$ of standard methods

Proof (Proposition 6) The parallel transport of Y_0 along γ is the solution $Y(t) = Y^1(t)e_1(\gamma(t)) + Y^2(t)e_2(\gamma(t)) + Y^3(t)e_3(\gamma(t))$ of the differential equation

$$\begin{cases} \nabla_{\dot{\gamma}}^E Y(t) = 0 \\ Y(0) = Y_0 \end{cases} \quad (28)$$

$$\nabla_{\dot{\gamma}}^E Y(t) = \nabla_{\dot{\gamma}}^E (\sum_{i=1}^3 Y^i e_i)(t)$$

$$\begin{aligned}
&= \sum_{i=1}^3 \sum_{j=1}^2 \left[\frac{\partial Y^i}{\partial t}(t) e_i(\gamma(t)) + Y^i(t) \left(\dot{\gamma}_j(t) \nabla_{\partial_{x_j}}^E e_i(\gamma(t)) \right) \right] \\
&= \sum_{i=1}^3 \sum_{j=1}^2 \left[\frac{\partial Y^i}{\partial t}(t) e_i(\gamma(t)) + Y^i(t) \left(\dot{\gamma}_j(t) \mathcal{I}_j(t) e_i(\gamma(t)) \right) \right]
\end{aligned}$$

We obtain three ordinary differential equation

$$\begin{cases} \frac{\partial Y^i}{\partial t}(t) = -(\dot{\gamma}_1(t) \mathcal{I}_1(t) + \dot{\gamma}_2(t) \mathcal{I}_2(t)) Y^i(t) \\ Y^i(0) = Y_0^i \end{cases}$$

from which we deduce that the solution of (28) is

$$\sum_{i=1}^3 Y_0^i \exp \left(- \int_0^t \dot{\gamma}_1(s) \mathcal{I}_1(s) + \dot{\gamma}_2(s) \mathcal{I}_2(s) ds \right) e_i(\gamma(t))$$

□

Proof (of Corollary 4) Let $x \in \Omega$. From the fact that the functions c_1, c_2 and the components $\xi_1, \xi_2, \eta_1, \eta_2$ of the mobile frame are constant on U_x , the symbols L_{ij}^k of the Levi-Cevita connection on (TU_x, g) , and the symbols $\mathcal{I}_{ij}^k$ of the connection ∇^E on E are zero. It follows that the kernel $K_t^0(x, y, H)$ is given on U_x by $(1/4\pi t) \times$

$$\exp \left(- \frac{g((y_1 - x_1, y_2 - x_2), (y_1 - x_1, y_2 - x_2))}{4t} \right) Id(x, y)$$

where $Id(x, y): E_y \longrightarrow E_x$ is the Identity operator on the components.

Indeed, a simple computation leads to

$$\exp_{(x_1, x_2)}(\mathbf{y}_1, \mathbf{y}_2) = (x_1 + \mathbf{y}_1, x_2 + \mathbf{y}_2)$$

where $\exp_x$ is the exponential map of X at x and $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2)$ are the normal coordinates of the point y around x . Then from the fact that $d(x, \exp_x \mathbf{y}) = \sqrt{g_x(\mathbf{y}, \mathbf{y})}$ on a Riemannian manifold of metric g , we have

$$d(x, y) = \sqrt{g_x(y_1 - x_1, y_2 - x_2), (y_1 - x_1, y_2 - x_2)}$$

□

G. Generalized Laplacians of parameter α

Proof (of Proposition 8) In the frames (e_1, e_2, e_3) of E and $(\partial/\partial x_1, \partial/\partial x_2)$ of TX , the symbols Υ_{ij}^k of ∇^E are given by

$$\Upsilon_{ij}^k = \alpha \delta_{kj}$$

Computing the equation of transport parallel map with respect to ∇^E

$$\begin{cases} \nabla_{\dot{\gamma}}^E Y(t) = 0 \\ Y(0) = Y_0 \end{cases}$$

we obtain 3 ordinary differential equations

$$\begin{cases} \frac{\partial Y^i}{\partial t}(t) = -\alpha(t) (\dot{\gamma}_1(t) + \dot{\gamma}_2(t)) Y^i(t) \\ Y^i(0) = Y_0^i \end{cases}$$

for $i = 1, 2, 3$, from which follows that the parallel transport of Y_0 along γ is

$$Y(t) = \sum_{i=1}^3 \exp \left(- \int_0^t \alpha(s) (\dot{\gamma}_1(s) + \dot{\gamma}_2(s)) ds \right) Y_0^i e_i(\gamma(t))$$

H. Generalized Laplacians of parameters (h, α, v)

Proof (of Proposition 9) In the global frames (e_1, e_2, e_3) of E and $(\partial/\partial x_1, \partial/\partial x_2)$ of TX , the symbol Υ_{ij}^k of the connection ∇^E is the e_k component of

$$\begin{aligned} \omega(\partial/\partial x_i \otimes e_j) &= (dx_1 + dx_2)(\partial/\partial x_i) P_v^\perp e_j \\ &= P_v^\perp e_j \end{aligned}$$

Hence we have

$$\begin{aligned} \Upsilon_{i1}^1 &= \alpha (v_1^2 h_{11} + v_1 v_2 h_{12} + v_1 v_3 h_{13}) / \|v\|^2 \\ \Upsilon_{i1}^2 &= \alpha (v_1 v_2 h_{11} + v_2^2 h_{12} + v_2 v_3 h_{13}) / \|v\|^2 \\ \Upsilon_{i1}^3 &= \alpha (v_1 v_3 h_{11} + v_2 v_3 h_{12} + v_3^2 h_{13}) / \|v\|^2 \end{aligned}$$

$$\begin{aligned} \Upsilon_{i2}^1 &= \alpha (v_1^2 h_{12} + v_1 v_2 h_{22} + v_1 v_3 h_{23}) / \|v\|^2 \\ \Upsilon_{i2}^2 &= \alpha (v_1 v_2 h_{12} + v_2^2 h_{22} + v_2 v_3 h_{23}) / \|v\|^2 \\ \Upsilon_{i2}^3 &= \alpha (v_1 v_3 h_{12} + v_2 v_3 h_{22} + v_3^2 h_{23}) / \|v\|^2 \end{aligned}$$

$$\begin{aligned} \Upsilon_{i3}^1 &= \alpha (v_1^2 h_{13} + v_1 v_2 h_{23} + v_1 v_3 h_{33}) / \|v\|^2 \\ \Upsilon_{i3}^2 &= \alpha (v_1 v_2 h_{13} + v_2^2 h_{23} + v_2 v_3 h_{33}) / \|v\|^2 \\ \Upsilon_{i3}^3 &= \alpha (v_1 v_3 h_{13} + v_2 v_3 h_{23} + v_3^2 h_{33}) / \|v\|^2 \end{aligned}$$

Then computing the equation of transport parallel map with respect to ∇^E

$$\begin{cases} \nabla_{\dot{\gamma}}^E Y(t) = 0 \\ Y(0) = Y_0 \end{cases}$$

we obtain the differential equation

$$\begin{pmatrix} \frac{\partial Y^1}{\partial t} \\ \frac{\partial Y^2}{\partial t} \\ \frac{\partial Y^3}{\partial t} \end{pmatrix} = -\alpha \frac{(\dot{\gamma}_1 + \dot{\gamma}_2)}{\|v\|^2} B \begin{pmatrix} Y^1 \\ Y^2 \\ Y^3 \end{pmatrix} \quad (29)$$

of initial condition

$$(Y^1(0), Y^2(0), Y^3(0)) = (Y_0^1, Y_0^2, Y_0^3),$$

with $B = (B)_{ij}$ a 3×3 matrix field given by

$$\begin{aligned} B_{11} &= v_1^2 h_{11} + v_1 v_2 h_{12} + v_1 v_3 h_{13} \\ B_{12} &= v_1^2 h_{12} + v_1 v_2 h_{22} + v_1 v_3 h_{23} \\ B_{13} &= v_1^2 h_{13} + v_1 v_2 h_{23} + v_1 v_3 h_{33} \end{aligned}$$

$$\begin{aligned} B_{21} &= v_1 v_2 h_{11} + v_2^2 h_{12} + v_2 v_3 h_{13} \\ B_{22} &= v_1 v_2 h_{12} + v_2^2 h_{22} + v_2 v_3 h_{23} \\ B_{23} &= v_1 v_2 h_{13} + v_2^2 h_{23} + v_2 v_3 h_{33} \end{aligned}$$

$$\begin{aligned} B_{31} &= v_1 v_3 h_{11} + v_2 v_3 h_{12} + v_3^2 h_{13} \\ B_{32} &= v_1 v_3 h_{12} + v_2 v_3 h_{22} + v_3^2 h_{23} \\ B_{33} &= v_1 v_3 h_{13} + v_2 v_3 h_{23} + v_3^2 h_{33} \end{aligned}$$

It proves *ii*.

Let us suppose that the coefficients h_{ij} of the metric h and the components (v_1, v_2, v_3) of v are constant in the global frame (e_1, e_2, e_3) . Let (v, w, z) be an orthonormal global frame of E with respect to h , where w and z are of constant components (w_1, w_2, w_3) and (z_1, z_2, z_3) in the frame (e_1, e_2, e_3) .

In the frames (v, w, z) of E and $(\partial/\partial x_1, \partial/\partial x_2)$ of TX , the symbols $\Upsilon_{ij}^{k'}$ of the connection ∇^E are

$$\begin{cases} \Upsilon_{ij}^{k'} = \alpha & \text{if } k = j = 1 \\ \Upsilon_{ij}^{k'} = 0 & \text{otherwise} \end{cases}$$

Computing the equation of the transport parallel map of Y_0 along γ in the frames (v, w, z) and $(\partial/\partial x_1, \partial/\partial x_2)$ with respect to ∇^E , we obtain the following differential equation in $\mathbb{R}^3$

$$\begin{pmatrix} \frac{\partial X^1}{\partial t} \\ \frac{\partial X^2}{\partial t} \\ \frac{\partial X^3}{\partial t} \end{pmatrix} = \begin{pmatrix} -\alpha A & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X^1 \\ X^2 \\ X^3 \end{pmatrix}$$

where $A : t \rightarrow \dot{\gamma}_1(t) + \dot{\gamma}_2(t)$, of initial condition

$$\begin{pmatrix} X^1(0) \\ X^2(0) \\ X^3(0) \end{pmatrix} = P^{-1} \begin{pmatrix} Y_0^1 \\ Y_0^2 \\ Y_0^3 \end{pmatrix}$$

where P is the change frame field from (e_1, e_2, e_3) to (v, w, z) .

Then, in the frame (v, w, z) , the solution of the parallel transport of Y_0 along γ is given by

$$\begin{pmatrix} X^1(t) \\ X^2(t) \\ X^3(t) \end{pmatrix} = C(t) \begin{pmatrix} X^1(0) \\ X^2(0) \\ X^3(0) \end{pmatrix}$$

where $C(t)$ is the matrix

$$\begin{pmatrix} \exp\left(-\int_0^t \alpha(s) A(s) ds\right) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence, the solution is

$$\begin{pmatrix} Y^1(t) \\ Y^2(t) \\ Y^3(t) \end{pmatrix} = P C(t) P^{-1} \begin{pmatrix} Y_0^1 \\ Y_0^2 \\ Y_0^3 \end{pmatrix}$$

in the frame (e_1, e_2, e_3) . It proves *i*. $\square$

I. Generalized Laplacians of parameters $(h, \alpha, \mathcal{P}_{v,w})$

Proof (of Proposition 10) In the frames (e_1, e_2, e_3) of E and $(\partial/\partial x_1, \partial/\partial x_2)$ of TX , the symbol Υ_{ij}^k of the connection ∇^E is the e_k component of

$$\begin{aligned} \omega(\partial/\partial x_i \otimes e_j) &= (dx_1 + \dots + dx_m)(\partial/\partial x_i) P_{\mathcal{P}}^\perp e_j \\ &= P_{\mathcal{P}_{v,w}}^\perp e_j \end{aligned}$$

Hence, putting

$$\begin{aligned} B_{11} &= -[2h_{12}(v_1 w_2 - v_2 w_1) + 2h_{13}(v_1 w_3 - v_3 w_1) + 2h_{23}(v_2 w_3 - w_2 v_3)](-h_{12}(v_1 w_2 - v_2 w_1) - h_{13}(v_1 w_3 - v_3 w_1) + [2h_{22}(v_1 w_2 - v_2 w_1)](h_{11}(v_1 w_2 - v_2 w_1) - h_{13}(v_2 w_3 - w_2 v_3)) + [4h_{23}(v_1 w_2 - w_1 v_2) + 2h_{33}(v_1 w_3 - v_3 w_1)] \times (h_{11}(v_1 w_3 - v_3 w_1) + h_{12}(v_2 w_3 - w_2 v_3)) \end{aligned}$$

$$\begin{aligned} B_{21} &= [-2h_{11}(v_1 w_2 - v_2 w_1)](h_{12}(v_1 w_2 - v_2 w_1) + h_{13}(v_1 w_3 - v_3 w_1)) + [2h_{12}(v_1 w_2 - v_2 w_1) - 2h_{13}(v_1 w_3 - v_3 w_1) - 2h_{23}(v_2 w_3 - w_2 v_3)](h_{11}(v_1 w_2 - v_2 w_1) - h_{13}(v_2 w_3 - w_2 v_3)) + [4h_{13}(v_1 w_2 - w_1 v_2) - 2h_{33}(v_2 w_3 - w_2 v_3)](h_{11}(v_1 w_3 - \end{aligned}$$

$$v_3w_1) + h_{12}(v_2w_3 - w_2v_3))$$

$$B_{31} = [-2h_{11}(v_1w_3 - v_3w_1)](h_{12}(v_1w_2 - v_2w_1) + h_{13}(v_1w_3 - v_3w_1)) + [4h_{12}(v_1w_3 - v_3w_1) + 2h_{22}(v_2w_3 - w_2v_3)] \times (h_{11}(v_1w_2 - v_2w_1) - h_{13}(v_2w_3 - w_2v_3)) + [2h_{13}(v_1w_3 - v_3w_1) + 2h_{23}(v_2w_3 - w_2v_3) - 2h_{12}(v_1w_2 - w_1v_2)] \times (h_{11}(v_1w_3 - v_3w_1) + h_{12}(v_2w_3 - w_2v_3))$$

$$B_{12} = -[2h_{12}(v_1w_2 - v_2w_1) + 2h_{13}(v_1w_3 - v_3w_1) + 2h_{23}(v_2w_3 - w_2v_3)](-h_{22}(v_1w_2 - v_2w_1) - h_{23}(v_1w_3 - v_3w_1)) + [2h_{22}(v_1w_2 - v_2w_1)](h_{12}(v_1w_2 - v_2w_1) - h_{23}(v_2w_3 - w_2v_3)) + [4h_{23}(v_1w_2 - w_1v_2) + 2h_{33}(v_1w_3 - v_3w_1)] \times (h_{12}(v_1w_3 - v_3w_1) + h_{22}(v_2w_3 - w_2v_3))$$

$$B_{22} = [-2h_{11}(v_1w_2 - v_2w_1)](h_{22}(v_1w_2 - v_2w_1) + h_{23}(v_1w_3 - v_3w_1)) + [2h_{12}(v_1w_2 - v_2w_1) - 2h_{13}(v_1w_3 - v_3w_1) - 2h_{23}(v_2w_3 - w_2v_3)](h_{12}(v_1w_2 - v_2w_1) - h_{23}(v_2w_3 - w_2v_3)) + [4h_{13}(v_1w_2 - w_1v_2) - 2h_{33}(v_2w_3 - w_2v_3)](h_{12}(v_1w_3 - v_3w_1) + h_{22}(v_2w_3 - w_2v_3))$$

$$B_{32} = [-2h_{11}(v_1w_3 - v_3w_1)](h_{22}(v_1w_2 - v_2w_1) + h_{23}(v_1w_3 - v_3w_1)) + [4h_{12}(v_1w_3 - v_3w_1) + 2h_{22}(v_2w_3 - w_2v_3)] \times (h_{12}(v_1w_2 - v_2w_1) - h_{23}(v_2w_3 - w_2v_3)) + [2h_{13}(v_1w_3 - v_3w_1) + 2h_{23}(v_2w_3 - w_2v_3) - 2h_{12}(v_1w_2 - w_1v_2)] \times (h_{12}(v_1w_3 - v_3w_1) + h_{22}(v_2w_3 - w_2v_3))$$

$$B_{13} = -[2h_{12}(v_1w_2 - v_2w_1) + 2h_{13}(v_1w_3 - v_3w_1) + 2h_{23}(v_2w_3 - w_2v_3)](-h_{23}(v_1w_2 - v_2w_1) - h_{33}(v_1w_3 - v_3w_1)) + [2h_{22}(v_1w_2 - v_2w_1)](h_{13}(v_1w_2 - v_2w_1) - h_{33}(v_2w_3 - w_2v_3)) + [4h_{23}(v_1w_2 - w_1v_2) + 2h_{33}(v_1w_3 - v_3w_1)] \times (h_{13}(v_1w_3 - v_3w_1) + h_{23}(v_2w_3 - w_2v_3))$$

$$B_{23} = [-2h_{11}(v_1w_2 - v_2w_1)](h_{23}(v_1w_2 - v_2w_1) + h_{33}(v_1w_3 - v_3w_1)) + [2h_{12}(v_1w_2 - v_2w_1) - 2h_{13}(v_1w_3 - v_3w_1) - 2h_{23}(v_2w_3 - w_2v_3)](h_{13}(v_1w_2 - v_2w_1) - h_{33}(v_2w_3 - w_2v_3)) + [4h_{13}(v_1w_2 - w_1v_2) - 2h_{33}(v_2w_3 - w_2v_3)](h_{13}(v_1w_3 - v_3w_1) + h_{23}(v_2w_3 - w_2v_3))$$

$$B_{33} = [-2h_{11}(v_1w_3 - v_3w_1)](h_{23}(v_1w_2 - v_2w_1) + h_{33}(v_1w_3 - v_3w_1)) + [4h_{12}(v_1w_3 - v_3w_1) + 2h_{22}(v_2w_3 - w_2v_3)] \times (h_{13}(v_1w_2 - v_2w_1) - h_{33}(v_2w_3 - w_2v_3)) + [2h_{13}(v_1w_3 - v_3w_1) + 2h_{23}(v_2w_3 - w_2v_3) - 2h_{12}(v_1w_2 - w_1v_2)](h_{13}(v_1w_3 - v_3w_1) + h_{23}(v_2w_3 - w_2v_3))$$

we have $\Upsilon_{ij}^k = \alpha B_{kj} / \|v \wedge w\|^2$. Then, computing the parallel transport of Y_0 along γ , we obtain

$$\begin{pmatrix} \frac{\partial Y^1}{\partial t} \\ \frac{\partial Y^2}{\partial t} \\ \frac{\partial Y^3}{\partial t} \end{pmatrix} = -(\dot{\gamma}_1 + \dot{\gamma}_2) \begin{pmatrix} \Upsilon_{11}^1 & \Upsilon_{12}^1 & \Upsilon_{13}^1 \\ \Upsilon_{11}^2 & \Upsilon_{12}^2 & \Upsilon_{13}^2 \\ \Upsilon_{11}^3 & \Upsilon_{12}^3 & \Upsilon_{13}^3 \end{pmatrix} \begin{pmatrix} Y^1 \\ Y^2 \\ Y^3 \end{pmatrix}$$

which proves *ii*.

Let us suppose that the coefficients h_{ij} of the metric h and the components (v_1, v_2, v_3) and (w_1, w_2, w_3) of v and w are constant in the global frame (e_1, e_2, e_3) . Let $z \in \Gamma(E)$ of constant components (z_1, z_2, z_3) such that z is orthogonal to v and w with respect to the metric h .

In the frames (v, w, z) of E and $(\partial/\partial x_1, \partial/\partial x_2)$ of TX , the symbols $\Upsilon_{ij}^{k'}$ of the connection ∇^E are

$$\begin{cases} \Upsilon_{ij}^{k'} = \alpha & \text{if } k = j \in \{1, 2\} \\ \Upsilon_{ij}^{k'} = 0 & \text{otherwise} \end{cases}$$

Computing the equation of the transport parallel map of Y_0 along γ in the frames (v, w, z) and $(\partial/\partial x_1, \partial/\partial x_2)$ with respect to ∇^E , we obtain the following differential equation in $\mathbb{R}^3$

$$\begin{pmatrix} \frac{\partial X^1}{\partial t} \\ \frac{\partial X^2}{\partial t} \\ \frac{\partial X^3}{\partial t} \end{pmatrix} = \begin{pmatrix} -\alpha A & 0 & 0 \\ 0 & -\alpha A & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} X^1 \\ X^2 \\ X^3 \end{pmatrix}$$

where $A : t \rightarrow \dot{\gamma}_1(t) + \dot{\gamma}_2(t)$, of initial condition

$$\begin{pmatrix} X^1(0) \\ X^2(0) \\ X^3(0) \end{pmatrix} = P^{-1} \begin{pmatrix} Y_0^1 \\ Y_0^2 \\ Y_0^3 \end{pmatrix}$$

where P is the change frame field from (e_1, e_2, e_3) to (v, w, z) .

Then, in the frame (v, w, z) , the solution is given by

$$\begin{pmatrix} Z^1(t) \\ Z^2(t) \\ Z^3(t) \end{pmatrix} = C(t) \begin{pmatrix} Z^1(0) \\ Z^2(0) \\ Z^3(0) \end{pmatrix}$$

where $C(t)$ is the matrix

$$\begin{pmatrix} \exp\left(-\int_0^t \alpha(s)A(s) ds\right) & 0 & 0 \\ 0 & \exp\left(-\int_0^t \alpha(s)A(s) ds\right) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence, the solution is

$$\begin{pmatrix} Y^1(t) \\ Y^2(t) \\ Y^3(t) \end{pmatrix} = P C(t) P^{-1} \begin{pmatrix} Y_0^1 \\ Y_0^2 \\ Y_0^3 \end{pmatrix}$$

in the frame (e_1, e_2, e_3) , which proves *i*. $\square$

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